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Research article

Some generic hypersurfaces in a Euclidean space

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Abstract: In this paper, we find three nontrivial characterizations of Euclidean spheres. In the first result, we show that the existence of a nonzero nontrivial concircular vector field ω on a compact and connected hypersurface N of the Euclidean space R^{m+1} with a mean curvature α constant along the integral curves of ω and a shape operator T satisfying $T(\omega) = \alpha\omega$ implies that α is a constant and N is isometric to a sphere, and the converse also holds. In the second result, we show that the presence of a unit Killing vector field $\mathbf{v}$ on a compact and connected hypersurface N of a Euclidean space R^{m+1} gives a nonzero function $\sigma = g(T\mathbf{v}, \mathbf{v})$ with shape operator T, and the integral of the function $m\alpha\sigma Ric(\mathbf{v}, \mathbf{v})$ has a certain lower bound, and is isometric to an odd-dimensional sphere, and the converse holds too. Finally, we show that for a compact and connected hypersurface N with support ρ and basic vector field $\mathbf{u}$, the integral of the Ricci curvature $Ric(\mathbf{u}, \mathbf{u})$ has a specific lower bound and is necessarily isometric to a sphere, and the converse also holds.

Keywords: hypersurfaces; Killing vector fields; concircular vector fields; n-sphere; Ricci curvature
Mathematics Subject Classification: 53C20, 53C21, 53B50

1. Introduction

The geometry of hypersurfaces lies at the foundation of differential geometry, it started with the theory of curves and surfaces in the Euclidean 3-space R^3 [11]. Given an orientable immersed hypersurface N in the Euclidean space R^{m+1} with immersion $\varphi : N \to R^{m+1}$, we have the unit normal ζ, the shape operator T, the support $\rho = \langle \varphi, \zeta \rangle$ a smooth function defined on the hypersurface N and the mean curvature α, given by $m\alpha = trT$ being trace of the shape operator T [11]. If the hypersurface N of the Euclidean space R^{m+1} is compact, then we have the following well-known Minkowski's formula:

$$\int_N (1 + \rho\alpha) = 0. \tag{1.1}$$

As an outcome of Minkowski's formula, we conclude that there are no compact minimal hypersurfaces (hypersurfaces with mean curvature $\alpha = 0$) in the Euclidean space R^{m+1}.

Among compact hypersurfaces of Euclidean spaces, important are the Euclidean spheres $S^m(c)$ of constant curvature c, with the imbedding $\varphi : S^m(c) \rightarrow R^{m+1}$, $\varphi(x) = x$, shape operator $T = -\sqrt{c}I$, and unit normal $\zeta = \sqrt{c}\varphi$. Taking $\mathbf{a}$ as a nonzero constant vector field on R^{m+1}, we can express it as $\mathbf{a} = \mathbf{u} + f\zeta$, where $f = \langle \mathbf{a}, \zeta \rangle$ and $\mathbf{u}$ is the tangential projection of $\mathbf{a}$ on the sphere $S^m(c)$. Letting g be the induced metric and ∇ the Riemannian connection on the sphere $S^m(c)$ and differentiating the equation $\mathbf{a} = \mathbf{u} + f\zeta$ with respect to the vector field E on $S^m(c)$, we have

$$\nabla_E \mathbf{u} = -\sqrt{c} f E, \quad \nabla f = \sqrt{c}\mathbf{u}, \tag{1.2}$$

where ∇f is the gradient of f.

On an odd dimensional sphere $S^{2m-1}(c)$ with imbedding $\varphi : S^{2m-1}(c) \rightarrow R^{2m}$ with unit normal $\zeta = \sqrt{c}\varphi$, shape operator $T = -\sqrt{c}I$, apart from the above vector field $\mathbf{u}$, there is a unit vector field $\mathbf{v}$ defined on $S^{2m-1}(c)$ by

$$\mathbf{v} = J\zeta, \tag{1.3}$$

where J is the complex structure on the Euclidean space R^{2m}. Differentiating the above equation using the Euclidean connection D with respect to a vector field E on $S^{2m-1}(c)$, one confirms

$$\nabla_E v - \sqrt{c}\,\langle E, \mathbf{v} \rangle\, \zeta = \sqrt{c} J E,$$

that is,

$$\nabla_E \mathbf{v} = \sqrt{c}\,(JE)^T, \tag{1.4}$$

where $(JE)^T$ is the tangential projection of JE on $S^{2m-1}(c)$.

Given an immersed hypersurface N of the Euclidean space R^{m+1}, the natural tools for studying the geometry of N are the shape operator T, the mean curvature α, the curvature tensor R, the Ricci tensor Ric, the Ricci operator S, and the scalar curvature τ of N. In [8], it is shown that a compact hypersurface M of the Euclidean space R^{m+1} satisfies the inequality

$$\|T\|^2 \tau \geq \frac{1}{2} \|R\|^2 + \|S\|^2 + 2m(m-1) \|\nabla\alpha\|^2,$$

if and only if α is a constant and N is isometric to the n-sphere $S^m\left(\alpha^2\right)$. Also, in [9], the position vector field φ of a compactly immersed hypersurface N in the Euclidean space R^{m+1} with immersion $\varphi : N \rightarrow R^{m+1}$ and unit normal ζ was used to define a vector field $\mathbf{u}$ on the hypersurface N as the tangential projection of the position vector field φ that leads to the integral formula

$$\int_N \left\{ Ric\,(\mathbf{u}, \mathbf{u}) + m(m-1) - \rho^2\tau \right\} = 0,$$

where $\rho = \langle \varphi, \zeta \rangle$ is the support of N. In [7,9], the above integral was used, which led to many important geometric implications on the compact hypersurface N of the Euclidean space R^{m+1}. Moreover, in [8], it is shown that a compact hypersurface N of positive Ricci curvature in the Euclidean space R^{m+1} with scalar curvature $\tau \leq \lambda_1(m-1)$ is necessarily isometric to the sphere $S^m(c)$, where λ_1 is the first nonzero eigenvalue of the Laplace operator Δ of N with respect to the induced metric.

