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Piecewise implicit coupled system under ABC fractional differential equations with variable order

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Abstract: This research paper presented a novel investigation into an implicit coupled system of fractional variable order, which has not been previously studied in the existing literature. The study focused on establishing and developing sufficient conditions for the existence and uniqueness of solutions, as well as the Ulam-Hyers stability, for the proposed coupled system without using semigroup property. By extending the existing conclusions examined for the Atangana-Baleanu-Caputo (ABC) operator, we contributed to advancing the understanding of variable-order fractional differential equations. The paper provided a solid theoretical foundation for further analysis, numerical simulations, and practical applications. The obtained results have implications for designing and controlling systems modeled using fractional variable order equations and serve as a basis for addressing a wide range of dynamical problems. The transformation techniques, qualitative analysis, and illustrative examples presented in this work highlight its unique contributions and potential to serve as a foundation for future research.

1. Introduction

The fundamental concept of fractional calculus involves replacing natural numbers with rational numbers in the order of differentiation operators. Although this concept may initially appear undeveloped and straightforward, it carries profound implications and yields significant results that effectively describe phenomena in various fields, including bioengineering, dynamics, modeling,

control theory, and medicine. For more detailed information on this topic, we refer the reader to the following references [1–4]. Fractional differential equations (FDEs) play a critical role in modeling the growth of systems, particularly through nonlocal Cauchy-type problems that describe nonnegative quantities like species concentrations or temperature distributions. Before applying these models to real-world phenomena, it is crucial to establish the existence of solutions, which can be addressed using fixed point theory. Furthermore, it is essential to develop stable algorithms and reliable numerical procedures to obtain accurate results and ensure the practicality and validity of the models in real-life applications. The articles [5–8] and their references explore a wide range of interesting and innovative findings related to the existence and stability of coupled systems in various types of FDEs. These studies delve into different aspects of coupled systems and contribute to our understanding of their dynamics and properties. They provide valuable insights into the behavior and solutions of coupled FDEs, opening up avenues for further research and applications in this field. New strategies and methodologies have been developed to describe the dynamics of real problems. In particular, Caputo and Fabrizio [9] introduced a new class of fractional derivatives within the exponential kernel, which Losada and Nieto [10] explored further, uncovering intriguing characteristics. Atangana and Baleanu [11] investigated a novel type of fractional derivatives using Mittag-Leffler (ML) kernels. Abdeljawad [12] extended the ABC fractional derivative to higher arbitrary orders and included their associated integral operators. The reader may find theoretical works on the ABC fractional operator in references [13–16].

It is evident that the challenges posed by operators of variable order are more complex compared to operators of constant order. Recently, several authors have explored the applications of variable order derivatives in various scientific fields, including multi-fractional Gaussian noises, mechanical applications, and anomalous diffusion models. Among these, numerous studies have focused on developing numerical techniques for a specific class of variable-order FDEs, as exemplified by [17–20]. In this regard, Lorenzo et al. [21], utilizing fractional operators with variable order, addressed the problem of fractional diffusion. Subsequently, fractional variable order has found applications in various areas of science and engineering (see [22–24]). Sun et al. [25] conducted a comparative study between constant and variable order models to describe systems with memory. Various techniques are available for approximating solutions to fractional boundary value problems with variable orders. For instance, the authors in [26] developed a nonlinear model of alcoholism within the framework of FDEs with variable order and examined its solutions numerically and analytically. Recently, Li et al [27] investigated the following fractional problem using a novel numerical strategy

$$\begin{cases} {}^{ML}\mathbf{D}_{0^+}^{\omega(\iota)}y(\iota) + a(\iota)y(\iota) = \mathcal{H}(\iota, y(\iota)), & \iota \in [0, 1], \\ \qquad\qquad B(y) = 0, \end{cases} \tag{1.1}$$

where, ${}^{ML}\mathbf{D}_{0^+}^{\omega(\iota)}$ is the Mittag-Leffler fractional derivative of order $\omega(\iota)$. Kaabar et al. [28] examined the qualitative characteristics of the following implicit FDEs with variable order

$$\begin{cases} {}^{C}\mathbf{D}_{0^+}^{\omega(\iota)}y(\iota) = \mathcal{H}(\iota, y(\iota), {}^{C}\mathbf{D}_{0^+}^{\omega(\iota)}y(\iota)), & \iota \in [0, b], \\ \qquad\qquad y(0) = 0, y(b) = 0, \end{cases} \tag{1.2}$$

where, ${}^{C}\mathbf{D}_{0^+}^{\omega(\iota)}$ is the Caputo fractional differential (FD) of variable order $\omega(\iota)$. Jeelani et al. [29] employed generalized intervals, Krasnoselskii and Banach's fixed point theorems, and piecewise constant functions to conduct qualitative analyses on a nonlinear fractional integrodifferential equation.

Based on the provided justifications, we are motivated to evaluate and investigate the necessary conditions for a coupled system within the context of the ABC-fractional derivative with a variable order as the following

$$\begin{cases} {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_1(\iota) = \mathcal{H}_1(\iota, y_2(\iota), {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_1(\iota)), & \iota \in \mathcal{J} := (0, b], \\ {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_2(\iota) = \mathcal{H}_2(\iota, y_1(\iota), {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_2(\iota)), & \iota \in \mathcal{J} := (0, b], \\ y_1(0) = 0, y_1(b) = \sum_{k=1}^{n} y_1(\tau_k), \ \tau_k \in (0, b), \\ y_2(0) = 0, y_2(b) = \sum_{k=1}^{n} y_2(\gamma_k), \ \gamma_k \in (0, b), \end{cases} \tag{1.3}$$

where

i) ${}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)}$ is the ABC-FD of variable order $\omega(\iota) \in (1, 2]$.

ii) $\tau_k, \gamma_k \in \mathbb{R}$, $k = 1, 2, ..., n$, are prefixed points satisfying $0 < \kappa_1 \leq \kappa_2 \leq ... \leq \kappa_n < b$ and $0 < \gamma_1 \leq \gamma_2 \leq ... \leq \gamma_n < b$.

iii) $\mathcal{H}_1, \mathcal{H}_2 : \mathcal{J} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions that comply with a few later-described presumptions.

To the best of our knowledge, no studies in the aforementioned body of research address an implicit coupled system of fractional variable order. The proposed implicit coupled system (1.3) is a more general form compared to the previous problems (1.1) and (1.2) in [26, 28, 29]. The proposed implicit coupled system (1.3) contains nonlocal conditions and variable order $\omega(\iota) \in (1, 2]$. The complexity of calculations and partitioning the principal time interval has limited the number of studies specifically addressing the stability and existence of fractional integrodifferential boundary value problems with variable order. To bridge this gap, we conduct a comprehensive qualitative analysis of a new implicit coupled system (1.3) with a variable order within the framework of ABC fractional operators. By employing extended intervals and piecewise constant functions, we accurately transform the coupled system into an equivalent standard system with constant order in the context of the ABC-fractional derivative. The existence and uniqueness of solutions to the coupled system (1.3) are then demonstrated using fixed point techniques such as Krasnoselskii and Banach theorems. The Ulam-Hyers stability result of the suggested problem is also discussed. Additionally, we establish suitable conditions for the existence of effective solutions to the coupled system (1.3). The contributions of this work lie in providing a framework to address the challenges associated with variable-order FDEs involving the ABC fractional operator. The study offers transformation techniques, investigation of qualitative properties, existence and uniqueness analysis, stability analysis, and illustrative examples to support its findings. The results obtained in this study will contribute to understanding the stability properties of the considered implicit coupled system of fractional variable order. This knowledge is valuable for designing and controlling systems modeled using such equations, ensuring their robustness, and predicting their behavior accurately.

The paper's structure is as follows: Section 2 offers an overview of fractional calculus notations, definitions, and relevant lemmas, including a crucial lemma that converts the coupled system of ABC-fractional derivative into an equivalent integral equation. Section 3 focuses on establishing the existence and uniqueness of solutions for the coupled system, while Section 4 investigates the Ulam-Hyers stability of the system. To illustrate the obtained results, Section 5 presents numerical examples. Finally, the paper concludes with a summary of the findings, concluding remarks, and suggestions for potential future research directions.

2. Preliminary results and essential concepts

Let $C(\mathcal{J}, \mathbb{R})$ be a Banach space consisting of all continuous functions $y : \mathcal{J} \to \mathbb{R}$, where $\mathcal{J} := (0, b]$ with the following supremum norm

$$\|y\| = \sup\{|y(\iota)| : \iota \in \mathcal{J}\}.$$

Define the product space $\Upsilon = C(\mathcal{J}, \mathbb{R}) \times C(\mathcal{J}, \mathbb{R})$. Thus, Υ is Banach space with the following norm

$$\|(y_1, y_2)\|_{\Upsilon} = \|y_1\| + \|y_2\|,$$

for each $(y_1, y_2) \in \Upsilon$.

Derivatives and integrals that have orders as functions of specific variables belong to the category of operators with variable order, which is a more complex classification. Variable order fractional integrals and derivatives can be defined in various ways.

Definition 2.1. *[17] Let $\omega(\iota) \in (n-1, n]$, $y \in H^1(\mathcal{J})$, then, the left ABC fractional derivative of order $\omega(\iota)$ for function $y(\iota)$ with the lower limit zero are defined as*

$$^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y(\iota) = \frac{\Lambda(\omega(\iota))}{1 - \omega(\iota)} \int_0^\iota E_{\omega(\iota)} \left(\frac{\omega(\iota)}{\omega(\iota) - 1} (\iota - \theta)^\omega \right) y'(\theta) d\theta, \quad \iota > 0. \tag{2.1}$$

The normalization function $\Lambda(\omega(\iota))$ satisfies $\Lambda(0) = \Lambda(1) = 1$, where $E_{\omega(\iota)}$ is the ML function defined by

$$\mathbf{E}_{\omega(\iota)}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k\omega(\iota) + 1)}, \quad Re(\omega) > 0, y \in \mathbb{C}.$$

The left Atangana-Baleanu (AB) fractional integral of order $\omega(\iota)$ for function $y(\iota)$ are defined as

$$^{AB}\mathbf{I}_{0^+}^{\omega(\iota)} y(\iota) = \frac{1 - \omega(\iota)}{\Lambda(\omega(\iota))} y(\iota) + \frac{\omega(\iota)}{\Lambda(\omega(\iota))\Gamma(\omega(\iota))} \int_0^\iota (\iota - s)^{\omega(\iota) - 1} y(s) ds.$$

Lemma 2.2. *[12] Let $y(\iota)$ be a function defined on the interval $[0, b]$ and $n < \omega \le n + 1$, $n \in \mathbb{N}_0$. Then, we have*

$$\left(^{AB}\mathbf{I}_{0^+}^{\omega ABC}\mathbf{D}_{0^+}^{\omega} y \right)(\iota) = y(\iota) - \sum_{k=0}^{n} \frac{y^{(k)}(0)}{k!} \iota^k.$$

Theorem 2.3. *[30] Let X be a Banach space and $S \subset X$ be a nonempty, closed, convex and bounded subset. If there exist two operators Φ^1, Φ^2 satisfying the following conditions:*

(1) $\Phi^1 u + \Phi^2 v \in X$, for all $u, v \in X$.
(2) Φ^1 is completely continuous and Φ^2 is a contraction mapping.

Then, there exists a function $z \in S$ such that $z = \Phi^1 z + \Phi^2 z$.

