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*Research article*

# An iterative approach for the solution of fully fuzzy linear fractional programming problems via fuzzy multi-objective optimization

**Sema Akin Bas**[1,*] **and Beyza Ahlatcioglu Ozkok**[2]

[1] Department of Mathematics,Yildiz Technical University, Istanbul, 34220, Türkiye
[2] Department of Business Administration,Yildiz Technical University, Istanbul, 34220, Türkiye

* **Correspondence:** Email: akins@yildiz.edu.tr.

**Abstract:** The primary goal of optimization theory is to formulate solutions for real-life challenges that play a fundamental role in our daily lives. One of the most significant issues within this framework is the Linear Fractional Programming Problem (LFrPP). In practical situations, such as production planning and financial decision-making, it is often feasible to express objectives as a ratio of two distinct objectives. To enhance the efficacy of these problems in representing real-world scenarios, it is reasonable to utilize fuzzy sets for expressing variables and parameters. In this research, we have worked on the Fully Fuzzy Linear Fractional Linear Programming Problem (FFLFrLPP). In our approach to problem-solving, we simplified the intricate structure of the FFLFrLPP into a crisp Linear Programming Problem (LPP) while accommodating the inherent fuzziness. Notably, unlike literature, our proposed technique avoided variable transformation, which is highly competitive in addressing fuzzy-based problems. Our methodology also distinguishes itself from the literature in preserving fuzziness throughout the process, from problem formulation to solution. In this study, we conducted a rigorous evaluation of our proposed methodology by applying it to a selection of numerical examples and production problems sourced from the existing literature. Our findings revealed significant improvements in performance when compared to established solution approaches. Additionally, we presented comprehensive statistical analyses showcasing the robustness and effectiveness of our algorithms when addressing large-scale problem instances. This research underscores the innovative contributions of our methods to the field, further advancing the state-of-the-art in problem-solving techniques.

**Keywords:** continuity; fully fuzzy programming; iterative optimization; linear fractional programming; multiobjective optimization
**Mathematics Subject Classification:** 90C05, 90C06, 90C32, 90C70

## 1. Introduction

The procedures that benefit the attainment of the optimal solution for LFrPPs constitute one of the important areas of study within optimization theory. In this context, there can be various interpretations of the objective function, which is expressed as the ratio of individual objectives. Today, effective decision-making in inherently complex and uncertain environments has become crucial for numerous business and engineering applications. Uncertainties frequently encountered in decision-making processes may arise from the natural structure of the decision-making environment as well as external factors. These uncertainties complicate the optimization problems and make it challenging to address the limitations of traditional mathematical optimization models. However, addressing these challenges adds significant value.

The fuzzy set theory was developed to address situations characterized by uncertainty, offering the flexibility required in decision-making processes. It enables the construction of mathematical models in a more realistic manner. An FFLFrLPP emerges when the coefficients and variables within the LFrPP are represented using fuzzy numbers, harnessing the principles of fuzzy set theory. Pop and Stancu-Minasian [14] examined an LFrPP constructed with Triangular Fuzzy Numbers (TFNs) and employed variable transformation to address it within the context of crisp Multiple Objective Linear Programming Problems (MOLPPs). Stanojević and Stancu-Minasian [19] focused on the FFLFrPP, utilizing the concept of fuzzy inequalities and employing the technique from [4] as their solution strategy. Singh and Yadav [17] tackled a fuzzy LFrPP, where all parameters were defined using intuitionistic TFNs and variables as crisp, transforming it into a crisp multi-objective LFrPP, and using Charnes-Cooper transformation [4]. Chinnadurai and Muthukumar [5] implemented an algorithm that leverages alpha cuts for upper and lower bounds to solve the FFLFrPP. Ebrahimnejad et al. [7] presented an extended version of this aforementioned work. Nayak and Maharana [11] devised a unique linearization procedure for the fuzzy multi-objective LFrPP concept of the centroid of TFNs for defuzzification. Anukokila et al. [1] not only addressed a transportation problem expressed as LFrPP by converting it to a multi-objective LFrPP but also proposed lexicographic ordering to get a solution of the equivalent multi-objective LFrPP. Srinivasan [18] tackled a problem related to a wooden company as a fuzzy LFrPP with TFN parameters, solving it by transforming it into a crisp LFrPP and using centroid ranking and LU decomposition. Manesh et al. [9] researched the solution of multi-objective LFrPPs with uncertain data using parametric approaches and the robust optimization techniques. Bhatia et al. [3] conducted a detailed examination of the transportation problem modeled as FFLFrPP in literature and argued that the Fully Fuzzy Linear Programming Problem (FFLPP) obtained by the transformation approach of Charnes-Cooper in the context of uncertainty would not be equivalent to the investigated FFLFrPP. Thus, they proposed a Mehar technique for the optimal solution of the transportation problem modeled as FFLFrPP. In the study conducted by Das [6], they offered a solution method that utilized the transformation technique from [4] and lexicographic order for FFLFrPP. A transportation model was employed to illustrate the presented technique. Mitlif [10] proposed a solution methodology to address FFLFrPPs using Pentagonal fuzzy numbers and three distinct ranking functions. In a different approach, Stanojević and Stanojević [20] relied on Monte Carlo Simulation for solving FFLFrPP.

To the best of our knowledge, there has been no attempt to solve FFLFrPP iteratively without variable transformation. In this study, we present an iterative approach for solving FFLFrPP while retaining uncertainty. We believe this approach makes sense within a fuzzy framework, where variable conversion is often impractical.

The subsequent sections of this article are structured as follows: Definitions and preliminary information are provided in Section 2, the solution approach is described in Section 3, Section 4 includes several numerical examples and applications, and finally, Sections 5 and 6 compare our findings to existing methodologies and provide conclusions, respectively.

## 2. Preliminaries and problem definition

To begin with, certain fundamental concepts, operations, and definitions related to fuzzy numbers, which are essential for addressing the FFLFrPP using the proposed method, will be reintroduced. For more comprehensive information, please refer to [2].

Zadeh described a fuzzy set $\tilde{K}$ in $\mathcal{X}$, which has an association with a real number in the interval $[0, 1]$, with the value of $h_{\tilde{K}}(x)$ described by a characteristic function corresponding the "degree of membership" of $x$ in $\tilde{K}$ [21].

**Definition 1.** A triplet that has a membership function defined as below and expressed with $\tilde{T} = \left(t^l, t^m, t^u\right)$ is called a TFN.

$$h_{\tilde{T}}(x) = \begin{cases} \dfrac{x - t^l}{t^m - t^l}, & t^l \le x < t^m, \\ \dfrac{x - t^u}{t^m - t^u}, & t^m \le x \le t^u, \\ 0, & \text{others}, \end{cases}$$

where $t^l, t^m, t^u \in \mathbf{R}$, and $\tilde{T} \in F(\mathbf{R})$ (fuzzy set on real number).

**Definition 2.** Let $\tilde{T} = \left(t^l,\ t^m,\ t^u\right)$ and $\tilde{S} = \left(s^l,\ s^m,\ s^u\right)$ be TFNs

$(i)$ $\tau\tilde{T} = \left(\tau t^l,\ \tau t^m,\ \tau t^u\right)$, where $\tau \in \mathbf{R}^+$.

$(ii)$ $\tau\tilde{T} = \left(\tau t^u,\ \tau t^m,\ \tau t^l\right)$, where $\tau \in \mathbf{R}^-$.

$(iii)$ $\tilde{T} \oplus \tilde{S} = \left(t^l + s^l,\ t^m + s^m,\ t^u + s^u\right)$.

$(iv)$ $\tilde{T} \ominus \tilde{S} = \left(t^l - s^u,\ t^m - s^m,\ t^u - s^l\right)$.

$(v)$ $\tilde{T} \otimes \tilde{S} = \left(\min\{t^l s^l,\ t^l s^u,\ t^u s^l,\ t^u s^u\},\ t^m s^m,\ \max\{t^l s^l,\ t^l s^u,\ t^u s^l,\ t^u s^u\}\right)$.

$(vi)$ $\frac{\tilde{T}}{\tilde{S}} = \left(\min\left\{\frac{t^l}{s^l},\ \frac{t^l}{s^u},\ \frac{t^u}{s^l},\ \frac{t^u}{s^u}\right\},\ \frac{t^m}{s^m},\ \max\left\{\frac{t^l}{s^l},\ \frac{t^l}{s^u},\ \frac{t^u}{s^l},\ \frac{t^u}{s^u}\right\}\right)$ where $b^\gamma \neq 0$, $\forall\gamma = l, m, u$.

**Definition 3.** [8] Let $\tilde{T} = \left(t^l, t^m, t^u\right)$ be a TFN, $\mathcal{R}(\tilde{T}) = \frac{t^l + 2t^m + t^u}{4}$ is a ranking function mapped from $F(\mathbf{R})$ to $\mathbf{R}$ had a natural order.

**Definition 4.** [8] Assuming $\tilde{T}, \tilde{S}, \tilde{T}_j \in F(\mathbf{R})$ $\forall \mathbf{j}$, then

- $\mathcal{R}\left(\tilde{T} \oplus \tilde{S}\right) = \mathcal{R}\left(\tilde{T}\right) + \mathcal{R}\left(\tilde{S}\right)$.

- Generally, $\mathcal{R}\left(\sum_{j=1}^{n} \tilde{T}_j\right) = \sum_{j=1}^{n} \mathcal{R}\left(\tilde{T}_j\right)$.

**Definition 5.** [8] The formulation of FFLPPs, characterized by $m$ mixed constraints and $n$ fuzzy variables, can be described as follows:

$$\begin{aligned} &\max \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j, \\ &\sum_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j \approx \tilde{b}_p \text{ for } p = 1, \ldots, t, \\ &\sum_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \geq \tilde{b}_q \text{ for } q = t+1, \ldots, s, \\ &\sum_{j=1}^{n} \tilde{a}_{rj} \otimes \tilde{x}_j \leq \tilde{b}_r \text{ for } r = s+1, \ldots, m, \\ &\tilde{\mathbf{x}} \geq \tilde{0}. \end{aligned} \tag{2.1}$$

Using matrix notation,

$$\begin{aligned} &\text{maximize (or minimize) } \widetilde{A}^T \otimes \widetilde{X}, \\ &\widetilde{A} \otimes \widetilde{X} \preccurlyeq, \approx, \succcurlyeq \widetilde{b}, \end{aligned} \tag{2.2}$$

is obtained. Where $\widetilde{C}^T = [\widetilde{c}_j]_{1\times n}, \widetilde{X} = [\widetilde{x}_j]_{1\times n}, \widetilde{A} = [\widetilde{a}_{ij}]_{m\times n}, \widetilde{b} = [\widetilde{b}_i]_{m\times 1}$ and $\widetilde{a}_{ij}, \widetilde{c}_j, \widetilde{b}_i \in F(\mathbf{R})$ and $\widetilde{x}_j$ is a non-negative fuzzy number.

