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Research article

Fixed/Prescribed stability criterions of stochastic system with time-delay

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Abstract: In this paper, the fixed/prescribed-time stability issues were considered for stochastic systems with time delay. First, some new fixed-time stability and prescribed-time stability criteria for stochastic systems with delay and multi-delay were established. Second, based on the new fixed/prescribed stability criteria, the fixed-time stabilization of the stochastic system with time-delay and the prescribed-time stabilization of the stochastic reaction-diffusion system with multi-delay were investigated, respectively. Third, two new fixed/prescribed-time delay-independent control mechanisms were designed. The primary advantage of the innovative fixed/prescribed-time controller lies in its independence from delayed states. This makes the controller applicable to systems with unknown delays. Finally, three numerical examples were provided to illustrate the feasibility of the stated theoretical results.

Keywords: fixed-time stability; prescribed-time stability; stochastic system; time-delay
Mathematics Subject Classification: 93D05, 93E03

1. Introduction

Stochastic nonlinear systems, as a distinctive type of nonlinear systems, have garnered substantial interest from researchers across several prominent domains over the past decades, such as biology, finance, and engineering; see [1, 2]. Compared to non-stochastic systems, stochastic differential equations can provide a more accurate description of the dynamics of practical systems that are subject to environmental noise and uncertain disturbances. Very recently, there has been a significant amount of fruitful and excellent research focusing on the application of stochastic nonlinear systems in the literature; see [3–5]. In [3], Li et al. designed a feedback controller for discussing the prescribed-time stability problem of a stochastic strict-feedback nonlinear system. In [4], by using the stochastic analysis technology and inequality method, sufficient conditions are derived to ensure the synchronization of the coupled reaction-diffusion neural networks with delays and multiple weights. In [5], Liu et al. designed a pinning controller and proposed a unified theoretical framework to study the finite/fixed-time synchronization for stochastic complex networks.

In many real systems, including multi-agent systems and complex networks, the presence of time delay is an inevitable phenomenon in the transmission of information among different components of nonlinear systems. This delay is a result of limitations in information transfer or switching speeds and can give rise to chaotic, divergent, oscillatory, and even unstable behaviors; see [6, 7]. Therefore, it is crucial and meaningful to take into account the influence of time delay when investigating stability-related issues. As time goes on, more and more stability analysis tools for the time-delayed systems have emerged [8, 9], and the Halanay inequality-based tool [10] is one of the most appealing. Halanay inequality-based tool was first established by Halanay in 1966 and was successfully explored to investigate the stability of delay stochastic systems, delay impulsive systems, delay complex networks, and so on. Subsequently, this technique has been further developed and extended, leading to the formulation of numerous generalized Halanay inequalities and their applications, as documented in [11–13]. In [11], Li et al. gave improvements on the Halanay inequalities with time-varying coefficients, and the sufficient conditions of stability for time-varying time-delay systems were established via the Lyapunov Razumikhin approach. In [12], based on the stochastic analysis technology, Ruan et al investigated a new type of generalized Halanay inequalities and derived the stability and dissipativity criterion of stochastic differential equations. In [13], Du et al. presented a novel fractional-order finite-time convergence principle, and the finite-time synchronization issue was investigated for a class of fractional-order delayed complex networks. However, the aforementioned discussions have been confined to exponential or finite-time convergence only.

The past few years have witnessed sustained growing interest in finite-time control of stochastic time-varying delay systems, leading to fruitful results [14–16]. However, when the system reaches finite-time stability, the setting time heavily depends on the initial values, which can be challenging to measure accurately due to practical constraints imposed by sensor technology. To address this challenge, the concept of fixed-time control theory was introduced later. The key distinction between fixed-time control and finite-time control lies in the fact that fixed-time control guarantees a maximum settling time, which is independent of the initial values [17]. Because of those benefits, fixed-time control has garnered significant attention in the past decades. As a result, various principles of fixed-time stability have been developed specifically for stochastic nonlinear systems [18] and reaction-diffusion systems [19], solving consensus issues [20], synchronization issues [21], and optimization issues [22]. In [23], the fixed-time stability criterion $\dot{V}(t) \le -aV^{\alpha}(t) - bV^{\beta}(t)$, $\alpha > 1$, $1 > \beta > 0$ was investigated to reach the synchronization problem of complex-valued neural networks. In [24], Hu et al. used the Beta function to give a more accurate estimation for the upper bound of setting time. In [25], the fixed-time stability criterion was extended to the general form $\dot{V}(t) \le -aV^{\alpha}(t) - bV^{\beta}(t) - cV(t)$, $\alpha > 1$, $1 > \beta > 0$ and was used to handle the synchronization issue of discontinuous neural networks with switching mode. In [26], Xu et al. generalized the differential operator of the fixed-time stability criterion into the Itô operator and derived a novel stability criterion concerning the stochastic system. Despite the advantages of fixed-time control in terms of estimating the settling time, there are still two problems that need to be addressed: First, in practical application, because there is no obvious relationship between the setting time and its upper bound. As a result, the settling time under the fixed-time control is often overestimated, leading to an inaccurate depiction of the system's performance. Second, the settling time is not a directly modifiable parameter as it depends on other controller design parameters, making it challenging to optimize and fine-tune for specific system requirements [27]. To address these two problems, the concept of predefined-time control was introduced in [28], where the upper bound of the settling time can be predetermined according to the specific circumstances, and it remains unaffected by the initial values of the system [29, 30]. Additionally, by using the time-varying transformation, the prescribed-time control was presented and has been becoming increasingly popular due to it allowing for presetting the settling time precisely and inheriting the advantages of finite-time control and fixed-time control [31, 32].