Remark 2.4. *Let $u(\iota), v(\iota) \in C(\mathcal{J}, \mathbb{R})$ be two continuous functions. Then, we have*

$$^{AB}\mathbf{I}_{0^+}^{u(\iota)} {}^{AB}\mathbf{I}_{0^+}^{v(\iota)} y(\iota) \ne {}^{AB}\mathbf{I}_{0^+}^{u(\iota) + v(\iota)} y(\iota).$$

Lemma 2.5. *[12] Let $\omega \in (1, 2]$ and $\hbar \in C([a, b], \mathbb{R})$. Then, the following integral equation*

$$y(\iota) = c_1 + c_2(\iota - a) + {}^{AB}\mathbf{I}_{a^+}^{\omega}\hbar(\iota),$$

is an equivalent to the following linear problem

$$\begin{cases} {}^{ABC}\mathbf{D}_{a^+}^{\omega}y(\iota) = \hbar(\iota), \\ y(a) = c_1, y'(a) = c_2, \end{cases}$$

where

$${}^{AB}\mathbf{I}_{a^+}^{\omega}\hbar(\iota) = \frac{2 - \omega}{\Lambda(\omega - 1)}\int_a^{\iota}\hbar(s)ds + \frac{\omega - 1}{\Lambda(\omega - 1)\Gamma(\omega)}\int_a^{\iota}(\iota - s)^{\omega - 1}\hbar(s)ds. \tag{2.2}$$

2.1. Notations

For the sake of simplicity in our analysis, we will use the following notations:

$$\Theta_1 = (b_i - b_{i-1}) - \sum_{k=1}^{n}(\tau_k - b_{i-1}) \neq 0,$$

$$\Theta_2 = (b_i - b_{i-1}) - \sum_{k=1}^{n}(\gamma_k - b_{i-1}) \neq 0,$$

$$P_1 = \frac{2 - \omega_i}{\Lambda(\omega_i - 1)}, P_2 = \frac{\omega_i - 1}{\Lambda(\omega_i - 1)}, P_3 = \frac{2 - \omega_i}{\Lambda(\omega_i - 1)}, P_4 = \frac{\omega_i - 1}{\Lambda(\omega_i - 1)\Gamma(\omega_i)},$$

$$Z_{\Pi} = \left(P_3(b_i - b_{i-1}) + \frac{P_4}{\Gamma(\omega_i + 1)}(b_i - b_{i-1})^{\omega_i}\right),$$

$$\begin{aligned} \mathcal{G}_{\Pi_1} &= \frac{P_1(b_i - b_{i-1})}{\Theta_1}\left(\sum_{k=1}^{n}(\tau_k - b_{i-1}) + (b_i - b_{i-1})\right) \\ &+ \frac{P_2(b_i - b_{i-1})}{\Theta_1\Gamma(\omega_i + 1)}\left(\sum_{k=1}^{n}(\tau_k - b_{i-1})^{\omega_i} + (b_i - b_{i-1})^{\omega_i}\right), \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{\Pi_2} &= \frac{P_1(b_i - b_{i-1})}{\Theta_2}\left(\sum_{k=1}^{n}(\gamma_k - b_{i-1}) + (b_i - b_{i-1})\right) \\ &+ \frac{P_2(b_i - b_{i-1})}{\Theta_2\Gamma(\omega_i + 1)}\left(\sum_{k=1}^{n}(\gamma_k - b_{i-1})^{\omega_i} + (b_i - b_{i-1})^{\omega_i}\right), \end{aligned}$$

$$\mathcal{R}_{\Pi_1} = \mathcal{G}_{\Pi_1} + Z_{\Pi} \text{ and } \mathcal{R}_{\Pi_2} = \mathcal{G}_{\Pi_2} + Z_{\Pi},$$

and

$$\Omega_1 = \mathcal{R}_{\Pi_1}\frac{\mathfrak{N}_{\mathcal{H}_1}}{1 - \mathfrak{N}_{\mathcal{H}_1}} \text{ and } \Omega_2 = \mathcal{R}_{\Pi_2}\frac{\mathfrak{N}_{\mathcal{H}_2}}{1 - \mathfrak{N}_{\mathcal{H}_2}}. \tag{2.3}$$

2.2. Equivalent integral equations

In this part, we convert the coupled system (1.3) with variable-order $\omega(\iota)$ into an equivalent integral equations. To begin, we transform the coupled system with variable order into a coupled system with constant order by using generalized intervals and piecewise constant functions. Let $\mathcal{B}$ be a partition of the interval $\mathcal{J}$ defined as

$$\mathcal{B} = \{\mathcal{J}_1 = [b_0, b_1], \mathcal{J}_2 = (b_1, b_2], \mathcal{J}_3 = (b_2, b_3], \ldots, \mathcal{J}_n = (b_{n-1}, b_n]\},$$

such that $b_0 = 0, b_n = b$ and $n \in \mathbb{N}$. Let $\omega(\iota) : \mathcal{J} \to (1, 2]$ be a piecewise constant function concerning $\mathcal{B}$. This means that the function $\omega(\iota)$ takes values in the interval $(1, 2]$ and is constant over different intervals or "pieces" defined by the set $\mathcal{B}$. In other words, for each interval in $\mathcal{B}$, the function $\omega(\iota)$ remains constant within that interval. The specific values of $\omega(\iota)$ within each interval can vary, but they will always be within the range of $(1, 2]$. The set $\mathcal{B}$ defines the partition or division of the domain $\mathcal{J}$ into different intervals, and within each interval, the function $\omega(\iota)$ takes on a constant value. So, the function $\omega(\iota)$ can be a representation as

$$\omega(\iota) = \sum_{i=1}^{n} \omega_i \mathcal{U}_i(\iota) = \begin{cases} \omega_1, & if \ \iota \in \mathcal{J}_1, \\ \omega_2, & if \ \iota \in \mathcal{J}_2, \\ & . \\ & . \\ & . \\ \omega_n, & if \ \iota \in \mathcal{J}_n, \end{cases}$$

where, $\omega_i \in (1, 2]$ are constants numbers, and $\mathcal{U}_i$ is the indicator of the interval $\mathcal{J}_i := (b_{i-1}, b_i]$, $i = 1, 2, ..., n$ such that

$$\mathcal{U}_i(\iota) = \begin{cases} 1, & if \ \iota \in \mathcal{J}_i, \\ 0, & for \ elsewhere. \end{cases}$$

Let $C(\mathcal{J}_i, \mathbb{R})$, $i \in \{1, 2, ..., n\}$ be a Banach space of all continuous functions defined as $y_i : \mathcal{J}_i \to \mathbb{R}$ with the following norm

$$\|y_i\| = \sup\{|y(\iota)| : \iota \in \mathcal{J}_i\}.$$

By (2.1), the coupled system (1.3) can be expressed as

$$\begin{cases} \frac{\Lambda(\omega(\iota))}{1-\omega(\iota)} \int_0^{\iota} E_{\omega(\iota)} \left(\frac{\omega(\iota)}{\omega(\iota)-1} (\iota - \theta)^{\omega} \right) y_1'(\theta) d\theta = \mathcal{H}_1(\iota, y_2(\iota), {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_1(\iota)), \\ \frac{\Lambda(\omega(\iota))}{1-\omega(\iota)} \int_0^{\iota} E_{\omega(\iota)} \left(\frac{\omega(\iota)}{\omega(\iota)-1} (\iota - \theta)^{\omega} \right) y_2'(\theta) d\theta = \mathcal{H}_2(\iota, y_1(\iota), {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_2(\iota)). \end{cases} \tag{2.4}$$

Then, for $\iota \in \mathcal{J}_i, i = 1, 2, ..., n$ the coupled system (2.4) on $\mathcal{J}_i$ can express as a sum of left ABC-coupled systems of constant-orders $\omega_1, \omega_2, ... \omega_n$ as follows

$$\begin{cases} \frac{\Lambda(\omega_1)}{1-\omega_1} \int_0^{b_1} E_{\omega_1} \left(\frac{\omega_1(\iota-\theta)^{\omega_1}}{\omega_1-1} \right) y_1'(\theta) d\theta + ... + \frac{\Lambda(\omega_i)}{1-\omega_i} \int_{b_{i-1}}^{\iota} E_{\omega_i} \left(\frac{\omega_i(\iota-\theta)^{\omega_i}}{\omega_i-1} \right) y_1'(\theta) d\theta = \mathcal{H}_1(\iota, y_2(\iota), {}^{ABC}\mathbf{D}_{0^+}^{\omega_i} y_1(\iota)), \\ \\ \frac{\Lambda(\omega_1)}{1-\omega_1} \int_0^{b_1} E_{\omega_1} \left(\frac{\omega_1(\iota-\theta)^{\omega_1}}{\omega_1-1} \right) y_2'(\theta) d\theta + ... + \frac{\Lambda(\omega_i)}{1-\omega_i} \int_{b_{i-1}}^{\iota} E_{\omega_i} \left(\frac{\omega_i(\iota-\theta)^{\omega_i}}{\omega_i-1} \right) y_2'(\theta) d\theta = \mathcal{H}_2(\iota, y_1(\iota), {}^{ABC}\mathbf{D}_{0^+}^{\omega_i} y_2(\iota)). \end{cases} \tag{2.5}$$

Let the function (y_1, y_2) be such that $y_1(\iota) \equiv y_2(\iota) \equiv 0$ on $\iota \in [0, b_{i-1}], i = 1, 2, ..., n$, and such that it solves the system (2.5). Hence, the coupled system (1.3) with variable-order $\omega(\iota)$ is reduced to the following system

$$
\begin{cases}
{}^{ABC}\mathbf{D}_{b_{i-1}^+}^{\omega_i} y_1(\iota) = \mathcal{H}_1(\iota, y_2(\iota), {}^{ABC}\mathbf{D}_{b_{i-1}^+}^{\omega_i} y_1(\iota)), \iota \in \mathcal{J}_i, \\
{}^{ABC}\mathbf{D}_{b_{i-1}^+}^{\omega_i} y_2(\iota) = \mathcal{H}_2(\iota, y_1(\iota), {}^{ABC}\mathbf{D}_{b_{i-1}^+}^{\omega_i} y_2(\iota)), \iota \in \mathcal{J}_i, \\
y_1(b_{i-1}) = 0, \ y_1(b_i) = \sum_{k=1}^n y_1(\tau_k), \ \tau_k \in (0, b), \\
y_2(b_{i-1}) = 0, \ y_2(b_i) = \sum_{k=1}^n y_2(\gamma_k), \ \gamma_k \in (0, b).
\end{cases}
\tag{2.6}
$$

Now, to analyze the properties of solutions using fixed point theorems, we transform the coupled system (2.6) into an equivalent integral equation. For that, we present the following theorem.