Recently, there has been a trend toward studying the geometry of the hypersurfaces in R^{m+1}, as the graphs of the smooth functions $h : R^{m+1} \to R$ are called the translation hypersurfaces. The focus, in translation hypersurface N of the Euclidean space R^{m+1}, is on the property function $h : R^{m+1} \to R$, whose graph is N. In [18], translation hypersurfaces of R^{m+1} are studied, whose Gauss-Kronecker curvature depends on either its first p variables or on the rest q variables, where $m = p + q$, and conditions on a translation hypersurface to have Gauss-Kronecker zero curvature are found. If a translation hypersurface N is defined as the graph of the function $h : R^{m+1} \to R$ with h satisfying certain additional conditions, then it is called a separable hypersurface. Separable hypersurfaces in the Euclidean space R^{m+1} have an interesting geometry, as studied in [6, 12, 13, 19]. A complete classification of separable hypersurfaces with zero Gauss-Kronecker curvature in the Euclidean space R^{m+1} is obtained in [6].

In this paper, we are interested in studying the impact of the existence of a concircular vector field as well as a Killing vector field on the immersed hypersurface N of the Euclidean space R^{m+1}. A vector field ω on a Riemannian manifold (N, g) is a concircular vector field if

$$\nabla_E \omega = \sigma E, \quad E \in \Psi(N),$$

where σ is a function on N and $\Psi(N)$ is the space of smooth vector fields on N. We shall use the abbreviation *CLVF* for a concircular vector field. It is known that a *CLVF* ω on a Riemannian manifold (N, g) influences the geometry of (N, g) [4,5]. Moreover, a *CLVF* ω has a role in general relativity [3].To understand the role of *CLVF* in relativity, recall that m-dimensional generalized Robertson-Walker space-time, $m > 3$, is the warped product $I \times_{h^2} M$, with Lorentz metric $g = -dt^2 + h^2 g^*$, where I is an interval $h : I \to R$ is a positive smooth function and (M, g^*) is a Riemannian manifold with $\dim M = (m - 1)$. In [3], Chen has proved a very significant result involving a *CLVF*, namely: A Lorentzian manifold admits a nontrivial timelike *CLVF* if and only if it is a generalized Robertson-Walker space-time. Note that Eq (1.2) shows that the vector field $\mathbf{u}$ is a *CLVF* on the sphere $S^m(c)$ with potential function $\sigma = -\sqrt{c} f$ and naturally the shape operator T of the sphere $S^m(c)$ as a hypersurface of the Euclidean space R^{m+1} satisfies $T(\mathbf{u}) = \alpha \mathbf{u}$, where $\alpha = -\sqrt{c}$ is the mean curvature of $S^m(c)$. This naturally raises a question: Is a compact and connected hypersurface N with shape operator T and mean curvature α of the Euclidean space R^{m+1} admitting a nonzero *CLVF* $\mathbf{u}$ satisfying $T(\mathbf{u}) = \alpha \mathbf{u}$, $\mathbf{u}(\alpha) = 0$, necessarily isometric to $S^m(c)$? In Section 3, we show that this question has an affirmative answer, and indeed, we show that the converse is also true.

Similarly, a vector field ω on an m-dimensional Riemannian manifold (N, g) is said to be a Killing vector field if

$$£_\omega g = 0,$$

and we shall use the abbreviation *KGVF* for a Killing vector field. Note that the presence of a *KGVF* ω on (N, g) influences its geometry as well as topology [2, 14, 17, 21]. Note that the unit vector field $\mathbf{v}$ on the sphere $S^{2m-1}(c)$ satisfies Eq (1.4), which leads to

$$£_{\mathbf{v}} g = 0,$$

that is, $\mathbf{v}$ is a unit *KGVF* on the sphere $S^{2m-1}(c)$. We see that $\sigma = g(T\mathbf{v}, \mathbf{v}) = -\sqrt{c}$ is a constant, and the following holds:

$$\int_{S^{2m-1}(c)} m\alpha\sigma Ric(\mathbf{v}, \mathbf{v}) = \int_{S^{2m-1}(c)} \left(m(m-1)\alpha^2\sigma^2 - \|\nabla\sigma\|^2 \right). \tag{1.7}$$

This raises the next question: Does a compact and connected hypersurface N with shape operator T, mean curvature α, induced metric g, admitting a unit *KGVF* $\mathbf{v}$, of a Euclidean space R^{m+1} with nonzero function $\sigma = g(T\mathbf{v}, \mathbf{v})$ satisfying Eq (1.7) necessarily imply m is odd, α a constant, and M isometric to $S^{2m-1}(c)$? In Section 4 of this paper, we answer this question and find a characterization of the sphere $S^{2m-1}(c)$.

Finally, in the last section, we consider an immersed compact and connected hypersurface N in the Euclidean space R^{m+1} with immersion $\varphi : N \to R^{m+1}$, unit normal ζ, and shape operator T. Then, we express the position vector field φ as $\varphi = \mathbf{u} + f\zeta$, where $f = \langle \varphi, \zeta \rangle$ is the support function of the hypersurface. In the last section, we shall prove that for a compact and connected hypersurface N with nonzero support function and if the following condition holds

$$\int_N Ric(\mathbf{u}, \mathbf{u}) \geq \frac{m-1}{m} \int_N (div\mathbf{u})^2,$$

then the mean curvature α is a constant and N is the sphere $S^m\left(\alpha^2\right)$.

