Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 14658–14680.
DOI: 10.3934/math.2024713
Received: 23 February 2024
Revised: 27 March 2024
Accepted: 16 April 2024
Published: 23 April 2024

Research article

Regularity and uniqueness of 3D compressible magneto-micropolar fluids

Mingyu Zhang*

School of Mathematics and Statistics, Weifang University, Weifang 261061, China

* **Correspondence:** Email: wfumath@126.com.

Abstract: This article established the global existence and uniqueness of solutions for the 3D compressible magneto-micropolar fluid system with vacuum. The remarkable thing is that in the context of small initial energy, we got a new result with a lower regularity than we ever have before.

Keywords: magneto-micropolar fluids; Cauchy problem; global regularity; uniqueness vacuum
Mathematics Subject Classification: 35B65, 35Q35, 76N10

1. Introduction

This paper focuses on the three-dimensional compressible magneto-micropolar fluid system in $\mathbb{R}^3$ [17, 18, 24]:

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) - (\eta+\beta)\Delta\mathbf{u} - (\eta+\kappa-\beta)\nabla\mathrm{div}\mathbf{u} + \nabla P(\rho) \\ \qquad = 2\beta\mathrm{rot}\mathbf{w} + (\nabla\times\mathbf{b})\times\mathbf{b}, \\ (\rho\mathbf{w})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{w}) - \mu'\Delta\mathbf{w} - (\eta'+\kappa')\nabla\mathrm{div}\mathbf{w} + 4\beta\mathbf{w} = 2\beta\mathrm{rot}\mathbf{u}, \\ \mathbf{b}_t + \nabla\times(\alpha\nabla\times\mathbf{b}) = \nabla\times(\mathbf{u}\times\mathbf{b}), \\ \mathrm{div}\mathbf{b} = 0. \end{cases} \tag{1.1}$$

Here, $\rho \geqslant 0$ is the density, $\mathbf{u}$ is the velocity, $\mathbf{w}$ is the micro-rotational velocity, $\mathbf{b}$ is the magnetic field and $P(\rho) = A\rho^\gamma$ $(A > 0,\ \gamma > 1)$ is the pressure. The parameters $\eta, \kappa, \beta, \eta', \kappa'$ and α are constants satisfying

$$\eta, \beta, \eta', \alpha > 0, \quad 2\eta + 3\kappa - 4\beta \geqslant 0, \quad 2\eta' + 3\kappa' \geqslant 0.$$

For the completeness of equations (1.1), we supplement the conditions

$$(\rho, \mathbf{u}, \mathbf{w}, \mathbf{b})(x, t)\big|_{t=0} = (\rho_0, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \tag{1.2}$$

and

$$(\rho, \mathbf{u}, \mathbf{w}, \mathbf{b})(x,t)\Big|_{|x|\to\infty} \to (1,0,0,0) \quad \text{as } t \geqslant 0. \tag{1.3}$$

When we only consider the effects of the velocity field and the magnetic field in (1.1) (i.e., $\mathbf{w} = 0, \beta = 0$), the system (1.1) becomes the compressible magnetohydrodynamic (MHD) system, which has been widely discussed; see [4,5,9,11,12,25,26] and references therein. When we lose sight of the effects of magnetic fields, the system (1.1) becomes a micropolar model whose theory was introduced by Eringen [10] and Lukaszewicz [16]. Chen-Xu-Zhang in [6] proved the existence of global weak and smooth solutions in cases of small energy. In bounded or unbounded domain $\Omega \subset \mathbb{R}^3$, Chen [8] also got the local existence of strong solutions in the context of large initial data.

For system (1.1), there are still many papers discussing this model [3,7,20]. Wei, Guo, and Li [18] found the global existence and decay properties of smooth solutions with small initial perturbation [23]. Later, Tong and Tan [17] improved the results of [18], and obtained the same decay property between the linearized equations and nonlinear system. Later, Xu, Tan, and Wang in [21] considered the system in a bounded region and derived the global well-posedness strong solutions. Zhang-Cai in [24] considered the system (1.1) in a periodic region and derived the well-posedness of solutions. Moreover, in the context of small conditions and symmetrical external forces, the periodic solution was attained. Utilizing Hoff's method [13], Xu, Tan, and Wang in [22] derived the global weak solutions in context of discontinuous initial data. Very recently, Xu and Zhong in [20] proved the local well-posedness of strong solutions to (1.1) with vacuum. Chen, Sun, and Zhong in [3] deduced the well-posedness of global classical solutions in case of vacuum states. However, the results obtained in [6–8] still require some compatibility conditions

$$\begin{cases} -(\eta+\beta)\Delta\mathbf{u}_0 - (\eta+\kappa-\beta)\nabla\mathrm{div}\mathbf{u}_0 + \nabla P(\rho_0) - 2\beta\mathrm{rot}\mathbf{w}_0 = \rho_0^{1/2} g_1, \\ -\eta'\Delta\mathbf{w}_0 - (\eta'+\kappa')\nabla\mathrm{div}\mathbf{w}_0 + 4\beta\mathbf{w}_0 - 2\beta\mathrm{rot}\mathbf{u}_0 = \rho_0^{1/2} g_2, \end{cases} \tag{1.4}$$

for some $(g_1, g_2) \in L^2$. In case of density including vacuum, it is known from (1.4) that

$$\lim_{t\to 0+} (\rho^{1/2}\dot{\mathbf{u}})(x,t) \in L^2, \quad \lim_{t\to 0+} (\rho^{1/2}\dot{\mathbf{w}})(x,t) \in L^2, \tag{1.5}$$

where, $\dot{\mathbf{g}} = \mathbf{g}_t + \mathbf{u}\cdot\nabla\mathbf{g}$ stands for the material derivative. If $\rho_0 > 0$, then (1.4) becomes

$$\lim_{t\to 0+} \Big(\dot{\mathbf{u}}(x,t), \dot{\mathbf{w}}(x,t)\Big) \in L^2. \tag{1.6}$$

To make sure (1.4)/(1.5) or (1.6) are true, the initial velocity and micro-rotational velocity satisfy

$$(\mathbf{u}_0, \mathbf{w}_0) \in H^2. \tag{1.7}$$

Thus,

$$\sup_{t\in[0,T]} \|(\dot{\mathbf{u}}, \dot{\mathbf{w}})\|_{L^2} + \int_0^T \|(\nabla\dot{\mathbf{u}}, \nabla\dot{\mathbf{w}})\|_{L^2}^2 dt \leqslant C(T), \quad \forall\, T \in (0,\infty), \tag{1.8}$$

where, $\|(\mathbf{g}, \mathbf{h})\|_{L^p} \triangleq \|\mathbf{g}\|_{L^p} + \|\mathbf{h}\|_{L^p}$. Higher-order estimates can be deduced from (1.8).

Similar to [13,15], we set

$$\begin{cases} F_1 \triangleq (2\eta+\kappa)\mathrm{div}\mathbf{u} - \Big(P(\rho) - P(1)\Big), \qquad G_1 \triangleq \mathrm{rot}\mathbf{u}, \\ F_2 \triangleq (2\eta'+\kappa')\mathrm{div}\mathbf{w}, \qquad G_2 \triangleq \mathrm{rot}\mathbf{w}, \end{cases} \tag{1.9}$$

where, the quantities F_1 and F_2 represent the "effective viscous flux", and G_1 and G_2 stand for the vorticity [6]. One can deduce from (1.9) that

$$\begin{cases}\Delta F_1 = \mathrm{div}(\rho\dot{\mathbf{u}}) - \mathrm{divdiv}(\mathbf{b}\otimes\mathbf{b}),\\ \Delta F_2 = \mathrm{div}(\rho\dot{\mathbf{w}}) - 4\beta\mathrm{div}\mathbf{w},\\ (\eta+\beta)\Delta G_1 = \nabla\times(\rho\dot{\mathbf{u}}) - 2\beta\nabla\times G_2,\\ \eta'\Delta G_2 - 4\beta G_2 = \nabla\times(\rho\dot{\mathbf{w}}) - 2\beta\nabla\times G_1.\end{cases} \tag{1.10}$$

Our purpose in this article is to modify the method of [19] and set up the following theorem.

Theorem 1.1. *For $s\in[9/2,6)$, assume that $(\rho_0,\mathbf{u}_0,\mathbf{w}_0,\mathbf{b}_0)$ satisfies*

$$\inf\rho_0(x)>0,\quad \rho_0-1\in H^1\cap W^{1,s},\quad (\mathbf{u}_0,\mathbf{w}_0)\in H^1\cap W^{1,3},\quad \mathbf{b}_0\in H^1. \tag{1.11}$$

There exists a constant $\varepsilon>0$, depending on $\eta,\kappa,\beta,\eta',\kappa',\gamma,\alpha,A,\inf\rho_0$, $\sup\rho_0$, $\|\nabla\mathbf{u}_0\|_{L^2}$, $\|\nabla\mathbf{w}_0\|_{L^2}$, and $\|\nabla\mathbf{b}_0\|_{L^2}$, such that if

$$\mathcal{S}_0 \triangleq \|(\rho_0-1,\mathbf{u}_0,\mathbf{w}_0,\mathbf{b}_0)\|_{L^2}^2 \leqslant \varepsilon, \tag{1.12}$$

the systems (1.1)–(1.3) possess a global uniqueness solution $(\rho,\mathbf{u},\mathbf{w},\mathbf{b})$ in $\mathbb{R}^3\times(0,\infty)$ satisfying

$$\begin{cases}\rho-1\in C([0,T];H^1\cap W^{1,s}),\quad \inf\rho(x,t)>0,\\ (\mathbf{u},\ \mathbf{w},\mathbf{b})\in C([0,T];L^2\cap L^a)\quad 2\leqslant a<6,\\ (\mathbf{u},\mathbf{w})\in L^\infty(0,T;H^1\cap W^{1,3})\cap L^2(0,T;H^2)\cap L^\ell(0,T;W^{1,\infty}),\\ \mathbf{b}\in L^\infty([0,T];H^1)\cap L^2(0,T;H^2),\\ (t^{1/2}\dot{\mathbf{u}},t^{1/2}\dot{\mathbf{w}})\in L^\infty(0,T;L^2),\quad (t^{1/2}\nabla\dot{\mathbf{u}},t^{1/2}\nabla\dot{\mathbf{w}})\in L^2(0,T;L^2),\\ (t^{1/2}\mathbf{b}_t,\ t^{1/2}\nabla^2\mathbf{b})\in L^\infty([0,T];L^2),\quad (t^{1/2}\nabla\mathbf{b}_t,\ t^{1/2}\nabla^3\mathbf{b})\in L^2([0,T];L^2),\end{cases} \tag{1.13}$$

with $1<\ell<(4s)/(5s-6)$.

2. Preliminaries

In this section, we begin to hammer at the derivations to obtain the global a priori estimates. Let $(\rho,\mathbf{u},\mathbf{b},\mathbf{w})$ stand for a smooth solution of (1.1)–(1.4) on $\mathbb{R}^3\times[0,T]$ for some $0<T<\infty$, then the system (1.1) becomes

$$\begin{cases}\rho_t+\mathrm{div}(\rho\mathbf{u})=0,\\ \rho(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})+\nabla P(\rho)-(\eta+\beta)\Delta\mathbf{u}-(\eta+\kappa-\beta)\nabla\mathrm{div}\mathbf{u}\\ \qquad\qquad = 2\beta\mathrm{rot}\mathbf{w}+\mathbf{b}\cdot\nabla\mathbf{b}-\frac{1}{2}\nabla|\mathbf{b}|^2,\\ \rho(\mathbf{w}_t+\mathbf{u}\cdot\nabla\mathbf{w})-\mu'\Delta\mathbf{w}-(\eta'+\kappa')\nabla\mathrm{div}\mathbf{w}=4\beta\mathbf{w}+2\beta\mathrm{rot}\mathbf{u},\\ \mathbf{b}_t+\mathbf{u}\cdot\nabla\mathbf{b}-\mathbf{b}\cdot\nabla\mathbf{u}+\mathbf{b}\mathrm{div}\mathbf{u}-\alpha\Delta\mathbf{b}=0,\quad \mathrm{div}\mathbf{b}=0.\end{cases} \tag{2.1}$$

Thus, we have the following lemma, which can be found in [6, 15].