Theorem 2.6. *For $i = 1, 2, ..., n$, let $\omega_i \in (1, 2]$, and $\mathcal{H} : \mathcal{J}_i \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and $\Theta_1, \Theta_2 \neq 0$. If $(y_1, y_2) \in \Upsilon$ is a solution of coupled system (2.6), then, (y_1, y_2) satisfies the following fractional integral equations*

$$
y_1(\iota) = \frac{(\iota - b_{i-1})}{\Theta_1} \left[\sum_{k=1}^n {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(\tau_k) - {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(b_i) \right] + {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(\iota),
\tag{2.7}
$$

and

$$
y_2(\iota) = \frac{(\iota - b_{i-1})}{\Theta_2} \left[\sum_{k=1}^n {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_2}(\gamma_k) - {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_2}(b_i) \right] + {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_2}(\iota),
\tag{2.8}
$$

where

$$
\mathcal{H}_{y_1}(\iota) = \mathcal{H}_1(\iota, y_2(\iota), {}^{ABC}\mathbf{D}_{b_{i-1}^+}^{\omega_i} y_1(\iota)),
$$
$$
\mathcal{H}_{y_2}(\iota) = \mathcal{H}_2(\iota, y_1(\iota), {}^{ABC}\mathbf{D}_{b_{i-1}^+}^{\omega_i} y_2(\iota)).
$$

Proof. We assume that $(y_1, y_2) \in \Upsilon$ is a solution of coupled system (2.6). Apply the operator ${}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i}$ to both sides of (2.6) and using Lemma 2.5, we get

$$
y_1(\iota) = c_1 + c_2(\iota - b_{i-1}) + {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(\iota),
\tag{2.9}
$$

and

$$
y_2(\iota) = c_3 + c_4(\iota - b_{i-1}) + {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_2}(\iota).
\tag{2.10}
$$

By conditions $y_1(b_{i-1}) = 0$ and $y_2(b_{i-1}) = 0$, we get $c_1 = c_3 = 0$. Hence. Eqs (2.9) and (2.10) become as

$$
y_1(\iota) = c_2(\iota - b_{i-1}) + {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(\iota),
$$

and

$$
y_2(\iota) = c_4(\iota - b_{i-1}) + {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_2}(\iota).
$$

By second conditions $y_1(b_i) = \sum_{k=1}^n y_1(\tau_k)$ and $y_2(b_i) = \sum_{k=1}^n y_2(\gamma_k)$, we get

$$
c_2 = \frac{1}{\Theta_1} \left[\sum_{k=1}^n {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(\tau_k) - {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(b_i) \right],
$$

and

$$
c_4 = \frac{1}{\Theta_2} \left[\sum_{k=1}^n {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_2}(\gamma_k) - {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_2}(b_i) \right].
$$

Substitute the values of c_1, c_2, c_3, c_4 into (2.9) and (2.10), and we get (2.7) and (2.8). Conversely, assume that $(y_1, y_2) \in \Upsilon$ satisfies (2.7) and (2.8). Then, by applying ${}^{ABC}\mathbf{D}_{b_{i-1}^+}^{\omega_i}$ on both sides of (2.7) and (2.8), with using the fact ${}^{ABC}\mathbf{D}_{b_{i-1}^+}^{\omega_i}(\iota - b_{i-1}) = 0$, we get (2.6). Next, take $\iota \to \tau_k$ in (2.7), and we get

$$
\begin{aligned}
\sum_{k=1}^{n} y_1(\tau_k) &= \frac{\sum_{k=1}^{n}(\tau_k - b_{i-1})}{\Theta_1} \left[\sum_{k=1}^{n} {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(\tau_k) - {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(b_i) \right] \\
&\quad + \sum_{k=1}^{n} {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(\tau_k) \\
&= \frac{(b_i - b_{i-1}) - \Theta_1}{\Theta_1} \left[\sum_{k=1}^{n} {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(\tau_k) - {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(b_i) \right] \\
&\quad + \sum_{k=1}^{n} {}^{AB}\mathbf{I}_{b_{i-1}^+}^{\omega_i} \mathcal{H}_{y_1}(\tau_k) \\
&= y_1(b_i).
\end{aligned}
$$

In the same manner, we get $y_2(b_i) = \sum_{k=1}^{n} y_2(\gamma_k)$. Thus, the nonlocal conditions are satisfied. □

For further analysis, we simplify the solutions (2.7) and (2.8) as

$$
\begin{aligned}
y_1(\iota) &= \frac{P_1(\iota - b_{i-1})}{\Theta_1} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} \mathcal{H}_{y_1}(s)\,ds - \int_{b_{i-1}}^{b_i} \mathcal{H}_{y_1}(s)\,ds \right) \\
&\quad + \frac{P_2(\iota - b_{i-1})}{\Theta_1 \Gamma(\omega_i)} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} (\tau_k - s)^{\omega_i - 1} \mathcal{H}_{y_1}(s)\,ds - \int_{b_{i-1}}^{b_i} (b_i - s)^{\omega_i - 1} \mathcal{H}_{y_1}(s)\,ds \right) \\
&\quad + P_3 \int_{b_{i-1}}^{\iota} \mathcal{H}_{y_1}(s)\,ds + \frac{P_4}{\Gamma(\omega_i)} \int_{b_{i-1}}^{\iota} (\iota - s)^{\omega_i - 1} \mathcal{H}_{y_1}(s)\,ds,
\end{aligned}
$$

and

$$
\begin{aligned}
y_2(\iota) &= \frac{P_1(\iota - b_{i-1})}{\Theta_2} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\gamma_k} \mathcal{H}_{y_2}(s)\,ds - \int_{b_{i-1}}^{b_i} \mathcal{H}_{y_2}(s)\,ds \right) \\
&\quad + \frac{P_2(\iota - b_{i-1})}{\Theta_2 \Gamma(\omega_i)} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\gamma_k} (\gamma_k - s)^{\omega_i - 1} \mathcal{H}_{y_2}(s)\,ds - \int_{b_{i-1}}^{b_i} (b_i - s)^{\omega_i - 1} \mathcal{H}_{y_2}(s)\,ds \right) \\
&\quad + P_3 \int_{b_{i-1}}^{\iota} \mathcal{H}_{y_2}(s)\,ds + \frac{P_4}{\Gamma(\omega_i)} \int_{b_{i-1}}^{\iota} (\iota - s)^{\omega_i - 1} \mathcal{H}_{y_2}(s)\,ds.
\end{aligned}
$$

3. Existence and uniqueness of solutions

In this part, we will examine the necessary conditions for the existence and uniqueness for the coupled system (2.6) by using some fixed point theorems.

Theorem 3.1. *Suppose that*

(H_1) *The functions* $\mathcal{H}_1, \mathcal{H}_2 : \mathcal{J}_i \times \mathbb{R}^2 \rightarrow \mathbb{R}$ *are continuous and there exist constant numbers* $\mathfrak{N}_{\mathcal{H}_1}, \mathfrak{N}_{\mathcal{H}_2} > 0$ *such that*

$$|\mathcal{H}_1(\iota, x, y) - \mathcal{H}_1(\iota, \overline{x}, \overline{y})| \leq \mathfrak{N}_{\mathcal{H}_1}(|x - \overline{x}| + |y - \overline{y}|), \mathfrak{N}_{\mathcal{H}_1} > 0,$$
$$|\mathcal{H}_2(\iota, x, y) - \mathcal{H}_2(\iota, \overline{x}, \overline{y})| \leq \mathfrak{N}_{\mathcal{H}_2}(|x - \overline{x}| + |y - \overline{y}|), \mathfrak{N}_{\mathcal{H}_2} > 0,$$

for all $x, y, \overline{x}, \overline{y} \in C(\mathcal{J}_i, \mathbb{R})$. *Then, the coupled system (2.6) has a unique solution provided that* $\max\{\Omega_1, \Omega_2\} < 1$, *where* Ω_1 *and* Ω_2 *are defined by (2.3).*

Proof. On the light of Theorem 2.6, we define the operator $\mathbf{\Pi} : \Upsilon \rightarrow \Upsilon$ as

$$\mathbf{\Pi}(y_1, y_2)(\iota) = (\mathbf{\Pi}_1(y_1, y_2)(\iota), \mathbf{\Pi}_2(y_1, y_2)(\iota)),$$

where

$$\begin{aligned}
\mathbf{\Pi}_1(y_1, y_2)(\iota) &= \frac{P_1(\iota - b_{i-1})}{\Theta_1}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\tau_k}\mathcal{H}_{y_1}(s)\,ds - \int_{b_{i-1}}^{b_i}\mathcal{H}_{y_1}(s)\,ds\right) \\
&\quad + \frac{P_2(\iota - b_{i-1})}{\Theta_1\Gamma(\omega_i)}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\tau_k}(\tau_k - s)^{\omega_i - 1}\mathcal{H}_{y_1}(s)\,ds - \int_{b_{i-1}}^{b_i}(b_i - s)^{\omega_i - 1}\mathcal{H}_{y_1}(s)\,ds\right) \\
&\quad + P_3\int_{b_{i-1}}^{\iota}|\mathcal{H}_{y_1}(s)|\,ds + \frac{P_4}{\Gamma(\omega_i)}\int_{b_{i-1}}^{\iota}(\iota - s)^{\omega_i - 1}\mathcal{H}_{y_1}(s)\,ds,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{\Pi}_2(y_1, y_2)(\iota) &= \frac{P_1(\iota - b_{i-1})}{\Theta_2}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\gamma_k}\mathcal{H}_{y_2}(s)\,ds - \int_{b_{i-1}}^{b_i}\mathcal{H}_{y_2}(s)\,ds\right) \\
&\quad + \frac{P_2(\iota - b_{i-1})}{\Theta_2\Gamma(\omega_i)}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\gamma_k}(\gamma_k - s)^{\omega_i - 1}\mathcal{H}_{y_2}(s)\,ds - \int_{b_{i-1}}^{b_i}(b_i - s)^{\omega_i - 1}\mathcal{H}_{y_2}(s)\,ds\right) \\
&\quad + P_3\int_{b_{i-1}}^{\iota}|\mathcal{H}_{y_2}(s)|\,ds + \frac{P_4}{\Gamma(\omega_i)}\int_{b_{i-1}}^{\iota}(\iota - s)^{\omega_i - 1}\mathcal{H}_{y_2}(s)\,ds.
\end{aligned}$$

Let β_{η_i} be a closed ball defined as

$$\beta_{\eta_i} = \{(y_1, y_2) \in \Upsilon : \|(y_1, y_2)\|_{\Upsilon} \leq \eta_i\},$$

with radius $\eta_i \geq \frac{(\Phi_1 + \Phi_2)}{1 - \max\{\Omega_1, \Omega_2\}}$, and

$$\Phi_i = \mathcal{R}_{\Pi_i}\mathcal{L}_{\mathcal{H}_i}, i = 1, 2,$$

where $\mathcal{L}_{\mathcal{H}_i} = \max_{\iota \in \mathcal{J}_i}|\mathcal{H}_i(\iota, 0, 0)|$. Now, we show that $\mathbf{\Pi}\beta_{\eta_i} \subset \beta_{\eta_i}$. For the operator $\mathbf{\Pi}_1$, let $(y_1, y_1) \in \beta_{\eta_i}$ and $\iota \in \mathcal{J}_i$. Then, we have

$$\begin{aligned}
|\mathbf{\Pi}_1(y_1, y_2)(\iota)| &= \frac{P_1(\iota - b_{i-1})}{\Theta_1}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\tau_k}|\mathcal{H}_{y_1}(s)|\,ds - \int_{b_{i-1}}^{b_i}|\mathcal{H}_{y_1}(s)|\,ds\right) \\
&\quad + \frac{P_2(\iota - b_{i-1})}{\Theta_1\Gamma(\omega_i)}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\tau_k}(\tau_k - s)^{\omega_i - 1}|\mathcal{H}_{y_1}(s)|\,ds - \int_{b_{i-1}}^{b_i}(b_i - s)^{\omega_i - 1}|\mathcal{H}_{y_1}(s)|\,ds\right)
\end{aligned}$$

$$+P_3 \int_{b_{i-1}}^{\iota} \left|\mathcal{H}_{y_1}(s)\right| ds + \frac{P_4}{\Gamma(\omega_i)} \int_{b_{i-1}}^{\iota} (\iota - s)^{\omega_i - 1} \left|\mathcal{H}_{y_1}(s)\right| ds.$$