2. Preliminaries

Let N be an orientable hypersurface of the Euclidean space R^{m+1} with unit normal ζ, shape operator T. We denote the Euclidean metric by $\langle , \rangle$ and by g the induced metric on N, and by ∇ and D, the Riemannian connection with respect to g and the Euclidean connection, respectively. Then, we have [11]

$$D_E F = \nabla_E F + g(TE, F)\zeta, \quad D_E \zeta = -TE, \quad E, F \in \Psi(N), \tag{2.1}$$

where $\Psi(N)$ is the space of smooth vector fields on N. The curvature tensor field of the hypersurface N is given by

$$R(E, F)G = g(TF, G)TE - g(TE, G)TF, \quad E, F, G \in \Psi(N), \tag{2.2}$$

and the Ricci tensor of N has the expression

$$Ric(E, F) = m\alpha g(TE, F) - g(TE, TF), \tag{2.3}$$

where α is the mean curvature of the hypersurface N, given by $m\alpha = trT$, the trace of the shape operator T. For a local orthonormal frame $\{w_k\}_1^m$ on the hypersurface, the scalar curvature τ of the hypersurface N is given by

$$\tau = \sum_{k=1}^{m} Ric(w_k, w_k),$$

and combining the above equation with (2.3), gives

$$\tau = m^2\alpha^2 - \|T\|^2, \tag{2.4}$$

where

$$\|T\|^2 = \sum_{k=1}^{m} g(Tw_k, Tw_k).$$

The Codazzi equation of the hypersurface N is given by

$$(\nabla_E T)F = (\nabla_F T)E, \quad E, F \in \Psi(N), \tag{2.5}$$

where $(\nabla_E T) F = \nabla_E TF - T(\nabla_E F)$. Note that, as the shape operator T is symmetric, we have for $E \in \Psi(N)$ and a local frame $\{w_k\}_1^m$,

$$\begin{aligned} mE(\alpha) &= \sum_{k=1}^{m} Eg(Tw_k, w_k) = \sum_{k=1}^{m} g((\nabla_E T)(w_k), w_k) + 2\sum_{k=1}^{m} g(Tw_k, \nabla_E w_k) \\ &= \sum_{k=1}^{m} g((\nabla_{w_k} T)(E), w_k) + 2\sum_{k=1}^{m} g(Tw_k, \nabla_E w_k) \\ &= \sum_{k=1}^{m} g(E, (\nabla_{w_k} T)(w_k)) + 2\sum_{k=1}^{m} g(Tw_k, \nabla_E w_k), \end{aligned} \tag{2.6}$$

and using the facts that

$$Tw_k = \sum_{j=1}^{m} \lambda_k^j w_j, \quad \nabla_E w_k = \sum_{i=1}^{m} \omega_k^i(E) w_i,$$

where $\left(\lambda_k^j\right)$ is a symmetric matrix and ω_k^i are connection forms, which are skew symmetric, that is, $\omega_k^i + \omega_i^k = 0$; in Eq (2.6), we conclude

$$mE(\alpha) = \sum_{k=1}^{m} g(E, (\nabla_{w_k} T)(w_k)).$$

Therefore, the gradient of α has the expression

$$\nabla\alpha = \frac{1}{m}\sum_{k=1}^{m} (\nabla_{w_k} T)(w_k). \tag{2.7}$$

Let ω be a *CLVF* on an m-dimensional Riemannian manifold (N, g). Then, we have

$$\nabla_E \omega = \sigma E, \quad E \in \Psi(N), \tag{2.8}$$

where σ is the potential function of the *CLVF* ω. A *CLVF* ω on (N, g) is said to be nontrivial if it is not parallel. We have the following expression for the curvature tensor field of (N, g) involving the *CLVF* ω

$$R(E, F)\omega = E(\sigma)F - F(\sigma)E, \quad E, F \in \Psi(N).$$

Taking the trace in the above equation, we see that the Ricci tensor of (N, g) is given by

$$Ric(E, \omega) = -(m-1)E(\sigma), \quad E \in \Psi(N). \tag{2.9}$$

The Ricci operator S of the Riemannian manifold (N, g) is given by

$$Ric(E, F) = g(SE, F),$$

and thus, using Eq (2.9), we see that the Ricci operator S operating on the *CLVF* ω is given by

$$S(\omega) = -(m-1)\nabla\sigma, \tag{2.10}$$

where $\nabla\sigma$ is the gradient of σ.

Now, consider a *KGVF* $\mathbf{v}$ on an m-dimensional Riemannian manifold (N, g) that satisfies [11]

$$\pounds_{\mathbf{v}} g = 0. \tag{2.11}$$

Note that the flow of a *KGVF* on a Riemannian manifold consists of isometries, and therefore, its presence influences both the topology and geometry of the manifold on which they live. For instance, if $\mathbf{v}$ is a *KGVF* on a Riemannian manifold (N, g), then the scalar curvature τ of (N, g) is constant along the integral curves of $\mathbf{v}$. It is known that, if a positively curved Riemannian manifold (N, g) admits a nontrivial *KGVF*, then its fundamental group contains a cyclic subgroup of constant index depending on $\dim N$ [17]. Also, the presence of a nontrivial *KGVF* influences the dimension of the Riemannian manifold on which they live. For instance, on the even-dimensional unit sphere S^{2m} there does not exist a unit *KGVF*, where as on S^{2m+1} a unit *KGVF* exists [2,11]. Moreover, the presence of a nontrivial *KGVF* on a compact Riemannian manifold (N, g) does not allow it to have a non-positive Ricci curvature [11].

