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Research article

Generalized Pareto distribution and income inequality: an extension of Gibrat's law

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Abstract: Motivated by empirical observations, we proposed a possible extension of Gibrat's law. By applying it into the random growth theory of income distribution, we found that the income distribution is described by a generalized Pareto distribution (GPD) with three parameters. We observed that there is a parameter η in the GPD that plays a key role in determining the shape of income distribution. By using the Kolmogorov-Smirnov test, we empirically showed that, for typical market-economy countries, η is significantly close to 0, indicating that the income distribution is characterized by a two-class pattern: The bottom 90% of the population is approximated by an exponential distribution, while the richest 1%~3% is approximated by an asymptotic power law. However, we empirically found that in China during the period of the planned economy and the early stages of market reform (from 1978 to 1990), η deviated significantly from 0, indicating that the bottom of the population no longer conformed to an exponential distribution.

Keywords: random growth theory; Kolmogorov forward equation; generalized Pareto distribution; Kolmogorov-Smirnov test; generalized double-Pareto distribution
Mathematics Subject Classification: 60H15, 60E05

1. Introduction

Global income inequality in the 21st century is growing, and top earners are taking hold of a larger and larger fraction of total income [1,2]. The upper tail of the income distribution, which is referred to as the richest 1%~3% of the population, has been long known to be approximated by the Pareto distribution [3,4]. Based on the random growth theory of income distribution (RGTID) [5–9], it has been acknowledged that the Pareto distribution arises because the dynamics of income obey the Gibrat's law in the stochastic process. This law was first observed by Gibrat [10] while investigating the growth of a firm's size, stating that firm growth is a purely random effect, independent of firm size. When this law is applied to the RGTID, the resulting distribution of income is governed by the Pareto law [5,9,11,12]. For this reason, Gibrat's law is widely used to understand the rise in top-income inequality. However, the singular focus on the top-income class of households overlooks the component of earnings inequality that is arguably most consequential for the low and middle-income classes of citizens [13]. Empirical observations [14–18] have indicated that, in market-economy countries, the income structure of the bottom 90% of the population is approximated by an exponential distribution. Therefore, it has been proposed [14,15] that the income distribution is characterized by a two-class pattern, in which the bottom 90% of the population is approximated by an exponential distribution and the richest 1%~3% is approximated by a Pareto distribution. In this paper, we attempt to reproduce this two-class pattern of income distribution within the framework of the RGTID by using a possible extension of Gibrat's law.

Although Gibrat's law accounts for the emergence of the Pareto distribution, it is not exact. In fact, literature on firm size has observed that small firms may grow faster than large ones [19,20]. Moreover, the independence between firm size and growth becomes clearer as time passes [21]. This suggests that Gibrat's law holds asymptotically for firms above a certain size threshold. Likewise, in the literature on income distribution, Blanchet et al. [22] proposed to identify the Pareto distribution of income as an asymptotic law above a certain high level, while Gabaix et al. [23] considered possible deviations from Gibrat's law to explore the extension of the Pareto distribution. By combining these two strands of literature, in this paper we use a possible extension of Gibrat's law to study the income distribution. As with empirical observations in the literature on firm size [19–21], the extension of Gibrat's law is required to satisfy that the growth rate of a person's income is asymptotically independent of their income above some high level. By applying this extension of Gibrat's law to the RGTID, we find that the resulting distribution of income is denoted by a generalized Pareto distribution (GPD) with three parameters. In particular, we observe that there is a key parameter η in the GPD to determine the shape of income distribution. As $\eta = 0$, the GPD becomes an exponential distribution. However, as long as $\eta > 0$, the GPD always has an asymptotic power-law tail (or the Pareto tail) above some high level. This implies that when η is close to 0, the GPD may exhibit a two-class pattern in income distribution. In this pattern, the bottom of the distribution is approximated by an exponential law, while the upper tail is approximated by a Pareto law.

The rest of the paper is organized as follows. Section 2 introduces a possible extension of Gibrat's law motivated by empirical observations. Section 3 applies this extension of Gibrat's law into the RGTID to derive income distribution, where we find that the resulting distribution of income is denoted by the GPD with three parameters. Section 4 shows that there is a key parameter η in the GPD to determine the shape of income distribution. In particular, we show that, as $\eta \approx 0$, the GPD yields a two-class pattern of income distribution, in which the bottom of the distribution is approximated by an exponential law, and the upper tail of the distribution is approximated by the Pareto law. Section 5 employs the Kolmogorov-Smirnov test to examine if the parameter η is significantly close to 0 by using the data from the United States, the United Kingdom, China, and Canada. Section 6 concludes.

2. The extension of Gibrat's law

Gibrat's law is a rule of proportionate growth. If one denotes the size of a firm at the time t by X_t, this law states that the increment of firm size is a linear function of firm size [19–21]

$$X_{t+\Delta t} - X_t = \Delta\varepsilon_t X_t, \tag{1}$$

where Δt denotes the increment of time, and the proportional coefficient $\Delta\varepsilon_t$ is assumed to be independent of X_t. In the simplest form, $\Delta\varepsilon_t$ can be considered as a constant, but a more realistic model allows for randomness.