The previously mentioned results regarding fixed/prescribed-time stability criteria have a common limitation. They do not apply to address fixed/prescribed-time stability issues in time-delay systems. To the best of our knowledge, there are currently no established fixed/prescribed-time stability criteria specifically designed for time-delay systems in the existing literature. This is our main motivation for composing this manuscript. Compared with the stability analysis of previously mentioned results, the complexity arises primarily from two factors: 1) The commonly used Halanay's inequality fails to achieve fixed-time stability because it can only yield conclusions regarding asymptotic stability. 2) When developing criteria for fixed-time stability, the incorporation of stochastic effects introduces additional complexity.

Drawing inspiration from the preceding discussion, this paper addresses the challenge of stochastic fixed/prescribed-time stability in stochastic time-delay systems, leveraging stochastic analysis techniques and the Lyapunov stability theory. This article presents three main contributions, which are delineated as follows.

1) Some new fixed-time stability and prescribed-time stability criteria for stochastic delay and multi-delay systems are established. In contrast to previous works [18–20], their conclusions are limited to non-delay and non-stochastic systems only. Thus, these fixed-time stability criteria are specific cases in this paper.
   We extend the differential inequality to a more general form $dV(\zeta) \leq [-a(\zeta)V(\zeta) + b(\zeta)\sup_{\zeta-\tau(\zeta)\leq s\leq\zeta} V(s) - f(V(\zeta))]d\zeta V + p(V(\zeta), V(\zeta - \tau(\zeta)))dw$, which offers a fresh perspective for exploring fixed/prescribed stability concerns in the context of stochastic delay systems.
2) The sufficient conditions of fixed-time stabilization for the stochastic time-delay system and the prescribed-time stabilization for the multi-delay stochastic reaction-diffusion system are given with the help of the Lyapunov functional theory and the stochastic analysis techniques.
3) Two novel fixed/prescribed-time controllers are proposed in this paper. Compared with some previous works [33–35], these controllers require information with delayed states. However, the controller designed in this paper is independent of delayed states. For situations where only the upper bound of the unknown delay is known, the control proposed in this paper remains effective.

The remainder of this paper is structured as follows. Section 2 introduces essential lemmas, definitions, and stochastic models. Section 3 addresses fixed/prescribed-time stability concerns for stochastic delay systems. Section 4 focuses on the fixed-time stabilization of a stochastic time-delay system and the prescribed-time stabilization of a multi-delay stochastic reaction-diffusion system. In Section 5, three numerical examples are presented to validate the theoretical findings. Lastly, Section 6 provides the concluding remarks of this article.

2. Preliminaries

2.1. Notations

See Table 1.

Table 1. Notations.

Symbol	Stand for
R	Real numbers set
R^+	Positive real numbers set
Z^+	Set of positive integers
$\mathcal{E}$	Mathematical expectation
$a \wedge b$	$\min\{a, b\}$
$B(\cdot, \cdot)$	Beta function
$\lambda_{min}(\cdot)$	The minimum eigenvalue of matrix
$\lambda_{max}(\cdot)$	The maximum eigenvalue of matrix
$\circ$	Hadamard product of matrices
$O(a(s)) = b(s)$	$\lim_{s\to\infty} \frac{a(s)}{b(s)} = c < +\infty$
$PC^b_{\mathcal{F}_\zeta}$	The family of all $\mathcal{F}_\zeta$ measurable function

2.2. Some definitions and lemmas

Definition 2.1. *[32] (**Incomplete beta function**). The the incomplete beta function is defined as*

$$I(\lambda, x, y) = \frac{1}{B(x, y)} \int_0^\lambda t^{x-1}(1 - t)^{y-1} dt,$$

where $\lambda \in [0, 1]$, x,$y > 0$. $B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt$.

Definition 2.2. *(Ω **type function**):*
If $\Omega : R \to R^+$ satisfies the following properties

1) $\Omega(x)$ is a monotonically increasing function,
2) The improper integral $\int_0^{+\infty} \frac{dz}{\Omega(z)} \leq +\infty$,

then we said $\Omega(x)$ is a Ω type function.

Remark 1. *Ω type function exists, for example, $\Omega(x) = \frac{\Gamma(1-\frac{1}{p})}{p} e^{x^p} x^{2-p}$ belongs to the Ω type function, where $(0 < p \leq 1)$. We notice that $\Omega(x) = \frac{\Gamma(1-\frac{1}{p})}{p} e^{x^p} x^{2-p}$ is a monotonically increasing function, and $\int_0^{+\infty} p z^{p-2} e^{-z^p} dx \overset{x^p=t}{=} \int_0^{+\infty} t^{(1-\frac{1}{p})-1} e^{-t} dt = \Gamma(1 - \frac{1}{p})$. It is worth pointing out $\Omega(x) = ax^\alpha + bx^\beta$, $a, b > 0$, $\alpha > 1$, and $0 < \beta < 1$; $\Omega(x) = (ax^\alpha + bx^\beta)^\gamma$, $a, b > 0$, $\alpha\gamma > 1$, and $0 < \beta\gamma < 1$ are also some commonly used Ω type functions.*

Definition 2.3. *(**Time-varying scaling function**) The time-varying scaling function $\Gamma(\zeta)$ is defined as*

$$\Gamma(\zeta) = \begin{cases} \sec^{\rho}(\frac{\pi\zeta}{2(\zeta_0+T)}), & \zeta \in [\zeta_0, \zeta_0 + T), \\ 0, & \zeta \in [\zeta_0 + T, \infty], \end{cases}$$

where T, ρ, and ζ_0 are positive parameters to be designed. $\Gamma(\zeta)$ plays a regulating role in the prescribed-time control. It is easy to see that $\Gamma(\zeta)$ is monotonically increasing on $[\zeta_0, \zeta_0 + T]$, and $\Gamma(\zeta_0) = \sec^{\rho}(\frac{\pi\zeta_0}{2(\zeta_0+T)})$ and $\lim_{\zeta\to\zeta_0+T}\Gamma(\zeta) = +\infty$. Moreover,

$$\dot{\Gamma}(\zeta) = \begin{cases} \frac{\pi\rho}{2(\zeta_0+T)}\sec^{\rho}(\frac{\pi\zeta}{2(\zeta_0+T)})\tan(\frac{\pi\zeta}{2(\zeta_0+T)}), & \zeta \in [\zeta_0, \zeta_0 + T), \\ 0, & \zeta \in [\zeta_0 + T, \infty]. \end{cases}$$