**Definition 6.** [8] The Optimal Fuzzy Solution (OFS) of the FFLPP (2.4) is denoted as $\widetilde{X}^* = [\widetilde{x}_j^*]_{n\times 1}$, if it satisfies the following criteria:

- $\widetilde{x}_j^*$ is a non-negative fuzzy number.
- $\widetilde{A} \otimes \widetilde{X}^* \prec, =, \succ \widetilde{b}$.

Furthermore, if there exists any non-negative $\widetilde{X} = [\widetilde{x}_j]_{n\times 1}$ such that $\widetilde{A} \otimes \widetilde{X}^* \prec, =, \succ \widetilde{b}$, then the following inequalities hold:

- In the case of a maximization problem: $\mathcal{R}(\widetilde{C}^T \otimes \widetilde{X}^*) \geq \mathcal{R}(\widetilde{C}^T \otimes \widetilde{X})$.
- In the case of a minimization problem: $\mathcal{R}(\widetilde{C}^T \otimes \widetilde{X}^*) \leq \mathcal{R}(\widetilde{C}^T \otimes \widetilde{X})$.

Kumar and Kaur [8] offered a methodology for determining the OFS of FFLPP with mixed constraints. In the method, parameters are performed by arbitrary TFNs, and decision variables are handled with non-negative TFNs. The infeasibility situation of the FFLPP cannot be studied by the provided technique. Therefore, Ozkok et al. [12] presented an expansion of the technique of Kumar and Kaur [8] for solution FFLPPs by claiming the infeasibility situation.
**Remark 1.** [8] The solution of the FFLPP expressed with (2.1) is obtained by finding OFS of the corresponding FFLPP (2.3) by means of the crisp system (2.4).

$$
\begin{aligned}
&\max \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j,\\
&\sum_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j \approx \tilde{b}_p \text{ for } p = 1, \ldots, t,\\
&\sum_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \oplus \tilde{K}_q \approx \tilde{b}_q \oplus \tilde{N}_q \text{ for } q = t+1, \ldots, s,\\
&\sum_{j=1}^{n} \tilde{a}_{rj} \otimes \tilde{x}_j \oplus \tilde{K}_r \approx \tilde{b}_r \oplus \tilde{N}_r \text{ for } r = s+1, \ldots, m,\\
&\mathcal{R}\left(\tilde{K}_q\right) - \mathcal{R}\left(\tilde{N}_q\right) \leq 0 \text{ for } q = t+1, \ldots, s,\\
&\mathcal{R}\left(\tilde{K}_r\right) - \mathcal{R}\left(\tilde{N}_r\right) \geq 0 \text{ for } r = s+1, \ldots, m,\\
&\tilde{\mathbf{x}} \geq \tilde{0},
\end{aligned}
\tag{2.3}
$$

where $\tilde{a}_{ij}, \tilde{c}_j, \tilde{b}_i = (b_i^\gamma), \tilde{K}_q = (k_q^\gamma), \tilde{N}_q = (n_q^\gamma), \tilde{K}_r = (k_r^\gamma)$ and $\tilde{N}_r = (n_r^\gamma)$ are arbitrary TFNs for $\gamma = l, m, u$. Using Zadeh's expansion principle and presuming $\tilde{a}_{ij} \otimes \tilde{x}_j = \left(y_{ij}^l,\ y_{ij}^m,\ y_{ij}^u\right)$ for $j = 1, \ldots, n$, and $i = 1, \ldots, m$.

$$
\begin{aligned}
&\max \mathcal{R}\left(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j\right),\\
&\sum_{j=1}^{n} y_{pj}^\gamma = b_p^\gamma \text{ for } p = 1, \ldots, t \text{ and } \forall\gamma,\\
&\sum_{j=1}^{n} y_{qj}^\gamma + k_q^\gamma = b_q^\gamma + n_q^\gamma \text{ for } q = t+1, \ldots, s \text{ and } \forall\gamma,\\
&\sum_{j=1}^{n} y_{rj}^\gamma + k_r^\gamma = b_r^\gamma + n_r^\gamma \text{ for } r = s+1, \ldots, m \text{ and } \forall\gamma,\\
&\mathcal{R}\left(\tilde{K}_q\right) - \mathcal{R}\left(\tilde{N}_q\right) \leq 0 \text{ for } q = t+1, \ldots, s,\\
&\mathcal{R}\left(\tilde{K}_r\right) - \mathcal{R}\left(\tilde{N}_r\right) \geq 0 \text{ for } r = s+1, \ldots, m,\\
&x_j^u - x_j^m \geq 0,\ x_j^m - x_j^l \geq 0,\ x_j^l \geq 0 \quad \forall j,\\
&k_\xi^u - k_\xi^m \geq 0,\ k_\xi^m - k_\xi^l \geq 0 \text{ for } \xi = t+1, \ldots, m,\\
&n_\xi^u - n_\xi^m \geq 0,\ n_\xi^m - n_\xi^l \geq 0 \text{ for } \xi = t+1, \ldots, m.
\end{aligned}
\tag{2.4}
$$

**Remark 2.** If an infeasible case arises while solving the system 2.4, the following FFLPP (2.5) is solved as an expansion procedure [12] of the Kumar and Kaur's technique [8] to obtain an approximate OFS of Problem 2.1.

$$
\begin{aligned}
&\max \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j,\\
&\sum_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j \oplus \tilde{K}_p = \tilde{b}_p \oplus \tilde{N}_p \text{ for } p = 1, \ldots, t,\\
&\sum_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \oplus \tilde{K}_q = \tilde{b}_q \oplus \tilde{N}_q \text{ for } q = t+1, \ldots, s,\\
&\sum_{j=1}^{n} \tilde{a}_{rj} \otimes \tilde{x}_j \oplus \tilde{K}_r = \tilde{b}_r \oplus \tilde{N}_r \text{ for } r = s+1, \ldots, m,\\
&\mathcal{R}\left(\tilde{K}_q\right) - \mathcal{R}\left(\tilde{N}_q\right) \leq 0 \text{ for } q = t+1, \ldots, s,\\
&\mathcal{R}\left(\tilde{K}_r\right) - \mathcal{R}\left(\tilde{N}_r\right) \geq 0 \text{ for } r = s+1, \ldots, m,\\
&\tilde{\mathbf{x}} \geq \tilde{0}.
\end{aligned}
\tag{2.5}
$$

where $\tilde{K}_\xi$ and $\tilde{N}_\xi$'s are arbitrary TFNs.

### 2.1. Fully fuzzy linear fractional programming problem

Utilizing matrix notation, the mathematical formula for any FFLFrPP is constructed as follows.

$$
\begin{aligned}
&\max\ \tilde{Z} = \frac{\tilde{N}(\tilde{\mathbf{x}})}{\tilde{D}(\tilde{\mathbf{x}})} = \frac{\tilde{\mathbf{c}^{\mathbf{T}}}\tilde{\mathbf{x}} + \tilde{\alpha}}{\tilde{\mathbf{d}^{\mathbf{T}}}\tilde{\mathbf{x}} + \tilde{\beta}},\\
&\tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} \succeq, =\preceq, \tilde{\mathbf{b}},\\
&\tilde{\mathbf{x}} \geq \tilde{0},
\end{aligned}
\tag{2.6}
$$

where $\tilde{Z}(\tilde{\mathbf{x}}) = (Z^l, Z^m, Z^u)$, $\tilde{\mathbf{c}}^T = \left[\tilde{c}_j\right]_{1\times n}$, $\tilde{\mathbf{d}}^T = \left[\tilde{d}_j\right]_{1\times n}$, $\tilde{\mathbf{x}} = \left[\tilde{x}_j\right]_{n\times 1}$, $\tilde{\mathbf{A}} = \left[\tilde{a}_{ij}\right]_{m\times n}$, $\tilde{\mathbf{b}} = \left[\tilde{b}_i\right]_{m\times 1}$. Furthermore, $\tilde{x}_j$ are nonnegative TFNs, $\tilde{a}_{ij}, \tilde{c}_j, \tilde{d}_j, \tilde{b}_i, \tilde{\alpha}, \tilde{\beta} \in F(\mathbf{R})$, and $T$ represents the transpose. In general, it is accepted that $\tilde{\mathbf{d}^{\mathbf{T}}}\tilde{\mathbf{x}} + \tilde{\beta} \succ \tilde{0}$ in the feasible region $\triangle = \left\{\tilde{\mathbf{x}} \mid \tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} \succeq, \approx, \preceq \tilde{\mathbf{b}}, \tilde{\mathbf{x}} \succeq \tilde{0}\right\}$.

Some definitions below have already been defined in our previous research [13]. The following notations and definitions are presented for brevity. Readers can refer to [13] for details.

**Remark 3.** With the assumption $\tilde{\mathbf{d}}^T\tilde{\mathbf{x}} + \tilde{\beta} \succ \tilde{0}$, each component of the fractional objective function is continuous on the domain. Namely, $Z^\gamma$ is continuous at $\tilde{\mathbf{x}}^i \in \triangle$, $\forall i \in \mathbf{N}$, the superscript $i$ denotes the iteration counter. Note that $Z^\gamma(\tilde{\mathbf{x}}) = Z^\gamma$ and $Z^\gamma\left(\tilde{\mathbf{x}}^i\right) = Z^{\gamma,i}$.

**Definition 7.** For all $\gamma$, each component of the fractional objective function $Z^\gamma$ is continuous on $\mathbf{R}$ satisfied that for all point $\tilde{\mathbf{x}}^i \in \triangle$ and $\varepsilon > 0$, a fuzzy number $\delta > 0$ is obtained that $Z^\gamma$ provides $\left|Z^\gamma - Z^{\gamma,i}\right| < \varepsilon$ whenever $\tilde{\mathbf{x}} \in \triangle$ and $\left|\mathbf{x}^\gamma - \mathbf{x}^{\gamma,i}\right| < \delta$.