To extend Gibrat's law (1), we write the increment of firm size as a general function of firm size

$$X_{t+\Delta t} - X_t = \Delta\varepsilon_t f(X_t), \tag{2}$$

where $f(X_t)$ is assumed to be a smooth function of X_t.

When $f(X_t) = X_t$, Eq (2) returns to Gibrat's law (1). Mathematically, the smooth function $f(X_t)$ can be always expanded as a Taylor series:

$$f(X_t) = a_0 + a_1 X_t + a_2 X_t^2 + a_3 X_t^3 + \cdots. \tag{3}$$

By using Eq (3), Eq (2) can be written as

$$X_{t+\Delta t} - X_t = \Delta\varepsilon_t (a_0 + a_1 X_t + a_2 X_t^2 + a_3 X_t^3 + \cdots). \tag{4}$$

Using Eq (4), the growth rate of firm size, $r_{t+\Delta t}$, is equal to

$$r_{t+\Delta t} = \frac{X_{t+\Delta t} - X_t}{X_t} = \Delta\varepsilon_t \left(\frac{a_0}{X_t} + a_1 + a_2 X_t + a_3 X_t^2 + \cdots\right). \tag{5}$$

Empirical observations indicate that small firms may grow faster than large firms [19,20], and the independence between size and growth becomes clearer as time passes [21]. This can be summarized as a stylized fact as below:

Stylized fact 1: The growth rate of firm size is asymptotically independent of the size above a certain size threshold.

By observing Eq (5), we find that $a_2 = a_3 = a_4 = \cdots = 0$ is the unique choice to satisfy the stylized fact 1. In this way, Eq (5) reads

$$r_{t+\Delta t} = \frac{X_{t+\Delta t} - X_t}{X_t} = \Delta\varepsilon_t \left(\frac{a_0}{X_t} + a_1\right), \tag{6}$$

which is independent of X_t as $X_t \to \infty$.

Therefore, we assume $a_2 = a_3 = a_4 = \cdots = 0$, by which Eq (4) can be written as

$$X_{t+\Delta t} - X_t = \Delta\varepsilon_t (a_0 + a_1 X_t). \tag{7}$$

In particular, according to Eq (6), when $\Delta\varepsilon_t a_0 > 0$ and $\Delta\varepsilon_t a_1 > 0$, $r_{t+\Delta t}$ is a decreasing function of X_t. This implies that for firms whose size increases over time, small firms tend to grow faster than large ones, which aligns with empirical observations [19,20].

Let us order $\Delta W_t = \Delta\varepsilon_t a_0$ and $\eta = a_1/a_0$. Thus, Eq (7) can be rewritten as

$$X_{t+\Delta t} - X_t = (1 + \eta X_t)\Delta W_t. \tag{8}$$

Equation (8) is a possible extension of Gibrat's law (1). It asymptotically returns to Gibrat's law (1) as[1] $X_t \to \infty$.

To obtain the continuous-time form of Eq (8), we assume that ΔW_t denotes the Brownian motion with drift, i.e., $\Delta W_t \sim N(\mu\Delta t, \sigma^2\Delta t)$. Consequently, as $\Delta t \to dt$, Eq (8) yields

$$dX_t = (1 + \eta X_t)(\mu dt + \sigma dZ_t), \tag{9}$$

where μ and σ are two constants, and $dZ_t \sim N(0, dt)$ denotes the standard Brownian motion.

In this paper, we assume $\eta \geq 0$. In Appendix B, we discuss its economic implications. Under this assumption, $\mu < 0$ is allowed. Empirical observations [19,20] only indicate that for firms whose size increases over time, small firms tend to grow faster than large ones. This does not rule out the possibility of $\mu < 0$. Here, we propose using Eq (9) with $\mu < 0$ to describe the dynamics of firms whose size decreases over time. In fact, if $\eta \geq 0$, $\mu < 0$ has been found to be necessary in the random growth process with a reflecting barrier, particularly when the birth and death processes of firms are ignored; see Gabaix's discussion regarding Eq (11) below, specifically on page 263 in [6].

Equation (9) can be rewritten in the form

$$dX_t = \mu(1 + \eta X_t)dt + \sigma(1 + \eta X_t)dZ_t. \tag{10}$$

Equation (10) is the starting point of this paper. It is easy to check that Eq (10) satisfies the asymptotical scale invariance as $X_t \to \infty$. To see this, we order $X_t \to \infty$ so that Eq (10) asymptotically yields

$$dX_t = \mu\eta X_t dt + \sigma\eta X_t dZ_t. \tag{11}$$

Distinguishing from Eq (10), the proportional increment dX_t/X_t in Eq (11) during dt has a systematic drift component $\mu\eta dt$ and a stochastic diffusion component $\sigma\eta dZ_t$, both of which are independent of X_t. Therefore, Eq (11) is invariant under the scaling change $X_t{}' \to \lambda X_t$. The scale invariance implies a certain kind of fractal property. Based on the RGTID, it has been known [5–9,11,12] that if the dynamics of income obey the random process (11), the resulting distribution of income is denoted by a power law (or the Pareto law). Jones and Kim [9] have pointed out that top-income inequality is well characterized by this fractal property. However, Eq (10) indicates that the scale invariance does not hold exactly but only asymptotically, such that the income distribution of top earners should be approximated by an asymptotic power law. Next, we apply Eq (10) into the RGTID.