Lemma 2.1. *Given positive numbers N (not necessarily small) and $\hat{\rho} > 2$, assume that* $(\rho_0, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)$ *satisfies*

$$0 \leqslant \inf \rho_0 \leqslant \sup \rho_0 \leqslant \hat{\rho}, \qquad \|(\nabla \mathbf{u}_0,\ \nabla \mathbf{w}_0,\ \nabla \mathbf{b}_0)\|_{L^2} \leqslant N, \tag{2.2}$$

then there exist constants $L > 0$ and $\varepsilon > 0$, depending on $\eta, \kappa, \beta, \eta', \kappa', \gamma, N, \alpha, A,$ and $\hat{\rho}$, such that if

$$\mathcal{S}_0 \triangleq \int \Big(J(\rho_0) + \frac{1}{2}\rho_0 |\mathbf{w}_0|^2 + \frac{1}{2}\rho_0 |\mathbf{u}_0|^2 + \frac{1}{2}|\mathbf{b}_0|^2 \Big) dx \leqslant \varepsilon, \tag{2.3}$$

where $J(\cdot)$ stands for the potential energy density

$$J(\rho) \triangleq \rho \int_1^{\rho} \frac{P(\varsigma) - P(1)}{\varsigma^2} d\varsigma,$$

then

$$0 \leqslant \rho(x,t) \leqslant 2\hat{\rho}, \quad \forall\, x \in \mathbb{R}^3,\ t \in [0,T], \tag{2.4}$$

$$\sup_{t\in[0,T]} \|\mathbf{b}\|_{L^3}^3 + \int_0^T \|\mathbf{b}\|_{L^9}^3 dx \leqslant \mathcal{S}_0^{1/9}, \tag{2.5}$$

$$\sup_{t\in[0,T]} \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_{L^2}^2 + \int_0^T \|(\rho^{1/2}\dot{\mathbf{u}}, \rho^{1/2}\dot{\mathbf{w}}, \nabla^2 \mathbf{b}, \mathbf{b}_t)\|_{L^2}^2 dt \leqslant L. \tag{2.6}$$

$$\sup_{t\in[0,T]} \|(\rho - 1, \mathbf{b}, \rho^{1/2}\mathbf{u}, \rho^{1/2}\mathbf{w})\|_{L^2}^2 + \int_0^T \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \mathrm{rot}\mathbf{u} - 2\mathbf{w}, \nabla \mathbf{b})\|_{L^2}^2 \leqslant C\mathcal{S}_0. \tag{2.7}$$

Remark 2.1. *The estimations given by* (2.4)–(2.7) *do not depend on T and* $\inf \rho_0$. *In addition, if* $\inf \rho_0 > 0$*, then $J(\cdot)$ implies*

$$E_0 \sim \|(\rho_0 - 1, \boldsymbol{u}_0, \boldsymbol{b}_0, \boldsymbol{w}_0)\|_{L^2}^2. \tag{2.8}$$

Remark 2.2. *We infer from* (2.7) *that*

$$\int_0^T \|\boldsymbol{w}\|_{L^2}^2 dt \leqslant C \int_0^T \big(\|\nabla \boldsymbol{u}\|_{L^2}^2 + \|\mathrm{rot}\mathbf{u} - 2\mathbf{w}\|_{L^2}^2\big) dt \leqslant C\mathcal{S}_0. \tag{2.9}$$

Lemma 2.2. *Under the circumstance of* (2.2) *and* (2.3)*, then there exists positive constant ε_1, such that*

$$\sup_{t\in[0,T]} \|t^{1/2}(\rho^{1/2}\dot{\mathbf{u}}, \rho^{1/2}\dot{\mathbf{w}}, \nabla^2\mathbf{b}, \mathbf{b}_t)\|_{L^2}^2 + \int_0^T t\Big(\|(\nabla\dot{\mathbf{u}}, \nabla\dot{\mathbf{w}}, \nabla\mathbf{b}_t)\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{H^2}^2\Big) dt \leqslant C(T), \tag{2.10}$$

provided $\mathcal{S}_0 \leqslant \varepsilon_1$.

Proof. First, operate $t\dot{u}^j[\frac{d}{dt} + \mathrm{div}(\mathbf{u}\cdot)]$ and $t\dot{w}^j[\frac{d}{dt} + \mathrm{div}(\mathbf{u}\cdot)]$ to $(2.1)_2^j$ and $(2.1)_3^j$, respectively. Integration by parts gives

$$\begin{aligned}
&\Big[\frac{t}{2}\|(\rho^{1/2}\dot{\mathbf{u}}, \rho^{1/2}\dot{\mathbf{w}})\|_{L^2}^2\Big]_t - \frac{1}{2}\|(\rho^{1/2}\dot{\mathbf{u}}, \rho^{1/2}\dot{\mathbf{w}})\|_{L^2}^2 \\
&= (\eta + \beta)\int t\dot{u}^j[\Delta u_t^j + \mathrm{div}(\Delta u^j \mathbf{u})]dx + \eta' \int t\dot{w}^j[\Delta w_t^j + \mathrm{div}(\Delta w^j \mathbf{u})]dx \\
&\quad + (\kappa + \eta - \beta)\int t\dot{u}^j[\partial_t\partial_j(\mathrm{div}\mathbf{u}) + \mathrm{div}(\mathbf{u}\partial_j(\mathrm{div}\mathbf{u}))]dx \\
&\quad + (\kappa' + \eta')\int t\dot{w}^j[\mathrm{div}[\mathbf{u}\partial_j(\mathrm{div}\mathbf{w})] + \partial_t\partial_j(\mathrm{div}\mathbf{w})]dx
\end{aligned} \tag{2.11}$$

$$
\begin{aligned}
&- \int [\mathrm{div}(\mathbf{u}\partial_j P) + \partial_j P_t] t\dot{u}^j dx + 2\beta \int [\partial_i (u^i \mathrm{rot}\mathbf{w}) + \mathrm{rot}\mathbf{w}_t] \cdot t\dot{\mathbf{u}} dx \\
&+ 2\beta \int [\partial_i (u^i \mathrm{rot}\mathbf{u}) + \mathrm{rot}\mathbf{u}_t] \cdot t\dot{\mathbf{w}} dx - 4\beta \int [\mathrm{div}(w^j \mathbf{u}) + w^j_t] t\dot{\mathbf{w}} dx \\
&+ \int \Big[\mathrm{div}(\mathbf{u}\mathbf{b}\cdot\nabla b^j) + \partial_t(\mathbf{b}\cdot\nabla b^j) - \frac{1}{2}\mathrm{div}(\mathbf{u}\partial_j(|\mathbf{b}|^2)) - \partial_t\partial_j(|\mathbf{b}|^2)\Big] t\dot{u}^j dx \triangleq \sum_{i=1}^{10} I_i.
\end{aligned}
$$

Based upon the integration by parts, one has

$$
\begin{aligned}
I_1 &= -(\eta+\beta) \int t \left(\partial_l \dot{u}^i \partial_l u^i_t - \partial_m \partial_l \dot{u}^j u^m \partial_l u^j - \partial_m \dot{u}^j \partial_l u^m \partial_l u^j\right) dx \\
&= -(\eta+\beta) \int t \left(|\nabla\dot{\mathbf{u}}|^2 - \partial_m \dot{u}^j \partial_m u^l \partial_l u^j + \partial_m \dot{u}^j \partial_l u^l \partial_m u^j - \partial_m \dot{u}^j \partial_l u^m \partial_l u^j\right) dx \qquad (2.12)\\
&\leqslant -\frac{3\eta+4\beta}{4}\left(t\|\nabla\dot{\mathbf{u}}\|^2_{L^2}\right) + Ct\|\nabla\mathbf{u}\|^4_{L^4},
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= -\eta' \int t \left(\partial_m \dot{w}^j \partial_m w^j_t - \partial_i \partial_m \dot{w}^j u^i \partial_m w^j - \partial_i \dot{w}^j \partial_m u^i \partial_m u^j\right) dx \\
&= -\eta' \int t \left(|\nabla\dot{\mathbf{w}}|^2 - \partial_m \dot{w}^j \partial_m u^l \partial_l w^j - \partial_m \dot{w}^j \partial_l u^m \partial_l w^j + \partial_m \dot{w}^j \partial_l u^l \partial_m w^j\right) dx \qquad (2.13)\\
&\leqslant -\frac{3\eta'}{4}\left(t\|\nabla\dot{\mathbf{w}}\|^2_{L^2}\right) + Ct\|\nabla\mathbf{u}\|^4_{L^4} + Ct\|\nabla\mathbf{w}\|^4_{L^4}.
\end{aligned}
$$

Similarly

$$
I_3 \leqslant -\frac{\mu+\lambda-\zeta}{2}\left(t\|\mathrm{div}\dot{\mathbf{u}}\|^2_{L^2}\right) + C(t\|\nabla\mathbf{u}\|^4_{L^4}), \qquad (2.14)
$$

and

$$
I_4 \leqslant -\frac{\mu'+\lambda'}{2}\left(t\|\mathrm{div}\dot{\mathbf{w}}\|^2_{L^2}\right) + Ct\|(\nabla\mathbf{u},\nabla\mathbf{w})\|^4_{L^4}. \qquad (2.15)
$$

Taking notice of $(1.1)_1$

$$
\Big[P(\rho)-P(1)\Big]_t + \mathbf{u}\cdot\nabla\left[P(\rho)-P(1)\right] = -\gamma \mathrm{div}\mathbf{u} P(\rho). \qquad (2.16)
$$

The inequality (2.4), together with the integration by parts, yields

$$
\begin{aligned}
I_5 &= \int \Big[-\partial_k(\partial_i \dot{u}^k u^i)P - \mathrm{div}\dot{\mathbf{u}}\left[\gamma \mathrm{div}\mathbf{u} P + \mathbf{u}\cdot\nabla P\right]\Big] t dx \\
&= \int tP \Big[\partial_k \dot{u}^k \partial_i u^i - \partial_i \dot{u}^k \partial_k u^i - \gamma(\mathrm{div}\dot{\mathbf{u}})(\mathrm{div}\mathbf{u})\Big] dx \qquad (2.17)\\
&\leqslant \frac{\eta}{8}\left(t\|\nabla\dot{\mathbf{u}}\|^2_{L^2}\right) + Ct\|\nabla\mathbf{u}\|^2_{L^2}.
\end{aligned}
$$

Next, for I_6 and I_7, we see that

$$
\begin{aligned}
I_6 + I_7 &= 4\beta \int \mathrm{rot}(\dot{\mathbf{w}}\dot{\mathbf{u}})\, t\, dx - 2\beta \int (\mathbf{u}\cdot t\nabla\mathbf{w})\cdot \mathrm{rot}\dot{\mathbf{u}}\, dx \\
&- 2\beta \int (\mathbf{u}\cdot t\nabla\mathbf{u})\cdot\ \mathrm{rot}\dot{\mathbf{w}} dx - 2\beta \int \partial_i \dot{\mathbf{u}}\cdot \mathrm{rot}\mathbf{w}\, t\, u^i dx - 2\beta \int \partial_i \dot{\mathbf{w}} \cdot \mathrm{rot}\mathbf{u}\, t\, u^i dx \qquad (2.18)\\
&\leqslant \beta t\|\nabla\dot{\mathbf{u}}\|^2_{L^2} + \frac{\eta'}{8} t\|\nabla\dot{\mathbf{w}}\|^2_{L^2} + Ct\|\rho\dot{\mathbf{w}}\|^2_{L^2} + C_1 t S_0^{2/3}\|\nabla\dot{\mathbf{w}}\|^2_{L^2} + Ct\|\nabla\mathbf{u}\|^6_{L^2} + Ct\|(\nabla\mathbf{u},\nabla\mathbf{w})\|^3_{L^3}.
\end{aligned}
$$

$$
\begin{aligned}
I_8 &= -4\beta\|t^{1/2}\dot{\mathbf{w}}\|_{L^2}^2 + 4\beta\int [(\mathbf{u}\cdot t\nabla\dot{\mathbf{w}})\cdot\mathbf{w} + (\mathbf{u}\cdot t\nabla\mathbf{w})\cdot\dot{\mathbf{w}}]dx \\
&= -4\beta\|t^{1/2}\dot{\mathbf{w}}\|_{L^2}^2 - 4\beta\int \operatorname{div}(\mathbf{u}t\mathbf{w})\cdot\dot{\mathbf{w}}dx \qquad (2.19)\\
&\leqslant -2\beta\|t^{1/2}\dot{\mathbf{w}}\|_{L^2}^2 + Ct\Big(\|\nabla\mathbf{w}\|_{L^2}^6 + \|\nabla\mathbf{u}\|_{L^3}^3\Big).
\end{aligned}
$$

Next, for I_9, the integration by parts gives

$$
\begin{aligned}
I_9 &= \int t\left(\partial_j\dot{u}^j b^i b^i_t + \partial_i\dot{u}^j u^i b^k\partial_j b^k\right)dx \\
&\leqslant Ct\|\nabla\dot{\mathbf{u}}\|_{L^2}\left(\|\mathbf{b}_t\|_{L^6}\|\mathbf{b}\|_{L^3} + \|\mathbf{u}\|_{L^6}\|\nabla\mathbf{b}\|_{L^3}\|\mathbf{b}\|_{L^\infty}\right) \qquad (2.20)\\
&\leqslant \frac{\eta}{8}\left(t\|\nabla\dot{\mathbf{u}}\|_{L^2}^2\right) + Ct\|\mathbf{b}\|_{L^3}^2\|\nabla\mathbf{b}_t\|_{L^2}^2 + Ct\|\nabla^2\mathbf{b}\|_{L^2}^2\left(\|\nabla\mathbf{b}\|_{L^2}^4 + \|\nabla\mathbf{u}\|_{L^2}^4\right),
\end{aligned}
$$

where, we have used the Gagliardo-Nirenberg's inequality:

$$
\|\mathbf{g}\|_{L^\beta} \leqslant C\|\mathbf{g}\|_{L^2}^{\frac{6-\beta}{2\beta}}\|\nabla\mathbf{g}\|_{L^2}^{\frac{3\beta-6}{2\beta}}, \quad \forall\, \mathbf{g}\in H^1 \text{ and } \beta\in[2,6]. \qquad (2.21)
$$

Integration by parts, together with (2.11), yields

$$
\begin{aligned}
I_{10} &= \int t\left(\dot{u}^j\partial_i b^j b^i_t + \dot{u}^j\partial_i b^j_t b^i - \partial_k\dot{u}^j u^k b^i\partial_i b^j\right)dx \\
&\leqslant Ct\|\nabla\dot{\mathbf{u}}\|_{L^2}\|\nabla\mathbf{b}_t\|_{L^2}\|\mathbf{b}\|_{L^3} + Ct\|\nabla\dot{\mathbf{u}}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{b}\|_{L^\infty}\|\nabla\mathbf{b}\|_{L^3} \qquad (2.22)\\
&\leqslant \frac{\eta}{8}\left(t\|\nabla\dot{\mathbf{u}}\|_{L^2}^2\right) + Ct\|\nabla\mathbf{b}_t\|_{L^2}^2\|\mathbf{b}\|_{L^3}^2 + Ct\left(\|\nabla\mathbf{u}\|_{L^2}^4 + \|\nabla\mathbf{b}\|_{L^2}^4\right)\|\nabla^2\mathbf{b}\|_{L^2}^2.
\end{aligned}
$$