By (H_1), we have

$$\begin{aligned}
\left|\mathcal{H}_{y_1}(\iota)\right| &= \left|\mathcal{H}_1(\iota)\right| = \left|\mathcal{H}_1(\iota, y_2(\iota), {}^{ABC}\mathbf{D}_{b_{i-1}^+}^{\omega_i} y_1(\iota)) - \mathcal{H}_1(\iota, 0, 0)\right| + \left|\mathcal{H}_1(\iota, 0, 0)\right| \\
&\leq \frac{\mathfrak{N}_{\mathcal{H}_1}}{1 - \mathfrak{N}_{\mathcal{H}_1}} \left|y_2(\iota)\right| + \mathcal{L}_{\mathcal{H}_1}.
\end{aligned} \tag{3.1}$$

Hence

$$\begin{aligned}
\|\mathbf{\Pi}_1(y_1, y_2)\| &\leq \frac{P_1(\iota - b_{i-1})}{\Theta_1} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} \left|\mathcal{H}_{y_1}(s)\right| ds - \int_{b_{i-1}}^{b_i} \left|\mathcal{H}_{y_1}(s)\right| ds \right) \\
&\quad + \frac{P_2(\iota - b_{i-1})}{\Theta_1 \Gamma(\omega_i)} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} (\tau_k - s)^{\omega_i - 1} \left|\mathcal{H}_{y_1}(s)\right| ds - \int_{b_{i-1}}^{b_i} (b_i - s)^{\omega_i - 1} \left|\mathcal{H}_{y_1}(s)\right| ds \right) \\
&\quad + P_3 \int_{b_{i-1}}^{\iota} \left|\mathcal{H}_{y_1}(s)\right| ds + \frac{P_4}{\Gamma(\omega_i)} \int_{b_{i-1}}^{\iota} (\iota - s)^{\omega_i - 1} \left|\mathcal{H}_{y_1}(s)\right| ds \\
&\leq \Omega_1 \|y_2\| + \Phi_1.
\end{aligned}$$

In the same manner, for the operator $\mathbf{\Pi}_2$, we get

$$\|\mathbf{\Pi}_2(y_1, y_2)\| \leq \Omega_2 \|y_1\| + \Phi_2.$$

Hence, by the fact $\|\mathbf{\Pi}(y_1, y_2)\|_\Upsilon = \|\mathbf{\Pi}_1(y_1, y_2)\| + \|\mathbf{\Pi}_2(y_1, y_1)\|$, we get

$$\begin{aligned}
\|\mathbf{\Pi}(y_1, y_2)\|_\Upsilon &\leq \Omega_1 \|y_1\| + \Omega_2 \|y_2\| + (\Phi_1 + \Phi_2) \\
&\leq \max\{\Omega_1, \Omega_2\} (\|y_1\| + \|y_2\|) + (\Phi_1 + \Phi_2) \\
&\leq \max\{\Omega_1, \Omega_2\} \|(y_1, y_2)\|_\Upsilon + (\Phi_1 + \Phi_2) \\
&\leq \eta_i.
\end{aligned}$$

Thus $\mathbf{\Pi}(y_1, y_2) \in \beta_{\eta_i}$. Now, we shall prove that $\mathbf{\Pi}(y_1, y_2)$ is a contraction. Let $(y_1, y_2), (\widehat{y_1}, \widehat{y_2}) \in \beta_{\eta_i}$ and $\iota \in \mathcal{J}_i$. Then, for the operator $\mathbf{\Pi}_1$, we have

$$\begin{aligned}
&\left|\mathbf{\Pi}_1(y_1, y_2)(\iota) - \mathbf{\Pi}_1(\widehat{y_1}, \widehat{y_2})(\iota)\right| \\
&\leq \frac{P_1(\iota - b_{i-1})}{\Theta_1} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} \left|\mathcal{H}_{y_1}(s) - \mathcal{H}_{\widehat{y_1}}(s)\right| ds + \int_{b_{i-1}}^{b_i} \left|\mathcal{H}_{y_1}(s) - \mathcal{H}_{\widehat{y_1}}(s)\right| ds \right) \\
&\quad + \frac{P_2(\iota - b_{i-1})}{\Theta_1 \Gamma(\omega_i)} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} (\tau_k - s)^{\omega_i - 1} \left|\mathcal{H}_{y_1}(s) - \mathcal{H}_{\widehat{y_1}}(s)\right| ds + \int_{b_{i-1}}^{b_i} (b_i - s)^{\omega_i - 1} \left|\mathcal{H}_{y_1}(s) - \mathcal{H}_{\widehat{y_1}}(s)\right| ds \right) \\
&\quad + P_3 \int_{b_{i-1}}^{\iota} \left|\mathcal{H}_{y_1}(s) - \mathcal{H}_{\widehat{y_1}}(s)\right| ds + \frac{P_4}{\Gamma(\omega_i)} \int_{b_{i-1}}^{\iota} (\iota - s)^{\omega_i - 1} \left|\mathcal{H}_{y_1}(s) - \mathcal{H}_{\widehat{y_1}}(s)\right| ds.
\end{aligned}$$

By (H_1), we obtain

$$\left|\mathcal{H}_{y_1}(s) - \mathcal{H}_{\widehat{y_1}}(s)\right| = \left|\mathcal{H}_1(s, y_2(s), {}^{ABC}\mathbf{D}^{\omega_i}_{b^+_{i-1}} y_1(s)) - \mathcal{H}_1(s, \widehat{y_2}(s), {}^{ABC}\mathbf{D}^{\omega_i}_{b^+_{i-1}} \widehat{y_1}(s))\right|$$

$$\leq \mathfrak{N}_f \left(\left|y_2(s) - \widehat{y_2}(s)\right| + \left|\mathcal{H}_{y_1}(s) - \mathcal{H}_{\widehat{y_1}}(s)\right|\right)$$

$$\leq \frac{\mathfrak{N}_f}{1 - \mathfrak{N}_f} \left(\left\|y_2 - \widehat{y_2}\right\|\right). \tag{3.2}$$

Hence

$$\left\|\mathbf{\Pi}_1(y_1, y_2) - \mathbf{\Pi}_1(\widehat{y_1}, \widehat{y_2})\right\| \leq \Omega_1 \left(\left\|y_2 - \widehat{y_2}\right\|\right).$$

In the same manner, for the operator $\mathbf{\Pi}_1$, we get

$$\left\|\mathbf{\Pi}_2(y_1, y_2) - \mathbf{\Pi}_2(\widehat{y_1}, \widehat{y_2})\right\| \leq \Omega_2 \left(\left\|y_1 - \widehat{y_1}\right\|\right).$$

Thus, we have

$$\left\|\mathbf{\Pi}(y_1, y_2) - \mathbf{\Pi}(\widehat{y_1}, \widehat{y_2})\right\|_\Upsilon = \left\|\mathbf{\Pi}_1(y_1, y_1) - \mathbf{\Pi}_1(\widehat{y_1}, \widehat{y_1})\right\| + \left\|\mathbf{\Pi}_2(y_1, y_1) - \mathbf{\Pi}_2(\widehat{y_1}, \widehat{y_1})\right\|$$

$$\leq \Omega_1 \left\|y_2 - \widehat{y_2}\right\| + \Omega_2 \left\|y_1 - \widehat{y_1}\right\|$$

$$\leq \max\{\Omega_1, \Omega_2\} \left\|(y_1, y_2) - (\widehat{y_1}, \widehat{y_2})\right\|_\Upsilon.$$

Due to $\max\{\Omega_1, \Omega_2\} < 1$, we deduce that the operator $\mathbf{\Pi}$ is a contraction mapping. Thus, $\mathbf{\Pi}$ has a unique fixed point. By Banach contraction principle, we deduce that the coupled system (2.6) has a unique solution. $\qquad\square$

Theorem 3.2. *Under the hypothesis (H_1) mentioned in Theorem 3.1. If $Z_\Pi \max\left(\frac{\mathfrak{N}_{\mathcal{H}_1}}{1-\mathfrak{N}_{\mathcal{H}_1}}, \frac{\mathfrak{N}_{\mathcal{H}_2}}{1-\mathfrak{N}_{\mathcal{H}_2}}\right) < 1$, then, the coupled system (2.6) has at least one solution*

Proof. Define the operators $\mathbf{F}_1, \mathbf{G}_1, \mathbf{F}_2, \mathbf{G}_2 : \Upsilon \to \Upsilon$ as

$$\mathbf{F}_1(y_1, y_2)(\iota) = P_3 \int_{b_{i-1}}^{\iota} \left|\mathcal{H}_{y_1}(s)\right| ds + \frac{P_4}{\Gamma(\omega_i)} \int_{b_{i-1}}^{\iota} (\iota - s)^{\omega_i - 1} \mathcal{H}_{y_1}(s) ds,$$

$$\mathbf{F}_2(y_1, y_2)(\iota) = P_3 \int_{b_{i-1}}^{\iota} \left|\mathcal{H}_{y_2}(s)\right| ds + \frac{P_4}{\Gamma(\omega_i)} \int_{b_{i-1}}^{\iota} (\iota - s)^{\omega_i - 1} \mathcal{H}_{y_2}(s) ds,$$

$$\mathbf{G}_1(y_1, y_2)(\iota) = \frac{P_1(\iota - b_{i-1})}{\Theta_1} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} \mathcal{H}_{y_1}(s) ds - \int_{b_{i-1}}^{b_i} \mathcal{H}_{y_1}(s) ds\right)$$

$$+ \frac{P_2(\iota - b_{i-1})}{\Theta_1 \Gamma(\omega_i)} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} (\tau_k - s)^{\omega_i - 1} \mathcal{H}_{y_1}(s) ds - \int_{b_{i-1}}^{b_i} (b_i - s)^{\omega_i - 1} \mathcal{H}_{y_1}(s) ds\right),$$

$$\mathbf{G}_2(y_1, y_2)(\iota) = \frac{P_1(\iota - b_{i-1})}{\Theta_2} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\gamma_k} \mathcal{H}_{y_2}(s) ds - \int_{b_{i-1}}^{b_i} \mathcal{H}_{y_2}(s) ds\right)$$

$$+ \frac{P_2(\iota - b_{i-1})}{\Theta_2 \Gamma(\omega_i)} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\gamma_k} (\gamma_k - s)^{\omega_i - 1} \mathcal{H}_{y_2}(s) ds - \int_{b_{i-1}}^{b_i} (b_i - s)^{\omega_i - 1} \mathcal{H}_{y_2}(s) ds\right).$$

From the above mentioned operators, we can write

$$\mathbf{\Pi}_1(y_1, y_2)(\iota) = \mathbf{F}_1(y_1, y_2)(\iota) + \mathbf{G}_1(y_1, y_2)(\iota),$$

and

$$\mathbf{\Pi}_2\left(y_1, y_2\right)(\iota) = \mathbf{F}_2\left(y_1, y_2\right)(\iota) + \mathbf{G}_2\left(y_1, y_2\right)(\iota).$$

Thus, the operator $\mathbf{\Pi}$ can be expressed as

$$\mathbf{\Pi}\left(y_1, y_2\right)(\iota) = \left(\mathbf{\Pi}_1\left(y_1, y_2\right)(\iota), \mathbf{\Pi}_2\left(y_1, y_2\right)(\iota)\right).$$

We split the proof into the following steps:

Step 1: In this step, we shall prove that Π is continuous. The continuity of $\mathcal{H}_{y_1}$ and $\mathcal{H}_{y_2}$ implies the continuity of $\mathbf{F}_1, \mathbf{G}_1, \mathbf{F}_2$ and $\mathbf{G}_2$. Consequently, $\mathbf{\Pi}$ is continuous.