3. The basic model

By using the well-established result in the RGTID literature for generating the income distribution [5,9,11,12], if the dynamics of income x obey the random process (10), the density of the income distribution, $f(x,t)$, satisfies a Kolmogorov forward equation[2] (also known as the Fokker-Planck equation):

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial[\mu(1+\eta x)f(x,t)]}{\partial x} + \frac{1}{2}\frac{\partial^2[\sigma^2(1+\eta x)^2 f(x,t)]}{\partial x^2}. \tag{12}$$

[1] It is easy to check that, as $X_t \to \infty$, $1 + \eta X_t$ yields ηX_t asymptotically. Then, Eq (8) can be asymptotically written as $X_{t+\Delta t} - X_t = \eta \Delta W_t X_t$, which is the same as Eq (1).

[2] The Kolmogorov forward equation, as applied to standard Brownian motion, demonstrates how the probability density function of the motion shifts over time while considering dispersion and drift.

The derivation of Eq (12) can be found in Appendix A.

To obtain the steady-state solution of Eq (12), we consider a lower bound on income so that a person cannot go below a given level x_{min}. In real economies, the unemployment compensation plays the role of the lower bound x_{min}. Thus, Eq (12) describes a random growth with a reflecting barrier x_{min} [5]. For the steady-state distribution of income, $f(x,t) = f(x)$, we should have $\partial f(x,t)/\partial t = 0$, such that Eq (12) yields

$$-\frac{\partial[\mu(1+\eta x)f(x)]}{\partial x} + \frac{1}{2}\frac{\partial^2[\sigma^2(1+\eta x)^2 f(x)]}{\partial x^2} = 0, \tag{13}$$

It is easy to get a solution of Eq (13) as below:

$$\begin{cases} f(x) = \frac{c_0}{\sigma^2}(1+\eta x)^{\frac{2\mu}{\sigma^2\eta}-2} \\ x \geq x_{min} \end{cases}, \tag{14}$$

where c_0 denotes an integral constant.

The existence of the lower bound x_{min} requires the normalization equation:

$$\int_{x_{min}}^{\infty} f(x)dx = 1, \tag{15}$$

which is used to determine the integral constant c_0.

By substituting Eq (14) into Eq (15), we get the density distribution of income

$$\begin{cases} f(x) = \left(\frac{\eta+\frac{1}{\theta}}{1+\eta x_{min}}\right)\left(\frac{1+\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-2} \\ x \geq x_{min} \end{cases}, \tag{16}$$

with

$$\theta = -\frac{\sigma^2}{2\mu}, \tag{17}$$

where $\mu < 0$ and $\eta \geq 0$ are used to satisfy the normalization Eq (15). Here, $\mu < 0$ arises because entry and exit processes (or birth and death processes) of agents in markets are ignored in the Kolmogorov forward equation indicated by Eq (12), as discussed by Gabaix [6]. Later, we will explore the Kolmogorov forward equation when the entry and exit processes of agents are taken into account.

We denote the cumulative distribution of income by

$$F_\eta(X \leq x) = \int_{x_{min}}^{x} f(z)dz, \tag{18}$$

where $F_\eta(X \leq x)$ denotes the fraction of the population with income not exceeding x.

Substituting Eq (16) into Eq (18) yields

$$F_\eta(X \geq x) = \left(\frac{1+\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-1}, \tag{19}$$

where $x \geq x_{min}$, and $F_\eta(X \geq x) = 1 - F_\eta(X \leq x)$ denotes the counter-cumulative distribution, which is known as the survival function.

Equation (19) can be written in the standard form of the generalized Pareto distribution [24–27][3]:

$$F_\eta(X \geq x) = \left[1 + A\left(\frac{x - x_{min}}{B}\right)\right]^{-\frac{1}{A}}, \tag{20}$$

where

$$A = \frac{\theta\eta}{1+\theta\eta} \text{ and } B = \frac{\theta(1+\eta x_{min})}{1+\theta\eta}.$$

Therefore, we call Eq (19) the generalized Pareto distribution (GPD). Equation (19) is the main result of this paper. It has three parameters to determine the shape of income distribution.

In particular, if $\eta > 0$, Eq (19) always has an asymptotical power-law tail (i.e., the Pareto distribution); that is, as $x \to \infty$, one has[4]

$$F_{\eta>0}(X \geq x) \sim \left(\frac{x}{x_0}\right)^{-\frac{1}{\theta\eta}-1}, \tag{21}$$

where $x_0 = (1 + \eta x_{min})/\eta$.

To account for the entry and exit processes of agents, the Kolmogorov forward equation indicated by Eq (12) can be revised as follows:

$$\frac{\partial f(x,t)}{\partial t} = -\delta f(x,t) - \frac{\partial[\mu(1+\eta x)f(x,t)]}{\partial x} + \frac{1}{2}\frac{\partial^2[\sigma^2(1+\eta x)^2 f(x,t)]}{\partial x^2}, \tag{22}$$

where $\delta \geq 0$ is a constant, signifying that agents may exit (or retire) at rate δ [23]. In [9], δ is also interpreted as the exogenous destruction rate, where the existing agents (entrepreneurs) are replaced by new “young” agents (entrepreneurs).