Substituting (2.12)–(2.15), (2.17)–(2.20), and (2.22) into (2.11), we obtain

$$
\begin{aligned}
&\left[\frac{t}{2}\|(\rho^{1/2}\dot{\mathbf{u}},\rho^{1/2}\dot{\mathbf{w}})\|_{L^2}^2\right]_t + \frac{\eta}{2}t\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + \frac{\eta'}{2}t\|\nabla\dot{\mathbf{w}}\|_{L^2}^2 + 2\beta t\|\dot{\mathbf{w}}\|_{L^2}^2 \\
&\leqslant \left(\frac{1}{2} + Ct\right)\int(\rho|\dot{\mathbf{u}}|^2 + \rho|\dot{\mathbf{w}}|^2)dx + Ct\|\nabla\mathbf{b}_t\|_{L^2}^2\|\mathbf{b}\|_{L^3}^2 \qquad (2.23)\\
&\quad + Ct\Big(\|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{L^4}^4 + \|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{L^3}^3 + \|\nabla\mathbf{u}\|_{L^2}^2 + \|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{L^2}^4\|\nabla^2\mathbf{b}\|_{L^2}^2\Big),
\end{aligned}
$$

provided

$$
\mathcal{S}_0 \leqslant \varepsilon_1 \triangleq \left(\frac{\eta'}{8C_1}\right)^{3/2}.
$$

Taking note of $\|\nabla\mathbf{b}_t\|_{L^2}$, then

$$
\mathbf{b}_{tt} - \alpha\Delta\mathbf{b}_t = (\mathbf{b}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{b} - \mathbf{b}\operatorname{div}\mathbf{u})_t.
$$

Thanks to $\mathbf{u}_t = \dot{\mathbf{u}} - \mathbf{u}\cdot\nabla\mathbf{u}$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(t\|\mathbf{b}_t\|_{L^2}^2\right) + \alpha t\|\nabla\mathbf{b}_t\|_{L^2}^2 - \frac{1}{2}\|\mathbf{b}_t\|_{L^2}^2 \\
&= \int t\,(\mathbf{b}_t\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{b}_t - \mathbf{b}_t\operatorname{div}\mathbf{u} + \mathbf{b}\cdot\nabla\dot{\mathbf{u}} - \dot{\mathbf{u}}\cdot\nabla\mathbf{b} - \mathbf{b}\operatorname{div}\dot{\mathbf{u}})\cdot\mathbf{b}_t\,dx \qquad (2.24)\\
&\quad + \int t\Big[(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) + \mathbf{b}\operatorname{div}(\mathbf{u}\cdot\nabla\mathbf{u})\Big]\cdot\mathbf{b}_t\,dx = \sum_{i=1}^{3} J_i.
\end{aligned}
$$

We utilize (2.21) to get that

$$\begin{aligned} J_1 &\leqslant Ct\|\mathbf{b}_t\|_{L^3}\|\mathbf{b}_t\|_{L^6}\|\nabla\mathbf{u}\|_{L^2} + Ct\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{b}_t\|_{L^2}\|\mathbf{b}_t\|_{L^3} \\ &\leqslant Ct\|\nabla\mathbf{b}_t\|_{L^2}^{\frac{3}{2}}\|\mathbf{b}_t\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2} \\ &\leqslant \frac{\alpha}{8}\left(t\|\nabla\mathbf{b}_t\|_{L^2}^2\right) + Ct\|\mathbf{b}_t\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^4, \end{aligned} \tag{2.25}$$

$$\begin{aligned} J_2 &= \int t\,\mathbf{b}_t\cdot(-\dot{\mathbf{u}}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\dot{\mathbf{u}} - \mathbf{b}\cdot\operatorname{div}\dot{\mathbf{u}})dx \\ &\leqslant Ct\|\mathbf{b}\|_{L^3}\|\nabla\dot{\mathbf{u}}\|_{L^2}\|\nabla\mathbf{b}_t\|_{L^2} \\ &\leqslant \frac{\alpha}{8}\left(t\|\nabla\mathbf{b}_t\|_{L^2}^2\right) + Ct\|\mathbf{b}\|_{L^3}^2\|\nabla\dot{\mathbf{u}}\|_{L^2}^2, \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} J_3 &= \int t\left(b^iu^m\partial_mu^k\partial_ib_t^k + u^k\partial_ku^i\partial_ib^mb_t^m - \partial_mb^iu^k\partial_ku^mb_t^i - b^iu^k\partial_ku^m\partial_mb_t^i\right)dx \\ &\leqslant \|\nabla\mathbf{b}_t\|_{L^2}\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^6} \\ &\leqslant \frac{\alpha}{8}\left(t\|\nabla\mathbf{b}_t\|_{L^2}^2\right) + Ct\|\nabla\mathbf{b}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^6}^2. \end{aligned} \tag{2.27}$$

Putting (2.25)–(2.27) into (2.24), one has

$$\begin{aligned} \frac{d}{dt}\left(t\|\mathbf{b}_t\|_{L^2}^2\right) + t\|\nabla\mathbf{b}_t\|_{L^2}^2 &\leqslant Ct\|\mathbf{b}\|_{L^3}^2\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\mathbf{b}_t\|_{L^2}^2 + Ct\|\mathbf{b}_t\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^4 \\ &\quad + Ct\|\nabla\mathbf{b}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^6}^2. \end{aligned} \tag{2.28}$$

The inequalities (2.23), (2.28), (2.5), and (2.6) yield

$$\begin{aligned} &\sup_{t\in[0,T]}\left(t\|(\rho^{1/2}\dot{\mathbf{u}},\rho^{1/2}\dot{\mathbf{w}},\mathbf{b}_t)\|_{L^2}^2\right) + \int_0^T t\|(\nabla\dot{\mathbf{u}},\nabla\dot{\mathbf{w}},\nabla\mathbf{b}_t)\|_{L^2}^2dt \\ &\leqslant C(T) + C\int_0^T t\Big(\|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{L^4}^4 + \|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{L^3}^3 + \|\nabla\mathbf{u}\|_{L^6}^2\Big)dt. \end{aligned} \tag{2.29}$$

We deduce from (1.9) and (1.10) that

$$\|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{L^6} \leqslant C\|(\rho\dot{\mathbf{u}},\rho\dot{\mathbf{w}})\|_{L^2} + C\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2} + C\|P(\rho)-P(1)\|_{L^6}, \tag{2.30}$$

which can be found in [3]. The inequalities (2.4)–(2.8), (2.21), and (2.30) give

$$\begin{aligned} &C\int_0^T t\Big(\|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{L^4}^4 + \|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{L^3}^3 + \|\nabla\mathbf{u}\|_{L^6}^2\Big)dt \\ &\leqslant C\int_0^T t\Big(\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^6}^3 + \|\nabla\mathbf{w}\|_{L^2}\|\nabla\mathbf{w}\|_{L^6}^3 + \|\nabla\mathbf{u}\|_{L^6}^2\Big)dt \\ &\quad + C\int_0^T t\Big(\|\nabla\mathbf{u}\|_{L^2}^{3/2}\|\nabla\mathbf{u}\|_{L^6}^{3/2} + \|\nabla\mathbf{w}\|_{L^2}^{3/2}\|\nabla\mathbf{w}\|_{L^6}^{3/2}\Big)dt \end{aligned} \tag{2.31}$$

$$
\begin{aligned}
&\leqslant C(T) + C\int_0^T t\Big(\|\nabla\mathbf{u}\|_{L^6}^3 + \|\nabla\mathbf{w}\|_{L^6}^3\Big)dt\\
&\leqslant C(T) + C\int_0^T t\left(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^3 + \|\rho^{1/2}\dot{\mathbf{w}}\|_{L^2}^3\right) + C\int_0^T t\|\mathbf{b}\|_{L^\infty}^3\|\nabla\mathbf{b}\|_{L^2}^3 dt\\
&\leqslant C(T) + C\sup_{0\leqslant t\leqslant T} t^{1/2}\left(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2} + \|\rho^{1/2}\dot{\mathbf{w}}\|_{L^2}\right)\int_0^T t^{1/2}\left(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\rho^{1/2}\dot{\mathbf{w}}\|_{L^2}^2\right)dt\\
&\quad + C\sup_{0\leqslant t\leqslant T} t^{1/2}\|\nabla^2\mathbf{b}\|_{L^2}\int_0^T t^{1/2}\|\nabla\mathbf{b}\|_{L^2}^2 dt\\
&\leqslant C(T) + \frac{1}{2}\sup_{0\leqslant t\leqslant T} t\left(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\rho^{1/2}\dot{\mathbf{w}}\|_{L^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\right).
\end{aligned}
$$

The inequality $(2.1)_4$ gives

$$
\|\nabla^2\mathbf{b}\|_{L^2} \leqslant C\left(\|\mathbf{b}_t\|_{L^2} + \|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^2\right),
$$

thus

$$
\sup_{0\leqslant t\leqslant T}\left(t\|\nabla^2\mathbf{b}\|_{L^2}^2\right) \leqslant C\sup_{0\leqslant t\leqslant T} t\left(\|\mathbf{b}_t\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^4\right) \leqslant C + C\int_0^T t\|\nabla\mathbf{u}\|_{L^4}^4 dt. \tag{2.32}
$$

Based upon $(2.1)_4$, it is easy to get that

$$
\|\nabla\mathbf{b}\|_{H^2} \leqslant C + C\left(\|\nabla\mathbf{b}_t\|_{L^2} + \|\nabla^2\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}^{1/2} + \|\nabla^2\mathbf{b}\|_{L^2} + \|\nabla^2\mathbf{u}\|_{L^2}\right),
$$

thus

$$
\begin{aligned}
\int_0^T t\|\nabla\mathbf{b}\|_{H^2}^2 dt &\leqslant C + C\int_0^T t\|\nabla\mathbf{b}_t\|_{L^2}^2 dt + C\sup_{0\leqslant t\leqslant T}\left(t\|\nabla^2\mathbf{u}\|_{L^2}^2\right)\int_0^T \|\nabla^2\mathbf{b}\|_{L^2} dt\\
&\quad + C\int_0^T t\left(\|\nabla^2\mathbf{b}\|_{L^2}^2 + \|\nabla^2\mathbf{u}\|_{L^2}^2\right) \leqslant C(T).
\end{aligned} \tag{2.33}
$$

Therefore, putting (2.31)–(2.33) into (2.29), we obtain (2.10). □

Lemma 2.3. *Under the circumstance of* (2.2) *and* (2.3)*,*

$$
\int_0^T \left(\|(\rho^{1/2}\dot{\mathbf{u}}, \rho^{1/2}\dot{\mathbf{w}}, \mathbf{b}\cdot\nabla\mathbf{b})\|_{L^r}^s + \|(\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{w}, \mathrm{rot}\mathbf{u}, \mathrm{rot}\mathbf{w})\|_{\mathrm{L}^\infty}^s\right)\mathrm{dt} \leqslant \mathrm{C(T)}, \tag{2.34}
$$

where (r, s) *satisfies*

$$
r \in (3, 6) \quad \textit{and} \quad 1 < s < \frac{4r}{5r-6} < \frac{4}{3}. \tag{2.35}
$$

Proof. The inequality (2.4) together with (2.21) leads to

$$
\begin{aligned}
&\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^r}^s + \|\rho^{1/2}\dot{\mathbf{u}}\|_{L^r}^s + \|\rho^{1/2}\dot{\mathbf{w}}\|_{L^r}^s\\
&\leqslant C\|\nabla\mathbf{b}\|_{L^2}^{\frac{3s}{r}}\|\nabla^2\mathbf{b}\|_{L^2}^{\frac{s(2r-3)}{r}} + C\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^{\frac{s(6-r)}{2r}}\|\nabla\dot{\mathbf{u}}\|_{L^2}^{\frac{s(3r-6)}{2r}} + C\|\rho^{1/2}\dot{\mathbf{w}}\|_{L^2}^{\frac{s(6-r)}{2r}}\|\nabla\dot{\mathbf{w}}\|_{L^2}^{\frac{s(3r-6)}{2r}},
\end{aligned}
$$

thus, by using Hölder's inequality and the inequalities (2.6) and (2.10) gives

$$
\begin{aligned}
&\int_0^T \left(\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^r}^s + \|\rho^{1/2}\dot{\mathbf{u}}\|_{L^r}^s + \|\rho^{1/2}\dot{\mathbf{w}}\|_{L^r}^s\right) dt \leqslant C \sup_{0\leqslant t\leqslant T}\left(t\|\nabla^2\mathbf{b}\|_{L^2}^2\right)^{\frac{s(2r-3)}{2r}} \int_0^T t^{-\frac{s(2r-3)}{2r}} dt \\
&+ C\sup_{0\leqslant t\leqslant T}\left[t\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + t\|\rho^{1/2}\dot{\mathbf{w}}\|_{L^2}^2\right]^{\frac{s(6-r)}{4r}} \int_0^T t^{-\frac{s}{2}}\left(t\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + t\|\nabla\dot{\mathbf{w}}\|_{L^2}^2\right)^{\frac{s(3r-6)}{4r}} dt \\
&\leqslant C\left(\int_0^T t^{-\frac{2rs}{4r-3sr+6s}} dt\right)^{\frac{4r-3rs+6s}{4r}} \left(\int_0^T (t\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + t\|\nabla\dot{\mathbf{w}}\|_{L^2}^2) dt\right)^{\frac{s(3r-6)}{4r}} \\
&+ C\int_0^T t^{-\frac{s(2r-3)}{2r}} dt \leqslant C(T).
\end{aligned}
\tag{2.36}
$$

Thanks to $r\in(3,6)$ and $1 < s < \frac{4r}{5r-6} < 2$, one can deduce

$$
0 < \frac{s(2r-3)}{2r} < 1, \qquad 0 < \frac{2rs}{4r-3rs+6s} < 1, \qquad 0 < \frac{s(3r-6)}{4r} < 1.
$$