There is a skew-symmetric operator ϕ associated with the *KGVF* $\mathbf{v}$ on (N, g) that satisfies

$$\nabla_E \mathbf{v} = \phi E, \quad E \in \Psi(N), \tag{2.12}$$

and that the covariant derivative of the operator ϕ is given by

$$(\nabla_E \phi)(F) = R(E, \mathbf{v}) F, \quad E, F \in \Psi(N). \tag{2.13}$$

It is clear from Eq (2.12) that $\mathbf{v}$, being a unit *KGVF* on (N, g), satisfies

$$\phi \mathbf{v} = 0. \tag{2.14}$$

Note that the flow of a *KGVF* $\mathbf{v}$ on an m-dimensional Riemannian manifold (N, g) consists of isometries of (N, g). Now suppose that N is an orientable hypersurface of the Euclidean space R^{m+1} with shape operator T, mean curvature α, and induced metric. Suppose that there is a unit *KGVF* $\mathbf{v}$ on the hypersurface N. We say that the shape operator T of the hypersurface is invariant under the unit *KGVF* $\mathbf{v}$ if

$$\psi_t^* (T) = T \circ d\psi_t, \tag{2.15}$$

where $\{\psi_t\}$ is the flow of the unit *KGVF* $\mathbf{v}$.

Lemma 1. *Let* $\mathbf{v}$ *be a unit KGVF on the hypersurface N of the Euclidean space* R^{m+1} *such that the shape operator T is invariant under* $\mathbf{v}$. *Then the shape operator satisfies*

$$(\nabla_E T)(\mathbf{v}) = \phi(TE) - T(\phi E), \quad E \in \Psi(N).$$

Proof. Since T is invariant under $\mathbf{v}$, Eq (2.15) implies

$$\pounds_{\mathbf{v}} T = 0,$$

which gives

$$[\mathbf{v}, TE] = T[\mathbf{v}, E], \quad E \in \Psi(N),$$

that is, in view of Eq (2.12), we have

$$(\nabla_{\mathbf{v}} T)(E) = \phi(TE) - T(\phi E), \quad E \in \Psi(N).$$

Combining the above equation with Eq (2.5), we get the result. □

3. Hypersurfaces with a concircular vector field

In this section, we are interested in studying the impact of a nonzero *CRVF* ω with potential σ on a compact hypersurface N of the Euclidean space R^{m+1}. We would like to recall that given a smooth curve $\beta : I \to N$ on the hypersurface N with mean curvature α, we get a smooth function $f : I \to R$ defined by $f = \alpha \circ \beta$ and if f is a constant function, we say the mean curvature α is a constant along the curve β on the hypersurface. Naturally, if the mean curvature α is a constant, then it will be constant along each curve on the hypersurface. However, mean curvature α being constant along some curves on hypersurface N does not imply that α is a constant on N. In the following result, we shall assume that the mean curvature α is a constant along the integral curves of the *CRVF* ω, which is a weaker condition than asking if the mean curvature α is a constant. Indeed, we prove the following:

Theorem 1. *A compact and connected hypersurface N of the Euclidean space R^{m+1}, $m > 1$, admits a nonzero nontrivial CRVF ω such that the mean curvature α is constant along the integral curves of ω and the shape operator T satisfies $T(\omega) = \alpha\omega$, if and only if α is a constant and N is isometric to $S^m\left(\alpha^2\right)$.*

Proof. Suppose that the compact and connected hypersurface N of R^{m+1}, $m > 1$, admits a nonzero nontrivial *CRVF* ω with potential σ, such that the mean curvature α is constant along the integral curves of ω and the shape operator T satisfies

$$T(\omega) = \alpha\omega. \tag{3.1}$$

Then we have

$$\omega(\alpha) = 0. \tag{3.2}$$

Using Eqs (2.8) and (3.1), we get

$$(\nabla_E T)(\omega) = E(\alpha)\omega + \sigma\alpha E - \sigma T E, \quad E \in \Psi(N),$$

that is,

$$\sigma(TE - \alpha E) = E(\alpha)\omega - (\nabla_E T)(\omega), \quad E \in \Psi(N). \tag{3.3}$$

Now, using a local frame $\{w_k\}_1^m$ on the hypersurface N, we have

$$\sigma^2 \|T - \alpha I\|^2 = \sum_{k=1}^m g\left(\sigma(Tw_k - \alpha w_k), \sigma(Tw_k - \alpha w_k)\right),$$

and employing Eq (3.3) in the above equation leads to

$$\begin{aligned}\sigma^2 \|T - \alpha I\|^2 &= \sum_{k=1}^m g\left(w_k(\alpha)\omega - (\nabla_{w_k}T)(\omega), w_k(\alpha)\omega - (\nabla_{w_k}T)(\omega)\right) \\ &= \|\nabla\alpha\|^2\|\omega\|^2 + \sum_{k=1}^m g\left((\nabla_{w_k}T)(\omega), (\nabla_{w_k}T)(\omega)\right) - 2g\left(\nabla\alpha, (\nabla_\omega T)(\omega)\right).\end{aligned} \tag{3.4}$$

Moreover, Eqs (3.1) and (3.2) give

$$(\nabla_\omega T)(\omega) = \nabla_\omega(\alpha\omega) - T(\sigma\omega) = 0. \tag{3.5}$$

Next, using Eq (3.1), we compute

$$\left(\nabla_{w_k} T\right)(\omega) = w_k(\alpha)\,\omega + \alpha\sigma w_k - \sigma T(w_k),$$

which, on using Eq (3.2), on some simplifications, gives

$$\sum_{k=1}^{m} g\left(\left(\nabla_{w_k} T\right)(\omega), \left(\nabla_{w_k} T\right)(\omega)\right) = \|\nabla\alpha\|^2 \|\omega\|^2 + \sigma^2 \|T\|^2 - m\sigma^2\alpha^2. \tag{3.6}$$

Thus, Eqs (3.4)–(3.6), yield

$$\sigma^2 \|T - \alpha I\|^2 = 2\|\nabla\alpha\|^2 \|\omega\|^2 + \sigma^2\left(\|T\|^2 - m\alpha^2\right). \tag{3.7}$$

Also, we have

$$\begin{aligned} \|T - \alpha I\|^2 &= \sum_{k=1}^{m} g\left((Tw_k - \alpha w_k), (Tw_k - \alpha w_k)\right) \\ &= \|T\|^2 + m\alpha^2 - 2\alpha \sum_{k=1}^{m} g(Tw_k, w_k) \\ &= \|T\|^2 - m\alpha^2. \end{aligned}$$