By solving Eq (22), the steady-state solution can be obtained:

$$f(x) = C_-(1+\eta x)^{-\phi(\delta)_- - 1} + C_+(1+\eta x)^{-\phi(\delta)_+ - 1}, \tag{23}$$

where

$$\phi(\delta)_\pm = \frac{1}{\eta}\left[-\frac{\mu - \frac{1}{2}\sigma^2\eta}{\sigma^2} \pm \sqrt{\left(\frac{\mu - \frac{1}{2}\sigma^2\eta}{\sigma^2}\right)^2 + \frac{2\delta}{\sigma^2}}\right],$$

C_- and C_+ are two constants.

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[3] In fact, Eq (20) represents the survival function of the generalized Pareto distribution. However, for simplicity, we do not distinguish between the terms “survival function” and “counter-cumulative distribution”. Therefore, we refer to Eq (20) directly as the generalized Pareto distribution. Furthermore, this distribution is also known as the Pareto type II distribution.

[4] As $x \to \infty$, one has $1 + \eta x \sim \eta x$, such that $F_{\eta>0}(X \geq x) \sim \left(\frac{\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-1} = \left(\frac{x}{x_0}\right)^{-\frac{1}{\theta\eta}-1}$, which represents the survival function of the Pareto type I distribution. For simplicity, in this paper, we refer to Eq (21) directly as the Pareto distribution.

Equation (23) indicates a generalized double-Pareto distribution[5], which will be discussed elsewhere.

4. The parameter η

Here, we show that the parameter η in the GPD (19) plays a key role in determining the shape of income distribution. To this end, we observe that, when $\eta = 0$, the GPD (19) becomes an exponential distribution

$$F_{\eta=0}(X \geq x) = \lim_{\eta \to 0} F_\eta(X \geq x) = exp\left(-\frac{x-x_{min}}{\theta}\right), \tag{24}$$

where $x_{min} > 0$.

Empirical observations [14–18] have indicated that, in market-economy countries, the income distribution is characterized by a two-class pattern, in which the bottom 90% of the population is approximated by the exponential law (24) and the richest 1%~3% is approximated by the power law (21). Next, we show that empirical observations can be explained by the GPD (19) as long as η is close to zero. To this end, we investigate the conditions in which the GPD (19) can be replaced by Eqs (21) and (24).

First, we consider $\eta \approx 0$ so that $|\eta x| \ll 1$ and $|\eta x_{min}| \ll 1$. Thus, one has

$$(1+\eta x)^{-\frac{1}{\theta\eta}-1} \approx exp(-x/\theta) \text{ and } (1+\eta x_{min})^{-\frac{1}{\theta\eta}-1} \approx exp(-x_{min}/\theta).$$

This means that the GPD (19) for $x_{min} \leq x \ll 1/\eta$ can be replaced by the exponential distribution (24); that is,

$$F_{\eta\approx 0}(X \geq x) = \left(\frac{1+\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-1} \approx exp\left(-\frac{x-x_{min}}{\theta}\right), \tag{25}$$

Therefore, we expect that as long as η is sufficiently close to zero, the exponential law (24) is roughly valid for the majority of the population, in which each person's income is lower than $1/\eta$.

Second, we consider that x is sufficiently large so that $\eta x \gg 1$. In this way, one has

$$F_{\eta\approx 0}(X \geq x) = \left(\frac{1+\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-1} \approx \left(\frac{\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-1} = \left(\frac{x}{x_0}\right)^{-\frac{1}{\theta\eta}-1}, \tag{26}$$

where $x_0 = (1+\eta x_{min})/\eta$ as denoted by Eq (21).

Equation (26) means that the GPD (19) for $x \gg 1/\eta$ can be replaced by the Pareto distribution (21). Thus, we expect that as long as η is sufficiently close to zero, the Pareto distribution (21) is roughly valid for top earners, in which each person's income is higher than $1/\eta$.

Based on the discussion above, we conclude that when η is close to zero, the GPD (19) can be approximated by the following two-class pattern:

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[5] For example, the steady-state solution (23) can be written in a two-class form as follows: $f(x) \sim \begin{cases} (1+\eta x)^{-\phi(\delta)_- -1}, & x < x_{min} \\ (1+\eta x)^{-\phi(\delta)_+ -1}, & x > x_{min} \end{cases}$, where $\mu > \frac{1}{2}\sigma^2\eta$. We call it the generalized double-Pareto distribution.

$$F_{\eta\approx 0}(X \geq x) \approx \begin{cases} exp\left(-\frac{x-x_{min}}{\theta}\right) & x_{min} \leq x < x_0 \approx \frac{1}{\eta} \\ \left(\frac{x}{x_0}\right)^{-\frac{1}{\theta\eta}-1} & x > x_0 \approx \frac{1}{\eta} \end{cases}, \quad (27)$$

where $x_0 = (1 + \eta x_{min})/\eta \approx 1/\eta$ arises because $|\eta x_{min}| \ll 1$.