With the help of (1.9) and (1.10), we get

$$
\|(F_1, G_1, F_2, G_2)\|_{L^2} \leqslant C\|(\nabla\mathbf{u}, \nabla\mathbf{w}, P(\rho) - P(1))\|_{L^2}, \tag{2.37}
$$

and for any $m\in[2,6]$,

$$
\|(\nabla F_1, \nabla G_1, \nabla F_2, \nabla G_2)\|_{L^m} + \|\nabla G_1\|_{L^m} \leqslant C\|(\rho\dot{\mathbf{u}}, \rho\dot{\mathbf{w}}, \mathbf{b}\cdot\nabla\mathbf{b}, \nabla\mathbf{u}, \nabla\mathbf{w}, \mathbf{w})\|_{L^m}, \tag{2.38}
$$

thus, we have from (2.4)–(2.7), (2.37), (2.30), (2.38) that

$$
\begin{aligned}
&\|(\mathrm{rot}\mathbf{u}, \mathrm{div}\mathbf{u}, \mathrm{rot}\mathbf{w}, \mathrm{div}\mathbf{w})\|_{L^\infty} \\
&\leqslant C\left(\|F_1\|_{L^\infty} + \|G_1\|_{L^\infty} + \|P(\rho) - P(1)\|_{L^\infty} + \|\mathbf{b}^2\|_{L^\infty} + \|F_2\|_{L^\infty} + \|G_2\|_{L^\infty}\right) \\
&\leqslant C + C\left(\|F_1\|_{L^2} + \|G_1\|_{L^2} + \|F_2\|_{L^2} + \|G_2\|_{L^2} + \|\nabla\mathbf{b}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}\right) \\
&\quad + C\left(\|\nabla F_1\|_{L^r} + \|\nabla G_1\|_{L^r} + \|\nabla F_2\|_{L^r} + \|\nabla G_2\|_{L^r}\right) \\
&\leqslant C + C\left(\|\nabla\mathbf{u}\|_{L^2} + \|P(\rho) - P(1)\|_{L^2} + \|\nabla\mathbf{w}\|_{L^2} + \|\nabla^2\mathbf{b}\|_{L^2}\right) \\
&\quad + C\left(\|\rho\dot{\mathbf{u}}\|_{L^r} + \|\rho\dot{\mathbf{w}}\|_{L^r} + \|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^r} + \|\nabla\mathbf{u}\|_{L^r} + \|\nabla\mathbf{w}\|_{L^r} + \|\mathbf{w}\|_{L^r}\right) \\
&\leqslant C + C\left(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^r} + \|\rho^{1/2}\dot{\mathbf{w}}\|_{L^r}\right) + C\left(\|\nabla\mathbf{u}\|_{L^2} + \|\nabla\mathbf{w}\|_{L^2} + \|\mathbf{w}\|_{L^2}\right) \\
&\quad + C\left(\|\nabla\mathbf{u}\|_{L^6} + \|\nabla\mathbf{w}\|_{L^6} + \|\nabla^2\mathbf{b}\|_{L^2}\right) + C\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^r} \\
&\leqslant C + C\left(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^r} + \|\rho^{1/2}\dot{\mathbf{w}}\|_{L^r}\right) + C\left(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2} + \|\rho^{1/2}\dot{\mathbf{w}}\|_{L^2}\right) \\
&\quad + C\left(\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2} + \|P(\rho) - P(1)\|_{L^6}\right) + C\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^p} + C\|\nabla^2\mathbf{b}\|_{L^2} \\
&\leqslant C + C\left(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^r} + \|\rho^{1/2}\dot{\mathbf{w}}\|_{L^r}\right) + C\left(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2} + \|\rho^{1/2}\dot{\mathbf{w}}\|_{L^2}\right) \\
&\quad + C\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^r} + C\|\nabla^2\mathbf{b}\|_{L^2}.
\end{aligned}
\tag{2.39}
$$

Due to $s\in(1, 4/3)$, one has

$$
\int_0^T \|\nabla^2\mathbf{b}\|_{L^2}^s dt \leqslant \left(t\|\nabla^2\mathbf{b}\|_{L^2}^2\right)^{\frac{s}{2}} \int_0^T t^{-\frac{s}{2}} dt \leqslant C(T),
$$

which, together with (2.36) and (2.39), gives (2.34). □

In the next step, we mainly focus on estimating that $\|\nabla\mathbf{u}\|_{L^p}$ holds for $p \in (1,\infty)$ and $p = \infty$, respectively. Due to $-\Delta\mathbf{u} = \nabla\mathrm{div}\mathbf{u} - \nabla \times \mathrm{rot}\mathbf{u}$, for $0 < p < \infty$, $\|\nabla\mathbf{u}\|_{L^p} \leqslant C(\|\mathrm{div}\mathbf{u}\|_{L^p} + \|\mathrm{rot}\mathbf{u}\|_{L^p})$. However, for $p = \infty$, we have the following lemma.

Lemma 2.4. *(Beale-Kato-Majda type inequality (cf. [1, 14])) For $k \in \mathbb{Z}^+$ and $s \in (1,+\infty)$, let $D^{k,s} \triangleq \left\{\mathbf{g} \in L^1_{loc} | \partial^k\mathbf{g} \in L^s\right\}$ and $D^1 \triangleq D^{1,2}$ be the homogeneous Sobolev spaces. For $\mathbf{g} \in D^1 \cap D^{2,s}$ with $s \in (3,+\infty)$, there exists a positive constant $C(s) > 0$, such that for all $\nabla\mathbf{g} \in L^2 \cap D^{1,s}$,*

$$\|\nabla\mathbf{g}\|_{L^\infty} \leqslant C(1 + \|\nabla\mathbf{g}\|_{L^2}) + C(\|\mathrm{div}\mathbf{g}\|_{L^\infty} + \|\mathrm{rot}\mathbf{g}\|_{L^\infty}) \ln\left(e + \|\nabla^2\mathbf{g}\|_{L^s}\right). \tag{2.40}$$

Lemma 2.5. *Let* (1.11) *and* (1.12) *be in force, then*

$$\sup_{t\in[0,T]} (\|\rho_t\|_{L^2} + \|\nabla\rho\|_{L^2\cap L^r}) + \int_0^T \left(\|\nabla\mathbf{u}\|^s_{L^\infty} + \|\nabla^2\mathbf{u}\|^s_{L^r}\right) dt \leqslant C(T), \tag{2.41}$$

where $r \in (3,6)$ and $s \in (1,\infty)$ are the same ones as in (2.35).

Proof. We operate ∇ to both sides of $(2.1)_1$ and multiply the result equation by $|\nabla\rho|^{r-2}\nabla\rho$ for $r \in [2,6]$. One can deduce from (2.4) and integration by parts that

$$\frac{d}{dt}\|\nabla\rho\|_{L^r} \leqslant C\left(\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^r} + \|\nabla^2\mathbf{u}\|_{L^r}\right). \tag{2.42}$$

Recalling that $\mathcal{G} \triangleq -(\eta+\beta)\Delta - (\eta+\kappa-\beta)\nabla\mathrm{div}$ is a strong elliptic operator ([2]), the inequalities $(2.1)_2$ and (2.4) give

$$\|\nabla^2\mathbf{u}\|_{L^r} \leqslant C\|(\nabla\rho, \rho^{1/2}\dot{\mathbf{u}}, \mathbf{b}\cdot\nabla\mathbf{b}, \mathrm{rot}\mathbf{w})\|_{L^r} \quad \forall\, r \in (3,6), \tag{2.43}$$

which, combined with (2.42), gives

$$\frac{d}{dt}\|\nabla\rho\|_{L^r} \leqslant C(\|\nabla\mathbf{u}\|_{L^\infty} + 1)\|\nabla\rho\|_{L^r} + C\left(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^r} + \|\mathrm{rot}\mathbf{w}\|_{L^r} + \|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^r}\right). \tag{2.44}$$

The combination of (2.40) and (2.43) gives

$$\begin{aligned} \|\nabla\mathbf{u}\|_{L^\infty} \leqslant{}& C + C\|(\mathrm{rot}\mathbf{u}, \mathrm{div}\mathbf{u})\|_{L^\infty} \ln(\|\nabla\rho\|_{L^r} + e) \\ &+ C(\|(\mathrm{rot}\mathbf{u}, \mathrm{div}\mathbf{u})\|_{L^\infty} \ln\left(\|(\rho^{1/2}\dot{\mathbf{u}}, \mathrm{rot}\mathbf{w}, \mathbf{b}\cdot\nabla\mathbf{b})\|_{L^r} + e\right), \end{aligned} \tag{2.45}$$

for any $3 < r < 6$. Taking (2.45) into (2.44), one gets

$$\frac{d}{dt}(\|\nabla\rho\|_{L^r} + e) \leqslant C\mathfrak{I}(t)\ln(\|\nabla\rho\|_{L^r} + e), \tag{2.46}$$

where

$$\begin{aligned} \mathfrak{I}(t) \triangleq{}& C\left(1 + \|(\mathrm{div}\mathbf{u}, \mathrm{rot}\mathbf{u})\|_{L^\infty} + \|(\rho^{1/2}\dot{\mathbf{u}}, \mathrm{rot}\mathbf{w}, \mathbf{b}\cdot\nabla\mathbf{b})\|_{L^r}\right) \\ &+ C\|(\mathrm{div}\mathbf{u}, \mathrm{rot}\mathbf{u})\|_{L^\infty} \ln\left(\|(\rho^{1/2}\dot{\mathbf{u}}, \mathrm{rot}\mathbf{w}, \mathbf{b}\cdot\nabla\mathbf{b})\|_{L^r} + e\right). \end{aligned}$$

Noting that the relationship $\ln(e+z) \leqslant (e+z)^\tau$ holds for any $z \geqslant 0$ and $\tau > 0$, the inequality (2.34) gives

$$\int_0^T \mathfrak{I}(t)dt \leqslant C(T),$$

which, combined with (2.46), leads to

$$\sup_{t\in[0,T]} \|\nabla\rho(t)\|_{L^r} \leqslant C(T), \quad \forall\, r \in (3,6). \tag{2.47}$$

The inequalities (2.7) and (2.34) give

$$\begin{aligned}\int_0^T \left(\|\nabla\mathbf{u}\|_{L^\infty}^s + \|\nabla^2\mathbf{u}\|_{L^r}^s\right) dt &\leqslant C\int_0^T \left(\|\nabla^2\mathbf{u}\|_{L^r}^s + 1\right) dt \\ &\leqslant C\int_0^T \left(\|(\rho^{1/2}\dot{\mathbf{u}}, \nabla P, \mathbf{b}\cdot\nabla\mathbf{b}, \mathrm{rot}\mathbf{w})\|_{L^r}^s + 1\right) dt \leqslant C(T),\end{aligned} \tag{2.48}$$

for any r, s being as the ones in (2.31). Next, taking $r = 2$ in (2.42), one has

$$\begin{aligned}\frac{d}{dt}\|\nabla\rho\|_{L^2} &\leqslant C\left(\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^2} + \|\nabla^2\mathbf{u}\|_{L^2}\right) \\ &\leqslant C\left(\|\nabla\mathbf{u}\|_{L^\infty} + 1\right)\|\nabla\rho\|_{L^2} + C\left(1 + \|(\rho^{1/2}\dot{\mathbf{u}}, \nabla\mathbf{w})\|_{L^2}\right),\end{aligned}$$

thus

$$\sup_{t\in[0,T]} \|\nabla\rho\|_{L^2} \leqslant C(T). \tag{2.49}$$

Note that the Eq $(2.1)_1$ gives

$$\begin{aligned}\|\rho_t\|_{L^2} &\leqslant C\left(\|\nabla\mathbf{u}\|_{L^2} + \|\nabla\rho\|_{L^3}\right) \leqslant C\left(\|\nabla\rho\|_{L^3} + 1\right) \\ &\leqslant C\left(\|\nabla\rho\|_{L^r} + 1 + \|\nabla\rho\|_{L^2}\right) \leqslant C(T), \quad \forall r \in (3,6),\end{aligned}$$

which, combined with (2.47)–(2.49), gives (2.41). □

Lemma 2.6. *Assume that ρ_0 satisfies $\inf_{x\in\mathbb{R}^3}\rho_0(x) \geqslant \check{\rho} > 0$, then there exists a constant $c > 0$, depending on $\check{\rho}$ and T, such that*

$$\rho(x,t) \geqslant c, \quad \forall\, x \in \mathbb{R}^3, \quad t \in [0,T], \tag{2.50}$$

and

$$\sup_{t\in[0,T]} \left(t\|(\nabla^2\mathbf{u}, \mathbf{u}_t)\|_{L^2}^2\right) + \int_0^T \|(\nabla^2\mathbf{u}, \nabla^2\mathbf{w}, \mathbf{u}_t, t^{1/2}\nabla\mathbf{u}_t)\|_{L^2}^2 dt \leqslant C(T). \tag{2.51}$$

Proof. The inequality (2.34) gives

$$\rho(x,t) \geqslant \inf_{x\in\mathbb{R}^3}\rho_0(x)e^{\left\{-\int_0^t \|\mathrm{div}\mathbf{u}\|_{L^\infty} ds\right\}} \geqslant c(\check{\rho}, T),$$

which yields (2.50).