Substituting this last equation in Eq (3.7), we arrive at

$$2\|\nabla\alpha\|^2 \|\omega\|^2 = 0,$$

and as ω is a nonzero vector field on the connected hypersurface N, we conclude that α is a constant. Now, using Eq (3.1) in the expression of the Ricci operator S of the hypersurface N, we get

$$S(\omega) = m\alpha T(\omega) - T^2(\omega) = (m-1)\alpha^2\omega.$$

Combining this equation with Eq (2.10), we have

$$\nabla\sigma = -\alpha^2\omega.$$

Differentiating the above equation with respect to a vector field E on N, and using Eq (2.8), we get

$$\nabla_E \nabla\sigma = -\alpha^2\sigma E, \quad E \in \Psi(N). \tag{3.8}$$

The mean curvature α is a constant; it has to be a nonzero constant as N is a compact hypersurface by virtue of the fact that there are no compact minimal hypersurfaces in the Euclidean space R^{m+1}, which is guaranteed by Minkowski's formula (1.1). Now, it remains to show that the potential σ cannot be a constant. To achieve it, we see that Eq (2.8) implies $div\omega = m\sigma$, which, on integration, yields

$$\int_N \sigma = 0,$$

and if σ were a constant, it should give $\sigma = 0$, which would make ω a trivial *CRVF*, which is a contradiction. Hence, σ is a non-constant function. Hence, Eq (3.8) is Obata's differential equation [15,16], which confirms that N is isometric to $S^n\left(\alpha^2\right)$.

Conversely, suppose N is isometric to $S^n(c)$. Then, by Eq (1.2), there is a *CRVF* $\mathbf{u}$ on $S^m(c)$ with potential $\sigma = -\sqrt{c}f$. We claim that $\mathbf{u}$ is a nonzero and nontrivial *CRVF* on $S^m(c)$. If $\mathbf{u} = 0$, then by Eq (1.2), it will follow that $f = 0$, and consequently, the constant vector $\mathbf{a} = 0$, which is contrary to our assumption that $\mathbf{a}$ is a nonzero constant vector field on the Euclidean space R^{m+1}. Similarly, if $\mathbf{u}$ is parallel, then by Eq (1.2), we have $f = 0$, and the second equation in Eq (1.2) will imply $\mathbf{u} = 0$, which is a contradiction. Hence, $\mathbf{u}$ is a nonzero and nontrivial *CRVF* on $S^m(c)$, which satisfies $T(\mathbf{u}) = \alpha\mathbf{u}$ and $\mathbf{u}(\alpha) = 0$. This completes the proof. □

4. Hypersurfaces with a Killing vector field

In this section, we are interested in studying hypersurfaces of the Euclidean space R^{m+1}, which admit a unit *KGVF*. Let N be an orientable hypersurface of the Euclidean space R^{m+1} with shape operator T, mean curvature α, and $\mathbf{v}$ be a unit *KGVF* on N with respect to which the shape operator T is invariant. We prove the following:

Theorem 2. *A compact and connected hypersurface N of the Euclidean space R^{m+1}, $m > 1$, with mean curvature α and shape operator T, admits a unit KGVF $\mathbf{v}$ such that the shape operator T is invariant under $\mathbf{v}$ and the function $\sigma = g(T\mathbf{v}, \mathbf{v})$ is nonzero and satisfies*

$$\int_N m\alpha\sigma Ric(\mathbf{v}, \mathbf{v}) \geq \int_N \left(m(m-1)\sigma^2\alpha^2 - \|\nabla\sigma\|^2\right),$$

if and only if m is odd, $m = (2n-1)$, α is a constant, and N is isometric to $S^{2n-1}\left(\alpha^2\right)$.

Proof. Suppose N is a compact and connected hypersurface of the Euclidean space R^{m+1}, $m > 1$, that admits a unit *KGVF* $\mathbf{v}$ such that the shape operator T is invariant under $\mathbf{v}$ and the function $\sigma = g(T\mathbf{v}, \mathbf{v})$ is nonzero and satisfies the condition

$$\int_N m\alpha\sigma Ric(\mathbf{v}, \mathbf{v}) \geq \int_N \left(m(m-1)\sigma^2\alpha^2 - \|\nabla\sigma\|^2\right). \tag{4.1}$$

Define a vector field $\mathbf{u} = T\mathbf{v} - \sigma\mathbf{v}$; it follows that $g(\mathbf{u}, \mathbf{v}) = 0$, that is, the vector field $\mathbf{u}$ is orthogonal to the unit *KGVF* $\mathbf{v}$. Now, using Eq (2.12) and Lemma 1, we compute

$$\nabla_E\mathbf{u} = (\nabla_E T)(\mathbf{v}) + T(\phi E) - E(\sigma)\mathbf{v} - \sigma\phi E,$$

that is,

$$\nabla_E\mathbf{u} = \phi(TE) - E(\sigma)\mathbf{v} - \sigma\phi E, \quad E \in \Psi(N). \tag{4.2}$$

Taking the inner product in the above equation with the vector field $\mathbf{v}$ and using $g(\mathbf{u}, \mathbf{v}) = 0$ and Eqs (2.12) and (2.14), we get

$$-g(\mathbf{u}, \phi E) = -E(\sigma), \quad E \in \Psi(N),$$

that is,

$$\nabla\sigma = -\phi\mathbf{u}. \tag{4.3}$$

Differentiating the above equation with respect to $E \in \Psi(N)$ and using Eqs (4.2), (2.13), and (2.14), we get

$$\begin{aligned}\nabla_E \nabla\sigma &= -(\nabla_E \phi)(\mathbf{u}) - \phi(\phi(TE) - E(\sigma)\mathbf{v} - \sigma\phi E)\\ &= -R(E,\mathbf{v})u - \phi^2(TE) + \sigma\phi^2 E, \quad E \in \Psi(N).\end{aligned} \tag{4.4}$$