Moreover, the Definition 7 can be restated from the standpoint of traditional neighborhoods as follows:

For all $\varepsilon$ and $\gamma$, if $\delta > 0$ is obtained that $\forall \mathbf{x}^\gamma \in B\left(\mathbf{x}^{\gamma,i}, \delta\right)$ and $Z^\gamma \in B\left(Z^{\gamma,i}, \varepsilon\right)$, then $Z^\gamma$ is continuous at $\tilde{\mathbf{x}}^i \in F(\mathbf{R})$.

## 3. Proposed fuzzy approach

First, the inequality constraints are turned into the equality constraints in compliance with the types of constraints of FFLFrPP (2.6) using fuzzy numbers and Remark 1, and the corresponding FFLFrPP (3.1) is obtained.

$$\begin{aligned}
&\max \tilde{Z},\\
&\text{s.t.} \quad \sum_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j = \tilde{b}_p \text{ for } p = 1, \ldots, t,\\
&\sum_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \oplus \tilde{K}_q = \tilde{b}_q \oplus \tilde{N}_q \text{ for } q = t+1, \ldots, s,\\
&\sum_{j=1}^{n} \tilde{a}_{rj} \otimes \tilde{x}_j \oplus \tilde{K}_r = \tilde{b}_r \oplus \tilde{N}_r \text{ for } r = s+1, \ldots, m,\\
&\mathcal{R}\left(\tilde{K}_q\right) - \mathcal{R}\left(\tilde{N}_q\right) \leq 0 \text{ for } q = t+1, \ldots, s,\\
&\mathcal{R}\left(\tilde{K}_r\right) - \mathcal{R}\left(\tilde{N}_r\right) \geq 0 \text{ for } r = s+1, \ldots, m,\\
&\tilde{\mathbf{x}} \geq \tilde{0}.
\end{aligned} \tag{3.1}$$

Afterward, with the help of definitions and notations explained in Section 2, the Cartesian product of neighborhoods defined by $B\left(\mathbf{x}^{\gamma,i}, \delta\right)$ and $B\left(Z^{\gamma,i}, \varepsilon\right)$ is $B\left(\mathbf{x}^{\gamma,i}, \delta\right) \times B\left(Z^{\gamma,i}, \varepsilon\right) = \left\{(\mathbf{x}^{\gamma}, Z^{\gamma}) \;\middle|\; \left|\mathbf{x}^{\gamma} - \mathbf{x}^{\gamma,\mathbf{i}}\right| < \delta, \left|Z^{\gamma} - Z^{\gamma,i}\right| < \varepsilon\right\}$. For all ordered pairs $(Z^{\gamma}, \mathbf{x}^{\gamma})$ and $\forall\gamma$ in the Cartesian product region, the inequality

$$\left|Z^{\gamma} - Z^{\gamma,i}\right| \left|\mathbf{x}^{\gamma} - \mathbf{x}^{\gamma,\mathbf{i}}\right| < \bar{\varepsilon}, \tag{3.2}$$

holds where $\bar{\varepsilon} = \varepsilon.\delta$. Since the inequality (3.2) is satisfied for all $\bar{\varepsilon}$, our convergence constraint

$$\left(Z^{\gamma} - Z^{\gamma,i}\right)\left(\mathbf{x}^{\gamma} - \mathbf{x}^{\gamma,\mathbf{i}}\right) = 0, \tag{3.3}$$

is constructed. If the convergence condition 3.3 is rearranged and using the fact that the Fuzzy Fractional Objective (FFrO) function components can be written as $Z^{\gamma} = \frac{\mathbf{c}^{\gamma T}\mathbf{x}^{\gamma} + \alpha^{\gamma}}{\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}}$, the following linear equations

$$\left(Z^{\gamma,i}\mathbf{d}^{\gamma T} - \mathbf{c}^{\gamma T}\right)\mathbf{x}^{\gamma} + \left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \beta^{\gamma}\right)\bar{Z}^{\gamma} = Z^{\gamma,i}\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \alpha^{\gamma}, \quad \forall\gamma, \tag{3.4}$$

are attained, where the objective $(\bar{Z}^{\gamma})$ will be used for the fuzzy MOLPP. The following fuzzy MOLPP is a reduced version of the given FFLFrPP using (3.4).

$$\begin{aligned}
&\max \left\{\bar{Z}^{l}, \bar{Z}^{m}, \bar{Z}^{u}\right\},\\
&\text{s.t.} \quad \left(Z^{\gamma,i}\mathbf{d}^{\gamma T} - \mathbf{c}^{\gamma T}\right)\mathbf{x}^{\gamma} + \left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \beta^{\gamma}\right)\bar{Z}^{\gamma} = Z^{\gamma,i}\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \alpha^{\gamma}, \quad \forall\gamma,\\
&\tilde{\mathbf{x}} \in \triangle.
\end{aligned} \tag{3.5}$$

To solve fuzzy MOLPP via a fuzzy approach, linear membership functions are utilized for simplicity in the solution phase. For the determination of the membership, the lower $(Z^{\gamma-})$ and upper $(Z^{\gamma+})$

boundaries regarding the satisfaction levels of the elements of the Fuzzy Objective Function (FOF) are calculated on the feasible region as follows.

$$Z^{\gamma-} = \min_{\tilde{x}\in\triangle} Z^{\gamma}(\tilde{x}) \text{ and } Z^{\gamma+} = \max_{\tilde{x}\in\triangle} Z^{\gamma}(\tilde{x}), \quad \forall\gamma.$$

For the fuzzy objective, which has a maximization direction, the linear membership functions are identified as follows:

$$h_{\gamma}\left(Z^{\gamma}(\tilde{x})\right) = \begin{cases} 1 & Z^{\gamma}(\tilde{x}) > Z^{\gamma+}, \\ \dfrac{Z^{\gamma}(\tilde{x}) - Z^{\gamma-}}{Z^{\gamma+} - Z^{\gamma-}}, & Z^{\gamma-} \leq Z^{\gamma}(\tilde{x}) \leq Z^{\gamma+}, \\ 0, & Z^{\gamma}(\tilde{x}) < Z^{\gamma-}, \end{cases}$$

where $Z^{\gamma-} \neq Z^{\gamma+}, \ \forall\gamma$.

With the help of Zimmermann's minimum operator [22], the fuzzy MOLPP (3.5) can be expressed as:

$$\begin{aligned} &\max \min_{\gamma} h_{\gamma}\left(\bar{Z}^{\gamma}\right), \\ &\text{s.t.} \quad \left(Z^{\gamma,i}\mathbf{d}^{\gamma T} - \mathbf{c}^{\gamma T}\right)\mathbf{x}^{\gamma} + \left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \beta^{\gamma}\right)\bar{Z}^{\gamma} = Z^{\gamma,i}\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \alpha^{\gamma}, \quad \forall\gamma, \\ &\qquad \tilde{\mathbf{x}} \in \triangle. \end{aligned} \tag{3.6}$$

By the aid of an ancillary variable $\lambda$, (3.6) can be turned into the following equivalent the iterative LPP (3.7) and the fuzzy optimal point of (2.6) is attained by optimizing the iterative LPP (3.7)

$$\begin{aligned} &\max \lambda, \\ &\text{s.t.} \quad \left(Z^{\gamma,i}\mathbf{d}^{\gamma T} - \mathbf{c}^{\gamma T}\right)\mathbf{x}^{\gamma} + \left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \beta^{\gamma}\right)\bar{Z}^{\gamma} = Z^{\gamma,i}\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \alpha^{\gamma}, \ \forall\gamma, \\ &\qquad \min_{\gamma} h_{\gamma}\left(\bar{Z}^{\gamma}\right) \geq \lambda, \quad \forall\gamma, \\ &\qquad \sum_{j=1}^{n} y_{pj}^{\gamma} = b_{p}^{\gamma} \text{ for } p = 1, \ldots, t \text{ and } \forall\gamma, \\ &\qquad \sum_{j=1}^{n} y_{qj}^{\gamma} + k_{q}^{\gamma} = b_{q}^{\gamma} + n_{q}^{\gamma} \text{ for } q = t+1, \ldots, s \text{ and } \forall\gamma, \\ &\qquad \sum_{j=1}^{n} y_{rj}^{\gamma} + k_{r}^{\gamma} = b_{r}^{\gamma} + n_{r}^{\gamma} \text{ for } r = s+1, \ldots, m \text{ and } \forall\gamma, \\ &\qquad \mathcal{R}\left(\tilde{K}_{q}\right) - \mathcal{R}\left(\tilde{N}_{q}\right) \leq 0 \text{ for } q = t+1, \ldots, s, \\ &\qquad \mathcal{R}\left(\tilde{K}_{r}\right) - \mathcal{R}\left(\tilde{N}_{r}\right) \geq 0 \text{ for } r = s+1, \ldots, m, \\ &\qquad x_{j}^{u} - x_{j}^{m} \geq 0, \ x_{j}^{m} - x_{j}^{l} \geq 0, \ x_{j}^{l} \geq 0 \quad \forall j, \\ &\qquad k_{\xi}^{u} - k_{\xi}^{m} \geq 0, \ k_{\xi}^{m} - k_{\xi}^{l} \geq 0 \text{ for } \xi = t+1, \ldots, m, \\ &\qquad n_{\xi}^{u} - n_{\xi}^{m} \geq 0, \ n_{\xi}^{m} - n_{\xi}^{l} \geq 0 \text{ for } \xi = t+1, \ldots, m, \\ &\qquad 0 \leq \lambda \leq 1. \end{aligned} \tag{3.7}$$

If an infeasible case arises, the following process (3.8) is pursued by the aid of Remark 2 to obtain an approximate OFS of (2.6).

$$
\begin{aligned}
&\max \min_{\gamma} h_{\gamma}\left(\bar{Z}^{\gamma}\right), \\
&\text{s.t.} \quad \left(Z^{\gamma i}\mathbf{d}^{\gamma T} - \mathbf{c}^{\gamma T}\right)\mathbf{x}^{\gamma} + \left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \beta^{\gamma}\right)\bar{Z}^{\gamma} = Z^{\gamma i}\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \alpha^{\gamma}, \quad \forall\gamma, \\
&\sum_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j \oplus \tilde{K}_p = \tilde{b}_p \oplus \tilde{N}_p \text{ for } p = 1, \ldots, t, \\
&\sum_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \oplus \tilde{K}_q = \tilde{b}_q \oplus \tilde{N}_q \text{ for } q = t+1, \ldots, s, \\
&\sum_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j \oplus \tilde{K}_r = \tilde{b}_r \oplus \tilde{N}_r \text{ for } r = s+1, \ldots, m, \\
&\mathcal{R}\left(\tilde{K}_q\right) - \mathcal{R}\left(\tilde{N}_q\right) \leq 0 \text{ for } q = t+1, \ldots, s, \\
&\mathcal{R}\left(\tilde{K}_r\right) - \mathcal{R}\left(\tilde{N}_r\right) \geq 0 \text{ for } r = s+1, \ldots, m, \\
&\tilde{\mathbf{x}} \geq \tilde{0}.
\end{aligned}
\tag{3.8}
$$

Beginning with an incipiency fuzzy solution $(\tilde{\mathbf{x}}^0, \tilde{Z}^0)$, we can iterate the sub-problem (3.7) or (3.8) using the solution $(\tilde{\mathbf{x}}^{i^*}, \tilde{Z}^i)$ at $i$-th iteration for finding the fuzzy optimal point $(\tilde{\mathbf{x}}^{i+1^*}, \tilde{Z}^{i+1})$ at $(i+1)$-th iteration.