Consider the following stochastic system:

$$dx(\zeta) = f(\zeta, x(\zeta))d\zeta + g(\zeta, x(\zeta))dw, \tag{2.1}$$

where $x(\zeta) \in R^n$ is the system state at time ζ. Stochastic nonlinear system (2.1) is defined on $\zeta \geq 0$ with initial value $x_0 \in C_{F_0}(R^n)$, and $C_{F_0}(R^n)$ is the family of all F_0-measurable bounded $C(R^n)$-valued random variables. f and $g : R^n \to R^n$, are nonlinear functions; $w(\zeta)$ is n-dimensional wiener process in complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\zeta\}_{\zeta\geq 0}, P)$. Denote Itô operator by $\mathcal{L}$. Suppose that $V(\zeta, x) \in C^{1,2}([\zeta_0, \zeta_0 + T) \times R^n; R^+)$ is a locally Lipschitz continuous function, and $x(\zeta)$ is the state at time ζ of stochastic nonlinear system (2.1), then

$$\begin{aligned}\mathcal{L}V(\zeta, x(\zeta)) =& V_\zeta(\zeta, x(\zeta)) + V_x(\zeta, x(\zeta))f(\zeta, x(\zeta)) \\ &+ \frac{1}{2}tr\{g^T V_{xx}(\zeta, x(\zeta))g\},\end{aligned}$$

where $V_x(\zeta, x(\zeta)) = (\frac{\partial V(\zeta,x(\zeta))}{\partial x_1}, \cdots, \frac{\partial V(\zeta,x(\zeta))}{\partial x_n})_{1\times n}$, $V_\zeta(\zeta, x(\zeta)) = \frac{\partial V(\zeta,x(\zeta))}{\partial \zeta}$, and $V_{xx}(\zeta, x(\zeta)) = (\frac{\partial^2 V(\zeta,x(\zeta))}{\partial x_i \partial x_j})_{n\times n}$.

Definition 2.4. *[36] (**Finite-time stability in probability**). The trivial solution $x(\zeta, x_0) = 0$ of stochastic nonlinear system (2.1) is finite-time stability in probability, if the solution $x(\zeta, x_0)$ exists for any $x_0 \in C_{F_0}(R^n)$ and $x(\zeta, x_0)$ is finite-time attractiveness in probability. That is to say, the stochastic setting time $T(x_0) = \inf\{\zeta|\ x(\zeta, x_0) = 0\}$ is finite a.s. and for $\forall\varepsilon > 0$, $\exists\eta > 0$, such that $P\{|x(\zeta, x_0)| < \varepsilon, \forall\zeta > 0\} \geq 1 - \eta$.*

Definition 2.5. *[37] (**Fixed-time stability in probability**). If stochastic nonlinear system (2.1) is finite-time stability in probability and $T(x_0)$ is bounded, i.e., there exists a constant $T_{max} > 0$ such that $T(x_0) < T_{max}$ for any $x_0 \in C_{F_0}(R^n)$, then stochastic nonlinear system (2.1) is said to be fixed-time stability in probability.*

Definition 2.6. *[38] (**Prescribed-time quasi-stability in probability**). The stochastic nonlinear system (2.1) is said to be prescribed-time quasi-stability in probability with error bound $\varepsilon > 0$ if for the prescribed constant $\zeta_0 + T$ and any $x_0 \in C_{F_0}(R^n)$, there exists a compact set M such that $\lim\limits_{\zeta\to\zeta_0+T} \mathcal{E}x(\zeta, x_0)$ converges into the set $M = \{\mathcal{E}x(\zeta, x_0) \mid \|\mathcal{E}x(\zeta, x_0)\| < \varepsilon\}$, and $\mathcal{E}x(\zeta) \in M$, a.s. when $\zeta \geq \zeta_0 + T$.*

Lemma 2.1. *Suppose that $V : R^n \to R$ is a positive-definite, radially unbounded, and differentiable function. $x(\zeta)$ is the state at time ζ of system (2.1). If there exists Ω type function $\Omega(z)$, such that*

$$\mathcal{L}V(x(\zeta)) \leq -\Omega(V(x(\zeta))),$$

then the stochastic system (2.1) is fixed-time stable, and the setting time $T(x_0) \leq \Upsilon = \int_0^{+\infty} \frac{dz}{\Omega(z)}$.

Proof. Define a positive definite function $\Psi(V(x(\zeta)))$ as follows:

$$\Psi(V(x(\zeta))) = \int_0^{V(x(\zeta))} \frac{ds}{\Omega(s)}. \tag{2.2}$$

Define the stopping time as $\zeta_\varepsilon = \inf\{\zeta \geq 0 : |x(\zeta, x_0)| \leq \varepsilon\}$. In the light of Itô's formula, we have

$$\begin{aligned} \mathcal{E}\Psi(V(x(\zeta_\varepsilon \wedge \zeta))) =& \mathcal{E}\Psi(V(x(\zeta_0))) + \int_{\zeta_0}^{\zeta_\varepsilon \wedge \zeta} \mathcal{L}\Psi(V(x(\zeta)))d\zeta \\ &+ \int_{\zeta_0}^{\zeta_\varepsilon \wedge \zeta} \mathcal{H}\Psi(V(x(\zeta)))dw, \end{aligned} \tag{2.3}$$