Step 2: In this step, we shall prove that $\mathbf{G}_1$ and $\mathbf{G}_2$ are bounded. By (3.1), we have

$$
\begin{aligned}
\|\mathbf{G}_1\left(y_1, y_2\right)\| &\leq \frac{P_1(\iota - b_{i-1})}{\Theta_1}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\tau_k}\left|\mathcal{H}_{y_1}(s)\right|ds - \int_{b_{i-1}}^{b_i}\left|\mathcal{H}_{y_1}(s)\right|ds\right) \\
&\quad + \frac{P_2(\iota - b_{i-1})}{\Theta_1\Gamma(\omega_i)}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\tau_k}(\tau_k - s)^{\omega_i - 1}\left|\mathcal{H}_{y_1}(s)\right|ds - \int_{b_{i-1}}^{b_i}(b_i - s)^{\omega_i - 1}\left|\mathcal{H}_{y_1}(s)\right|ds\right) \\
&\leq \frac{P_1(b_i - b_{i-1})}{\Theta_1}\left(\sum_{k=1}^{n}(\tau_k - b_{i-1}) + (b_i - b_{i-1})\right)\left(\frac{\mathfrak{N}_{\mathcal{H}_1}}{1 - \mathfrak{N}_{\mathcal{H}_1}}\|y_1\| + \mathcal{L}_{\mathcal{H}_1}\right) \\
&\quad + \frac{P_2(b_i - b_{i-1})}{\Theta_1\Gamma(\omega_i + 1)}\left(\sum_{k=1}^{n}(b_i - b_{i-1})^{\omega_i} + (b_i - b_{i-1})^{\omega_i}\right)\left(\frac{\mathfrak{N}_{\mathcal{H}_1}}{1 - \mathfrak{N}_{\mathcal{H}_1}}\|y_1\| + \mathcal{L}_{\mathcal{H}_1}\right) \\
&\leq \mathcal{G}_{\Pi_1}\frac{\mathfrak{N}_{\mathcal{H}_1}}{1 - \mathfrak{N}_{\mathcal{H}_1}}\|y_1\| + \mathcal{G}_{\Pi_1}\mathcal{L}_{\mathcal{H}_1} \\
&= Q_1.
\end{aligned}
$$

In the same manner, we get

$$
\begin{aligned}
\|\mathbf{G}_2\left(y_1, y_2\right)\| &\leq \mathcal{G}_{\Pi_2}\frac{\mathfrak{N}_{\mathcal{H}_2}}{1 - \mathfrak{N}_{\mathcal{H}_2}}\|y_2\| + \mathcal{G}_{\Pi_2}\mathcal{L}_{\mathcal{H}_2} \\
&= Q_2,
\end{aligned}
$$

which implies

$$\|\mathbf{G}\left(y_1, y_2\right)\|_{\Upsilon} \leq \max\left(Q_1, Q_2\right),$$

where

$$Q_i = \mathcal{G}_{\Pi_i}[\frac{\mathfrak{N}_{\mathcal{H}_i}}{1 - \mathfrak{N}_{\mathcal{H}_i}}\|y_i\| + \mathcal{L}_{\mathcal{H}_i}].$$

This proves $\mathbf{G}_1, \mathbf{G}_2$ are uniformly bounded.

Step 3: In this step, we will prove that $\mathbf{G}_1$ and $\mathbf{G}_2$ are equicontinuous. For this purpose, let $(y_1, y_2) \in \beta_{\eta_l}$ and $\iota_1, \iota_2 \in \mathcal{J}$ such that $0 \leq \iota_2 < \iota_1 \leq b$. Then, we have

$$
\begin{aligned}
&\left|\mathbf{G}_1\left(y_1, y_2\right)(\iota_2) - \mathbf{G}_1\left(y_1, y_2\right)(\iota_1)\right| \\
&\leq \frac{P_1\left[(\iota_2 - b_{i-1}) - (\iota_1 - b_{i-1})\right]}{\Theta_1}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\tau_k}\left|\mathcal{H}_{y_1}(s)\right|ds - \int_{b_{i-1}}^{b_i}\left|\mathcal{H}_{y_1}(s)\right|ds\right)
\end{aligned}
$$

$$+ \frac{P_2(\iota_2 - b_{i-1}) - (\iota_1 - b_{i-1})}{\Theta_1 \Gamma(\omega_i)} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} (\tau_k - s)^{\omega_i - 1} \left| \mathcal{H}_{y_1}(s) \right| ds - \int_{b_{i-1}}^{b_i} (b_i - s)^{\omega_i - 1} \left| \mathcal{H}_{y_1}(s) \right| ds \right).$$

Thus

$$\left\| \mathbf{G}_1(y_1, y_2)(\iota_2) - \mathbf{G}_1(y_1, y_2)(\iota_1) \right\| \to 0 \text{ as } \iota_2 \to \iota_1.$$

In the same manner, we get

$$\left\| \mathbf{G}_2(y_1, y_2)(\iota_2) - \mathbf{G}_2(y_1, y_2)(\iota_1) \right\| \to 0 \text{ as } \iota_2 \to \iota_1.$$

We conclude that the operator $\mathbf{G}_1$ and $\mathbf{G}_2$ are relatively compact. Based on the previous steps and using the Arzela-Ascoli theorem, we deduce that $\mathbf{G}$ is completely continuous.

Step 4: In this step, we will prove that $\mathbf{F}_1$ and $\mathbf{F}_2$ are contractions. Let $(y_1, y_2), (\widehat{y_1}, \widehat{y_2}) \in \beta_{\eta_i}$ and $\iota \in \mathcal{J}_i$. Then, for the operator $\mathbf{F}_1$, we have

$$\left\| \mathbf{F}_1(y_1, y_2)(\iota) - \mathbf{F}_1(\widehat{y_1}, \widehat{y_2})(\iota) \right\|$$

$$\leq P_3 \int_{b_{i-1}}^{\iota} \left| \mathcal{H}_{y_1}(s) - \mathcal{H}_{\widehat{y_1}}(s) \right| ds + \frac{P_4}{\Gamma(\omega_i)} \int_{b_{i-1}}^{\iota} (\iota - s)^{\omega_i - 1} \left| \mathcal{H}_{y_1}(s) - \mathcal{H}_{\widehat{y_1}}(s) \right| ds$$

$$\leq \left(P_3(b_i - b_{i-1}) + \frac{P_4}{\Gamma(\omega_i + 1)} (b_i - b_{i-1})^{\omega_i} \right) \frac{\mathfrak{N}_{\mathcal{H}_1}}{1 - \mathfrak{N}_{\mathcal{H}_1}}.$$

Thus, we have

$$\left\| \mathbf{F}_1(y_1, y_2) - \mathbf{F}_1(\widehat{y_1}, \widehat{y_2}) \right\| \leq Z_\Pi \frac{\mathfrak{N}_{\mathcal{H}_1}}{1 - \mathfrak{N}_{\mathcal{H}_1}} \left(\left\| y_1 - \widehat{y_1} \right\| \right).$$

In the same manner, we get

$$\left\| \mathbf{F}_2(y_1, y_2) - \mathbf{F}_2(\widehat{y_1}, \widehat{y_2}) \right\| \leq Z_\Pi \frac{\mathfrak{N}_{\mathcal{H}_2}}{1 - \mathfrak{N}_{\mathcal{H}_2}} \left(\left\| y_2 - \widehat{y_2} \right\| \right).$$

Consequently,

$$\left\| \mathbf{F}(y_1, y_2) - \mathbf{F}(\widehat{y_1}, \widehat{y_2}) \right\|_\Upsilon \leq Z_\Pi \max \left(\frac{\mathfrak{N}_{\mathcal{H}_1}}{1 - \mathfrak{N}_{\mathcal{H}_1}}, \frac{\mathfrak{N}_{\mathcal{H}_2}}{1 - \mathfrak{N}_{\mathcal{H}_2}} \right) \left\| (y_1, y_2) - (\widehat{y_1}, \widehat{y_2}) \right\|_\Upsilon.$$

Due to $Z_\Pi \max \left(\frac{\mathfrak{N}_{\mathcal{H}_1}}{1 - \mathfrak{N}_{\mathcal{H}_1}}, \frac{\mathfrak{N}_{\mathcal{H}_2}}{1 - \mathfrak{N}_{\mathcal{H}_2}} \right) < 1$, we deduce that the operator $\mathbf{F}$ is contraction map. Hence, by Theorem 2.3, we see that Π at least one fixed point, which is the corresponding solution $(y_1, y_2)_i$ of the coupled system (2.6). For $i \in \{1, 2, ..., n\}$, we define the functions

$$(y_1, y_2)_i = \begin{cases} 0, & \iota \in [0, b_{i-1}], \\ (\widehat{y_1}, \widehat{y_2})_i, & \iota \in \mathcal{J}_i. \end{cases} \tag{3.3}$$

As a result of this, it's well-known that, $(y_1, y_2)_i \in C(\mathcal{J}_i, \mathbb{R}) \times C(\mathcal{J}_i, \mathbb{R})$ given by (3.3) satisfies the following problem

$$\frac{\Lambda(\omega(\iota))}{1 - \omega(\iota)} \int_0^{b_1} E_{\omega(\iota)} \left(\frac{\omega(\iota)(\iota_1 - \theta)^{\omega(\iota)}}{\omega(\iota) - 1} \right) y'_{1i}(\theta) d\theta$$

$$+ \ldots + \frac{\Lambda(\omega(\iota))}{1 - \omega(\iota)} \int_{b_{i-1}}^{\iota} E_{\omega(\iota)} \left(\frac{\omega(\iota) (b - \theta)^{\omega(\iota)}}{\omega(\iota) - 1} \right) y'_{1i}(\theta) d\theta$$
$$= \mathcal{H}_1(\omega, y_{1i}(\omega), {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_{2i}(\omega)),$$

and

$$\frac{\Lambda(\omega(\iota))}{1 - \omega(\iota)} \int_0^{b_1} E_{\omega(\iota)} \left(\frac{\omega(\iota) (\iota_1 - \theta)^{\omega(\iota)}}{\omega(\iota) - 1} \right) y'_{2i}(\theta) d\theta$$
$$+ \ldots + \frac{\Lambda(\omega(\iota))}{1 - \omega(\iota)} \int_{b_{i-1}}^{\iota} E_{\omega(\iota)} \left(\frac{\omega(\iota) (b - \theta)^{\omega(\iota)}}{\omega(\iota) - 1} \right) y'_{2i}(\theta) d\theta$$
$$= \mathcal{H}_2(\omega, y_{2i}(\omega), {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_{1i}(\omega)),$$

where $(y_1, y_2)_i$ is a solution to system (2.6) equipped with $(y_1, y_2)_i (0) \equiv (y_1, y_2)_i (b_i) \equiv (\widetilde{y_1}, \widetilde{y_2})_i (b_i) \equiv 0$. Then

$$(y_1, y_2) (\iota) = \begin{cases} (y_1, y_2)_1 (\iota), & \iota \in \mathcal{J}_1, \\ (y_1, y_2)_2 (\iota) = \begin{cases} 0, & \iota \in \mathcal{J}_1, \\ (\widetilde{y_1}, \widetilde{y_2})_2, & \iota \in \mathcal{J}_2, \end{cases} \\ \quad \cdot \\ \quad \cdot \\ \quad \cdot \\ (y_1, y_2)_i (\iota) = \begin{cases} 0, & \iota \in [0, b_{i-1}], \\ (\widetilde{y_1}, \widetilde{y_2})_n, & \iota \in \mathcal{J}_i, \end{cases} \end{cases}$$

is the solution for coupled system (2.6). $\qquad \square$

4. Stability results of the coupled system (2.6)

In this section, we discuss Ulam-Hyers (UH) stability results for coupled system (2.6). For more information about the following definition see [31–33]. Let $\varepsilon_1, \varepsilon_2 > 0$, and y be a function such that $y(\iota) \in C (\mathcal{J}_i, \mathbb{R})$ satisfies the following inequalities:

$$\begin{cases} \left| {}^{ABC}\mathbf{D}_0^{\omega(\iota)} y_1(\iota) - \mathcal{H}_{y_1}(\iota) \right| \leq \varepsilon_1, & \iota \in \mathcal{J}, \\ \left| {}^{ABC}\mathbf{D}_0^{\omega(\iota)} y_2(\iota) - \mathcal{H}_{y_2}(\iota) \right| \leq \varepsilon_2, & \iota \in \mathcal{J}. \end{cases} \tag{4.1}$$