**Proposition 1.** *For all $\gamma$, the gradient vectors of each component of the FFrO function $Z^{\gamma}$ in* (2.6) *and its linear objective functions $\bar{Z}^{\gamma}$ in* (3.6) *have the same value at every solution $\tilde{\mathbf{x}}^i \in \triangle$.*

*Proof.* The gradient vectors of component functions of objective $Z^{\gamma}$ can be signified as follows:

$$
\begin{aligned}
\nabla Z^{\gamma}\left(\mathbf{x}^{\gamma}\right) &= \frac{\left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}\right)\nabla\left(\mathbf{c}^{\gamma T}\mathbf{x}^{\gamma} + \alpha^{\gamma}\right) - \left(\mathbf{c}^{\gamma T}\mathbf{x}^{\gamma} + \alpha^{\gamma}\right)\nabla\left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}\right)}{\left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}\right)^2} \\
&= \frac{\left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}\right)\mathbf{c}^{\gamma T} - \left(\mathbf{c}^{\gamma T}\mathbf{x}^{\gamma} + \alpha^{\gamma}\right)\mathbf{d}^{\gamma T}}{\left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}\right)^2} = \frac{\mathbf{c}^{\gamma T} - \frac{\left(\mathbf{c}^{\gamma T}\mathbf{x}^{\gamma} + \alpha^{\gamma}\right)}{\left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}\right)}\mathbf{d}^{\gamma T}}{\left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}\right)} \\
&= \frac{\mathbf{c}^{\gamma T} - Z_k(\mathbf{x}^{\gamma})\mathbf{d}^{\gamma T}}{\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}}.
\end{aligned}
$$

Hence, the values of the gradient vectors at the point $\mathbf{x}^{\gamma}$

$$
\nabla Z^{\gamma}\left(\mathbf{x}^{\gamma}\right) = \frac{\mathbf{c}^{\gamma T} - Z^{\gamma}(\mathbf{x}^{\gamma})\mathbf{d}^{\gamma T}}{\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}} = \frac{\mathbf{c}^{\gamma T} - Z^{\gamma,i}\mathbf{d}^{\gamma T}}{\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}},
$$

can be written. On the other hand, rearranging the Eq (3.4), the gradient vectors of components of fuzzy linear objective function $\bar{Z}^{\gamma}\left(\mathbf{x}^{\gamma}\right)$ are as follows:

$$
\left(Z^{\gamma,i}\mathbf{d}^{\gamma T} - \mathbf{c}^{\gamma T}\right)\nabla\mathbf{x}^{\gamma} + \left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma} + \beta^{\gamma}\right)\nabla\bar{Z}^{\gamma} = 0,
$$

$$\nabla \bar{Z}^{\gamma}\left(\mathbf{x}^{\gamma}\right)=\frac{\mathbf{c}^{\gamma T}-Z^{\gamma,i}\mathbf{d}^{\gamma T}}{\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma}+\beta^{\gamma}}.$$

Furthermore, the values of the gradient vectors at the point $\mathbf{x}^{\gamma}$

$$\nabla \bar{Z}^{\gamma}\left(\mathbf{x}^{\gamma}\right)=\frac{\mathbf{c}^{\gamma T}-Z^{\gamma,i}\mathbf{d}^{\gamma T}}{\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma}+\beta^{\gamma}},$$

can be written. □

**Remark 4.** The increment at components of fuzzy linear objective function such that $\bar{Z}^{\gamma}\left(\tilde{\mathbf{x}}^{i}\right)\leq\bar{Z}^{\gamma}(\tilde{\mathbf{x}}^{*})$ indicates that the increment at components of fuzzy linear fractional objective function such that $Z^{\gamma}\left(\tilde{\mathbf{x}}^{i}\right)\leq Z^{\gamma}(\tilde{\mathbf{x}}^{*})$.

**Proposition 2.** *The values of components of FFrO function $Z^{\gamma,i}$ generate a monotonic non-decreasing sequence at the consecutive OFSs $\tilde{\mathbf{x}}^{i}\in\triangle$ of the Problem* (2.6)*. Beginning with an incipiency fuzzy point $\tilde{\mathbf{x}}^{0}\in\triangle$, that is $Z^{\gamma 0}\leq Z^{\gamma 1}\leq\ldots.\leq Z^{\gamma i}\leq Z^{\gamma i+1}\leq\ldots,\ \forall\gamma\in\{l,m,u\}$ and $\forall i\in\mathbf{N}$.*

*Proof.* Let the nonempty set $\triangle^{i}$ be defined such that $\triangle^{i}=\triangle\cap B\left(\mathbf{x}^{\gamma,\mathbf{i}},\delta\right)$. The following inequality is satisfied for all $\tilde{\mathbf{x}}\in\triangle^{i}$.

$$Z^{\gamma}(\tilde{\mathbf{x}})\leq\max_{\tilde{\mathbf{x}}\in\triangle^{i}}Z^{\gamma}(\tilde{\mathbf{x}}).\tag{3.9}$$

Considering $\tilde{\mathbf{x}}^{i}\in\triangle^{i}$ and $Z^{\gamma,i}$, if $\max_{\tilde{\mathbf{x}}\in\triangle^{i}}Z^{\gamma}(\tilde{\mathbf{x}})=Z^{\gamma}(\tilde{\mathbf{x}}^{*})$, then the inequality (3.9)

$$Z^{\gamma}(\tilde{\mathbf{x}}^{i})\leq Z^{\gamma}(\tilde{\mathbf{x}}^{*}),\tag{3.10}$$

can be rewritten. In the proposed algorithm, $\tilde{\mathbf{x}}^{i+1}=\tilde{\mathbf{x}}^{*}$ can be defined due to the fact that the fuzzy optimal point $\tilde{\mathbf{x}}^{*}$ is used in the next iteration. Hence, the inequality (3.10) is denoted by $Z^{\gamma,i}\leq Z^{\gamma i+1}$. If the alike process is reiterated for the set $\triangle^{i+1}=\triangle\cap B\left(\mathbf{x}^{\gamma i+1},\delta\right)$, then

$$Z^{\gamma}(\tilde{\mathbf{x}}^{i+1})\leq Z^{\gamma}(\tilde{\mathbf{x}}^{i+2}),\tag{3.11}$$

is obtained for all $\tilde{\mathbf{x}}^{i+1}\in\triangle^{i+1}$, that is $Z^{\gamma i+1}\leq Z^{\gamma i+2}$.

If this procedure is continued, it is unfalteringly said that the values of fractional objective functions generate a non-decreasing sequence as

$$Z^{\gamma,0}\leq Z^{\gamma,1}\leq\cdots\leq Z^{\gamma,i}\leq Z^{\gamma,(i+1)}\leq\ldots,\ \forall^{\gamma},\ \ \forall i\in\mathbf{N}.$$

□

**Theorem 1.** *If a non-decreasing sequence $\left(Z^{\gamma,i}\right)_{i\in\mathbf{N}}$ is bounded above, the monotonic sequence is convergent.*

*Proof.* The proof is straightforward. □

### *3.1. Determining an incipiency point*

To start the algorithm, an arbitrary fuzzy value can be selected, or the incipiency fuzzy point $\tilde{\mathbf{x}}^{\mathbf{0}}$ can be determined by solving one of the problems that assumes one of the elements of the set $\left\{\max_{\tilde{\mathbf{x}}\in\triangle}\tilde{0},\ \max_{\tilde{\mathbf{x}}\in\triangle}\tilde{\mathbf{c}}^{T}\tilde{\mathbf{x}},\ \max_{\tilde{\mathbf{x}}\in\triangle}-\tilde{\mathbf{d}}^{T}\tilde{\mathbf{x}},\ \max_{\tilde{\mathbf{x}}\in\triangle}\left(\tilde{\mathbf{c}}^{T}-\tilde{\mathbf{d}}^{T}\right)\tilde{\mathbf{x}}\right\}$ as an objective function over the feasible region $\triangle$.