Next, the inequalities $(2.1)_1$, (2.4), and (2.5) give

$$\|\nabla^2\mathbf{u}\|_{L^2} \leqslant C\left(\|(\rho\dot{\mathbf{u}}, \nabla P, \mathrm{rot}\mathbf{w})\|_{L^2} + \|\mathbf{b}\|_{L^3}\|\nabla^2\mathbf{b}\|_{L^2}\right) \leqslant C\|(\dot{\mathbf{u}}, \nabla\rho, \nabla\mathbf{w}, \nabla^2\mathbf{b})\|_{L^2},$$

thus, using (2.4), (2.5), (2.10), (2.41), and (2.50), one has

$$\begin{aligned} &\sup_{t\in[0,T]}\left(t\|\nabla^2\mathbf{u}\|_{L^2}^2\right)+\int_0^T\|\nabla^2\mathbf{u}\|_{L^2}^2dt \\ &\leqslant C\sup_{t\in[0,T]}\left[t\|(\dot{\mathbf{u}},\nabla\rho,\nabla\mathbf{w},\nabla^2\mathbf{b})\|_{L^2}^2\right]+C\int_0^T\|(\dot{\mathbf{u}},\nabla\rho,\nabla\mathbf{w},\nabla^2\mathbf{b})\|_{L^2}^2dt\leqslant C(T). \end{aligned} \tag{2.52}$$

Taking note of $\dot{\mathbf{u}}$,

$$\|\mathbf{u}_t\|_{L^2}^2\leqslant\|\dot{\mathbf{u}}\|_{L^2}^2+\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2\leqslant C\left(\|\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{H^1}^2\right),$$

and the inequalities (2.4), (2.6), (2.10), (2.50), and (2.52) give

$$\sup_{t\in[0,T]}\left(t\|\mathbf{u}_t\|_{L^2}^2\right)+\int_0^T\|\mathbf{u}_t\|_{L^2}^2dt\leqslant C(T), \tag{2.53}$$

and

$$\begin{aligned} \int_0^T t\|\nabla\mathbf{u}_t\|_{L^2}^2dt &\leqslant\int_0^T t\|(\nabla\dot{\mathbf{u}},\nabla(\mathbf{u}\cdot\nabla\mathbf{u}))\|_{L^2}^2dt \\ &\leqslant C+C\int_0^T t\|\nabla^2\mathbf{u}\|_{L^2}^4dt+C\int_0^T t\|\mathbf{u}\|_{L^\infty}^2\|\nabla^2\mathbf{u}\|_{L^2}^2dt \\ &\leqslant C+C\int_0^T t\|\nabla^2\mathbf{u}\|_{L^2}^4dt\leqslant C. \end{aligned} \tag{2.54}$$

On the other hand, due to $(2.1)_3$, (2.4)–(2.6), and (2.9), one has

$$\int_0^T\|\nabla^2\mathbf{w}\|_{L^2}^2dt\leqslant C\int_0^T\left(\|\rho^{1/2}\dot{\mathbf{w}}\|_{L^2}^2+\|\mathbf{w}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2\right)dt\leqslant C. \tag{2.55}$$

Utilizing (2.52)–(2.55), we can obtain the estimate (2.51). □

Based upon the foregoing, the following estimation of $\|\nabla\mathbf{u}\|_{L^3}$ is very crucial for this article.

Lemma 2.7. *Under the circumstance of Theorem 1.1, we have*

$$\sup_{t\in[0,T]}\|\nabla\mathbf{u}\|_{L^3}^3+\int_0^T\|(|\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u},|\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u})\|_{L^2}^2dt\leqslant C(T). \tag{2.56}$$

Proof. Utilizing the inequality $\|\nabla\mathbf{u}\|_{L^m}\leqslant C\|(\mathrm{div}\mathbf{u},\mathrm{rot}\mathbf{u})\|_{L^m}$ for $1<m<\infty$, thus, we just have to estimate $\|\mathrm{div}\mathbf{u}\|_{L^m}$ and $\|\mathrm{rot}\mathbf{u}\|_{L^m}$. Operating *div* and *rot* to $(2.1)_2$, one gets

$$\begin{aligned} &\rho(\mathrm{div}\mathbf{u})_t+\rho\mathbf{u}\cdot\nabla(\mathrm{div}\mathbf{u})-(2\eta+\kappa)\Delta(\mathrm{div}\mathbf{u}) \\ &=-(\nabla\rho)\cdot\mathbf{u}_t-\partial_k(\rho u^i)\partial_i u^k-\Delta P+\partial_k b^i\partial_i b^k-\frac{1}{2}\Delta|\mathbf{b}|^2, \end{aligned} \tag{2.57}$$

and

$$\begin{aligned} &\rho(\mathrm{rot}\mathbf{u})_t+\rho\mathbf{u}\cdot\nabla(\mathrm{rot}\mathbf{u})-(\eta+\beta)\Delta(\mathrm{rot}\mathbf{u}) \\ &=-(\nabla\rho)\times\mathbf{u}_t-\nabla(\rho u^k)\times(\partial_k\mathbf{u})+2\beta\mathrm{rot}(\mathrm{rot}\mathbf{w})+(\nabla b^i)\times(\partial_i\mathbf{b})+\mathbf{b}\cdot\nabla(\mathrm{rot}\mathbf{b}). \end{aligned} \tag{2.58}$$

We first to estimate $\|\mathrm{div}\mathbf{u}\|_{L^3}$. Multiplying (2.57) by $|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u}$, one can deduce from integration by parts that

$$\begin{aligned}
&\left(\frac{1}{3}\|\rho^{1/3}\mathrm{div}\mathbf{u}\|_{L^3}^3\right)_t + (2\eta+\kappa)\|(|\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}, |\mathrm{div}\mathbf{u}|^{1/2}\nabla|\mathrm{div}\mathbf{u}|)\|_{L^2}^2 \\
&= -\int\left[\mathbf{u}_t\cdot\nabla\rho + \partial_k(\rho u^i)\partial_i u^k\right](|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})\,dx \\
&\quad + \int\nabla(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})\cdot\nabla P dx + \int\partial_j b^i\partial_i b^j(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})\,dx \\
&\quad + \int\nabla(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})\cdot\nabla|\mathbf{b}|^2 dx \triangleq \sum_{i=1}^5 J_i.
\end{aligned} \tag{2.59}$$

With the help of $(2.1)_2$ and the integration by parts, we get

$$\begin{aligned}
J_1 &= -\int\rho^{-1}\Big[((\eta+\beta)\Delta\mathbf{u} + (\eta+\kappa-\beta)\nabla\mathrm{div}\mathbf{u} - \nabla P - \rho\mathbf{u}\cdot\nabla\mathbf{u}\Big]\cdot\nabla\rho\,(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})\,dx \\
&\quad - \int\rho^{-1}\left(2\beta\mathrm{rot}\mathbf{w} + \mathbf{b}\cdot\nabla\mathbf{b} - \frac{1}{2}\nabla|\mathbf{b}|^2\right)\cdot\nabla\rho\,(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})\,dx \\
&= (\eta+\beta)\int\Big[\left(\partial_m u^k\partial_k(\ln\rho)\right)\partial_m(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u}) - (\partial_m\mathrm{div}\mathbf{u})(\partial_m(\ln\rho))(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})\Big]dx \\
&\quad - (\eta+\beta)\int\left(\partial_m u^k\partial_m(\ln\rho)\right)\partial_k(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})\,dx \\
&\quad - (\eta+\kappa-\beta)\int(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})\nabla\mathrm{div}\mathbf{u}\cdot\nabla(\ln\rho)dx \\
&\quad + \int(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})\rho^{-1}\nabla P\cdot\nabla\rho dx + \int(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\rho dx \\
&\quad - \int(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u})\Big[\left(2\beta\mathrm{rot}\mathbf{w} + \mathbf{b}\cdot\nabla\mathbf{b} - \frac{1}{2}\nabla|\mathbf{b}|^2\right)\cdot\nabla(\ln\rho)\Big]dx \triangleq \sum_{i=1}^5 J_{1,i}.
\end{aligned} \tag{2.60}$$

The inequality (2.6), together with (2.41) gives

$$\begin{aligned}
J_{1,1} &\leqslant C\|\nabla\rho\|_{L^3}\|\nabla\mathbf{w}\|_{L^3}\|\mathrm{div}\mathbf{u}\|_{L^6}^2 \\
&\leqslant C\|\nabla\mathbf{w}\|_{L^2}^{1/2}\|\nabla^2\mathbf{w}\|_{L^2}^{1/2}\|\mathrm{div}\mathbf{u}\|_{L^3}^{1/2}\|\mathrm{div}\mathbf{u}\|_{L^9}^{3/2} \\
&\leqslant C\|\nabla^2\mathbf{w}\|_{L^2}^{1/2}\|\mathrm{div}\mathbf{u}\|_{L^3}^{1/2}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2} \\
&\leqslant \frac{2\mu+\lambda}{16}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 + C\|\nabla^2\mathbf{w}\|_{L^2}^2\|\mathrm{div}\mathbf{u}\|_{L^3},
\end{aligned} \tag{2.61}$$

where, we used the following facts:

$$\begin{cases}
\|\mathrm{div}\mathbf{v}\|_{L^6} \leqslant C\|\mathrm{div}\mathbf{v}\|_{L^3}^{1/4}\|\mathrm{div}\mathbf{v}\|_{L^9}^{3/4}, \\
\|\mathrm{div}\mathbf{v}\|_{L^9} = \||\mathrm{div}\mathbf{v}|^{3/2}\|_{L^6}^{2/3} \leqslant C\||\mathrm{div}\mathbf{v}|^{1/2}\nabla\mathrm{div}\mathbf{v}\|_{L^2}^{2/3}.
\end{cases} \tag{2.62}$$

In terms of $J_{1,2}$ and the inequality (2.41), together with (2.62), gives

$$\begin{aligned} J_{1,2} &\leqslant C\|\nabla\rho\|_{L^p}\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}\|\mathrm{div}\mathbf{u}\|_{L^9}^{1/2}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2} \\ &\leqslant \frac{2\mu+\lambda}{32}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 + C\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}^2\|\mathrm{div}\mathbf{u}\|_{L^9} \\ &\leqslant \frac{2\mu+\lambda}{16}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 + C\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}^3, \end{aligned} \tag{2.63}$$

$$\begin{aligned} J_{1,3}+J_{1,4} &\leqslant C\|\nabla\rho\|_{L^3}^2\|\mathrm{div}\mathbf{u}\|_{L^6}^2 + C\|\nabla\rho\|_{L^3}\|\nabla^2\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\|\mathrm{div}\mathbf{u}\|_{L^6}^2 \\ &\leqslant C\|\nabla^2\mathbf{u}\|_{L^2}^2 + C\|\nabla^2\mathbf{u}\|_{L^2}\|\mathrm{div}\mathbf{u}\|_{L^6}^2 \\ &\leqslant C\|\nabla^2\mathbf{u}\|_{L^2}^2 + C\|\mathrm{div}\mathbf{u}\|_{L^3}^{1/2}\|\nabla^2\mathbf{u}\|_{L^2}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2} \\ &\leqslant \frac{2\eta+\kappa}{16}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 + C\|\mathrm{div}\mathbf{u}\|_{L^3}\|\nabla^2\mathbf{u}\|_{L^2}^2 + C\|\nabla^2\mathbf{u}\|_{L^2}^2, \end{aligned} \tag{2.64}$$

and

$$\begin{aligned} J_{1,5} &\leqslant C\|\nabla\rho\|_{L^3}\|\mathbf{b}\|_{L^6}\|\nabla\mathbf{b}\|_{L^6}\|\mathrm{div}\mathbf{u}\|_{L^6}^2 \\ &\leqslant C\|\mathrm{div}\mathbf{u}\|_{L^3}^{1/2}\|\nabla^2\mathbf{b}\|_{L^2}\|\mathrm{div}\mathbf{u}\|_{L^9}^{3/2} \\ &\leqslant C\|\mathrm{div}\mathbf{u}\|_{L^3}^{1/2}\|\nabla^2\mathbf{b}\|_{L^2}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2} \\ &\leqslant \frac{2\eta+\kappa}{16}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 + C\|\nabla^2\mathbf{b}\|_{L^2}^2\|\mathrm{div}\mathbf{u}\|_{L^3}. \end{aligned} \tag{2.65}$$

Putting (2.61), (2.63)–(2.65) into (2.60), one has

$$\begin{aligned} J_1 &\leqslant \frac{2\eta+\kappa}{4}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 + C\left(\|\nabla^2\mathbf{u}\|_{L^2}^2 + \left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}^3\right) \\ &\quad + C\left(\|(\nabla^2\mathbf{w},\nabla^2\mathbf{b},\nabla^2\mathbf{u})\|_{L^2}^2+1\right)\left(1+\|\mathrm{div}\mathbf{u}\|_{L^3}^2\right), \quad 3<r<6. \end{aligned} \tag{2.66}$$

Next, by virtue of (2.4), (2.6), (2.41), and (2.62),

$$\begin{aligned} J_2 &\leqslant C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\rho\|_{L^3}\|\nabla^2\mathbf{u}\|_{L^2}\|\mathrm{div}\mathbf{u}\|_{L^6}^2 + C\|\nabla^2\mathbf{u}\|_{L^2}^2\|\mathrm{div}\mathbf{u}\|_{L^3}^2 \\ &\leqslant \frac{2\eta+\kappa}{8}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 + C\|\nabla^2\mathbf{u}\|_{L^2}^2\left(\|\mathrm{div}\mathbf{u}\|_{L^3}^2+1\right), \end{aligned} \tag{2.67}$$

and

$$\begin{aligned} J_3 &\leqslant \|\mathrm{div}\mathbf{u}\|_{L^3}^{1/2}\|\nabla\rho\|_{L^3}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2} \\ &\leqslant \frac{2\eta+\kappa}{8}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 + C\left(1+\|\mathrm{div}\mathbf{u}\|_{L^3}^2\right). \end{aligned} \tag{2.68}$$

Noticing that $\mathrm{div}\mathbf{b}=0$, the inequality of (2.6) gives

$$\begin{aligned} J_4+J_5 &= C\int[\partial_j b^i b^i - b^i\partial_i b^j]\partial_j\left(|\mathrm{div}\mathbf{u}|\mathrm{div}\mathbf{u}\right)dx \\ &\leqslant \|\mathbf{b}\|_{L^3}\|\mathrm{div}\mathbf{u}\|_{L^3}^{1/2}\|\nabla\mathbf{b}\|_{L^6}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2} \\ &\leqslant \frac{2\eta+\kappa}{8}\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 + C\|\nabla^2\mathbf{b}\|_{L^2}^2\left(1+\|\mathrm{div}\mathbf{u}\|_{L^3}^2\right). \end{aligned} \tag{2.69}$$

Substituting (2.66)–(2.69) into (2.59), one has

$$\begin{aligned}&\left(\|\rho^{1/3}\mathrm{div}\mathbf{u}\|_{L^3}^3\right)_t+\||\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2\\&\leqslant C\left(\|(\nabla^2\mathbf{u},\nabla^2\mathbf{w},\nabla^2\mathbf{b})\|_{L^2}^2+1\right)\left(\|\mathrm{div}\mathbf{u}\|_{L^3}^2+1\right)+C\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}^3.\end{aligned}\tag{2.70}$$