Now, we defined the functions $(y_1, y_2)_i (\iota), (\widehat{y_1}, \widehat{y_2})_i (\iota), (\Bbbk_1, \Bbbk_2)_i (\iota), \iota \in \mathcal{J}_i$ as

$$(y_1, y_2)_i (\iota) = \begin{cases} 0, & if \ \iota \in [0, b_{i-1}], \\ (y_1, y_2) (\iota) & if \ \iota \in \mathcal{J}_i, \end{cases} \tag{4.2}$$

$$(\widehat{y_1}, \widehat{y_2})_i (\iota) = \begin{cases} 0, & if \ \iota \in [0, b_{i-1}], \\ (\widehat{y_1}, \widehat{y_2}) (\iota) & if \ \iota \in \mathcal{J}_i, \end{cases} \tag{4.3}$$

and

$$(\Bbbk_1, \Bbbk_2)_i (\iota) = \begin{cases} 0, & if \ \iota \in [0, b_{i-1}], \\ (\Bbbk_1, \Bbbk_2) (\iota) & if \ \iota \in \mathcal{J}_i. \end{cases}$$

In the forthcoming analyses, we will use the above notations $(y_1, y_2) (\iota), (\widehat{y_1}, \widehat{y_2}) (\iota)$ and $(\Bbbk_1, \Bbbk_2) (\iota)$.

Definition 4.1. *The coupled system (2.6) is UH stable if there is a real number $C_{\mathcal{H}} = \max\{C_{\mathcal{H}_1}, C_{\mathcal{H}_2}\} > 0$ such that for each $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$, and each solution $(\widehat{y_1}, \widehat{y_2}) \in \Upsilon$ of coupled system (2.6) satisfying the inequalities (4.1), there exists a solution $(y_1, y_2) \in \Upsilon$ of coupled system (2.6) such that*

$$\left\| (\widehat{y_1}, \widehat{y_2}) - (y_1, y_2) \right\|_{\Upsilon} \leq C_{\mathcal{H}} \varepsilon,$$

where (y_1, y_2) and $(\widehat{y_1}, \widehat{y_2})$ defined in (4.2) and (4.3), respectively.

Remark 4.2. *Let $(\widehat{y_1}, \widehat{y_2}) \in \Upsilon$ be solutions of the inequalities (4.1) if and only if there is a function $(\Bbbk_1, \Bbbk_2)(\iota) \in \Upsilon$ (which depends on (y_1, y_2)) satisfying the following conditions,*

i) $\begin{cases} |\Bbbk_1(\iota)| \leq \varepsilon_1 \ \ for \ all \ \iota \in \mathcal{J}, \\ |\Bbbk_2(\iota)| \leq \varepsilon_2 \ \ for \ all \ \iota \in \mathcal{J}, \end{cases}$

ii) $\begin{cases} {}^{ABC}\mathbf{D}^{\omega_i}_{b^+_{i-1}} \widehat{y_1}(\iota) = \mathcal{H}_{\widehat{y_1}}(\iota) + \Bbbk_1(\iota), \iota \in \mathcal{J}, \\ {}^{ABC}\mathbf{D}^{\omega_i}_{b^+_{i-1}} \widehat{y_2}(\iota) = \mathcal{H}_{\widehat{y_2}}(\iota) + \Bbbk_2(\iota), \iota \in \mathcal{J}. \end{cases}$

Lemma 4.3. *Let $(\widehat{y_1}, \widehat{y_2}) \in \Upsilon$ be satisfying (4.1), then $(\widehat{y_1}, \widehat{y_2})$ satisfy the following inequalities*

$$\begin{cases} \left| \widehat{y_1}(\iota) - \Psi_{\widehat{y_1}} - P_3 \int_{b_{i-1}}^{\iota} \mathcal{H}_{\widehat{y_1}}(s)\, ds - \frac{P_4}{\Gamma(\omega_i)} \int_{b_{i-1}}^{\iota} (\iota - s)^{\omega_i - 1} \mathcal{H}_{\widehat{y_1}}(s)\, ds \right| \leq \varepsilon_1 \mathcal{R}_{\Pi_1}, \\ \left| \widehat{y_2}(\iota) - \Psi_{\widehat{y_2}} - P_3 \int_{b_{i-1}}^{\iota} \mathcal{H}_{\widehat{y_2}}(s)\, ds - \frac{P_4}{\Gamma(\omega_i)} \int_{b_{i-1}}^{\iota} (\iota - s)^{\omega_i - 1} \mathcal{H}_{\widehat{y_2}}(s)\, ds \right| \leq \varepsilon_2 \mathcal{R}_{\Pi_2}, \end{cases}$$

where

$$\begin{aligned} \Psi_{\widehat{y_1}} &= \frac{P_1(\iota - b_{i-1})}{\Theta_1} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} \mathcal{H}_{\widehat{y_1}}(s)\, ds - \int_{b_{i-1}}^{b_i} \mathcal{H}_{\widehat{y_1}}(s)\, ds \right) \\ &+ \frac{P_2(\iota - b_{i-1})}{\Theta_1 \Gamma(\omega_i)} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} (\tau_k - s)^{\omega_i - 1} \mathcal{H}_{\widehat{y_1}}(s)\, ds - \int_{b_{i-1}}^{b_i} (b_i - s)^{\omega_i - 1} \mathcal{H}_{\widehat{y_1}}(s)\, ds \right), \end{aligned}$$

and

$$\begin{aligned} \Psi_{\widehat{y_2}} &= \frac{P_1(\iota - b_{i-1})}{\Theta_2} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\gamma_k} \mathcal{H}_{\widehat{y_2}}(s)\, ds - \int_{b_{i-1}}^{b_i} \mathcal{H}_{\widehat{y_2}}(s)\, ds \right) \\ &+ \frac{P_2(\iota - b_{i-1})}{\Theta_2 \Gamma(\omega_i)} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\gamma_k} (\gamma_k - s)^{\omega_i - 1} \mathcal{H}_{\widehat{y_2}}(s)\, ds - \int_{b_{i-1}}^{b_i} (b_i - s)^{\omega_i - 1} \mathcal{H}_{\widehat{y_2}}(s)\, ds \right). \end{aligned}$$

Proof. By Remark 4.2, we have

$$\begin{cases} {}^{ABC}\mathbf{D}^{\omega_i}_{b^+_{i-1}} \widehat{y_1}(\iota) = \mathcal{H}_{\widehat{y_1}}(\iota) + \Bbbk_1(\iota), \iota \in \mathcal{J}, \\ {}^{ABC}\mathbf{D}^{\omega_i}_{b^+_{i-1}} \widehat{y_2}(\iota) = \mathcal{H}_{\widehat{y_2}}(\iota) + \Bbbk_2(\iota), \iota \in \mathcal{J}. \end{cases} \tag{4.4}$$

Then, by Theorem 2.6, the solutions of the system (4.4) are given as

$$\begin{aligned} \widehat{y_1}(\iota) &= \Psi_{\widehat{y_1}} + P_3 \int_{b_{i-1}}^{\iota} \mathcal{H}_{y_1}(s)\, ds + \frac{P_4}{\Gamma(\omega_i)} \int_{b_{i-1}}^{\iota} (\iota - s)^{\omega_i - 1} \mathcal{H}_{y_1}(s)\, ds \\ &+ \frac{P_1(\iota - b_{i-1})}{\Theta_1} \left(\sum_{k=1}^{n} \int_{b_{i-1}}^{\tau_k} \Bbbk_1(s)\, ds - \int_{b_{i-1}}^{b_i} \Bbbk_1(s)\, ds \right) \end{aligned}$$

$$+\frac{P_2(\iota - b_{i-1})}{\Theta_1\Gamma(\omega_i)}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\tau_k}(\tau_k - s)^{\omega_i-1}\Bbbk_1(s)ds - \int_{b_{i-1}}^{b_i}(b_i - s)^{\omega_i-1}\Bbbk_1(s)ds\right)$$

$$+P_3\int_{b_{i-1}}^{\iota}\Bbbk_1(s)ds + \frac{P_4}{\Gamma(\omega_i)}\int_{b_{i-1}}^{\iota}(\iota - s)^{\omega_i-1}\Bbbk_1(s)ds,$$

and

$$\begin{aligned}\widehat{y_2}(\iota) &= \Psi_{\widehat{y_2}} + P_3\int_{b_{i-1}}^{\iota}\mathcal{H}_{y_2}(s)\,ds + \frac{P_4}{\Gamma(\omega_i)}\int_{b_{i-1}}^{\iota}(\iota - s)^{\omega_i-1}\mathcal{H}_{y_2}(s)\,ds\\ &+\frac{P_1(\iota - b_{i-1})}{\Theta_2}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\gamma_k}\Bbbk_2(s)ds - \int_{b_{i-1}}^{b_i}\Bbbk_2(s)ds\right)\\ &+\frac{P_2(\iota - b_{i-1})}{\Theta_2\Gamma(\omega_i)}\left(\sum_{k=1}^{n}\int_{b_{i-1}}^{\gamma_k}(\gamma_k - s)^{\omega_i-1}\Bbbk_2(s)ds - \int_{b_{i-1}}^{b_i}(b_i - s)^{\omega_i-1}\Bbbk_2(s)ds\right)\\ &+P_3\int_{b_{i-1}}^{\iota}\Bbbk_2(s)ds + \frac{P_4}{\Gamma(\omega_i)}\int_{b_{i-1}}^{\iota}(\iota - s)^{\omega_i-1}\Bbbk_2(s)ds.\end{aligned}$$

By some calculations, we get

$$\begin{cases}\left|\widehat{y_1}(\iota) - \Psi_{\widehat{y_1}} - P_3\int_{b_{i-1}}^{\iota}\mathcal{H}_{\widehat{y_1}}(s)\,ds - \frac{P_4}{\Gamma(\omega_i)}\int_{b_{i-1}}^{\iota}(\iota - s)^{\omega_i-1}\mathcal{H}_{\widehat{y_1}}(s)\,ds\right| \le \varepsilon_1\mathcal{R}_{\Pi_1},\\ \left|\widehat{y_2}(\iota) - \Psi_{\widehat{y_2}} - P_3\int_{b_{i-1}}^{\iota}\mathcal{H}_{\widehat{y_2}}(s)\,ds - \frac{P_4}{\Gamma(\omega_i)}\int_{b_{i-1}}^{\iota}(\iota - s)^{\omega_i-1}\mathcal{H}_{\widehat{y_2}}(s)\,ds\right| \le \varepsilon_2\mathcal{R}_{\Pi_2}.\end{cases}$$

$\square$

Theorem 4.4. *Suppose that* (H_1) *holds. If*

$$\left(P_3(b_i - b_{i-1}) + \frac{P_4(b_i - b_{i-1})^{\omega_i}}{\Gamma(\omega_i + 1)}\right)\frac{\mathfrak{N}_f(1 + \mathcal{M})}{1 - \mathfrak{N}_f} < 1,$$

then the coupled system (2.6) is UH stable.