where $\mathcal{H}\Psi(V(x(\zeta))) = \frac{V_x(x(\zeta))}{\Omega(Vx(\zeta))} g(\zeta, x(\zeta))$. Taking the exception on both sides of (2.3) and noticing $\int_{\zeta_0}^{\zeta} \mathcal{H}\Psi(V(x(\zeta)))dw(\zeta)$ is a square integrable martingale of zero mean, i.e., $\mathcal{E}\int_{\zeta_0}^{\zeta} \mathcal{H}\Psi(V(x(\zeta)))dw(\zeta) = 0$, one obtains

$$\mathcal{E}\Psi(V(x(\zeta_\varepsilon \wedge \zeta))) = \mathcal{E}\Psi(V(x(\zeta_0))) + \mathcal{E}\int_{\zeta_0}^{\zeta_\varepsilon \wedge \zeta} \mathcal{L}\Psi(V(x(\zeta)))d\zeta. \tag{2.4}$$

Based on $\left(\frac{V_x(x(\zeta))}{\Omega(V(x(\zeta)))}\right)' = \left(\frac{V_{xx}(x(\zeta))}{\Omega(V(x(\zeta)))} - \frac{\dot{\Omega}(V(x(\zeta)))}{\Omega^2(V(x(\zeta)))} V_x^2(x(\zeta))\right)$, we have

$$\begin{aligned} \mathcal{L}\Psi(V(x(\zeta))) =& \frac{V_x(x(\zeta))}{\Omega(V(x(\zeta)))} f(\zeta, x(\zeta)) + \\ & \frac{1}{2} tr\left\{\left(\frac{V_{xx}(x(\zeta))}{\Omega(V(x(\zeta)))} - \frac{\dot{\Omega}(V(x(\zeta)))}{\Omega^2(V(x(\zeta)))} V_x^2(x(\zeta))\right) \times g^T(\zeta, x(\zeta)) g(\zeta, x(\zeta))\right\} \\ =& \frac{1}{\Omega(V(x(\zeta)))} \left\{V_x(x(\zeta)) f(\zeta, x(\zeta)) + \frac{1}{2} tr\left\{g^T(\zeta, x(\zeta)) V_{xx}(x(\zeta)) g(\zeta, x(\zeta))\right\}\right\} \\ & - \frac{1}{2} tr\left\{\frac{\dot{\Omega}(V(x(\zeta)))}{\Omega^2(V(x(\zeta)))} V_x^2((x(\zeta))) g^T(\zeta, x(\zeta)) g(\zeta, x(\zeta))\right\} \\ =& \frac{\mathcal{L}V((x(\zeta)))}{\Omega(V(x(\zeta)))} - \frac{1}{2} tr\left\{\frac{\dot{\Omega}(V(x(\zeta)))}{\Omega^2(V(x(\zeta)))} V_x^2((x(\zeta))) g^T(\zeta, x(\zeta)) g(\zeta, x(\zeta))\right\}. \end{aligned}$$

Based on $\Omega(x)$ being a monotone increasing function, i.e., $\dot{\Omega}(V(x(\zeta))) \geq 0$,

$$\frac{1}{2} tr\left\{\frac{\dot{\Omega}(V(x(\zeta)))}{\Omega^2(V(x(\zeta)))} V_x^2((x(\zeta))) g^T(\zeta, x(\zeta)) g(\zeta, x(\zeta))\right\} > 0.$$

According to $\mathcal{L}V(x(\zeta)) \leq -\Omega(V(x(\zeta)))$, we have

$$\mathcal{L}\Psi(V(x(\zeta))) \leq -1. \tag{2.5}$$

Hence,

$$\mathcal{E}\Psi(V(x(\zeta_\varepsilon \wedge \zeta))) - \mathcal{E}\Psi(V(x(\zeta_0))) = \mathcal{E}\int_{\zeta_0}^{\zeta_\varepsilon \wedge \zeta} \mathcal{L}\Psi(V(x(\zeta)))d\zeta \leq -\mathcal{E}(\zeta_\varepsilon \wedge \zeta - \zeta_0). \tag{2.6}$$

Notice $V(0) = 0$, $\Psi(0) = 0$; thus, for $\forall\varepsilon \to 0$, there exist $\delta \geq 0$, $(\delta \to 0)$ such that $\Psi(V(\varepsilon)) \leq \delta$. Combining with (2.6) yields

$$\begin{aligned} T(x_0) =& \lim_{\varepsilon\to 0} \inf\{\zeta \geq 0 : |x(\zeta; x_0)| \leq \varepsilon\} = \inf\{\zeta \geq 0 | x(\zeta, x_0) = 0\} \\ =& \lim_{\varepsilon\to 0}(\zeta_\varepsilon \wedge \zeta - \zeta_0) \leq \lim_{\varepsilon\to 0}\left[\mathcal{E}\Psi(V(x(\zeta_0))) - \mathcal{E}\Psi(V(x(\zeta_\varepsilon \wedge \zeta)))\right] \\ \leq& \lim_{\varepsilon\to 0}\left[\mathcal{E}\Psi(V(x(\zeta_0))) + \Psi(V(\varepsilon))\right]) \leq \mathcal{E}\left(\int_0^{V(x(\zeta_0))} \frac{ds}{\Omega(s)} + \delta\right) \\ \leq& \mathcal{E}\left(\int_0^{+\infty} \frac{ds}{\Omega(s)} + \delta\right) = \Upsilon + \delta. \end{aligned}$$

When $\varepsilon \to 0$, then $\delta \to 0$. Thus, we have $T(x_0) \leq \Upsilon$, and the stochastic system (2.1) achieves fixed-time stability in probability. The proof is completed.