However, if η is sufficiently larger than 0, then Eq (24) may break down. For example, if $\eta x > 1$ for any $x \geq x_{min}$, then

$$(1 + \eta x)^{-\frac{1}{\theta\eta}-1} \approx exp(-x/\theta)$$

holds no longer. This implies that when η deviates significantly from 0, the bottom of the population cannot be approximated by the exponential distribution (24).

In practice, the statement "the parameter η is close to 0" may be highly subjective. To address this problem, we use the Kolmogorov-Smirnov test to identify if the parameter η is significantly close to 0.

5. Tests using the data from the US, the UK, China, and Canada

From a statistical perspective, the data on household income can be regarded as a sample drawn from the income distribution $F_\eta(X \geq x)$. Therefore, we selected household income data[6] from the latest years available for four representative market-economy countries to test if the parameter η is significantly close to 0. The four countries include three developed economies (the United States in 2020, the United Kingdom in 2018, and Canada in 2018) and one developing economy (China in 2015) that is at present the world's second-largest economy. Figure 1 shows that, for each country, the income distribution of the bottom 90% of the sample (circle) is well approximated by the exponential law (24) (red curve), while the upper tail of the sample is approximated by the power law (21) [i.e., the Pareto distribution] (black curve). By Eq (27), this implies that η is close to 0 for the four countries.

To strictly identify if the parameter η is significantly close to 0 for four countries, we use the Kolmogorov-Smirnov (KS) test. The hypothesis testing is written as:

$$H_0: \eta = 0,$$

$$H_1: \eta > 0,$$

where by Eq (24), the null hypothesis $\eta = 0$ indicates that the income distribution is described by the exponential law, in which x_{min} is chosen to be the first quantile of the sample. In our database, the data from the United States, the United Kingdom, and China constitute large samples, each comprising 99 quantiles[7]. This implies that the KS test is feasible for these three countries. Although the data from Canada only includes 12 quantiles, we still use the KS test to perform a rough examination.

[6] The data resource can be found in Data Availability Statement.

[7] Here, we also list the quantile functions of the Pareto distribution (21) and the exponential distribution (24), that is, $x(p) = \frac{1+\eta x_{min}}{\eta}(1-p)^{-\frac{\theta\eta}{1+\theta\eta}}$ and $x(p) = x_{min} - \theta ln(1-p)$, respectively, where $p = F_\eta(X \leq x)$. The derivation of both quantile functions can be found in page 10 in [28].

The testing result is listed in Table 1. For the United States in 2020, the United Kingdom in 2018, Canada in 2018, and China in 2015, $\eta = 0$ cannot be rejected at the significance level of 0.05. This means that η is significantly close to 0. However, η is not exactly equal to zero, as Figure 1 has shown that the income distribution of each country has a power-law tail as described by Eq (21). Furthermore, for the four countries depicted in Figure 1, by using Eqs (21) and (24) we have presented the least square estimations for the three parameters in the GPD (19) in Table 2. For the sake of simplicity, monetary units have been omitted from the estimation results presented in Table 2. Here, the parameter η is found to be approximately in the order of 10^{-6} to 10^{-5}, which is in accordance with the KS test. To perform the least square estimation, we use the top four quantile points for China, the United Kingdom, and the United States, as well as the top three quantile points for Canada to represent the upper tail areas (i.e., the Pareto area), as shown in Figure 1. To show top-income inequality in four countries, we estimate the Pareto exponent in Table 2, which is denoted by $1 + 1/(\theta\eta)$ according to Eq (21). In this regard, Jones and Kim [9] have pointed out that a larger Pareto exponent is associated with lower top-income inequality. This suggests that when the income distribution follows a Pareto distribution with a larger exponent, the concentration of income among the top earners is less pronounced.