Proof. Let $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and $(\widehat{y_1}, \widehat{y_2}) \in \Upsilon$ be a solution of coupled system (2.6) satisfying (4.1), and let $(y_1, y_2) \in \Upsilon$ be the unique solution of the coupled system (2.6). Then, by Theorem 2.6, the solution of coupled system (2.6) is given by

$$\begin{aligned}y_1(\iota) &= \Psi_{\widehat{y_1}} + P_3\int_{b_{i-1}}^{\iota}\mathcal{H}_{y_1}(s)\,ds + \frac{P_4}{\Gamma(\omega_i)}\int_{b_{i-1}}^{\iota}(\iota - s)^{\omega_i-1}\mathcal{H}_{y_1}(s)\,ds,\\ y_2(\iota) &= \Psi_{\widehat{y_2}} + P_3\int_{b_{i-1}}^{\iota}\mathcal{H}_{y_2}(s)\,ds + \frac{P_4}{\Gamma(\omega_i)}\int_{b_{i-1}}^{\iota}(\iota - s)^{\omega_i-1}\mathcal{H}_{y_2}(s)\,ds.\end{aligned}$$

Hence by Lemma 4.3 and (H_1), we have

$$\begin{aligned}\left|\widehat{y_1}(\iota) - y_2(\iota)\right| &\le \left|\widehat{y_1}(\iota) - \Psi_{\widehat{y_1}} - P_3\int_{b_{i-1}}^{\iota}\mathcal{H}_{\widehat{y_1}}(s)\,ds - \frac{P_4}{\Gamma(\omega_i)}\int_{b_{i-1}}^{\iota}(\iota - s)^{\omega_i-1}\mathcal{H}_{\widehat{y_1}}(s)\,ds\right|\\ &+P_3\int_{b_{i-1}}^{\iota}\left|\mathcal{H}_{\widehat{y_1}}(s) - \mathcal{H}_{y_1}(s)\right|\,ds + \frac{P_4}{\Gamma(\omega_i)}\int_{b_{i-1}}^{\iota}(\iota - s)^{\omega_i-1}\left|\mathcal{H}_{\widehat{y_1}}(s) - \mathcal{H}_{y_1}(s)\right|\,ds\end{aligned}$$

$$\leq \; \varepsilon_1 \mathcal{R}_{\Pi_1} + \left(P_3 \left(b_i - b_{i-1} \right) + \frac{P_4}{\Gamma(\omega_i + 1)} \left(b_i - b_{i-1} \right)^{\omega_i} \right) \frac{\mathfrak{N}_f}{1 - \mathfrak{N}_f} \left\| \widehat{y}_1 - y_1 \right\|_{\Upsilon}.$$

Thus

$$\left\| \widehat{y}_1 - y_1 \right\| \leq C_{\mathcal{H}_1} \varepsilon_1,$$

where

$$C_{\mathcal{H}_1} = \frac{\mathcal{R}_{\Pi_1}}{1 - \left(P_3 \left(b_i - b_{i-1} \right) + \frac{P_4}{\Gamma(\omega_i+1)} \left(b_i - b_{i-1} \right)^{\omega_i} \right) \frac{\mathfrak{N}_f}{1 - \mathfrak{N}_f}} > 0.$$

In the same manner, we have

$$\left\| \widehat{y}_2 - y_2 \right\|_{\Upsilon} \leq C_{\mathcal{H}_2} \varepsilon_2,$$

where

$$C_{\mathcal{H}_2} = \frac{\mathcal{R}_{\Pi_1}}{1 - \left(P_3 \left(b_i - b_{i-1} \right) + \frac{P_4}{\Gamma(\omega_i+1)} \left(b_i - b_{i-1} \right)^{\omega_i} \right) \frac{\mathfrak{N}_f}{1 - \mathfrak{N}_f}} > 0.$$

Thus, we have

$$
\begin{aligned}
\left\| (\widehat{y}_1, \widehat{y}_2) - (y_1, y_2) \right\|_{\Upsilon} &= \left\| \widehat{y}_1 - y_1 \right\| + \left\| \widehat{y}_2 - y_2 \right\| \\
&\leq C_{\mathcal{H}_1} \varepsilon_1 + C_{\mathcal{H}_2} \varepsilon_2 \\
&\leq 2 C_{\mathcal{H}} \varepsilon.
\end{aligned}
$$

Hence, coupled system (2.6) is UH stability. Now, by choosing $C_{\mathcal{H}}(\varepsilon) = C_{\mathcal{H}} \varepsilon$ such that $C_{\mathcal{H}}(0) = 0$, we conclude that coupled system (2.6) has Generalized UH stability. $\square$

5. Examples

To illustrate our key results, this section presents numerical examples addressing the coupled system (2.6) in the context of ABC-fractional derivative with variable order $\omega(\iota)$. These specific examples have been chosen based on the conditions outlined in the theorems used, the specific conditions mentioned in our proposed conclusions, and the values of parameters and fractional-order derivatives. These examples serve as an illustration of the various dynamic applications that can be created by combining the ABC-fractional derivative with variable order $\omega(\iota)$.

Example 5.1. *Let us consider the following coupled system of ABC fractional problem with variable order $\omega(\iota)$*

$$
\begin{cases}
{}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_1(\iota) = \frac{\iota^2}{5 e^\iota} \left(e^{-\iota} + \frac{|y_2(\iota)|}{1 + |y_2(\iota)|} + \frac{{}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_1(\iota)}{1 + {}^{ABC} D_{0^+}^{\omega(\iota)} y_1(\iota)} \right), & \iota \in \mathcal{J} := (0, 2], \\
{}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_2(\iota) = \frac{\iota^3}{2(e^\iota - 1)} \left(\frac{1}{e^\iota} + \frac{|y_1(\iota)|}{1 + |y_1(\iota)|} + \frac{{}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_2(\iota)}{1 + {}^{ABC} D_{0^+}^{\omega(\iota)} y_2(\iota)} \right), & \iota \in \mathcal{J} := (0, 2], \\
y_1(0) = 0, y_2(0) = 0, y_1(2) = \frac{1}{2}, y_2(2) = \frac{1}{3}.
\end{cases}
$$

Here, $a = 0, b = 2$, and

$$
\mathcal{H}_{y_1}(\iota) = \frac{\iota^2}{5 e^\iota} \left(e^{-\iota} + \frac{|y_2(\iota)|}{1 + |y_2(\iota)|} + \frac{{}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_1(\iota)}{1 + {}^{ABC} D_{0^+}^{\omega(\iota)} y_1(\iota)} \right),
$$

$$\mathcal{H}_{y_2}(\iota) = \frac{\iota^3}{2(e^{\iota} - 1)} \left(\frac{1}{e} + \frac{|y_1(\iota)|}{1 + |y_1(\iota)|} + \frac{{}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_2(\iota)}{1 + {}^{ABC}D_{0^+}^{\omega(\iota)} y_2(\iota)} \right).$$

Let $\iota \in [0, 1]$, $y_1, \bar{y}_1 \in C(\mathcal{J}_i, \mathbb{R})$. Then

$$\left| \mathcal{H}_{y_1}(\iota) - \mathcal{H}_{\bar{y}_1}(\iota) \right|$$
$$\leq \frac{1}{5} \left(|y_2(\iota) - \bar{y}_2(\iota)| + \left| {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_1(\iota) - {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} \bar{y}_1(\iota) \right| \right),$$

and

$$\left| \mathcal{H}_{y_2}(\iota) - \mathcal{H}_{\bar{y}_2}(\iota) \right|$$
$$\leq \frac{1}{2} \left(|y_1(\iota) - \bar{y}_1(\iota)| + \left| {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} y_2(\iota) - {}^{ABC}\mathbf{D}_{0^+}^{\omega(\iota)} \bar{y}_2(\iota) \right| \right).$$

Therefore, hypothesis (H_1) holds with $\mathfrak{N}_{\mathcal{H}_1} = \frac{1}{5}$ and $\mathfrak{N}_{\mathcal{H}_2} = \frac{1}{2}$. Let

$$\omega(\iota) = \begin{cases} \frac{3}{2}, & \text{if } \iota \in (0, 1], \\ \frac{5}{2} & \text{if } \iota \in (1, 2]. \end{cases}$$

Then, for $i = 1$, by some calculation, we get $\Omega_1 = 0.65$ and $\Omega_2 = 0.52$ and hence $\Omega = \max\{\Omega_1, \Omega_2\} = 0.65 < 1$. Thus, by Theorem 3.1, the coupled system of ABC (2.6) has a unique solution. For every $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each $(\widehat{y_1}, \widehat{y_2}) \in \Upsilon$ satisfying

$$\begin{cases} \left| {}^{ABC}\mathbf{D}_0^{\omega(\iota)} \widehat{y_1}(\iota) - \mathcal{H}_{\widehat{y_1}}(\iota) \right| \leq \varepsilon_1, \\ \left| {}^{ABC}\mathbf{D}_0^{\omega(\iota)} \widehat{y_2}(\iota) - \mathcal{H}_{\widehat{y_2}}(\iota) \right| \leq \varepsilon_2, \end{cases}$$

there exists a solution $(y_1, y_2) \in \Upsilon$ of the coupled system of ABC (2.6) with

$$\begin{cases} \left\| \widehat{y_1} - y_1 \right\| \leq C_{\mathcal{H}_1} \varepsilon_1, \\ \left\| \widehat{y_2} - y_2 \right\| \leq C_{\mathcal{H}_2} \varepsilon_2, \end{cases}$$

where

$$C_{\mathcal{H}_1} = \frac{\mathcal{R}_{\Pi_1}}{1 - \left(P_3 (b_i - b_{i-1}) + \frac{P_4}{\Gamma(\omega_i + 1)} (b_i - b_{i-1})^{\omega_i} \right) \frac{\mathfrak{N}_f}{1 - \mathfrak{N}_f}} > 0,$$

and

$$C_{\mathcal{H}_2} = \frac{\mathcal{R}_{\Pi_1}}{1 - \left(P_3 (b_i - b_{i-1}) + \frac{P_4}{\Gamma(\omega_i + 1)} (b_i - b_{i-1})^{\omega_i} \right) \frac{\mathfrak{N}_f}{1 - \mathfrak{N}_f}} > 0.$$

Thus,

$$\begin{aligned} \left\| (\widehat{y_1}, \widehat{y_2}) - (y_1, y_2) \right\|_{\Upsilon} &\leq \left\| \widehat{y_1} - y_1 \right\|_{\Upsilon} + \left\| \widehat{y_2} - y_2 \right\|_{\Upsilon} \\ &\leq C_{\mathcal{H}_1} \varepsilon_1 + C_{\mathcal{H}_2} \varepsilon_2 \\ &\leq 2 C_{\mathcal{H}} \varepsilon. \end{aligned}$$

Therefore, by Theorem 4.4, the coupled system (2.6) is UH stable. Now, let $C_{\mathcal{H}}(\varepsilon) = C_{\mathcal{H}} \varepsilon$ such that $C_{\mathcal{H}}(0) = 0$, then the coupled system of ABC-fractional derivative (2.6) is GUH stability.