Corollary 2.1. *Let $V : R^n \to R$ be a positive-definite, radially unbounded, and differentiable function. Consider the state $x(\zeta)$ at time ζ of the system described by Eq (2.1). If there exist constants $\alpha, \beta \geq 0$, $0 < p < 1$, and $q > 1$ such that*

$$\mathcal{L}V(x(\zeta)) \leq -\left(\alpha V^p(x(\zeta)) + \beta V^q(x(\zeta))\right), \tag{2.7}$$

then the stochastic system (2.1) is fixed-time stable, and the settling time $T(x_0) \leq \Upsilon = \left(\frac{\alpha}{\beta}\right)^{\frac{1-p}{q-p}} \frac{\pi}{\sin\left(\frac{1-p}{q-p}\pi\right)\alpha(q-p)}$.

Proof. By virtue of Lemma 2.1, our objective is to confirm that $\Omega(x) = \alpha x^p + \beta x^q$ is a Ω function. We observe that $\Omega(x)$ is monotonically increasing and

$$\begin{aligned} \int_0^{+\infty} \frac{dx}{\alpha x^p + \beta x^q} =& \int_0^{+\infty} \frac{x^{(1-p)-1}dx}{\alpha + \beta x^{q-p}} \overset{a}{=} (\frac{\alpha}{\beta})^{\frac{1-p}{q-p}} \frac{1}{\alpha} \frac{1}{q-p} \int_0^{+\infty} \frac{\zeta^{\frac{1-p}{q-p}-1}}{1+\zeta} d\zeta \\ =& (\frac{\alpha}{\beta})^{\frac{1-p}{q-p}} \frac{1}{\alpha} \frac{1}{q-p} B(\frac{1-p}{q-p}, 1 - \frac{1-p}{q-p}) = (\frac{\alpha}{\beta})^{\frac{1-p}{q-p}} \frac{\pi}{\sin(\frac{1-p}{q-p}\pi)\alpha(q-p)} \leq +\infty. \end{aligned}$$

Equation a is derived from $\beta x^{q-p} = \alpha\zeta$, then this concludes the proof.

3. Main results

In this section, we establish two novel criteria for fixed-time stability in stochastic delay systems and multi-delay systems. Furthermore, leveraging a new time-varying scaling function, we introduce two prescribed-time stability criteria for both stochastic delay systems and multi-delay systems.

Let us consider the subsequent stochastic delay system:

$$\begin{cases} dx(\zeta) = h(x(\zeta), x(\zeta - \tau(\zeta)))d\zeta + g(x(\zeta), x(\zeta - \tau(\zeta)))dw, \zeta \in [\zeta_0, \infty) \\ x(\zeta) = \mathbf{x}_0(\zeta), \quad \zeta \in [\zeta_0 - \tau, \zeta_0), \end{cases} \tag{3.1}$$

Here, $x(\zeta) \in R^n$ signifies the system state at time ζ, and $0 \le \tau(\zeta) \le \tau$ represents the bounded time-varying delay. The function $\mathbf{x}_0(\zeta) \in PC^b_{\mathcal{F}_{\zeta_0}}$ denotes the initial function defined on $[\zeta_0 - \tau, \zeta_0)$. h and $g : R^n \times R^n \to R^n$ are nonlinear functions; $w(\zeta)$ is the n-dimensional wiener process in complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\zeta\}_{\zeta \ge 0}, P)$.

Our goal in this article is to establish some new fixed-time stability and prescribed-time stability criteria for stochastic delay and multi-delay systems based on stochastic analysis techniques and the Lyapunov theory, which will be discussed further below.

3.1. Fixed-time stability of stochastic time-delay system

Theorem 3.1. *Suppose that $V(\zeta) : R^n \to [0, +\infty)$ is a positive-definite, radially unbounded, and differentiable function. Let $x(\zeta)$ represent the state of (3.1). Define $V(x(\zeta)) = \mathbf{V}(\zeta)$ and $\mathbf{v}_0(\zeta) = V(\mathbf{x}_0(\zeta))$. If there exist integrable function $a(\zeta) : [\zeta_0, +\infty) \to R^+$, bounded function $b(\zeta) : [\zeta_0, +\infty) \to [0, b]$, Ω type function $f(\zeta) : [\zeta_0, +\infty) \to R^n$, and a function $p(x, y) : [\zeta_0, +\infty) \times [\zeta_0, +\infty) \to R^n$, such that*

$$\begin{cases} d\mathbf{V}(\zeta) \le [-a(\zeta)\mathbf{V}(\zeta) + b(\zeta) \sup_{\zeta - \tau(\zeta) \le s \le \zeta} \mathbf{V}(s) - f(\mathbf{V}(\zeta)) - \mathbf{c}(\zeta)] d\zeta \\ \qquad\qquad + p(\mathbf{V}(\zeta), \mathbf{V}(\zeta - \tau(\zeta))) dw, \quad \zeta \in [\zeta_0, +\infty), \\ \mathbf{V}(\zeta) = \mathbf{v}_0(\zeta), \quad \zeta \in [\zeta_0 - \tau, \zeta_0), \end{cases} \tag{3.2}$$

supposing the following conditions hold:

(a). There exists a positive integrable function $\xi(\zeta)$ satisfying:

$$\begin{cases} \lim_{t \to \infty} \int_{\zeta_0}^{\zeta} \xi(s) ds \to +\infty, \\ \sup_{\zeta \ge \zeta_0} \left\{ \int_{\zeta - \tau(\zeta)}^{\zeta} \xi(l) dl \right\} := \xi, \\ -a(\zeta) + b(\zeta) e^{\xi} + \xi(\zeta) \le 0. \end{cases} \tag{3.3}$$

(b). For the Ω type function $f(\zeta)$, there exists a constant $\Im$ that satisfies: $f(\zeta_1) + f(\zeta_2) \ge \Im \cdot f(\zeta_1 + \zeta_2)$, $(\forall \zeta_1, \zeta_2 \in R^+)$.