Next, we estimate $\|\mathrm{rot}\mathbf{u}\|_{L^3}$. We first multiply (2.58) by $|\mathrm{rot}\mathbf{u}|\mathrm{rot}\mathbf{u}$ and then deduce from integration by parts that

$$\begin{aligned}&\left(\frac{1}{3}\|\rho^{1/3}\mathrm{rot}\mathbf{u}\|_{L^3}^3\right)_t+(\eta+\beta)\left(\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2}^2+\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla|\mathrm{rot}\mathbf{u}|\|_{L^2}^2\right)\\&=\int\left[2\beta\mathrm{rot}(\mathrm{rot}\mathbf{w})-\nabla\rho\times\mathbf{u}_t-\nabla(\rho u^k)\times(\partial_k\mathbf{u})\right]\cdot(|\mathrm{rot}\mathbf{u}|\mathrm{rot}\mathbf{u})dx\\&\quad+\int\left[(\nabla b^k)\times(\partial_k\mathbf{b})+\mathbf{b}\cdot\nabla(\mathrm{rot}\mathbf{b})\right]\cdot(|\mathrm{rot}\mathbf{u}|\mathrm{rot}\mathbf{u})dx\triangleq\sum_{i=1}^5 I_i.\end{aligned}\tag{2.71}$$

The Hölder's inequality gives

$$\begin{aligned}I_1&\leqslant C\int|\nabla^2\mathbf{w}||\nabla\mathbf{u}||\mathrm{rot}\mathbf{u}|dx\leqslant C\|\nabla^2\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{w}\|_{L^2}\|\mathrm{rot}\mathbf{u}\|_{L^3}\\&\leqslant C\|\nabla^2\mathbf{w}\|_{L^2}^2\|\mathrm{rot}\mathbf{u}\|_{L^3}^2+C\|\nabla^2\mathbf{u}\|_{L^2}^2.\end{aligned}\tag{2.72}$$

Noting that

$$(\eta+\beta)\Delta\mathbf{u}+(\eta+\kappa-\beta)\nabla\mathrm{div}\mathbf{u}=(2\eta+\kappa)\nabla\mathrm{div}\mathbf{u}-(\eta+\beta)\nabla\times(\mathrm{rot}\mathbf{u}),$$

thus

$$\begin{aligned}I_2=&-\int(|\mathrm{rot}\mathbf{u}|\mathrm{rot}\mathbf{u})\cdot(\nabla\ln\rho)\times(2\beta\mathrm{rot}\mathbf{w})\,dx\\&+\int(|\mathrm{rot}\mathbf{u}|\mathrm{rot}\mathbf{u})\cdot(\nabla\ln\rho)\times[(\eta+\beta)(\nabla\times\mathrm{rot}\mathbf{u})-(2\mu+\lambda)(\nabla\mathrm{div}\mathbf{u})]dx\\&+\int(|\mathrm{rot}\mathbf{u}|\mathrm{rot}\mathbf{u})\cdot(\nabla\ln\rho)\times(\nabla P+\rho\mathbf{u}\cdot\nabla\mathbf{u})\,dx\\&-\int(|\mathrm{rot}\mathbf{u}|\mathrm{rot}\mathbf{u})\cdot(\nabla\ln\rho)\times\left(\mathbf{b}\cdot\nabla\mathbf{b}-\frac{1}{2}\nabla|\mathbf{b}|^2\right)dx\triangleq\sum_{i=1}^6 I_{2,i}.\end{aligned}\tag{2.73}$$

By virtue of (2.6), (2.23), (2.41), (2.62), we get

$$\begin{aligned}I_{2,1}&\leqslant C\|\nabla\mathbf{w}\|_{L^3}\|\nabla\rho\|_{L^3}\|\nabla\mathrm{rot}\mathbf{u}\|_{L^2}^2\\&\leqslant\frac{\eta+\beta}{16}\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2}^2+C\left(\|\nabla^2\mathbf{w}\|_{L^2}^2+1\right)\left(\|\mathrm{rot}\mathbf{u}\|_{L^3}^2+1\right),\end{aligned}\tag{2.74}$$

and

$$\begin{aligned}I_{2,2}&\leqslant C\|\mathrm{rot}\mathbf{u}\|_{L^9}^{1/2}\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2}\|\nabla\rho\|_{L^r}\\&\leqslant\frac{\eta+\beta}{16}\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2}^2+C\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}^3.\end{aligned}\tag{2.75}$$

For $I_{2,3}$, we utilize the fact

$$\int (\nabla\phi \times \nabla\psi)\cdot v dx = -\int \psi(\nabla\phi)\cdot(\nabla\times v)dx,$$

and take $\phi = \ln\rho$, $\psi = \mathrm{div}\mathbf{u}$, and $v = |\mathrm{rot}\mathbf{u}|\mathrm{rot}\mathbf{u}$, then

$$\begin{aligned} I_{2,3} &\leqslant C\|\nabla\rho\|_{L^r}\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}\|\mathrm{rot}\mathbf{u}\|_{L^9}^{1/2}\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2} \\ &\leqslant \frac{\eta+\beta}{16}\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2}^2 + C\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}^3. \end{aligned} \tag{2.76}$$

The inequality (2.41), together with (2.4), leads to

$$I_{2,4} \leqslant C\int |\mathrm{rot}\mathbf{u}|^2|\nabla\rho|^2 dx \leqslant C\|\nabla^2\mathbf{u}\|_{L^2}^2\|\nabla\rho\|_{L^3}^2 \leqslant C\|\nabla^2\mathbf{u}\|_{L^2}^2, \tag{2.77}$$

$$\begin{aligned} I_{2,5} &\leqslant C\int |\nabla\mathbf{u}||\nabla\rho||\mathrm{rot}\mathbf{u}|^2|\rho||\mathbf{u}|dx \leqslant C\|\nabla^2\mathbf{u}\|_{L^2}\|\mathrm{rot}\mathbf{u}\|_{L^6}^2 dx \\ &\leqslant \frac{\eta+\kappa}{32}\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{u}\|_{L^2}^2\left(\|\mathrm{rot}\mathbf{u}\|_{L^3}^2+1\right), \end{aligned} \tag{2.78}$$

and

$$\begin{aligned} I_{2,6} &\leqslant C\int |\nabla\mathbf{b}||\mathrm{curl}\mathbf{u}|^2|\nabla\rho||\mathbf{b}|dx \leqslant C\|\mathbf{b}\|_{L^6}\|\nabla\rho\|_{L^3}\|\nabla\mathbf{b}\|_{L^6}\|\mathrm{curl}\mathbf{u}\|_{L^6}^2 \\ &\leqslant \frac{\eta}{32}\||\mathrm{curl}\mathbf{u}|^{1/2}\nabla\mathrm{curl}\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\left(\|\mathrm{curl}\mathbf{u}\|_{L^3}^2+1\right). \end{aligned} \tag{2.79}$$

Taking (2.74)–(2.79) into (2.73), one has

$$\begin{aligned} I_2 &\leqslant \frac{\eta+\beta}{4}\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2}^2 + C\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}^3 \\ &\quad + C\left(1+\|(\nabla^2\mathbf{u},\nabla^2\mathbf{w},\nabla^2\mathbf{B})\|_{L^2}^2\right)\left(\|\mathrm{rot}\mathbf{u}\|_{L^3}^2+1\right). \end{aligned} \tag{2.80}$$

Similarly,

$$\begin{aligned} I_3 &\leqslant C\int |\mathrm{rot}\mathbf{u}|^2\left(|\mathbf{u}||\nabla\rho||\nabla\mathbf{u}| + |\nabla\mathbf{u}|^2|\rho|\right)dx \leqslant C\|\nabla^2\mathbf{u}\|_{L^2}\|\mathrm{rot}\mathbf{u}\|_{L^6}^2 dx \\ &\leqslant \frac{\eta+\beta}{8}\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{u}\|_{L^2}^2\left(\|\mathrm{rot}\mathbf{u}\|_{L^3}^2+1\right), \end{aligned} \tag{2.81}$$

and

$$I_4 \leqslant C\int |\mathrm{rot}\mathbf{u}|^2|\nabla\mathbf{b}|^2 dx \leqslant C\|\nabla^2\mathbf{b}\|_{L^2}^2\|\mathrm{rot}\mathbf{u}\|_{L^3}^2. \tag{2.82}$$

We notice that $\mathrm{div}\mathbf{b} = 0$, thus

$$\begin{aligned} I_5 &\leqslant C\int |\mathrm{rot}\mathbf{u}||\mathbf{b}||\nabla\mathrm{rot}\mathbf{u}||\nabla\mathbf{b}|dx \\ &\leqslant C\|\mathbf{b}\|_{L^3}\|\nabla^2\mathbf{b}\|_{L^2}\|\mathrm{rot}\mathbf{u}\|_{L^3}^{1/2}\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2} \\ &\leqslant \frac{\eta}{8}\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\left(\|\mathrm{rot}\mathbf{u}\|_{L^3}^2+1\right). \end{aligned} \tag{2.83}$$

Putting (2.72), (2.73) and (2.80)–(2.83) into (2.71), we obtain

$$\begin{aligned}&\left(\|\rho^{1/3}\mathrm{rot}\mathbf{u}\|_{L^3}^3\right)_t+\||\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u}\|_{L^2}^2\\&\leqslant C\left(\|(\nabla^2\mathbf{u},\nabla^2\mathbf{w},\nabla^2\mathbf{b})\|_{L^2}^2+1\right)\left(\|\mathrm{rot}\mathbf{u}\|_{L^3}^2+1\right)+C\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}^3.\end{aligned}\tag{2.84}$$

We close the estimations. The inequality (2.70), together with (2.84), gives

$$\begin{aligned}&\left(\|(\rho^{1/3}\mathrm{div}\mathbf{u},\rho^{1/3}\mathrm{rot}\mathbf{u})\|_{L^3}^3\right)_t+\|(|\mathrm{div}\mathbf{u}|^{1/2}\nabla\mathrm{div}\mathbf{u},|\mathrm{rot}\mathbf{u}|^{1/2}\nabla\mathrm{rot}\mathbf{u})\|_{L^2}^2\\&\leqslant\left(1+\|(\nabla^2\mathbf{w},\nabla^2\mathbf{u},\nabla^2\mathbf{b})\|_{L^2}^2\right)\left(1+\|(\mathrm{div}\mathbf{u},\mathrm{rot}\mathbf{u})\|_{L^3}^2\right)+C\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}^3.\end{aligned}\tag{2.85}$$

Due to

$$1<\frac{18-2r}{r}\leqslant 2,\quad\text{and}\quad 1\leqslant\frac{5r-18}{r}<2,\quad\text{for}\quad\frac{9}{2}\leqslant r<6,$$

thus

$$\left\|\nabla\mathbf{u}\right\|_{L^{\frac{9r}{4r-9}}}^3\leqslant C\left\|\nabla\mathbf{u}\right\|_{L^3}^{\frac{5r-18}{r}}\left\|\nabla\mathbf{u}\right\|_{L^6}^{\frac{18-2r}{r}}\leqslant C\left(1+\|\nabla\mathbf{u}\|_{L^3}^2\right)\left(1+\|\nabla\mathbf{u}\|_{H^1}^2\right).\tag{2.86}$$

Thanks to $\|\nabla\mathbf{u}\|_{L^r}\leqslant C\left(\|\mathrm{div}\mathbf{u}\|_{L^r}+\|\mathrm{rot}\mathbf{u}\|_{L^r}\right)$ ($\forall\ r>1$), we obtain from (2.6), (2.51), (2.85), and (2.86) that

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T}\|\nabla\mathbf{u}\|_{L^3}^3\leqslant C\sup_{0\leqslant t\leqslant T}\left(\|\mathrm{div}\mathbf{u}\|_{L^3}^3+\|\mathrm{rot}\mathbf{u}\|_{L^3}^3\right)\\&\leqslant C(T)+C\int_0^T\|(\nabla^2\mathbf{w},\nabla^2\mathbf{u},\nabla^2\mathbf{b})\|_{L^2}^2\left(1+\|\nabla\mathbf{u}\|_{L^3}^2\right)dt\\&\quad+C\int_0^T\left(1+\|\nabla\mathbf{u}\|_{H^1}^2\right)\left(1+\|\nabla\mathbf{u}\|_{L^3}^2\right)dt\\&\leqslant C(T)+C\int_0^T\|(\nabla^2\mathbf{w},\nabla^2\mathbf{u},\nabla^2\mathbf{b})\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^3}^2dt,\end{aligned}$$

which, allied with (2.51), gives (2.56). □

Lemma 2.8. *Under the circumstance of Theorem 1.1, then*

$$\sup_{t\in[0,T]}\|\nabla\mathbf{w}\|_{L^3}^3+\int_0^T\|(|\mathrm{div}\mathbf{w}|^{1/2}\nabla\mathrm{div}\mathbf{w},|\mathrm{rot}\mathbf{w}|^{1/2}\nabla\mathrm{rot}\mathbf{w})\|_{L^2}^2dt\leqslant C(T).\tag{2.87}$$

Proof. Operating *div* and *rot* to $(2.1)_3$, one has

$$\rho(\mathrm{div}\mathbf{w})_t+\rho\mathbf{u}\cdot\nabla(\mathrm{div}\mathbf{w})-(2\eta'+\kappa')\Delta(\mathrm{div}\mathbf{w})+(\nabla\rho)\cdot\mathbf{w}_t+\partial_k(\rho u^m)\partial_m w^k+4\beta\mathrm{div}\mathbf{w}=0,\tag{2.88}$$

and

$$\rho(\mathrm{rot}\mathbf{w})_t+\rho\mathbf{u}\cdot\nabla(\mathrm{rot}\mathbf{w})-\eta'\Delta(\mathrm{rot}\mathbf{w})+(\nabla\rho)\times\mathbf{w}_t+\nabla(\rho u^m)\times(\partial_m\mathbf{w})+4\beta\mathrm{rot}\mathbf{w}-2\beta\mathrm{rot}(\mathrm{rot}\mathbf{u})=0.\tag{2.89}$$