Note that by Eqs (2.13) and (2.14), we have

$$R(E,\mathbf{v})\mathbf{v} = -\phi^2 E, \quad E \in \Psi(N), \tag{4.5}$$

and using it in Eq (4.4), we conclude

$$\nabla_E \nabla\sigma = -R(E,\mathbf{v})u + R(TE,\mathbf{v})\mathbf{v} - \sigma R(E,\mathbf{v})\mathbf{v}, \quad E \in \Psi(N).$$

Now, using $\mathbf{u} = T\mathbf{v} - \sigma\mathbf{v}$ to plug the first and last terms in the right-hand side of the above equation, we confirm

$$\nabla_E \nabla\sigma = -R(E,\mathbf{v})T\mathbf{v} + R(TE,\mathbf{v})\mathbf{v},$$

which, using Eq (2.2), yields

$$\nabla_E \nabla\sigma = -\|T\mathbf{v}\|^2 TE + \sigma T^2 E, \quad E \in \Psi(N).$$

Taking the trace in the above equation and using $\Delta\sigma = div(\nabla\sigma)$, we conclude

$$\Delta\sigma = -m\alpha\|T\mathbf{v}\|^2 + \sigma\|T\|^2,$$

that is,

$$\sigma\Delta\sigma = -m\alpha\sigma\|T\mathbf{v}\|^2 + \sigma^2\|T\|^2. \tag{4.6}$$

Using Eq (2.3), we have

$$\|T\mathbf{v}\|^2 = m\alpha g(T\mathbf{v},\mathbf{v}) - Ric(\mathbf{v},\mathbf{v}) = m\alpha\sigma - Ric(\mathbf{v},\mathbf{v}),$$

and inserting it in Eq (4.6), gives

$$\sigma\Delta\sigma = -m^2\alpha^2\sigma^2 + m\alpha\sigma Ric(\mathbf{v},\mathbf{v}) + \sigma^2\|T\|^2.$$

Integrating the above equation, yields

$$-\int_N \|\nabla\sigma\|^2 = \int_N \left(-m^2\alpha^2\sigma^2 + m\alpha\sigma Ric(\mathbf{v},\mathbf{v}) + \sigma^2\|T\|^2\right),$$

which is rearranged as

$$\int_N \sigma^2\left(\|T\|^2 - m\alpha^2\right) = \int_N \left(m(m-1)\alpha^2\sigma^2 - \|\nabla\sigma\|^2\right) - \int_N m\alpha\sigma Ric(\mathbf{v},\mathbf{v}).$$

Using the inequality (4.1) in the above equation, it confirms

$$\int_N \sigma^2\left(\|T\|^2 - m\alpha^2\right) \leq 0.$$

However, by Schwartz's inequality, we have $\|T\|^2 \geq m\alpha^2$ and therefore, the integrand on the left-hand side of the above inequality is non-negative. Hence, we have

$$\sigma^2 \left(\|T\|^2 - m\alpha^2\right) = 0,$$

with the function σ nonzero on connected N, which implies $\left(\|T\|^2 - m\alpha^2\right) = 0$. The equality $\|T\|^2 = m\alpha^2$ in Schwartz's inequality holds if and only if

$$T = \alpha I, \tag{4.7}$$

which gives

$$(\nabla_E T)(F) = E(\alpha) F, \quad E, F \in \Psi(N).$$

Taking a local frame $\{w_k\}_1^m$ on the hypersurface N, in the above equation, we have

$$\sum_{k=1}^{m} (\nabla_{w_k} T)(w_k) = \sum_{k=1}^{m} w_k(\alpha) w_k,$$

which, in view of Eq (2.7), implies

$$m\nabla\alpha = \nabla\alpha,$$

and as $m > 1$, it confirms that α is a constant. Then, by Eqs (2.2) and (4.7), we have

$$R(E, F) G = \alpha^2 \{g(F, G) E - g(E, G) F\}, \quad E, F, G \in \Psi(N).$$

Note that $\alpha \neq 0$, because compact minimal hypersurfaces in Euclidean space do not exist. Hence, $\alpha^2 > 0$, and N is isometric to $S^m\left(\alpha^2\right)$. Note that a Killing vector field on an even-dimensional compact Riemannian manifold of positive sectional curvature must vanish at some point [11]. Therefore, as $\mathbf{v}$ is a unit vector field, it never vanishes, and it announces that m cannot be even. Hence, $m = 2n - 1$, that is, N is isometric to $S^{2n-1}\left(\alpha^2\right)$.

Conversely, suppose N is isometric to $S^{2n-1}\left(\alpha^2\right)$. Then by Eqs (1.3) and (1.4), there is a unit vector field $\mathbf{v} = J\zeta$ on $S^{2n-1}\left(\alpha^2\right)$ that satisfies

$$\nabla_E \mathbf{v} = \alpha (JE)^T, \quad E \in \Psi\left(S^{2n-1}\left(\alpha^2\right)\right), \tag{4.8}$$

where J is the complex structure of the ambient Euclidean space R^{2n}, and ζ is the unit normal, and $(JE)^T$ is the tangential projection of the vector field JE to $S^{2n-1}\left(\alpha^2\right)$. Taking the inner product in Eq (4.8) by the vector field F on the sphere $S^{2n-1}\left(\alpha^2\right)$, we have

$$g(\nabla_E \mathbf{v}, F) = \alpha g\left((JE)^T (JE)^T, F\right) = \alpha \langle JE, F\rangle,$$

and we conclude

$$(\pounds_{\mathbf{v}} g)(E, F) = \alpha \langle JE, F\rangle + \alpha \langle JF, E\rangle = 0,$$