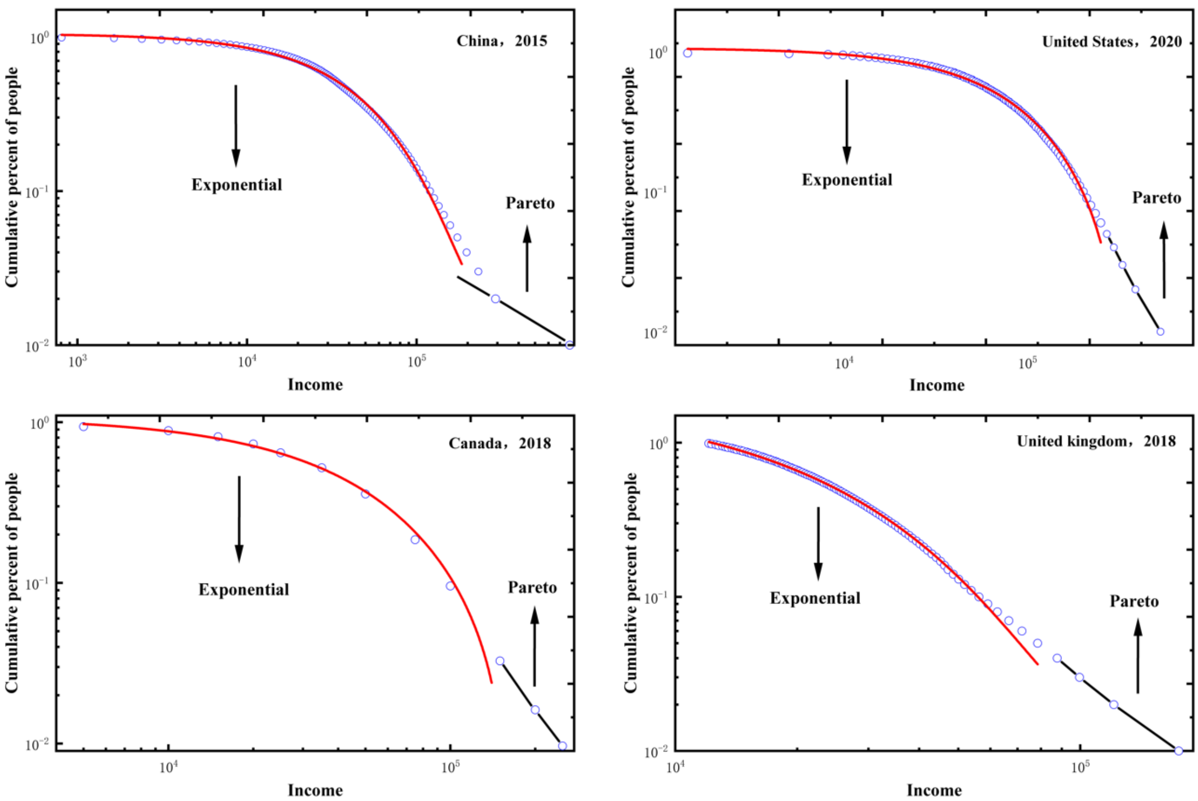

Figure 1. Income distributions of the US, the UK, China, and Canada.

Table 1. Kolmogorov-Smirnov (KS) test for the US, the UK, China, and Canada in the latest years available.

Year		2015	2018	2020
China	P-value (KS)	0.633	——	——
US		——	——	0.741
UK		——	0.970	——
Canada		——	0.593	——

Table 2. The least square estimation of three parameters in the GPD (19) and the Pareto exponent for four countries in Figure 1.

Country	$\boldsymbol{\eta}$	$\boldsymbol{x_{min}}$	$\boldsymbol{\theta}$	**1+1/($\boldsymbol{\theta\eta}$)**
China (2015)	5.72×10^{-5}	-1.07×10^{3}	5.23×10^{4}	1.33
Canada (2018)	1.72×10^{-5}	5.58×10^{3}	4.17×10^{4}	2.39
UK (2018)	4.92×10^{-5}	1.10×10^{4}	2.05×10^{4}	1.99
US (2020)	8.67×10^{-6}	7.79×10^{3}	8.53×10^{4}	2.35

China is a special sample that has undergone the transition from a centrally planned economy to a market economy. Here, we selected the data[8] from China in 1978, 1980, 1990, and 2000 to check if the parameter η was significantly close to 0 in the early stages of market-oriented economic reformation. As shown in Table 3, $\eta = 0$ is rejected at the significance level of 0.01 in 1978, 1980, and 1990, while it cannot be rejected in 2000. This suggests that for China during the period of the planned economy and the early stages of market reform (from 1978 to 1990), η deviates significantly from 0, such that the bottom of the population no longer conforms to an exponential distribution[9]. The KS test results are supported by the data fitting presented in Figure 2, which illustrates how the income distribution of the bottom 90% of the population in China deviates from an exponential distribution (represented by the red curve) between 1978 and 1990.

However, for all the years presented, the income distribution consistently exhibits a power-law tail, as indicated by the black curve in Figure 2. This empirical observation is in agreement with Eq (21).

[8] The data resource can be found in Data Availability Statement.

[9] In Appendix B, we further discuss the economic implication of the parameter η. Within the framework of the random growth theory of income distribution (RGTID), we find that the parameter η may have a significance on characterizing the inequality of earning opportunities. Ideally, $\eta = 0$ corresponds to the equality of earning opportunities. From this sense, $\eta = 0$ being rejected for China in the years 1978, 1980, and 1990 implies that the equality of earning opportunities was significantly disrupted during the period of the planned economy and the early stages of market reform.

Table 3. Kolmogorov-Smirnov (KS) test for China in the early stages of market-oriented economic reformation.