We multiply (2.88) by $|\mathrm{div}\mathbf{w}|\mathrm{div}\mathbf{w}$ and integrate by parts over $\mathbb{R}^3$, and one has

$$\begin{aligned}&\Big(\frac{1}{3}\|\rho^{1/3}\mathrm{div}\mathbf{w}\|_{L^3}^3\Big)_t+4\beta\|\mathrm{div}\mathbf{w}\|_{L^3}^3+(2\eta'+\kappa')\|(|\mathrm{div}\mathbf{w}|^{1/2}\nabla\mathrm{div}\mathbf{w},|\mathrm{div}\mathbf{w}|^{1/2}\nabla|\mathrm{div}\mathbf{w}|)\|_{L^2}^2\\&=-\int(|\mathrm{div}\mathbf{w}|\mathrm{div}\mathbf{w})\,(\mathbf{w}_t\cdot\nabla\rho)\,dx-\int(|\mathrm{div}\mathbf{w}|\mathrm{div}\mathbf{w})\,\partial_k(\rho u^m)\partial_m w^k dx\triangleq N_1+N_2.\end{aligned}\tag{2.90}$$

The inequalitis $(2.1)_2$ and (2.60) give

$$\begin{aligned}N_1\leqslant&\frac{3(2\eta'+\kappa')}{8}\||\mathrm{div}\mathbf{w}|^{1/2}\nabla\mathrm{div}\mathbf{w}\|_{L^2}^2+C\Big\|\nabla\mathbf{w}\Big\|_{L^{\frac{9r}{4r-9}}}^3\\&+C\|(\nabla^2\mathbf{w},\nabla^2\mathbf{u})\|_{L^2}^2\left(\|\mathrm{div}\mathbf{u}\|_{L^3}^2+1\right),\quad 3<r<6.\end{aligned}\tag{2.91}$$

Similar to the estimation of (2.67), we get

$$N_2\leqslant\frac{2\eta'+\kappa'}{8}\||\mathrm{div}\mathbf{w}|^{1/2}\nabla\mathrm{div}\mathbf{w}\|_{L^2}^2+C\|(\nabla^2\mathbf{u},\nabla^2\mathbf{w})\|_{L^2}^2\left(\|\mathrm{div}\mathbf{w}\|_{L^3}^2+1\right).\tag{2.92}$$

Substituting (2.91) and (2.92) into (2.90), we have

$$\begin{aligned}&\left(\|\rho^{1/3}\mathrm{div}\mathbf{w}\|_{L^3}^3\right)_t+\||\mathrm{div}\mathbf{w}|^{1/2}\nabla\mathrm{div}\mathbf{w}\|_{L^2}^2+\|\mathrm{div}\mathbf{w}\|_{L^3}^3\\&\leqslant C\left(\|\nabla^2\mathbf{w}\|_{L^2}^2+\|\nabla^2\mathbf{u}\|_{L^2}^2\right)\left(1+\|\mathrm{div}\mathbf{w}\|_{L^3}^2\right)+C\Big\|\nabla\mathbf{w}\Big\|_{L^{\frac{9p}{4p-9}}}^3.\end{aligned}\tag{2.93}$$

Multiplying (2.89) by $|\mathrm{rot}\mathbf{w}|\mathrm{rot}\mathbf{w}$ and integrating by parts over $\mathbb{R}^3$, we deduce

$$\begin{aligned}&\Big(\frac{1}{3}\|\rho^{1/3}\mathrm{rot}\mathbf{w}\|_{L^3}^3\Big)_t+\eta'\|(|\mathrm{rot}\mathbf{w}|^{1/2}\nabla\mathrm{rot}\mathbf{w},|\mathrm{rot}\mathbf{w}|^{1/2}\nabla|\mathrm{rot}\mathbf{w}|)\|_{L^2}^2+4\beta\|\mathrm{rot}\mathbf{w}\|_{L^3}^3\\&=\int\Big[2\beta\mathrm{rot}(\mathrm{rot}\mathbf{u})-\nabla\rho\times\mathbf{w}_t-\nabla(\rho u^k)\times(\partial_k\mathbf{w})\Big]\cdot(|\mathrm{rot}\mathbf{w}|\mathrm{rot}\mathbf{w})dx.\end{aligned}$$

Hence

$$\begin{aligned}&\left(\|\rho^{1/3}\mathrm{rot}\mathbf{w}\|_{L^3}^3\right)_t+\||\mathrm{rot}\mathbf{w}|^{1/2}\nabla\mathrm{rot}\mathbf{w}\|_{L^2}^2+\|\mathrm{rot}\mathbf{w}\|_{L^3}^3\\&\leqslant C\left(\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\nabla^2\mathbf{w}\|_{L^2}^2\right)\left(1+\|\mathrm{rot}\mathbf{w}\|_{L^3}^2\right)+C\Big\|\nabla\mathbf{w}\Big\|_{L^{\frac{9p}{4p-9}}}^3.\end{aligned}\tag{2.94}$$

Similar to the estimation of (2.85), we can get (2.87) from (2.93) and (2.94). □

3. Proof of Theorem 1.1

In this section, we mainly focus on proving in Theorem 1.1 holds, based on the global a priori estimates that have been obtained in Section 2. Actually, the global existence can be established by modifying the method in [6]. In the next step, by modifying the ideas of [19], we will prove the uniqueness of solutions holds.

Proof of uniqueness. Let $(\rho_1,\mathbf{u}_1,\mathbf{w}_1,\mathbf{b}_1)$ and $(\rho_2,\mathbf{u}_2,\mathbf{w}_2,\mathbf{b}_2)$ be two solutions to the system (2.1), (1.2) and (1.3) on $\mathbb{R}^3\times[0,T]$ that satisfy (1.13) and have the same initial data. Define

$$\varphi\triangleq\rho_1-\rho_2,\quad \mathbf{v}\triangleq\mathbf{u}_1-\mathbf{u}_2,\quad \varpi\triangleq\mathbf{w}_1-\mathbf{w}_2,\quad \phi\triangleq\mathbf{b}_1-\mathbf{b}_2,$$

then it follows from $(2.1)_1$ that

$$\varphi_t + \mathbf{u}_2 \cdot \nabla\varphi + \varphi \mathrm{div}\mathbf{u}_2 + \rho_1 \mathrm{div}\mathbf{v} + \mathbf{v} \cdot \nabla\rho_1 = 0. \tag{3.1}$$

Multiplying (3.1) by φ, one can deduce from integration by parts that

$$\begin{aligned} \frac{d}{dt}\|\varphi\|_{L^2}^2 &\leqslant C\|\mathrm{div}\mathbf{u}_2\|_{L^\infty}\|\varphi\|_{L^2}^2 + C\left(\|\nabla\mathbf{v}\|_{L^2} + \|\mathbf{v}\|_{L^6}\|\nabla\rho_1\|_{L^3}\right)\|\varphi\|_{L^2} \\ &\leqslant C\|\mathrm{div}\mathbf{u}_2\|_{L^\infty}\|\varphi\|_{L^2}^2 + C\|\nabla\mathbf{v}\|_{L^2}\|\varphi\|_{L^2}. \end{aligned} \tag{3.2}$$

Due to (1.13), we know that $\|\mathrm{div}\mathbf{u}_2\|_{L^\infty} \in L^1(0,T)$. Thus, the inequality (3.2) yields

$$\|\varphi(t)\|_{L^2} \leqslant C\int_0^t \|\nabla\mathbf{v}\|_{L^2} ds \leqslant Ct^{1/2}\left(\int_0^t \|\nabla\mathbf{v}\|_{L^2}^2 ds\right)^{1/2}, \quad \forall\, t \in [0,T]. \tag{3.3}$$

Since it holds that $\dot{\mathbf{u}}_2 = \mathbf{u}_{2t} + \mathbf{u}_2 \cdot \nabla\mathbf{u}_2$, we get

$$\begin{aligned} &\rho_1\mathbf{v}_t + \rho_1\mathbf{u}_1 \cdot \nabla\mathbf{v} - (\eta+\beta)\Delta\mathbf{v} - (\eta+\kappa-\beta)\nabla\mathrm{div}\mathbf{v} \\ &= -\varphi\dot{\mathbf{u}}_2 - \rho_1\mathbf{v}\cdot\nabla\mathbf{u}_2 - \nabla\left(P(\rho_1) - P(\rho_2)\right) + 2\beta\mathrm{rot}\varpi + \mathbf{b}_1 \cdot \nabla\phi - \frac{1}{2}\nabla(|\mathbf{b}_1|^2 - |\mathbf{b}_2|^2). \end{aligned} \tag{3.4}$$

Multiplying (3.4) by $\mathbf{v}$ and integrating by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\rho_1^{1/2}\mathbf{v}\|_{L^2}^2 + (\eta+\beta)\|\nabla\mathbf{v}\|_{L^2}^2 + (\eta+\kappa-\beta)\|\mathrm{div}\mathbf{v}\|_{L^2}^2 \\ &\quad\leqslant C\|\varphi\|_{L^2}\|\dot{\mathbf{u}}_2\|_{L^3}\|\mathbf{v}\|_{L^6} + C\|\mathbf{v}\|_{L^2}\|\nabla\mathbf{u}_2\|_{L^3}\|\mathbf{v}\|_{L^6} \\ &\qquad + C\|\varphi\|_{L^2}\|\nabla\mathbf{v}\|_{L^2} + C\|\varpi\|_{L^2}\|\nabla\mathbf{v}\|_{L^2} \\ &\qquad + C\left(\|\mathbf{b}_1\|_{L^\infty} + \|\mathbf{b}_2\|_{L^\infty}\right)\|\phi\|_{L^2}\|\nabla\mathbf{v}\|_{L^2} \\ &\leqslant \frac{\eta+\beta}{2}\|\nabla\mathbf{v}\|_{L^2}^2 + C\left(1 + \|\dot{\mathbf{u}}_2\|_{L^3}^2\right)\|\varphi\|_{L^2}^2 \\ &\quad + C\left(1 + \|\mathbf{b}_1\|_{L^\infty}^2 + \|\mathbf{b}_2\|_{L^\infty}^2 + \|\nabla\mathbf{u}_2\|_{L^3}^2\right)\left(\|\mathbf{v}\|_{L^2}^2 + \|\varpi\|_{L^2}^2 + \|\phi\|_{L^2}^2\right). \end{aligned}$$

Thus, the inequality (1.13), together with (2.50) and (2.56), leads to

$$\frac{d}{dt}\|\rho_1^{1/2}\mathbf{v}\|_{L^2}^2 + \|\nabla\mathbf{v}\|_{L^2}^2 \leqslant C\left(1 + \|\dot{\mathbf{u}}_2\|_{L^3}^2\right)\|\varphi\|_{L^2}^2 + C\left(1 + \|\rho_1^{1/2}\mathbf{v}\|_{L^2}^2 + \|\rho_1^{1/2}\varpi\|_{L^2}^2 + \|\phi\|_{L^2}^2\right). \tag{3.5}$$

On the other hand, the inequality $(2.1)_3$ gives

$$\rho_1\varpi_t + \rho_1\mathbf{u}_1 \cdot \nabla\varpi - \eta'\Delta\varpi - (\eta'+\kappa')\nabla\mathrm{div}\varpi = -\varphi\dot{\mathbf{w}}_2 - \rho_1\varpi\cdot\nabla\mathbf{w}_2 - 4\beta\varpi + 2\beta\mathrm{rot}\mathbf{v}. \tag{3.6}$$

Multiply (3.6) by ϖ and integrate by parts, and we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\rho_1^{1/2}\varpi\|_{L^2}^2 + \eta'\|\nabla\varpi\|_{L^2}^2 + (\eta'+\kappa')\|\mathrm{div}\varpi\|_{L^2}^2 \\ &\leqslant C\|\varphi\|_{L^2}\|\dot{\mathbf{w}}_2\|_{L^3}\|\varpi\|_{L^6} + C\|\mathbf{v}\|_{L^2}\|\nabla\mathbf{w}_2\|_{L^3}\|\varpi\|_{L^6} + C\|\mathbf{v}\|_{L^2}\|\nabla\varpi\|_{L^2} \\ &\leqslant \frac{\eta'}{2}\|\nabla\varpi\|_{L^2}^2 + C\|\dot{\mathbf{w}}_2\|_{L^3}^2\|\varphi\|_{L^2}^2 + C\left(1 + \|\nabla\mathbf{w}_2\|_{L^3}^2\right)\|\mathbf{v}\|_{L^2}^2, \end{aligned}$$

thus

$$\frac{d}{dt}\|\rho_1^{1/2}\varpi\|_{L^2}^2+\|\nabla\varpi\|_{L^2}^2\leqslant C\|\dot{\mathbf{w}}_2\|_{L^3}^2\|\varphi\|_{L^2}^2+C\left(1+\|\nabla\mathbf{w}_2\|_{L^3}^2\right)\|\mathbf{v}\|_{L^2}^2. \tag{3.7}$$

Note that

$$\phi_t-\alpha\Delta\phi=-\mathbf{u}_1\cdot\nabla\phi-\mathbf{v}\cdot\nabla\mathbf{b}_2+\phi\cdot\nabla\mathbf{u}_1+\mathbf{b}_2\cdot\nabla\mathbf{v}-\phi\mathrm{div}\mathbf{u}_1-\mathbf{b}_2\mathrm{div}\mathbf{v}. \tag{3.8}$$

Multiply (3.8) by ϕ, and one can get form integration by parts that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\phi\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2=&\int(-\mathbf{u}_1\cdot\nabla\phi-\mathbf{v}\cdot\nabla\mathbf{b}_2+\phi\cdot\nabla\mathbf{u}_1+\mathbf{b}_2\cdot\nabla\mathbf{v}\\&-\phi\mathrm{div}\mathbf{u}_1-\mathbf{b}_2\mathrm{div}\mathbf{v})\cdot\phi\,dx\triangleq\sum_{i=1}^{6}J_i.\end{aligned} \tag{3.9}$$