by virtue of the skew symmetry of the complex structure, that is, the Euclidean metric is a Hermitian metric. Hence, $\mathbf{v}$ is a unit $KGVF$ on $S^{2n-1}\left(\alpha^2\right)$. Note that, in this case the shape operator is $T = \alpha I$, and the function $\sigma = g(T\mathbf{v}, \mathbf{v}) = \alpha$ is a nonzero constant. Moreover, with $m = 2n - 1$

$$\int_{S^{2n-1}\left(\alpha^2\right)} m\alpha\sigma Ric(\mathbf{v}, \mathbf{v}) = \int_{S^{2n-1}\left(\alpha^2\right)} 2(2n-1)(n-1)\alpha^4 \tag{4.9}$$

and

$$\int_{S^{2n-1}(\alpha^2)} \left(m(m-1)\sigma^2\alpha^2 - \|\nabla\sigma\|^2\right) = \int_{S^{2n-1}(\alpha^2)} 2(2n-1)(n-1)\alpha^4, \tag{4.10}$$

as $\nabla\sigma = 0$. Hence, by Eqs (4.9) and (4.10), we get

$$\int_{S^{2n-1}(\alpha^2)} m\alpha\sigma Ric(\mathbf{v},\mathbf{v}) = \int_{S^{2n-1}(\alpha^2)} \left(m(m-1)\sigma^2\alpha^2 - \|\nabla\sigma\|^2\right),$$

and this finishes the proof. □

5. Hypersurfaces with a generic bound on Ricci curvature

Let N be an immersed hypersurface in the Euclidean space R^{m+1} with unit normal ζ, shape operator T, and mean curvature α. Let $\varphi : N \to R^{m+1}$ be the immersion and $\rho = \langle\varphi, \zeta\rangle$ be the support of N. The position vector field φ is expressed as

$$\varphi = \mathbf{u} + \rho\zeta, \tag{5.1}$$

and we call $\mathbf{u}$ the basic vector field of the hypersurface N. Differentiating Eq (5.1), using Eq (2.1), and equating similar components, we get

$$\nabla_E\mathbf{u} = E + \rho TE, \quad \nabla\rho = -T\mathbf{u}, \quad E \in \Psi(N). \tag{5.2}$$

The first equation in Eq (5.2), gives

$$div\mathbf{u} = m(1+\rho\alpha). \tag{5.3}$$

In this section, we prove the following result:

Theorem 3. *A compact and connected immersed hypersurface N of the Euclidean space R^{m+1}, $m > 1$, with nonzero support ρ and basic vector field $\mathbf{u}$ satisfies*

$$\int_N Ric(\mathbf{u},\mathbf{u}) \geq \frac{m-1}{m}\int_N (div\mathbf{u})^2,$$

if and only if, the mean curvature α is a constant and N is isometric to $S^m(\alpha^2)$.

Proof. Suppose that the immersed hypersurface N of the Euclidean space R^{m+1}, $m > 1$, has nonzero support ρ and the basic vector field $\mathbf{u}$ satisfy

$$\int_N Ric(\mathbf{u},\mathbf{u}) \geq \frac{m-1}{m}\int_N (div\mathbf{u})^2. \tag{5.4}$$

Using Eq (5.2), we have

$$\rho(TE - \alpha E) = \nabla_E\mathbf{u} - (1+\rho\alpha)E,$$

and using a local frame $\{w_k\}_1^m$ on the hypersurface N with the above equation, we get

$$\rho^2\|T - \alpha I\|^2 = \sum_{k=1}^{m} g(\rho(Tw_k - \alpha w_k), \rho(Tw_k - \alpha w_k))$$

$$
\begin{aligned}
&= \sum_{k=1}^{m} g\left(\nabla_{w_k}\mathbf{u} - (1+\rho\alpha)\, w_k, \nabla_{w_k}\mathbf{u} - (1+\rho\alpha)\, w_k\right) \\
&= \|\nabla\mathbf{u}\|^2 + m\,(1+\rho\alpha)^2 - 2\,(1+\rho\alpha)\, div\mathbf{u}.
\end{aligned}
$$

Using Eq (5.3) in the above equation, we have

$$
\rho^2 \|T - \alpha I\|^2 = \|\nabla\mathbf{u}\|^2 - \frac{1}{m}\,(div\mathbf{u})^2\,. \tag{5.5}
$$

Note that on using Eq (5.2), we have

$$
(\pounds_{\mathbf{u}} g)(E,F) = 2g(E,F) + 2\rho g(TE,F)\,, \quad E,F \in \Psi(N),
$$

which gives

$$
\begin{aligned}
|\pounds_{\mathbf{u}} g|^2 &= \sum_{jk} (\pounds_{\mathbf{u}} g)\left(w_j, w_k\right) = 4\sum_{jk}\left(g\left(w_j, w_k\right) + \rho g\left(Tw_j, w_k\right)\right)^2 \\
&= 4\left(m + 2m\rho\alpha + \rho^2 \|T\|^2\right).
\end{aligned}
$$

Integrating the last equation, while using Minkowski's formula, we have

$$
\frac{1}{2}\int_N |\pounds_{\mathbf{u}} g|^2 = 2\int_N \left(\rho^2 \|T\|^2 + m\rho\alpha\right). \tag{5.6}
$$

Next, we recall the following integral formula [20]

$$
\int_N \left(Ric\,(\mathbf{u},\mathbf{u}) + \frac{1}{2}\,|\pounds_{\mathbf{u}} g|^2 - \|\nabla\mathbf{u}\|^2 - (div\mathbf{u})^2\right) = 0,
$$

which holds for any vector field on the compact Riemannian manifold (N,g).