Year	P-value (KS)
1978	0.000
1980	0.000
1990	0.008
2000	0.658

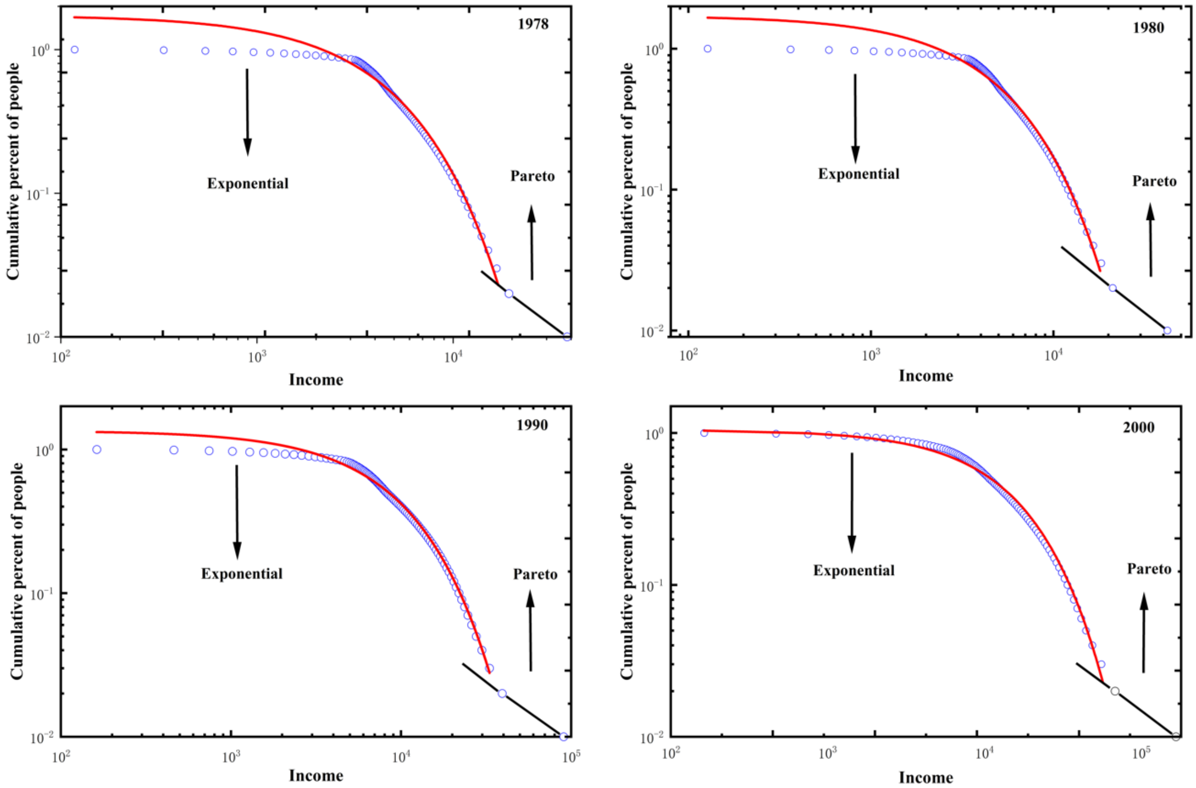

Figure 2. Income distribution in China from 1978 to 2000.

6. Conclusions

Motivated by empirical observations, we propose a possible extension of Gibrat's law, which states that the growth rate of firm size is asymptotically independent of the size above a certain size threshold. By applying this extension of Gibrat's law to the RGTID, we find that the income distribution is described by a generalized Pareto distribution (GPD) with three parameters. In particular, we observe that there is a key parameter η in the GPD to determine the shape of income distribution. As $\eta = 0$, the GPD becomes an exponential distribution. However, as long as $\eta > 0$, the GPD always has an asymptotic power-law tail (or the Pareto tail) above some high level. This implies that when η is close to 0, the GPD may exhibit a two-class pattern in income distribution. In this pattern, the bottom of the distribution is approximated by an exponential law, while the upper tail is approximated by a Pareto law.

By employing the Kolmogorov-Smirnov test, we empirically demonstrate that $\eta = 0$ cannot be rejected at the significance level of 0.05 for typical market-economy countries such as the United

States, the United Kingdom, Canada, and China (post-2000), while the income distributions in these countries exhibit a power-law tail without exception. This suggests that η is significantly close to 0 for these market-economy countries, indicating that the income distribution is characterized by a two-class pattern: The bottom 90% of the population is well approximated by an exponential distribution, while the richest 1%~3% of the population is approximated by the Pareto distribution. However, we empirically find that $\eta = 0$ is rejected at the significance level of 0.01 for China between 1978 and 1990. This suggests that for China during the period of the planned economy and the early stages of market reform, η deviates significantly from 0, indicating that the bottom of the population no longer conforms to an exponential distribution.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Data availability statement

This study analyzed publicly available datasets. These datasets can be accessed here:

Data resource for China: http://wid.world/data/;

Data resource for the United States: https://ipums.org/;

Data resource for Canada: https://www150.statcan.gc.ca/t1/tbl1/en/tv.action?pid=1110000801;

Data resource for the United Kingdom: https://www.gov.uk/government/statistics/percentile-points-from-1-to-99-for-total-income-before-and-after-tax.

Conflict of interest

The author declares no conflict of interest.

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Appendices

Appendix A. Derivation of Eq (12)

Let us consider the Ito process

$$dX_t = \mathcal{A}(t, X_t)dt + \mathcal{B}(t, X_t)dZ_t, \tag{A.1}$$

where $\mathcal{A}(t, X_t)$ and $\mathcal{B}(t, X_t)$ are two functions of X_t and t, and dZ_t denotes the standard Brownian motion.

By using the theory of stochastic differential equations, it has been known that when the variable x evolves according to the random process (A.1), the density distribution of x satisfies the Kolmogorov forward equation:

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial[\mathcal{A}(t,x)f(x,t)]}{\partial x} + \frac{1}{2}\frac{\partial^2[\mathcal{B}(t,x)^2 f(x,t)]}{\partial x^2}. \tag{A.2}$$

The derivation of Eq (A.2) can be found in any book on stochastic calculus, e.g., see [29, page 282] or [30, page 50].