*3.2. Statement of the presented algorithm*

Regarding the previous expressions and operations, the algorithm can be simply expressed as:
**Step 0.** Load the following FFLFrPP (2.6).

$$\begin{aligned}&\max\ \tilde{Z} = \frac{\tilde{N}(\tilde{\mathbf{x}})}{\tilde{D}(\tilde{\mathbf{x}})} = \frac{\tilde{\mathbf{c}}^{\mathbf{T}}\tilde{\mathbf{x}} + \tilde{\alpha}}{\tilde{\mathbf{d}}^{\mathbf{T}}\tilde{\mathbf{x}} + \tilde{\beta}},\\&\tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} \geq, =\leq, \tilde{\mathbf{b}},\\&\tilde{\mathbf{x}} \geq \tilde{0}.\end{aligned}$$

**Step 1.** Classify the constraints (in terms of "$\leq, =, \geq$") and transform inequality types constraints ("$\leq$ , $\geq$") into equality ones by aid of fuzzy numbers and Remark 1, attain the equivalent FFLFrPP (3.1) as.

$$\begin{aligned}&\max \tilde{Z},\\ \text{s.t.}\quad &\sum_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j = \tilde{b}_p \text{ for } p = 1, \ldots, t,\\ &\sum_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \oplus \tilde{K}_q = \tilde{b}_q \oplus \tilde{N}_q \text{ for } q = t+1, \ldots, s,\\ &\sum_{j=1}^{n} \tilde{a}_{rj} \otimes \tilde{x}_j \oplus \tilde{K}_r = \tilde{b}_r \oplus \tilde{N}_r \text{ for } r = s+1, \ldots, m,\\ &\mathcal{R}\left(\tilde{K}_q\right) - \mathcal{R}\left(\tilde{N}_q\right) \leq 0 \text{ for } q = t+1, \ldots, s,\\ &\mathcal{R}\left(\tilde{K}_r\right) - \mathcal{R}\left(\tilde{N}_r\right) \geq 0 \text{ for } r = s+1, \ldots, m,\\ &\tilde{\mathbf{x}} \geq \tilde{0}.\end{aligned}$$

**Step 2.** Select an incipiency fuzzy solution $\tilde{\mathbf{x}}^0$ for corresponding the value of FOF $\tilde{Z}^0$ and assign $i = 0$.
**Step 3.** Establish the following fuzzy MOLPP (3.5).

$$\begin{aligned}&\max \left\{\bar{Z}^l, \bar{Z}^m, \bar{Z}^u\right\},\\ \text{s.t.}\quad &\left(Z^{\gamma,i}\mathbf{d}^{\gamma T} - \mathbf{c}^{\gamma T}\right)\mathbf{x}^{\gamma} + \left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \beta^{\gamma}\right)\bar{Z}^{\gamma} = Z^{\gamma,i}\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}} + \alpha^{\gamma}, \quad \forall\gamma,\\ &\tilde{\mathbf{x}} \in \triangle.\end{aligned}$$

**Step 4.** Via a fuzzy approach, considering the following iterative LPP (3.7),

$$
\begin{aligned}
&\max \lambda,\\
&\text{s.t.}\quad \left(Z^{\gamma,i}\mathbf{d}^{\gamma T}-\mathbf{c}^{\gamma T}\right)\mathbf{x}^{\gamma}+\left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}}+\beta^{\gamma}\right)\bar{Z}^{\gamma}=Z^{\gamma,i}\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}}+\alpha^{\gamma},\ \forall\gamma,\\
&\qquad \min_{\gamma} h_{\gamma}\left(\bar{Z}^{\gamma}\right)\geq\lambda,\quad \forall\gamma,\\
&\qquad \sum_{j=1}^{n} y_{pj}^{\gamma}=b_p^{\gamma} \text{ for } p=1,\ldots,t \text{ and } \forall\gamma,\\
&\qquad \sum_{j=1}^{n} y_{qj}^{\gamma}+k_q^{\gamma}=b_q^{\gamma}+n_q^{\gamma} \text{ for } q=t+1,\ldots,s \text{ and } \forall\gamma,\\
&\qquad \sum_{j=1}^{n} y_{rj}^{\gamma}+k_r^{\gamma}=b_r^{\gamma}+n_r^{\gamma} \text{ for } r=s+1,\ldots,m \text{ and } \forall\gamma,\\
&\qquad \mathcal{R}\left(\tilde{K}_q\right)-\mathcal{R}\left(\tilde{N}_q\right)\leq 0 \text{ for } q=t+1,\ldots,s,\\
&\qquad \mathcal{R}\left(\tilde{K}_r\right)-\mathcal{R}\left(\tilde{N}_r\right)\geq 0 \text{ for } r=s+1,\ldots,m,\\
&\qquad x_j^u-x_j^m\geq 0,\ x_j^m-x_j^l\geq 0,\ x_j^l\geq 0\quad \forall j,\\
&\qquad k_\xi^u-k_\xi^m\geq 0,\ k_\xi^m-k_\xi^l\geq 0 \text{ for } \xi=t+1,\ldots,m,\\
&\qquad n_\xi^u-n_\xi^m\geq 0,\ n_\xi^m-n_\xi^l\geq 0 \text{ for } \xi=t+1,\ldots,m,\\
&\qquad 0\leq\lambda\leq 1.
\end{aligned}
$$

(a) Obtain the OFS $\tilde{\mathbf{x}}^*$, skip the Step 5.

(b) If above model (3.7) has no feasible solutions, the iterative LPP is infeasible. Then use following (3.8) for an approximate OFS of (2.1).

$$
\begin{aligned}
&\max \min_{\gamma} h_{\gamma}\left(\bar{Z}^{\gamma}\right),\\
&\text{s.t.}\quad \left(Z^{\gamma i}\mathbf{d}^{\gamma T}-\mathbf{c}^{\gamma T}\right)\mathbf{x}^{\gamma}+\left(\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}}+\beta^{\gamma}\right)\bar{Z}^{\gamma}=Z^{\gamma i}\mathbf{d}^{\gamma T}\mathbf{x}^{\gamma,\mathbf{i}}+\alpha^{\gamma},\quad \forall\gamma,\\
&\sum_{j=1}^{n}\tilde{a}_{pj}\otimes\tilde{x}_j\oplus\tilde{K}_p=\tilde{b}_p\oplus\tilde{N}_p \text{ for } p=1,\ldots,t,\\
&\sum_{j=1}^{n}\tilde{a}_{qj}\otimes\tilde{x}_j\oplus\tilde{K}_q=\tilde{b}_q\oplus\tilde{N}_q \text{ for } q=t+1,\ldots,s,\\
&\sum_{j=1}^{n}\tilde{a}_{pj}\otimes\tilde{x}_j\oplus\tilde{K}_r=\tilde{b}_r\oplus\tilde{N}_r \text{ for } r=s+1,\ldots,m,\\
&\mathcal{R}\left(\tilde{K}_q\right)-\mathcal{R}\left(\tilde{N}_q\right)\leq 0 \text{ for } q=t+1,\ldots,s,\\
&\mathcal{R}\left(\tilde{K}_r\right)-\mathcal{R}\left(\tilde{N}_r\right)\geq 0 \text{ for } r=s+1,\ldots,m,\\
&\tilde{\mathbf{x}}\geq\tilde{0}.
\end{aligned}
$$

**Step 5.** Construct $\tilde{\mathbf{x}}^{i+1}=\tilde{\mathbf{x}}^*$ and compute $\tilde{Z}^{i+1}=\tilde{Z}\left(\tilde{\mathbf{x}}^{i+1}\right)$.

**Step 6.** If $\tilde{Z}^{i+1}=\tilde{Z}^i$, the (approximate) OFS of (2.6) is achieved, that is $\tilde{\mathbf{x}}^*=\tilde{\mathbf{x}}^i$ and $\tilde{Z}^*=\tilde{Z}^i$. STOP. Otherwise, take $i=i+1$ and skip Step 4.

The proposed fuzzy method's flow diagram is shown in Figure 1. The maximal objective's direction is displayed for it. We also point out that, by virtue of the procedure's flexibility, the method may be renewed to work toward the minimal objective in terms of the FFLFrPP's structure.

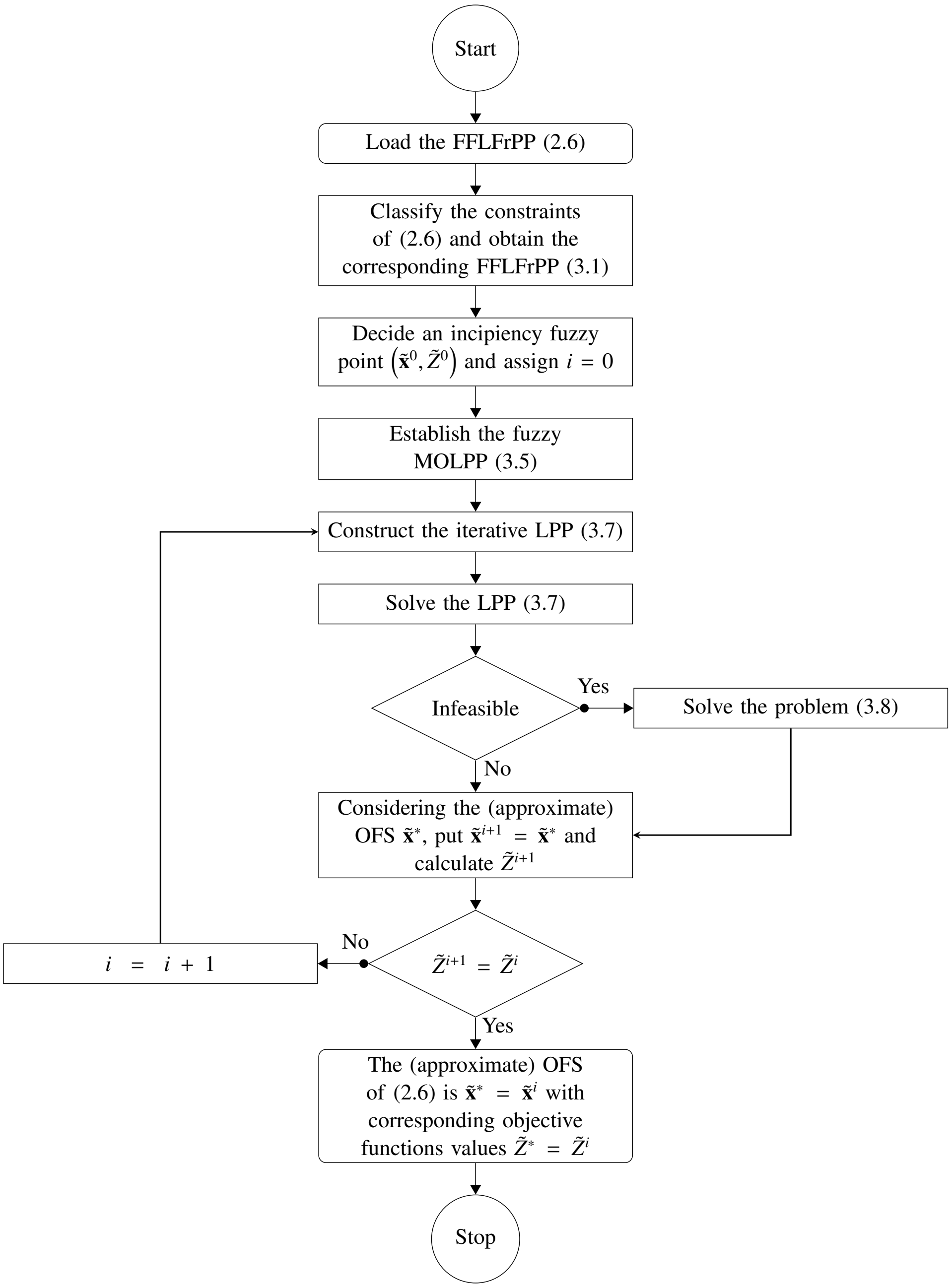


**Figure 1.** Flow diagram of our fuzzy approach.

## 4. Numeric Expressions

Two computational experiments and two small practical applications gathered from the literature are offered to examine the offered fuzzy technique in this section. The steps of the suggested algorithm are carried out for only the first example. Furthermore, all numerical findings are included in Table 1, together with comparisons to existing approaches.