Using the above integral formula with the integral of Eq (5.5), we get

$$
\int_N \rho^2 \|T - \alpha I\|^2 = \int_N \left(Ric\,(\mathbf{u},\mathbf{u}) + \frac{1}{2}\,|\pounds_{\mathbf{u}} g|^2 - (div\mathbf{u})^2 - \frac{1}{m}\,(div\mathbf{u})^2\right). \tag{5.7}
$$

Now, using Eq (1.1) in Eq (5.6), we have

$$
\begin{aligned}
\frac{1}{2}\int_N |\pounds_{\mathbf{u}} g|^2 &= 2\int_N \left(\rho^2\left(\|T\|^2 - m\alpha^2\right) + m\left(\rho^2\alpha^2 + \rho\alpha\right)\right) \\
&= 2\int_N \left(\rho^2\left(\|T\|^2 - m\alpha^2\right) + m\left(\rho^2\alpha^2 + 2\rho\alpha + 1\right)\right) \\
&= 2\int_N \left(\rho^2\left(\|T\|^2 - m\alpha^2\right) + m\,(1+\rho\alpha)^2\right).
\end{aligned}
$$

Employing (5.1), in the above equation, we conclude

$$
\frac{1}{2}\int_N |\pounds_{\mathbf{u}} g|^2 = 2\int_N \left(\rho^2\left(\|T\|^2 - m\alpha^2\right) + \frac{1}{m}\,(div\mathbf{u})^2\right).
$$

Inserting this equation in Eq (5.7), we find that

$$\int_N \rho^2 \|T - \alpha I\|^2 = \int_N \left(Ric(\mathbf{u}, \mathbf{u}) + 2\rho^2 \left(\|T\|^2 - m\alpha^2 \right) - \frac{m-1}{m} (div\mathbf{u})^2 \right). \tag{5.8}$$

Finally, observe that

$$\begin{aligned} \|T - \alpha I\|^2 &= \sum_{k=1}^{m} g(Tw_k - \alpha w_k, Tw_k - \alpha w_k) \\ &= \|T\|^2 - 2m\alpha^2 + m\alpha^2, \end{aligned}$$

that is,

$$\rho^2 \|T - \alpha I\|^2 = \rho^2 \left(\|T\|^2 - m\alpha^2 \right)$$

and utilizing the above equation in Eq (5.8), we obtain

$$\int_N \rho^2 \|T - \alpha I\|^2 = \frac{m-1}{m} \int_N (div\mathbf{u})^2 - \int_N Ric(\mathbf{u}, \mathbf{u}).$$

Using inequality (5.4) in the above equation, we get

$$\int_N \rho^2 \|T - \alpha I\|^2 \leq 0,$$

which gives $\rho^2 \|T - \alpha I\|^2 = 0$. However, the support $\rho \neq 0$ on connected N implies

$$T = \alpha I,$$

and as in the proof of Theorem 2, we realize that α is a constant, and by Eq (2.2), the curvature tensor of N is given by

$$R(E, F) G = \alpha^2 \{ g(F, G) E - g(E, G) F \}, \quad E, F, G \in \Psi(N),$$

with constant $\alpha \neq 0$ as there are no compact minimal hypersurfaces in the Euclidean space. Hence, N is isometric to $S^m \left(\alpha^2 \right)$.

Conversely, suppose N is isometric to $S^m \left(\alpha^2 \right)$. Then, the embedding $\varphi : S^m \left(\alpha^2 \right) \to R^{m+1}$ has shape operator $T = \alpha I$, unit normal $\zeta = -\alpha\varphi$ and support $\rho = -\frac{1}{\alpha} \neq 0$. Moreover, the basic vector field $\mathbf{u} = 0$. Hence, the condition (5.4) vacuously holds as an equality. □

6. Conclusions

In Sections 3 and 4, we have employed a *CLVF* and a *KGVF* on a compact hypersurface N, respectively, of the Euclidean space R^{m+1} to find a characterization of spheres $S^m(c)$ and $S^{2n-1}(c)$, respectively. This further increases the scope of the study of hypersurfaces in the Euclidean space R^{m+1}; for instance, one would be interested in analyzing the impact of the presence of a geodesic vector field ξ on an orientable hypersurface N of the Euclidean space R^{m+1} [10]. A vector field ξ on a Riemannian manifold (N, g) is said to be a geodesic vector field, if its integral curves are geodesics of

(N, g). A unit Killing vector field on (N, g) is a geodesic vector field, and the converse is not true. To support this fact that a geodesic vector field need not be a *KGVF*, we need to introduce a 3-dimensional trans-Sasakian manifold $(N, g, \phi, \zeta, \eta, f, h)$, where (N, g) is a 3-dimensional Riemannian manifold, ϕ is a $(1, 1)$ tensor field, ζ is a unit vector field (called Reeb vector field), η is 1-form dual to ζ, and f, h are smooth functions on M satisfying [1]

$$\phi^2 = -I + \eta \otimes \zeta, \quad \phi(\zeta) = 0, \quad \eta \circ \phi = 0, \quad g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F)$$

and

$$\begin{aligned} \nabla_E \zeta &= -f\phi E + h(E - \eta(E)\zeta), \\ (\nabla_E \phi)(F) &= f(g(E, F)\xi - \eta(F)E) + h(g(\phi E, F)\xi - \eta(F)\phi E), \end{aligned}$$

$E, F \in \Psi(N)$. A trans-Sasakian manifold $(N, g, \phi, \zeta, \eta, f, h)$ is said to be proper, if neither of the functions f nor h are zero. It is easy to see that $\nabla_\zeta \zeta = 0$, that is, ζ is a geodesic vector field. However, on a proper trans-Sasakian manifold $(N, g, \phi, \zeta, \eta, f, h)$

$$\left(£_\zeta g\right)(E, F) = 2hg(\phi E, \phi F) \neq 0,$$

that is, ζ is not a Killing vector field. Hence, on a proper trans-Sasakian manifold $(N, g, \phi, \zeta, \eta, f, h)$, the Reeb vector field ζ is a geodesic vector field that is not a *KGVF*. Thus, a geodesic vector field being a nontrivial generalization of a Killing vector field makes it a potential case for studying the impact of the presence of a geodesic vector field on the geometry of an orientable hypersurface of the Euclidean space R^{m+1}.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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