Comparing Eqs (10) and (A.1), one has

$$\mathcal{A}(t, X_t) = \mu(1 + \eta X_t), \tag{A.3}$$

and

$$\mathcal{B}(t, X_t) = \sigma(1 + \eta X_t). \tag{A.4}$$

Substituting Eqs (A.3) and (A.4) into Eq (A.2) yields Eq (12).

Appendix B. Economic implication of the parameter η

Here, we provide a possible economic implication for the parameter η within the framework of the RGTID. In the setting of random growth process (10), the resulting distribution of income is characterized by the random variable X with the survival function $F_\eta(X \geq x)$. This setting implies that the income distribution $F_\eta(X \geq x)$ can be understood as a probability distribution[10], which represents the likelihood of a person acquiring an income equal to x. Given this understanding, we propose a *special* definition for identifying the equality of earning opportunities. To this end, let us calculate the following conditional probability:

$$F_\eta(X \geq x + y | X \geq y) = \frac{F_\eta(\{X \geq x + y\} \cap \{X \geq y\})}{F_\eta(X \geq y)} = \frac{F_\eta(X \geq x + y)}{F_\eta(X \geq y)}, \tag{B.1}$$

which denotes the probability of a person acquiring an income equal to x, given that they have earned an income of y.

Definition B.1 (Equality of earning opportunities). *If the probability of a person acquiring earnings equal to x is denoted by $F_\eta(X \geq x)$, then earning opportunities are considered equal for everyone as long as $F_\eta(X \geq x + y | X \geq y)$ is independent of y for any x and y.*

To understand the implication of Definition B.1, we substitute Eq (19) into Eq (B.1) and obtain

$$F_\eta(X \geq x + y | X \geq y) = \left[\frac{1 + \eta(x + y)}{1 + \eta y}\right]^{-\frac{1}{\theta\eta} - 1}. \tag{B.2}$$

When $\eta = 0$, Eq (B.2) becomes

$$F_{\eta=0}(X \geq x + y | X \geq y) = \lim_{\eta \to 0} F_\eta(X \geq x + y | X \geq y) = exp\left(-\frac{x}{\theta}\right), \tag{B.3}$$

which means that a person's probability of acquiring future earnings of x is irrelevant to their past earnings of y. In other words, the probability of acquiring earnings of x is equal for everyone.

However, when $\eta > 0$, Eq (B.2) can be written as

$$F_{\eta>0}(X \geq x + y | X \geq y) = \left[1 + \frac{\eta x}{1 + \eta y}\right]^{-\frac{1}{\theta\eta} - 1}, \tag{B.4}$$

which is a monotonically increasing function of y. This means that when $\eta > 0$, a person's probability of acquiring future earnings of x is positively proportional to their past earnings y. This represents the manifestation of the Matthew effect [31] in income accumulation.

Based on Eqs (B.3) and (B.4) and according to Definition B.1, we identify the parameter η as an indicator that characterizes the inequality of earning opportunities. That is, $\eta = 0$ indicates the presence of equal earning opportunities, while $\eta > 0$ suggests a situation where equal opportunity is compromised. However, it should be clarified that Definition B.1 is only applied to the RGTID,

[10] This understanding is widely used in the literature of income distribution [9, 22].

wherein the income distribution is ideally equivalent to a probability distribution of income. It does not correspond to the general concept of "equality of earning opportunities". In fact, identifying the equality of earning opportunities in a general sense is a complex issue. For example, earnings mobility should be taken into account [32]. From this sense, identifying $\eta = 0$ as representing the equality of earning opportunities in a strict sense is naive. It holds strictly only if the income distribution is ideally equivalent to a probability distribution of income.

According to Definition B.1, one can explain why the richest 1%–3% of the population is capturing an increasingly larger share of total income, in stark contrast to the bottom 90%, thereby exacerbating global income inequality, as reported in the literature [33]. First, we demonstrate that if η is not equal to zero, the earning opportunities of top earners may be significantly enhanced. To do so, according to Eq (B.4), $F_\eta(X \geq x + y|X \geq y)$ is observed to be a monotonically increasing function with respect to y, provided that η remains greater than zero. In particular, as $y \to \infty$, by Eq (B.4) one has

$$\lim_{y\to\infty} F_{\eta>0}(X \geq x + y|X \geq y) = 1, \tag{B.5}$$

which means that if a person's past income y is sufficiently high, then they have the potential to earn any amount of income x in the future.

Second, based on our empirical observations presented in Section 5, it can be inferred that η is approximately equal to zero in typical market-economy countries. Under this assumption, by Eq (B.4), $F_\eta(X \geq x + y|X \geq y)$ is roughly independent of the variable y, provided that $\eta y \ll 1$. This implies that when η is approximately equal to zero, individuals[11] in the lower-income bracket of the population have roughly equal probabilities of earning an amount x, as indicated by Eq (B.3). However, when y becomes sufficiently large, Eq (B.5) suggests that the earning probabilities for top earners may be improved significantly, even if η is approximately zero. This, in turn, can lead to a sharp increase in the earnings opportunity gap between the top earners and those at the lower end of the population.

[11] The statement "individuals in the lower-income bracket of the population" means that their income in the past is much less than $1/\eta$.