(c). There exists a constant $c > 0$ that satisfies:$-\mathbf{c}(\zeta) \le c[-a(\zeta) + b(\zeta)\lambda(\zeta)] - hf(c)$,

then it follows that $\lim_{\zeta \to \zeta_0 + T} \mathcal{E}\mathbf{V}(\zeta) = 0$. For $\zeta \ge \zeta_0 + T$, we have $\mathcal{E}\mathbf{V}(\zeta) = 0$ almost surely, where $T = \int_0^{+\infty} \frac{ds}{hf(s)}$, and h is a positive constant, $0 < h \le \frac{1}{\tau \Im e^{\xi} b + 1} \le \frac{1}{\tau \Im e^{\xi} b(\zeta) + 1}$. Moreover, the system (3.1) is fixed-time stable.

Proof. At first, we construct the following auxiliary stochastic differential equation:

$$\begin{cases} d\mathbf{W}(\zeta) = \Big\{ [-a(\zeta) + b(\zeta)\lambda(\zeta)]\mathbf{W}(\zeta) - hf(\mathbf{W}(\zeta)) - \mathbf{c}(\zeta) \Big\} d\zeta \\ \qquad\qquad + p(\mathbf{W}(\zeta), \mathbf{W}(\zeta - \tau(\zeta))) dw, \zeta \in [\zeta_0, +\infty), \\ \mathbf{W}(\zeta) = \sup_{s \le \zeta_0} |\mathcal{E}\mathbf{v}_0(s)|, \zeta \in [\zeta_0 - \tau, \zeta_0), \end{cases} \tag{3.4}$$

where $\lambda(\zeta) = e^{\sup_{\zeta - \tau(\zeta) \le s \le \zeta} \int_s^{\zeta} \xi(s) ds}$. $a(\zeta), b(\zeta), f(\zeta), p(x, y)$ are the same as the above definition of (3.2). It is obvious that

$$\mathcal{L}\mathbf{W}(\zeta) = [-a(\zeta) + b(\zeta)\lambda(\zeta)]\mathbf{W}(\zeta) - hf(\mathbf{W}(\zeta)) - \mathbf{c}(\zeta), \tag{3.5}$$

then based on Itô's formula and the condition (b), (c), we have

$$\begin{aligned}\mathcal{E}\mathbf{W}(\zeta)-\mathcal{E}\mathbf{W}(\zeta_0)=&\mathcal{E}\int_{\zeta_0}^{\zeta}[-a(\zeta)+b(\zeta)\lambda(\zeta)]\mathbf{W}(\zeta)-hf(\mathbf{W}(\zeta)-\mathbf{c}(\zeta))ds\\ \leq&\mathcal{E}\int_{\zeta_0[-a(\zeta)+b(\zeta)\lambda(\zeta)][\mathbf{W}(\zeta)+c]-h\mathfrak{I}\cdot f(\mathbf{W}(\zeta)+c)}^{\zeta}\end{aligned}\tag{3.6}$$

For convenience, let's define $\hat{\mathbf{W}}(\zeta)=\mathbf{W}(\zeta)+c$. Take the derivative of (3.6), and use Condition (a), which yields

$$\begin{aligned}\frac{d\mathcal{E}\hat{\mathbf{W}}(\zeta)}{d\zeta}=&[-a(\zeta)+b(\zeta)\lambda(\zeta)]\mathcal{E}\hat{\mathbf{W}}(\zeta)-h\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\\ \leq&-\xi(\zeta)\mathcal{E}\hat{\mathbf{W}}(\zeta)-h\mathfrak{I}\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta)).\end{aligned}\tag{3.7}$$

Thus, from (3.7), we have

$$\begin{aligned}\mathcal{E}\hat{\mathbf{W}}(\zeta-\tau(\zeta))\leq&\mathcal{E}\hat{\mathbf{W}}(\zeta)e^{\int_{\zeta}^{\zeta-\tau(\zeta)}-\xi(s)ds}-\int_{\zeta}^{\zeta-\tau(\zeta)}e^{\int_{s}^{\zeta-\tau(\zeta)}-\xi(u)du}h\mathfrak{I}\cdot\mathcal{E}f(\hat{\mathbf{W}}(s))ds\\ =&\mathcal{E}\hat{\mathbf{W}}(\zeta)e^{\int_{\zeta-\tau(\zeta)}^{\zeta}\xi(s)ds}+\int_{\zeta-\tau(\zeta)}^{\zeta}e^{\int_{\zeta-\tau(\zeta)}^{s}\xi(u)du}h\mathfrak{I}\cdot\mathcal{E}f(\hat{\mathbf{W}}(s))ds.\end{aligned}\tag{3.8}$$

According to $\int_{\zeta-\tau(\zeta)}^{s}\xi(u)du\leq\underset{\zeta\geq\zeta_0}{sup}\left\{\int_{\zeta-\tau(\zeta)}^{\zeta}\xi(l)dl\right\}:=\xi$ and in the light of the integral mean value theorem, $\exists\theta\in(\zeta-\tau(\zeta),\zeta)$, such that

$$\begin{aligned}&\int_{\zeta-\tau(\zeta)}^{\zeta}e^{\int_{\zeta-\tau(\zeta)}^{s}\xi(u)du}h\mathfrak{I}\cdot\mathcal{E}f(\hat{\mathbf{W}}(s))ds=h\mathfrak{I}\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\int_{\theta}^{\zeta}e^{\int_{\zeta-\tau(\zeta)}^{s}\xi(u)du}ds\\ \leq&h\mathfrak{I}\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\int_{\theta}^{\zeta}e^{\xi}ds\leq h\mathfrak{I}\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta))(\zeta-\theta)e^{\xi}\\ \leq&h\mathfrak{I}\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\tau(\zeta)e^{\xi}\leq h\mathfrak{I}\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\tau e^{\xi}.\end{aligned}$$