For i = 2, with some calculation, we have $\Omega_1 = 0.55$ and $\Omega_2 = 0.43$, and hence, $\Omega = \max\{\Omega_1, \Omega_2\} = 0.55 < 1$. Then, by Theorem 3.1, the coupled system (2.6) has a unique solution. For every $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each $(\widehat{y_1}, \widehat{y_2}) \in \Upsilon$ satisfying

$$
\begin{cases}
\left| {}^{ABC}\mathbf{D}_0^{\omega(\iota)}\widehat{y_1}(\iota) - \mathcal{H}_{\widehat{y_1}}(\iota) \right| \leq \varepsilon_1, \\
\left| {}^{ABC}\mathbf{D}_0^{\omega(\iota)}\widehat{y_2}(\iota) - \mathcal{H}_{\widehat{y_2}}(\iota) \right| \leq \varepsilon_2,
\end{cases}
$$

there exists a solution $(y_1, y_2) \in \Upsilon$ of the coupled system of ABC-fractional derivative (2.6) with

$$
\begin{aligned}
\left\| (\widehat{y_1}, \widehat{y_2}) - (y_1, y_2) \right\|_\Upsilon &\leq \left\| \widehat{y_1} - y_1 \right\|_\Upsilon + \left\| \widehat{y_2} - y_2 \right\|_\Upsilon \\
&\leq 2C_{\mathcal{H}}\varepsilon,
\end{aligned}
$$

where $C_{\mathcal{H}} = \max\{C_{\mathcal{H}_1}, C_{\mathcal{H}_2}\} > 0$. Therefore, by Theorem 4.4, the coupled system (2.6) is UH stable. Now, by choosing $C_{\mathcal{H}}(\varepsilon) = C_{\mathcal{H}}\varepsilon$ such that $C_{\mathcal{H}}(0) = 0$, the coupled system (2.6) is a Generalized UH stability.

6. Conclusions

Recent research has emphasized the Atangana-Baleanu fractional derivative, leading to the study and development of various qualitative features of solutions to FDEs incorporating these operators. This research paper presents a novel investigation into an implicit coupled system of fractional variable order, filling a gap in the existing literature. In this context, we have established and examined the necessary conditions for the existence and uniqueness of solutions to the coupled system of the ABC fractional problem with variable order, without relying on the semigroup property. We utilized extended intervals, piecewise constant functions, and fixed point theorems of the Banach-type and Krasnoselskii-type to reduce the proposed ABC system into a fractional integral equation. Furthermore, we employed mathematical analysis tools to investigate the stability results in the UH sense. An illustrative example in two cases was provided to support the findings of the study. By extending the existing conclusions examined for the ABC operator, this research contributes to advancing the understanding of variable-order FDEs. The paper provides a solid theoretical foundation that can be further explored through analysis, numerical simulations, and practical applications. The transformation techniques, qualitative analysis, and illustrative examples presented in this work highlight its unique contributions and potential to serve as a foundation for future research in the field. Overall, this research paper expands the knowledge and understanding of variable-order FDEs, offering valuable insights and paving the way for further advancements in the field. We believe that the results obtained in this study will be significant for future research in the field of fractional calculus theory, given the recent extensive explorations and applications of the Mittag-Leffler power law.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

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References

1. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1998.

2. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives : Theory and applications*, Philadelphia: Gordon and Breach Science Publishers, 1993.

3. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.

4. R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World Scientific, 2000. https://doi.org/10.1142/3779

5. S. S. Redhwan, M. Han, M. A. Almalahi, M. Alsulami, M. A. Alyami, Boundary value problem for a coupled system of nonlinear fractional q-difference equations with Caputo fractional derivatives, *Fractal Fract.*, **8** (2024), 73. https://doi.org/10.3390/fractalfract8010073

6. S. Y. Al-Mayyahi, M. S. Abdo, S. S. Redhwan, B. N. Abood, Boundary value problems for a coupled system of Hadamard-type fractional differential equations, *IAENG Inter. J. Appl. Math*, **51** (2021), 1–10.

7. S. S. Redhwan, S. Y. Al-mayyahi, S. L. Shaikh, M. S. Abdo, A coupled non-separated system of Hadamard-type fractional differential equations, *Adv. Theory Nonlinear Anal. Appl.*, **6** (2021), 33–44. https://doi.org/10.31197/atnaa.925365

8. M. A. Almalahi, S. K. Panchal, T. A. Aljaaidi, F. Jarad, New results for a coupled system of ABR fractional differential equations with sub-strip boundary conditions, *AIMS Mathematics*, **7** (2022), 4386–4404. https://doi.org/10.3934/math.2022244

9. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85.

10. J. Losada, J. J. Nieto, Properties of a new fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 87–92.

11. A. Atangana, D. Baleanu, New fractional derivative with non-local and non-singular kernel, *Therm. Sci.*, **20** (2016), 763–769. https://doi.org/10.2298/TSCI160111018A

12. T. Abdeljawad, A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel, *J. Inequal. Appl.*, **2017** (2017), 130. https://doi.org/10.1186/s13660-017-1400-5

13. F. Jarad, T. Abdeljawad, Z. Hammouch, On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative, *Chaos Solitons Fractals*, **117** (2018), 16–20. https://doi.org/10.1016/j.chaos.2018.10.006

14. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and nonsingular kernel: Theory and application to heat transfer model, *arXiv:1602.03408*, 2016. https://doi.org/10.48550/arXiv.1602.03408

15. S. S. Redhwan, M. S. Abdo, K. Shah, T. Abdeljawad, S. Dawood, H. A. Abdo, et al., Mathematical modeling for the outbreak of the coronavirus (COVID-19) under fractional nonlocal operator, *Results Phys.*, **19** (2020) , 103610. https://doi.org/10.1016/j.rinp.2020.103610

16. M. A. Almalahi, A. B. Ibrahim, A. Almutairi, O. Bazighifan, T. A. Aljaaidi, J. Awrejcewicz, A qualitative study on second-order nonlinear fractional differential evolution equations with generalized ABC operator, *Symmetry*, **14** (2022), 207. https://doi.org/10.3390/sym14020207

17. C. J. Zúñiga-Aguilar, H. M. Romero-Ugalde, J. F. Gómez-Aguilar, R. F. Escobar-Jiménez, M. Valtierra-Rodríguez, Solving fractional differential equations of variable-order involving operators with Mittag-Leffler kernel using artificial neural networks, *Chaos Solitons Fractals*, **103** (2017), 382–403. https://doi.org/10.1016/j.chaos.2017.06.030

18. Z. Bouazza, M. S. Souid, H. Günerhan, Multiterm boundary value problem of Caputo fractional differential equations of variable order, *Adv. Differ. Equ.*, **2021** (2021), 400. https://doi.org/10.1186/s13662-021-03553-z

19. A. Benkerrouche, M. S. Souid, S. Chandok, A. Hakem, Existence and stability of a Caputo variable-order boundary value problem, *J. Math.*, **2021** (2021), 7967880. https://doi.org/10.1155/2021/7967880

20. B. P. Moghaddam, J. A. T. Machado, H. Behforooz, An integro quadratic spline approach for a class of variable-order fractional initial value problems, *Chaos Solitons Fractals*, **102** (2017), 354–360. https://doi.org/10.1016/j.chaos.2017.03.065

21. C. F. Lorenzo, T. T. Hartley, Initialized fractional calculus, *Int. J. Appl. Math.*, **3** (2000), 249–265.

22. H. G. Sun, W. Chen, Y. Q. Chen, Variable-order fractional differential operators in anomalous diffusion modeling, *Phys. A*, **388** (2009), 4586–4592. https://doi.org/10.1016/j.physa.2009.07.024

23. J. V. C. Sousa, E. C. de Oliverira, Two new fractional derivatives of variable order with non-singular kernel and fractional differential equation, *Comput. Appl. Math.*, **37** (2018), 5375–5394. https://doi.org/10.1007/s40314-018-0639-x

24. I. Suwan, M. S. Abdo, T. Abdeljawad, M. M. Mater, A. Boutiara, M. A. Almalahi, Existence theorems for Psi-fractional hybrid systems with periodic boundary conditions, *AIMS Mathematics*, **7** (2022), 171–186. https://doi.org/10.3934/math.2022010

25. H. G. Sun, W. Chen, H. Wei, Y. Q. Chen, A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems, *Eur. Phys. J. Spec. Top.*, **193** (2011), 185–192. https://doi.org/10.1140/epjst/e2011-01390-6

26. J. F. Gómez-Aguilar, Analytical and numerical solutions of nonlinear alcoholism model via variable-order fractional differential equations, *Phys. A*, **494** (2018), 52–75. https://doi.org/10.1016/j.physa.2017.12.007

27. X.Y. Li, Y. Gao, B.Y. Wu, Approximate solutions of Atangana-Baleanu variable order fractional problems, *AIMS Mathematics*, **5** (2020), 2285–2294. https://doi.org/10.3934/math.2020151

28. M. K. A. Kaabar, A. Refice, M. S. Souid, F. Martinez, S. Etemad, Z. Siri, et al., Existence and UHR stability of solutions to the implicit nonlinear FBVP in the variable order settings, *Mathematics*, **9** (2021), 1693. https://doi.org/10.3390/math9141693

29. M. B. Jeelani, A. S. Alnahdi, M. A. Almalahi, M. S. Abdo, H. A. Wahash, N. H. Alharthi, Qualitative analyses of fractional integro differential equations with a variable order under the Mittag-Leffler power law, *J. Funct. Spaces*, **2022** (2022), 6387351. https://doi.org/10.1155/2022/6387351

30. T. A. Burton, A fixed-point theorem of Krasnoselskii, *Appl. Math. Lett.*, **11** (1998), 85–88. https://doi.org/10.1016/S0893-9659(97)00138-9

31. O. Naifar, G. Rebiai, A. B. Makhlouf, M. A. Hammami, A. Guezane-Lakoud, Stability analysis of conformable fractional-order nonlinear systems depending on a parameter, *J. Appl. Anal.*, **26** (2020), 287–296. https://doi.org/10.1515/jaa-2020-2025

32. A. Ben Makhlouf, D. Boucenna, M. A. Hammami, Existence and stability results for generalized fractional differential equations, *Acta Math. Sci.*, **40** (2020), 141–154. https://doi.org/10.1007/s10473-020-0110-3

33. O. Naifar, A. Jmal, A. M. Nagy, A. Ben Makhlouf, Improved quasiuniform stability for fractional order neural nets with mixed delay, *Math. Probl. Eng.*, **2020** (2020), 8811226. https://doi.org/10.1155/2020/8811226