**Table 1.** Comparison results for examples.

| Example | Ref. | $\tilde{Z}$ | Ranking $\left(\mathcal{R}(\tilde{Z})\right)$ |
|---|---|---|---|
| 1 | Proposed Algorithm | $(-0.1336, 0.5371, 6.8975)$ | **1.9595** |
| | Pop and Stancu-Minasian [14] | $(-0.2108, 0.6667, 5.82)$ | 1.7356 |
| | Stanojevic and Stancu-Minasian [19] | $(0, 0.55, 1.09)$ | 0.5475 |
| 2 | Proposed Algorithm | $(0.1436, 2, 27.6393)$ | **7.9457** |
| | Safaei [15] | $(1.34, 2, 2.31)$ | 1.9125 |
| | Kumar, Mandal and Edalatpanah [16] | $(1.5, 2, 2.8)$ | 2.075 |

### *4.1. Examples*

**Example 1.** Handle the FFLFrPP [14]:

$$\begin{aligned} &\max \tilde{Z}(\tilde{\mathbf{x}}) = \frac{(0,1,2) \otimes \tilde{x}_1 \oplus (-2,-1,0) \otimes \tilde{x}_2 \oplus (0,1,2)}{(0,1,2) \otimes \tilde{x}_1 \oplus (0,1,2) \otimes \tilde{x}_2 \oplus (1,2,3)}, \\ &\text{s.t. } (0,1,2) \otimes \tilde{x}_1 \oplus (-2,-1,0) \otimes \tilde{x}_2 \preceq (0,1,2), \\ &\quad (0,1,2) \otimes \tilde{x}_1 \oplus (0,1,2) \otimes \tilde{x}_2 \preceq (1,2,3), \\ &\quad \tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}. \end{aligned} \tag{4.1}$$

The feasible region is denoted by $\triangle$.

**Step 1.** ransforming the constraints of FFLFrPP (4.1), the following equivalent FFLFrPP is constructed.

$$\begin{aligned} &\max \tilde{Z}(\tilde{\mathbf{x}}) = \frac{(0,1,2) \otimes \tilde{x}_1 \oplus (-2,-1,0) \otimes \tilde{x}_2 \oplus (0,1,2)}{(0,1,2) \otimes \tilde{x}_1 \oplus (0,1,2) \otimes \tilde{x}_2 \oplus (1,2,3)}, \\ &\text{s.t. } (0,1,2) \otimes \tilde{x}_1 \oplus (-2,-1,0) \otimes \tilde{x}_2 \oplus \tilde{K}_1 = (0,1,2) \oplus \tilde{N}_1, \\ &\quad (0,1,2) \otimes \tilde{x}_1 \oplus (0,1,2) \otimes \tilde{x}_2 \oplus \tilde{K}_2 = (1,2,3) \oplus \tilde{N}_2, \\ &\quad \mathcal{R}\left(\tilde{K}_r\right) - \mathcal{R}\left(\tilde{N}_r\right) \geq 0 \text{ for } r = 1,2 \\ &\quad \tilde{x}_j \succeq \tilde{0} \quad j = 1,2. \end{aligned} \tag{4.2}$$

**Step 2.** $\tilde{\mathbf{x}}^0 = [(0,1,1),(0,0,0)]$ the incipiency fuzzy solution and the value of FOF $\tilde{Z}^0 = (0, 0.6667, 4)$ are obtained by dealing with the FFLPP (4.3)

$$\begin{aligned} &\max \tilde{0}, \\ &\text{s.t. } (0,1,2) \otimes \tilde{x}_1 \oplus (-2,-1,0) \otimes \tilde{x}_2 \oplus \tilde{K}_1 = (0,1,2) \oplus \tilde{N}_1, \\ &\quad (0,1,2) \otimes \tilde{x}_1 \oplus (0,1,2) \otimes \tilde{x}_2 \oplus \tilde{K}_2 = (1,2,3) \oplus \tilde{N}_2, \\ &\quad \mathcal{R}\left(\tilde{K}_r\right) - \mathcal{R}\left(\tilde{N}_r\right) \geq 0 \text{ for } r = 1,2, \\ &\quad \tilde{x}_j \succeq \tilde{0} \quad j = 1,2, \end{aligned} \tag{4.3}$$

and its corresponding to the LPP (4.4).

$$
\begin{aligned}
&\max\ 0,\\
&\text{s.t.}\quad -2x_2^u + k_1^l = n_1^l\\
&\quad x_1^m - x_2^m + k_1^m = 1 + n_1^m\\
&\quad 2x_1^u + k_1^u = 2 + n_1^u\\
&\quad x_1^m + x_2^m + k_2^m = 2 + n_2^m\\
&\quad 2x_1^u + 2x_2^u + k_2^u = 3 + n_2^u\\
&\quad k_\xi^l + 2k_\xi^m + k_\xi^u - n_\xi^l - 2n_\xi^m - n_\xi^u \geq 0 \text{ for } \xi = 1,2\\
&\quad x_j^u - x_j^m \geq 0,\ x_j^m - x_j^l \geq 0,\ x_j^l \geq 0 \text{ for } j = 1,2\\
&\quad k_\xi^u - k_\xi^m \geq 0,\ k_\xi^m - k_\xi^l \geq 0 \text{ for } \xi = 1,2\\
&\quad n_\xi^u - n_\xi^m \geq 0,\ n_\xi^m - n_\xi^l \geq 0 \text{ for } \xi = 1,2
\end{aligned}
\tag{4.4}
$$

**Step 3.** Utilizing $\tilde{\mathbf{x}}^0, \tilde{Z}^0$ and FFLFrPP (4.5), the fuzzy MOLPP (4.6) corresponding to (3.5) is obtained.

$$
\begin{aligned}
&\max \tilde{Z}(\tilde{\mathbf{x}}) = \left(\frac{-2x_2^u}{2x_1^u + 2x_2^u + 3}, \frac{x_1^m - x_2^m + 1}{x_2^m + 2}, \frac{x_1^u + 2}{1}\right),\\
&\text{s.t.}\ (-2x_2^u, x_1^m - x_2^m, 2x_1^u) \oplus \left(k_1^l, k_1^m, k_1^u\right) = (0,1,2) \oplus \left(n_1^l, n_1^m, n_1^u\right),\\
&\quad (0, x_1^m + x_2^m, 2x_1^u + 2x_2^u) \oplus \left(k_2^l, k_2^m, k_2^u\right) = (1,2,3) \oplus \left(n_2^l, n_2^m, n_2^u\right),\\
&\quad k_\xi^l + 2k_\xi^m + k_\xi^u - n_\xi^l - 2n_\xi^m - n_\xi^u \geq 0 \text{ for } \xi = 1,2,\\
&\quad x_j^u - x_j^m \geq 0,\ x_j^m - x_j^l \geq 0,\ x_j^l \geq 0 \text{ for } j = 1,2,\\
&\quad k_\xi^u - k_\xi^m \geq 0,\ k_\xi^m - k_\xi^l \geq 0 \text{ for } \xi = 1,2,\\
&\quad n_\xi^u - n_\xi^m \geq 0,\ n_\xi^m - n_\xi^l \geq 0 \text{ for } \xi = 1,2.
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
&\max\ \left\{\bar{Z}^l, \bar{Z}^m, \bar{Z}^u\right\},\\
&\text{s.t.}\quad 2x_2^u + 5\bar{Z}^l = 0,\\
&\quad -0.3333x_1^m + 1.6667x_2^m + 3\bar{Z}^m = 1.6667,\\
&\quad -2{x_1}^u + \bar{Z}^u = 2,\\
&\quad (-2x_2^u, x_1^m - x_2^m, 2x_1^u) \oplus \left(k_1^l, k_1^m, k_1^u\right) = (0,1,2) \oplus \left(n_1^l, n_1^m, n_1^u\right),\\
&\quad (0, x_1^m + x_2^m, 2x_1^u + 2x_2^u) \oplus \left(k_2^l, k_2^m, k_2^u\right) = (1,2,3) \oplus \left(n_2^l, n_2^m, n_2^u\right),\\
&\quad k_\xi^l + 2k_\xi^m + k_\xi^u - n_\xi^l - 2n_\xi^m - n_\xi^u \geq 0 \text{ for } \xi = 1,2,\\
&\quad x_j^u - x_j^m \geq 0,\ x_j^m - x_j^l \geq 0,\ x_j^l \geq 0 \text{ for } j = 1,2,\\
&\quad k_\xi^u - k_\xi^m \geq 0,\ k_\xi^m - k_\xi^l \geq 0 \text{ for } \xi = 1,2,\\
&\quad n_\xi^u - n_\xi^m \geq 0,\ n_\xi^m - n_\xi^l \geq 0 \text{ for } \xi = 1,2.
\end{aligned}
\tag{4.6}
$$