We deduce from (2.56) that

$$J_1=-\frac{1}{2}\int\mathbf{u}_1\cdot\nabla(|\phi|^2)dx=\frac{1}{2}\int\mathrm{div}\mathbf{u}_1|\phi|^2dx\leqslant\frac{\alpha}{8}\|\nabla\phi\|_{L^2}^2+\|\phi\|_{L^2}^2, \tag{3.10}$$

$$J_2\leqslant C\|\mathbf{v}\|_{L^2}\|\nabla\mathbf{b}_2\|_{L^3}\|\phi\|_{L^6}\leqslant\frac{\alpha}{8}\|\nabla\phi\|_{L^2}^2+C\left(1+\|\nabla^2\mathbf{b}_2\|_{L^2}^2\right)\|\sqrt{\rho_1}\mathbf{v}\|_{L^2}^2, \tag{3.11}$$

and

$$J_3+J_4+J_5\leqslant\frac{\alpha}{4}\|\nabla\phi\|_{L^2}^2+C\left(1+\|\nabla^2\mathbf{b}_2\|_{L^2}^2\right)\|\sqrt{\rho_1}\mathbf{v}\|_{L^2}^2+\|\phi\|_{L^2}^2. \tag{3.12}$$

We have from (1.13) that

$$\begin{aligned}J_6&\leqslant C\|\mathbf{b}_2\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^2}\|\phi\|_{L^2}\leqslant C_1\|\nabla\mathbf{v}\|_{L^2}^2+C\|\mathbf{b}_2\|_{L^6}\|\nabla\mathbf{b}_2\|_{L^6}\|\phi\|_{L^2}^2\\&\leqslant C_1\|\nabla\mathbf{v}\|_{L^2}^2+C\left(1+\|\nabla^2\mathbf{b}_2\|_{L^2}^2\right)\|\phi\|_{L^2}^2.\end{aligned} \tag{3.13}$$

Taking (3.10)–(3.13) into (3.9), one has

$$\frac{d}{dt}\|\phi\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\leqslant C_1\|\nabla\mathbf{v}\|_{L^2}^2+C\left(1+\|\nabla^2\mathbf{b}_2\|_{L^2}^2\right)\left(\|\sqrt{\rho_1}\mathbf{v}\|_{L^2}^2+\|\phi\|_{L^2}^2\right),$$

which, combined with (3.3), (3.5), and (3.7), gives

$$\begin{aligned}&\left(\|\rho_1^{1/2}\mathbf{v}\|_{L^2}^2+\|\rho_1^{1/2}\varpi\|_{L^2}^2+\|\phi\|_{L^2}^2\right)_t+\|\nabla\mathbf{v}\|_{L^2}^2+\|\nabla\varpi\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\\&\leqslant C\left(\|\rho_1^{1/2}\mathbf{v}\|_{L^2}^2+\|\rho_1^{1/2}\varpi\|_{L^2}^2+\|\phi\|_{L^2}^2\right)+Ct\left(1+\|\dot{\mathbf{u}}_2\|_{L^3}^2+\|\dot{\mathbf{w}}_2\|_{L^3}^2\right)\left(\int_0^t\|\nabla\mathbf{v}\|_{L^2}^2ds\right).\end{aligned} \tag{3.14}$$

Let

$$\Phi(t)\triangleq\left(\|\rho_1^{1/2}\mathbf{v}\|_{L^2}^2+\|\rho_1^{1/2}\varpi\|_{L^2}^2+\|\phi\|_{L^2}^2\right)+\int_0^t\left(\|\nabla\mathbf{v}\|_{L^2}^2+\|\nabla\varpi\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right)ds,$$

then, the inequality (3.14) gives

$$\ell'(t)\leqslant\psi(t)\ell(t)\quad\text{with}\quad\ell(0)=0, \tag{3.15}$$

where

$$\psi(t) \triangleq Ct\Big(1 + \|(\dot{\mathbf{u}}_2, \dot{\mathbf{w}}_2)\|_{L^3}^2\Big) + C.$$

We know from (1.13) that $\psi(t) \in L^1(0,T)$; thus, the inequality (3.15), combined with Gronwall's inequality, gives

$$\Big(\|\mathbf{v}\|_{L^2}^2 + \|\varpi\|_{L^2}^2 + \|\phi\|_{L^2}^2\Big) + \int_0^T \Big(\|\nabla\mathbf{v}\|_{L^2}^2 + \|\nabla\varpi\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2\Big)\, dt = 0, \quad \forall\, t \in [0,T],$$

so

$$\mathbf{v}(x,t) = 0, \quad \varpi(x,t) = 0, \quad \phi(x,t) = 0, \quad \text{a.e. on } \mathbb{R}^3 \times [0,T],$$

which, combined with (3.3), gives

$$\varphi(x,t) = 0, \quad \text{a.e. on } \mathbb{R}^3 \times [0,T].$$

Thus, Theorem 1.1 is proved. □

4. Conclusions

From the discussion in the previous sections, we have concluded that the three-dimensional compressible magneto-micropolar fluid system (1.1) possesses a global and unique solution in $\mathbb{R}^3$, as follows:

For $s \in [9/2, 6)$, assume that $(\rho_0, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)$ satisfies

$$\inf \rho_0(x) > 0, \quad \rho_0 - 1 \in H^1 \cap W^{1,s}, \quad (\mathbf{u}_0, \mathbf{w}_0) \in H^1 \cap W^{1,3}, \quad \mathbf{b}_0 \in H^1.$$

There exists a constant $\varepsilon > 0$, depending on $\eta, \kappa, \beta, \eta', \kappa', \gamma, \alpha, A, \inf\rho_0, \sup\rho_0, \|\nabla\mathbf{u}_0\|_{L^2}, \|\nabla\mathbf{w}_0\|_{L^2}$, and $\|\nabla\mathbf{b}_0\|_{L^2}$, such that if

$$\mathcal{S}_0 \triangleq \|(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_{L^2}^2 \leqslant \varepsilon,$$

the systems (1.1)–(1.3) possess a global uniqueness solution $(\rho, \mathbf{u}, \mathbf{w}, \mathbf{b})$ in $\mathbb{R}^3 \times (0, \infty)$ satisfying

$$\begin{cases} \rho - 1 \in C([0,T]; H^1 \cap W^{1,s}), \quad \inf \rho(x,t) > 0, \\ (\mathbf{u},\ \mathbf{w}, \mathbf{b}) \in C([0,T]; L^2 \cap L^a) \quad 2 \leqslant a < 6, \\ (\mathbf{u}, \mathbf{w}) \in L^\infty(0,T; H^1 \cap W^{1,3}) \cap L^2(0,T; H^2) \cap L^\ell(0,T; W^{1,\infty}), \\ \mathbf{b} \in L^\infty([0,T]; H^1) \cap L^2(0,T; H^2), \\ (t^{1/2}\dot{\mathbf{u}}, t^{1/2}\dot{\mathbf{w}}) \in L^\infty(0,T; L^2), \quad (t^{1/2}\nabla\dot{\mathbf{u}}, t^{1/2}\nabla\dot{\mathbf{w}}) \in L^2(0,T; L^2), \\ (t^{1/2}\mathbf{b}_t,\ t^{1/2}\nabla^2\mathbf{b}) \in L^\infty([0,T]; L^2), \quad (t^{1/2}\nabla\mathbf{b}_t,\ t^{1/2}\nabla^3\mathbf{b}) \in L^2([0,T]; L^2), \end{cases}$$

with $1 < \ell < (4s)/(5s-6)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We sincerely thank the Associate Editor and the anonymous referees for their carefully reading and helpful suggestions that led to the improvement of the paper.

M.Y. Zhang was partially supported by the Natural Science Foundation of Shandong Province of China (Grant No. ZR2021QA049) and the Science and Technology Project of Weifang (2022GX006).

Conflict of interest

The author declares no conflict of interest.

References

1. J. T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, *Commun. Math. Phys.*, **94** (1984), 61–66. https://doi.org/10.1007/BF01212349
2. Y. Cho, H. J. Choe, H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, *J. Math. Pures Appl.*, **83** (2004), 243–275. https://doi.org/10.1016/j.matpur.2003.11.004
3. H. Chen, Y. M. Sun, X. Zhong, Global classical solutions to the 3D Cauchy problem of compressible magneto-micropolar fluid equations with far field vacuum, *Discrete Cont Dyn Sys. B*, **29** (2024), 282–318. https://doi.org/10.3934/dcdsb.2023096
4. G. Q. Chen, D. H. Wang, Global solution of nonlinear magnetohydrodynamics with large initial data, *J. Differ. Equ.*, **182** (2002), 344–376. https://doi.org/10.1006/jdeq.2001.4111
5. G. Q. Chen, D. Wang, Existence and continuous dependence of large solutions for the magnetohydrodynamic equations, *Z. Angew. Math. Phys.*, **54** (2003), 608–632. https://doi.org/10.1007/s00033-003-1017-z
6. M. T. Chen, X. Y. Xu, J. W. Zhang, Global weak solutions of 3D compressible micropolar fluids with discontinuous initial data and vacuum, *Commun. Math. Sci.*, **13** (2015), 225–247. https://doi.org/10.4310/CMS.2015.v13.n1.a11
7. M. T. Chen, Blowup criterion for viscous, compressible micropolar fluids with vacuum, *Nonlin. Anal. RWA*, **13** (2012), 850–859. https://doi.org/10.1016/j.nonrwa.2011.08.021
8. M. T. Chen, Unique solvability of compressible micropolar viscous fluids, *Bound. Value Probl.*, **2012** (2012), 232. https://doi.org/10.1186/1687-2770-2012-32
9. B. Ducomet, E. Feireisl, The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars, *Commun. Math. Phys.*, **266** (2006), 595–629. https://doi.org/10.1007/s00220-006-0052-y
10. A. C. Eringen, Theory of micropolar fluids, *J. Math. Mech.*, **16** (1966), 1–18.
11. J. S. Fan, S. Jiang, G. Nakamura, Vanishing shear viscosity limit in the magnetohydrodynamic equations, *Commun. Math. Phys.*, **270** (2007), 691–708. https://doi.org/10.1007/s00220-006-0167-1
12. J. S. Fan, W. H. Yu, Global variational solutions to the compressible magnetohydrodynamic equations, *Nonlinear Anal.*, **69** (2008), 3637–3600. https://doi.org/10.1016/j.na.2007.10.005

13. D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differ. Equ.*, **120** (1995), 215–254. https://doi.org/10.1006/jdeq.1995.1111
14. X. D. Huang, J. Li, Z. P. Xin, Serrin-type criterion for the three-dimensional viscous compressible flows, *SIAM J. Math. Anal.*, **43** (2011), 1872–1886. https://doi.org/10.1137/100814639
15. X. D. Huang, J. Li, Z. P. Xin, Global well-posedness of classical solutions with large oscil-lations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations, *Commun. Pure Appl. Math.*, **65** (2012), 549–585. https://doi.org/10.1002/cpa.21382
16. G. Lukaszewicz, *Micropolar Fluids: Theory and Applications Modeling and Simulation in Science, Engineering and Technology*, Boston: Brikhäuser, 1999. https://doi.org/10.1007/978-1-4612-0614-5
17. L. L. Tong, Z. Tan, Optimal decay rates of the compressible magneto-micropolar fluids system in $\mathbb{R}^3$, *Commun. Math. Sci.*, **17** (2019), 1109–1134. https://doi.org/10.4310/CMS.2019.v17.n4.a13
18. R. Y. Wei, B. L. Guo, Y. Li, Global existence and optimal convergence rates of solutions for 3D compressible magneto-micropolar fluid equations, *J. Differ. Equ.*, **263** (2017), 2457–2480. https://doi.org/10.1016/j.jde.2017.04.002
19. H. Xu, J. W. Zhang, Regularity and uniqueness for the compressible full Navier-Stokes equations, *J. Differ. Equ.*, **272** (2021), 46–73. https://doi.org/10.1016/j.jde.2020.09.036
20. Q. Xu, X. Zhong, Strong solutions to the three-dimensional barotropic compressible magneto-micropolar fluid equations with vacuum, *Z. Angew. Math. Phys.*, **73** (2022), 14. https://doi.org/10.1007/s00033-021-01642-3
21. Q. J. Xu, Z. Tan, H. Q. Wang, Global existence and asymptotic behavior for the 3D compressible magneto-micropolar fluids in a bounded domain, *J. Math. Phys.*, **61** (2020), 011506. https://doi.org/10.1063/1.5121247
22. Q. J. Xu, Z. Tan, H. Q. Wang, L. L. Tong, Global low-energy weak solutions of the compressible magneto-micropolar fluids in the half-space, *Z. Angew. Math. Phys.*, **73** (2022), 223. https://doi.org/10.1007/s00033-022-01860-3
23. P. X. Zhang, Decay of the compressible magneto-micropolar fluids, *J. Math. Phys.*, **59** (2018), 023102. https://doi.org/10.1063/1.5024795
24. X. L. Zhang, H. Cai, Existence and uniqueness of time periodic solutions to the compressible magneto-micropolar fluids in a periodic domain, *Z. Angew. Math. Phys.*, **71** (2020), 184. https://doi.org/10.1007/s00033-020-01409-2
25. J. W. Zhang, S. Jiang, F. Xie, Global weak solutions pf an initial boundary value problem for screw pinches in plasma physics, *Math. Model. Meth. Appl. Sci.*, **19** (2009), 833–875. https://doi.org/10.1142/S0218202509003644
26. J. W. Zhang, J. N. Zhao, Some decay estimates of solutions for the 3-D compressible isentropic magnetohydrodynamics, *Commun. Math. Sci.*, **8** (2010), 835–850. https://projecteuclid.org/euclid.cms/1288725260

AIMS Press