Thus, we derive $\mathcal{E}\hat{\mathbf{W}}(\zeta-\tau(\zeta))\leq\mathcal{E}\hat{\mathbf{W}}(\zeta)e^{\int_{\zeta-\tau(\zeta)}^{\zeta}\xi(u)du}+h\mathfrak{I}\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\tau e^{\xi}$. That is to say,

$$\begin{aligned}\sup_{\zeta-\tau(\zeta)\leq s\leq\zeta}\mathcal{E}\hat{\mathbf{W}}(s)\leq&\mathcal{E}\hat{\mathbf{W}}(\zeta)e^{\int_{\zeta-\tau(\zeta)}^{\zeta}\xi(u)du}+h\tau e^{\xi}\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\\ \leq&\mathcal{E}\hat{\mathbf{W}}(\zeta)\lambda(\zeta)+h\tau\mathfrak{I}\cdot e^{\xi}\mathcal{E}f(\hat{\mathbf{W}}(\zeta)),\zeta\in[\zeta_0,+\infty).\end{aligned}$$

Thus, we obtain

$$-\mathcal{E}\hat{\mathbf{W}}(\zeta)\lambda(\zeta)\leq-\sup_{\zeta-\tau(\zeta)\leq s\leq\zeta}\mathcal{E}\hat{\mathbf{W}}(s)+h\tau\mathfrak{I}\cdot e^{\xi}\mathcal{E}f(\hat{\mathbf{W}}(\zeta)),\zeta\in[\zeta_0,+\infty).\tag{3.9}$$

Next, we aim to prove

$$\mathcal{E}\mathbf{V}(\zeta)+c\leq\mathcal{E}\hat{\mathbf{W}}(\zeta),\zeta\in[\zeta_0,+\infty).\tag{3.10}$$

We notice that

$$\mathcal{E}\mathbf{V}(\zeta)+c\leq\sup_{s\leq\zeta_0}|\mathcal{E}\mathbf{v}_0(s)|+c=\mathcal{E}\hat{\mathbf{W}}(\zeta),\zeta\in[\zeta_0-\tau,\zeta_0).\tag{3.11}$$

Assume that there exists $\zeta^*>\zeta_0$ such that $\mathcal{E}\mathbf{V}(\zeta)+c<\mathcal{E}\hat{\mathbf{W}}(\zeta)$ for $\zeta\in[\zeta_0-\tau,\zeta^*)$, and $\mathcal{E}\mathbf{V}(\zeta^*)+c=\mathcal{E}\hat{\mathbf{W}}(\zeta^*)$, then we have $\left(\frac{d\mathcal{E}\mathbf{V}(\zeta)+c}{d\zeta}-\frac{d\mathcal{E}\hat{\mathbf{W}}(\zeta)}{d\zeta}\right)\Big|_{\zeta=\zeta^*}>0$. On the other hand, combine (3.2) with (3.4) and (3.9) and use Condition (a),(b) to yield

$$\begin{aligned}
&\left(\frac{d\mathcal{E}\mathbf{V}(\zeta)+c}{d\zeta}-\frac{d\mathcal{E}\hat{\mathbf{W}}(\zeta)}{d\zeta}\right)\Big|_{\zeta=\zeta^*}=\left(\frac{d\mathcal{E}\mathbf{V}(\zeta)}{d\zeta}-\frac{d\mathcal{E}\mathbf{W}(\zeta)}{d\zeta}\right)\Big|_{\zeta=\zeta^*}\\
\leq&-a(\zeta^*)\left(\mathcal{E}\mathbf{V}(\zeta^*)-\mathcal{E}\mathbf{W}(\zeta^*)\right)-\mathcal{E}f(\mathbf{V}(\zeta^*))+h\cdot\mathcal{E}f(\mathbf{W}(\zeta^*))\\
&+b(\zeta^*)\left[\left(\sup_{\zeta^*-\tau(\zeta^*)\leq s\leq\zeta^*}\mathcal{E}\mathbf{V}(s)\right)-\lambda(\zeta)\mathcal{E}\mathbf{W}(\zeta)\right]\\
\leq&-a(\zeta^*)(\mathcal{E}\mathbf{V}(\zeta^*)-\mathcal{E}\mathbf{W}(\zeta^*))+\left(h+h\tau\mathfrak{I}\cdot e^{\xi}b(\zeta^*)\right)\mathcal{E}f(\mathbf{W}(\zeta^*))\\
&-\mathcal{E}f(\mathbf{V}(\zeta^*))+b(\zeta^*)\sup_{\zeta^*-\tau(\zeta^*)\leq s\leq\zeta^*}[\mathcal{E}\mathbf{V}(s)-\mathcal{E}\mathbf{W}(s)]\\
\leq&-a(\zeta^*)(\mathcal{E}\mathbf{V}(\zeta^*)-\mathcal{E}\mathbf{W}(\zeta^*))+[\mathcal{E}f(\mathbf{W}(\zeta^*))-\mathcal{E}f(\mathbf{V}(\zeta^*))]\\
&+b(\zeta^*)\sup_{\zeta^*-\tau(\zeta^*)\leq s\leq\zeta^*}[\mathcal{E}\mathbf{V}(s)-\mathcal{E}\mathbf{W}(s)]\\
=&b(\zeta^*)\sup_{\zeta^*-\tau(\zeta^*)\leq s\leq\zeta^*}[\mathcal{E}\mathbf{V}(s)-\mathcal{E}\mathbf{W}(s)]\leq 0,
\end{aligned}\tag{3.12}$$

which leads to a contradiction. Thus, we establish the inequality $\mathcal{E}\mathbf{V}(\zeta)+c\leq\mathcal{E}\hat{\mathbf{W}}(\zeta)$ for $\zeta\in[\zeta_0,+\infty)$. Based on (3.5) and condition (c), one obtains

$$\begin{aligned}
\mathcal{L}\hat{\mathbf{W}}(\zeta)=&[-a(\zeta)+b(\zeta)\lambda(\zeta)]\hat{\mathbf{W}}(\zeta)-hf(\hat{\mathbf{W}}(\zeta))\\
\leq&-\xi(\zeta)\hat{\mathbf{W}}(\zeta)-hf(\hat{\mathbf{W}}(\zeta))\leq-hf(\hat{\mathbf{W}}(\zeta)).
\end{aligned}\tag{3.13}$$