**Step 4.** With the fuzzy approach, the LPP corresponding to (3.7) is constituted as:

$$\begin{aligned}
&\max \lambda,\\
\text{s.t.}\quad & 2x_2^u + 5\bar{Z}^l = 0,\\
& -0.3333x_1^m + 1.6667x_2^m + 3\bar{Z}^m = 1.6667,\\
& -2x_1{}^u + \bar{Z}^u = 2,\\
& \bar{Z}^l - 0.7272\lambda \geq -0.7272,\\
& \bar{Z}^m - 0.9643\lambda \geq -0.25,\\
& \bar{Z}^u - 6\lambda \geq 2,\\
& -2x_2^u + k_1^l = n_1^l,\\
& x_1^m - x_2^m + k_1^m = 1 + n_1^m,\\
& 2x_1^u + k_1^u = 2 + n_1^u,\\
& x_1^m + x_2^m + k_2^m = 2 + n_2^m,\\
& 2x_1^u + 2x_2^u + k_2^u = 3 + n_2^u,\\
& k_\xi^l + 2k_\xi^m + k_\xi^u - n_\xi^l - 2n_\xi^m - n_\xi^u \geq 0 \text{ for } \xi = 1,2,\\
& x_j^u - x_j^m \geq 0,\ x_j^m - x_j^l \geq 0,\ x_j^l \geq 0 \text{ for } j = 1,2,\\
& k_\xi^u - k_\xi^m \geq 0,\ k_\xi^m - k_\xi^l \geq 0 \text{ for } \xi = 1,2,\\
& n_\xi^u - n_\xi^m \geq 0,\ n_\xi^m - n_\xi^l \geq 0 \text{ for } \xi = 1,2,\\
& 0 \leq \lambda \leq 1.
\end{aligned} \tag{4.7}$$

The OFS of (4.7) is $\tilde{\mathbf{x}}^* = [(0,0,2.411),(0,0.055,0.357)]$.
**Step 5.** $\tilde{\mathbf{x}}^1 = \tilde{\mathbf{x}}^* = [(0,0,2.411),(0,0.055,0.357)]$ and the value of FFrO is $\tilde{Z}^1 = (-0.143, 0.525, 6.823)$.
**Step 6.** Due to $\tilde{Z}^0 \neq \tilde{Z}^1$, assign $i = 1$ and skip Step 4.

**Step 4.** With $\tilde{\mathbf{x}}^1$ and $\tilde{Z}^1$, the LPP corresponding to (3.7) is constituted as:

$$
\begin{aligned}
&\max \lambda,\\
\text{s.t.}\quad & -0.1671{x_1}^u + 1.8329x_2^u + 8.5363\bar{Z}^l = -0.4626,\\
& -0.54x_1^m + 1.46x_2^m + 2.0548\bar{Z}^m = 1.0252,\\
& -2{x_1}^u + \bar{Z}^u = 2,\\
& \bar{Z}^l - 0.7272\lambda \geq -0.7272,\\
& \bar{Z}^m - 0.9643\lambda \geq -0.25,\\
& \bar{Z}^u - 6\lambda \geq 2,\\
& -2x_2^u + k_1^l = n_1^l,\\
& x_1^m - x_2^m + k_1^m = 1 + n_1^m,\\
& 2x_1^u + k_1^u = 2 + n_1^u\\
& x_1^m + x_2^m + k_2^m = 2 + n_2^m,\\
& 2x_1^u + 2x_2^u + k_2^u = 3 + n_2^u,\\
& k_\xi^l + 2k_\xi^m + k_\xi^u - n_\xi^l - 2n_\xi^m - n_\xi^u \geq 0 \text{ for } \xi = 1, 2,\\
& x_j^u - x_j^m \geq 0,\ x_j^m - x_j^l \geq 0,\ x_j^l \geq 0 \text{ for } j = 1, 2,\\
& k_\xi^u - k_\xi^m \geq 0,\ k_\xi^m - k_\xi^l \geq 0 \text{ for } \xi = 1, 2,\\
& n_\xi^u - n_\xi^m \geq 0,\ n_\xi^m - n_\xi^l \geq 0 \text{ for } \xi = 1, 2,\\
& 0 \leq \lambda \leq 1.
\end{aligned} \tag{4.8}
$$

The OFS of (4.8) is $\tilde{\mathbf{x}}^* = [(0, 0.145, 2.449), (0, 0, 0.593)]$.
**Step 5.** $\tilde{\mathbf{x}}^2 = \tilde{\mathbf{x}}^* = [(0, 0.145, 2.449), (0, 0, 0.593)]$ and the value of FFrO is $\tilde{Z}^2 = (-0.1306, 0.5337, 6.8971)$.
**Step 6.** Due to $\tilde{Z}^1 \neq \tilde{Z}^2$, assign $i = 2$ and skip Step 4.

**Step 4.** With $\tilde{\mathbf{x}}^1$ and $\tilde{Z}^1$, the LPP corresponding to (3.7) is constituted as:

$$
\begin{aligned}
&\max \lambda,\\
\text{s.t.}\quad & -0.2613{x_1}^u + +1.7387x_2^u + 9.084\bar{Z}^l = -0.795,\\
& -0.4662x_1^m + 1.5338x_2^m + 2.145\bar{Z}^m = 1.0774,\\
& -2{x_1}^u + \bar{Z}^u = 2,\\
& \bar{Z}^l - 0.7272\lambda \geq -0.7272,\\
& \bar{Z}^m - 0.9643\lambda \geq -0.25,\\
& \bar{Z}^u - 6\lambda \geq 2,\\
& -2x_2^u + k_1^l = n_1^l,\\
& x_1^m - x_2^m + k_1^m = 1 + n_1^m,\\
& 2x_1^u + k_1^u = 2 + n_1^u\\
& x_1^m + x_2^m + k_2^m = 2 + n_2^m,\\
& 2x_1^u + 2x_2^u + k_2^u = 3 + n_2^u,\\
& k_\xi^l + 2k_\xi^m + k_\xi^u - n_\xi^l - 2n_\xi^m - n_\xi^u \geq 0 \text{ for } \xi = 1, 2,\\
& x_j^u - x_j^m \geq 0,\ x_j^m - x_j^l \geq 0,\ x_j^l \geq 0 \text{ for } j = 1, 2,\\
& k_\xi^u - k_\xi^m \geq 0,\ k_\xi^m - k_\xi^l \geq 0 \text{ for } \xi = 1, 2,\\
& n_\xi^u - n_\xi^m \geq 0,\ n_\xi^m - n_\xi^l \geq 0 \text{ for } \xi = 1, 2,\\
& 0 \leq \lambda \leq 1.
\end{aligned}
\tag{4.9}
$$

The OFS of (4.9) is $\tilde{\mathbf{x}}^* = [(0, 0.16, 2.449), (0, 0, 0.609)]$.
**Step 5.** $\tilde{\mathbf{x}}^3 = \tilde{\mathbf{x}}^* = [(0, 0.16, 2.449), (0, 0, 0.609)]$ and the value of FFrO is $\tilde{Z}^3 = (-0.1336, 0.5371, 6.8976)$.
**Step 6.** Due to $\tilde{Z}^2 \neq \tilde{Z}^3$, assign $i = 3$ and skip Step 4.

**Step 4.** With $\tilde{\mathbf{x}}^3$ and $\tilde{Z}^3$, the LPP corresponding to (3.7) is constituted as:

$$\begin{aligned}
&\max \lambda, \\
\text{s.t.}\quad & -0.2672{x_1}^u + 1.7328x_2^u + 9.1156\bar{Z}^l = -0.8171, \\
& -0.4629x_1^m + 1.5371x_2^m + 2.16\bar{Z}^m = 1.086, \\
& -2{x_1}^u + \bar{Z}^u = 2, \\
& \bar{Z}^l - 0.7272\lambda \geq -0.7272, \\
& \bar{Z}^m - 0.9643\lambda \geq -0.25, \\
& \bar{Z}^u - 6\lambda \geq 2, \\
& -2x_2^u + k_1^l = n_1^l, \\
& x_1^m - x_2^m + k_1^m = 1 + n_1^m, \\
& 2x_1^u + k_1^u = 2 + n_1^u, \\
& x_1^m + x_2^m + k_2^m = 2 + n_2^m, \\
& 2{x_1}^u + 2{x_2}^u + {k_2}^u = 3 + {n_2}^u, \\
& k_\xi^l + 2k_\xi^m + k_\xi^u - n_\xi^l - 2n_\xi^m - n_\xi^u \geq 0 \text{ for } \xi = 1,2, \\
& x_j^u - x_j^m \geq 0,\ x_j^m - x_j^l \geq 0,\ x_j^l \geq 0 \text{ for } j = 1,2, \\
& k_\xi^u - k_\xi^m \geq 0,\ k_\xi^m - k_\xi^l \geq 0 \text{ for } \xi = 1,2, \\
& n_\xi^u - n_\xi^m \geq 0,\ n_\xi^m - n_\xi^l \geq 0 \text{ for } \xi = 1,2, \\
& 0 \leq \lambda \leq 1.
\end{aligned} \tag{4.10}$$

The OFS of (4.10) is $\tilde{\mathbf{x}}^* = [(0, 0.16, 2.449), (0, 0, 0.609)]$.
**Step 5.** $\tilde{\mathbf{x}}^3 = \tilde{\mathbf{x}}^* = [(0, 0.16, 2.449), (0, 0, 0.609)]$ and the value of FFrO is $\tilde{Z}^3 = (-0.1336, 0.5371, 6.8976)$.
**Step 6.** Due to $\tilde{Z}^2 = \tilde{Z}^3$, $\tilde{\mathbf{x}}^2 = \tilde{\mathbf{x}}^* = [(0, 0.16, 2.449), (0, 0, 0.609)]$ and $\tilde{Z}^2 = (-0.1336, 0.5371, 6.8976)$ are the fuzzy solution of (4.1) and the FOF, respectively.

The proposed algorithm achieved a ranking score of 1.9595, which is higher than the ranking scores obtained by the Pop and Stancu-Minasian method [14] and the Stanojevic and Stancu-Minasian method [19] (See Table 1). This indicates that the proposed method outperforms the competing algorithms in this example.

**Example 2.** Consider the following FFLFrPP [15, 16]:

$$\begin{aligned}
&\max \tilde{Z}(\tilde{\mathbf{x}}) = \frac{(2,4,7) \otimes \tilde{x}_1 \oplus (1,3,4) \otimes \tilde{x}_2 \oplus (1,2,4)}{(1,2,3) \otimes \tilde{x}_1 \oplus (3,5,8) \otimes \tilde{x}_2 \oplus (0,1,2)}, \\
&\text{s.t. } (1,2,3) \otimes \tilde{x}_1 \oplus (0,1,2) \otimes \tilde{x}_2 \preceq (2,11,28), \\
&\quad\ (0,1,2) \otimes \tilde{x}_1 \oplus (1,2,3) \otimes \tilde{x}_2 \preceq (1,10,17), \\
&\quad\ \tilde{x}_j \geq \tilde{0} \quad j = 1,2.
\end{aligned} \tag{4.11}$$

The proposed algorithm demonstrated its superiority by achieving a remarkable ranking score of 7.9457, which significantly surpasses the ranking scores of the method of Safaei [15] and Kumar, Mandal, and Edalatpanah [16].

*4.2. Applications*

Linear fractional programming can be a highly valuable tool when it comes to creating models that allow us to make important decisions that arise in our lives. In these decision-making structures, it is possible to reflect the inherent uncertainty in decision-making by modeling the variables and parameters we consider with fuzzy numbers. Examples directly related to human life, such as logistics planning, production decisions, and portfolio optimization, can be cited as application areas where the full fuzzy integer linear programming problem can be addressed. In this context, we have provided solutions for these examples by addressing applications on production planning problems presented in Chinnadurai and Muthukumar [5] and Srinivasan [18].

**Example 3.** Take into account the production planning problem expressed as FFLFrPP [5]:

$$\begin{aligned} \max \tilde{Z}(\tilde{\mathbf{x}}) &= \frac{\tilde{10} \otimes \tilde{x}_1 \oplus \tilde{20} \otimes \tilde{x}_2 \oplus \tilde{10}}{\tilde{3} \otimes \tilde{x}_1 \oplus \tilde{4} \otimes \tilde{x}_2 \oplus \tilde{20}}, \\ \text{s.t. } & \tilde{1} \otimes \tilde{x}_1 \oplus \tilde{3} \otimes \tilde{x}_2 \preceq \tilde{50}, \\ & \tilde{3} \otimes \tilde{x}_1 \oplus \tilde{2} \otimes \tilde{x}_2 \preceq \tilde{80}, \\ & \tilde{x}_j \succeq \tilde{0} \quad j = 1, 2. \end{aligned} \tag{4.12}$$

The optimal value of FFrO is $\tilde{Z} = (0.7199, 2.804, 13.992)$.

**Example 4.** Tackle the material-related production planning problem of a wooden company in India expressed as FFLFrPP [18]:

$$\begin{aligned} \max \tilde{Z}(\tilde{\mathbf{x}}) &= \frac{(1,2,3) \otimes \tilde{x}_1 \oplus (0,1,2) \otimes \tilde{x}_2}{(1,3,5) \otimes \tilde{x}_1 \oplus (1,1,1) \otimes \tilde{x}_2 \oplus (3,6,9)}, \\ \text{s.t. } & (4,7,10) \otimes \tilde{x}_1 \oplus (0,1,2) \otimes \tilde{x}_2 \preceq (4,6,8), \\ & (1,5,9) \otimes \tilde{x}_1 \oplus (2,3,4) \otimes \tilde{x}_2 \preceq (5,6,7), \\ & \tilde{x}_j \succeq \tilde{0} \quad j = 1, 2. \end{aligned} \tag{4.13}$$

The optimal value of FFrO is $\tilde{Z} = (0.0356, 0.158, 2.2125)$.

**Example 5.** Furthermore, it is possible to fully embrace the fuzzy nature of large-scale real-world scenarios by incorporating individual objective ratios. In this context, our study goes beyond the limited applications found in the literature, generating large-scale Full Fuzzy Linear Fractional Programming Problems (FFLFrPPs) using randomly generated data and solving them through the proposed methodology. We investigate the impact of the Number of Fuzzy Variables (NoFV) and Number of Fuzzy Constraints (NoFC) on the overall execution time and iteration count of the newly developed approach, as outlined in Table 2. Test results demonstrate that our novel strategy for addressing real-world problems, even those of large-scale, efficiently handles the objective function solving process in terms of time and iterations. Additionally, Figure 2 presents the average elapsed time and average number of iterations for the designated (NoFV, NoFC) pairs. To ensure robustness, ten randomly generated test problems were solved for each class using the proposed approach. Furthermore, all computational tests were conducted using the GAMS 24.3.3 software on a computer equipped with an Intel 11th generation i7 CPU operating at 2.30 GHz and 16 GB of RAM.

**Table 2.** Results with the suggested algorithm for the created FFLFrPPs.

| Class | | Number of iterations | | | CPU Time (in seconds) | | |
|---|---|---|---|---|---|---|---|
| | $(NoFV, NoFC)$ | Min | Max | Average | Min | Max | Average |
| 1 | (10, 10) | 5 | 9 | 6.6667 | 1.931 | 2.3999 | 2.1273 |
| 2 | (25, 25) | 3 | 6 | 4.8 | 1.8199 | 2.223 | 2.0505 |
| 3 | (50, 50) | 4 | 6 | 5.5 | 2.0079 | 2.281 | 2.1899 |
| 4 | (75, 75) | 6 | 7 | 6.1 | 2.4119 | 2.7109 | 2.5754 |
| 5 | (100, 100) | 4 | 7 | 6 | 3.0989 | 4.2339 | 3.7125 |
| 6 | (150, 150) | 6 | 9 | 6.9 | 7.607 | 13.93 | 9.5591 |
| 7 | (200, 200) | 6 | 8 | 6.7 | 14.932 | 32.964 | 21.1138 |
| 8 | (250, 250) | 4 | 8 | 5.7 | 44.625 | 98.6399 | 67.2097 |
| 9 | (300, 300) | 5 | 8 | 6.2 | 88.6249 | 168.066 | 138.142 |
| 10 | (500, 500) | 5 | 8 | 6.5 | 733.247 | 1719.065 | 1141.71 |

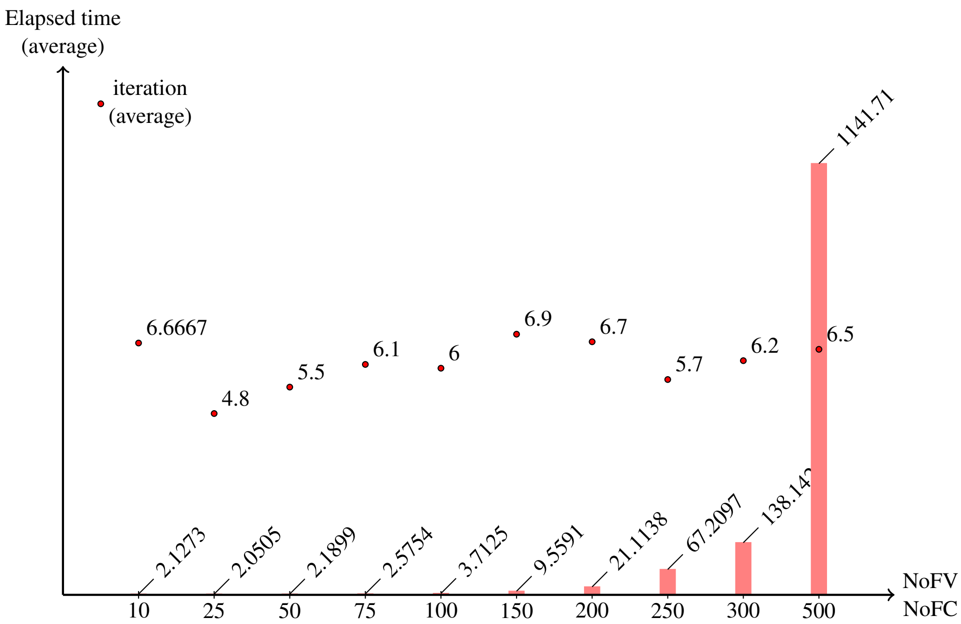


**Figure 2.** The average number of iterations and CPU time vs (NoFV, NoFC).

## 5. Comparison

In order to compare the proposed solution approach with the methods in the literature, examples taken from the existing literature have been solved. Then, the effectiveness of the proposed solution approach has been rigorously assessed through a comparative analysis with the results utilizing the ranking function, which is used in the literature to show the superiority of the algorithms. In particular, we have compared the outcomes achieved through our suggested methodology with those obtained from previously reported methods [14–16, 19]. As illustrated in Table 1, it is evident that our approach consistently yields higher ranking values for every numerical challenge when compared to alternative methods. Furthermore, due to the fact that our suggested approach solves only the LPP in each iteration, it lends itself to easy manual or computer-based implementation and management in terms of numerical processing load.

In addition, we have conducted a comprehensive series of tests, generating a large-scale set of test instances, which have never been addressed by the methods for FFLFrPP in the literature, at random to assess the adaptability of our proposed solution approach in real-world scenarios. The results of these tests are presented in Table 2. Notably, beyond the small-scale cases documented in the literature, our study reveals the substantial advantages of our approach in addressing fuzzy scenarios of varying scales. The proposed solution method differs from previous research in the field because it preserves the fuzziness from the establishment of the problem to the final stage of obtaining the solution and because it has been shown to be applicable to large-scale problems.

## 6. Conclusions

When dealing with problem structures designed using fuzzy set theory, researchers must navigate various challenges, including handling inverse fuzzy numbers. It is crucial to prioritize clarity over complexity to avoid potential misinterpretations. In response to these challenges, we introduced a straightforward, traditional continuity-based methodology for solving FFLFrPP. Unlike other approaches that require intricate fuzzy variable transformations, our method simplifies the process by incorporating Zimmermann's operator and transitioning to a simple, crisp LPP model, enabling us to obtain the optimal solution easily.

Our approach has led to improved objective function values compared to previously published studies. Additionally, through these simplifications, we have demonstrated the capability of our proposed algorithm to tackle large-scale FFLFrPPs, critical contribution that was previously unaddressed in the literature.

In addition, applying the proposed method to a practical situation that inherently involves uncertainty with real market data is a future research endeavor. As another future study, we are actively considering the expansion of our methodology to encompass multi-objective fuzzy linear fractional programming. This potential extension holds the promise of further advancing the applicability and scope of our proposed algorithm, addressing a critical research gap in the field.

## Author contributions

The authors contributed significantly, directly, and intellectually to the work and approved its publication. The published version of the work has been reviewed and approved by all authors.

## Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare no conflicts of interest.

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