Based on the preceding analysis, we derive the inequality $\mathcal{L}\hat{\mathbf{W}}(\zeta)\leq-hf(\hat{\mathbf{W}}(\zeta))$, and notice that $f(\zeta)$ is a Ω type function. In accordance with Lemma 2.1, we deduce that $\lim\limits_{\zeta\to T}\mathcal{E}\hat{\mathbf{W}}(\zeta)=0$ and $\mathcal{E}\hat{\mathbf{W}}(\zeta)=0$, almost surely, for $\zeta\geq\zeta_0+T$, where $T=\int_0^{+\infty}\frac{ds}{hf(s)}$.

However, considering the inequalities $0\leq\lim_{\zeta\to\zeta_0+T}\mathcal{E}[V(\zeta,x(\zeta))]+c\leq\lim_{\zeta\to\zeta_0+T}\mathcal{E}[\hat{\mathbf{W}}(x(\zeta))]\leq 0$, we arrive at the conclusion that there must be a $\vartheta<\zeta_0+T$ such that $\mathcal{E}[V(\zeta,x(\zeta))]=0$ for $\zeta\in[\vartheta,+\infty)$. If not, one derives $0\leq\mathcal{E}[V(\zeta_0+T,x(\zeta_0+T))]<-c<0$, which is a contradiction, then we have $\mathcal{E}[x(\zeta)]=0$, almost surely, for $\zeta\geq\vartheta$. Hence, the proof is concluded.

Remark 2. *In the (3.2), the term $b(\zeta)\sup_{\zeta-\tau(\zeta)\leq s\leq\zeta}\mathbf{V}(s)$ exerts a destabilizing effect on $\mathbf{V}(\zeta)$, while the term $-a(\zeta)\mathbf{V}(\zeta)$ has a stabilizing effect. Concerning Condition (a) as presented in Theorem 3.1, to achieve fixed-time stability for the system (3.1) in comparison to $b(\zeta)$, the parameter function $a(\zeta)$ must be sufficiently large in such a way that a suitable $\xi(\zeta)$ exists, satisfying the inequality $-a(\zeta)+b(\zeta)e^{\xi(\zeta)}+\xi(\zeta)\leq 0$. The function $\xi(\zeta)$ can be chosen to be a constant or exhibit behavior of the form $O\left(\frac{1}{(1+\zeta)^{\alpha}}\right)$, where $(\alpha\geq 1)$.*

For instance, given a function $b(\zeta)$*, one can choose* $a(\zeta) = 3b(\zeta) + \frac{\ln 3}{\tau}\frac{1}{1+\zeta}$ *and* $\xi(\zeta) = \frac{\ln 3}{\tau}\frac{1}{1+\zeta}$*. Notably, as* ζ *approaches infinity,* $\int_{\zeta_0}^{\zeta}\frac{\ln 3}{\tau}\frac{1}{1+s}ds$ *tends to infinity, and* $\sup_{\zeta\geq\zeta_0}\left\{\int_{\zeta-\tau(\zeta)}^{\zeta}\frac{\ln 3}{\tau}\frac{1}{1+l}dl\right\} < \tau(\zeta)\frac{\ln 3}{\tau} \leq \ln 3$*. Thus,* $\xi(\zeta) = \frac{\ln 3}{\tau}\frac{1}{1+\zeta}$ *satisfies Condition (a). Alternatively, one can select* $a(\zeta) \geq e^{4\tau}b(\zeta) + 4$*, in which case* $\xi(\zeta) = 4$ *satisfies Condition (a).*

Remark 3. *When* $a(\zeta)$*,* $b(\zeta)$ *become constants and* $f(\zeta) = g(\zeta) = 0$*, Theorem 3.1 degrades into the well-known Halanay inequalities. When* $f(x) = 0$*, by repeating the procedure of above proof, we can derive that* $\mathcal{E}\mathbf{V}(\zeta) \leq \sup_{s\leq\zeta_0}|\mathcal{E}\mathbf{v}_0(s)|e^{-\xi(\zeta-\zeta_0)}$*; in this case, the stochastic time-delay system (3.1) is exponentially stable in probability. In contrast to the works of [19, 20], which are not applicable to stochastic systems, Theorem 3.1 can be adapted to address the stability of stochastic systems. As a result, it offers a wider range of application scenarios compared to the aforementioned studies.*

Remark 4. *Compared to traditional stability criteria like [17, 27] and [18–21], which overlook the influence of time delays, Theorem 3.1 presents a new method for estimating convergence time in stochastic time-delay systems. It can be viewed as an extension of prior research, providing a fresh viewpoint and addressing the inherent challenges posed by time delays in system analysis.*

Remark 5. *It is worth it to point out that the fixed-time stability criteria presented in this paper has distinct advantages. First, the setting time* T_C *does not rely on any specific initial values, ensuring their applicability across various scenarios. Second, they are entirely independent of system delays. This implies that the setting times* T_C *are solely determined by the controller parameters and can be preassigned by the user.*

Corollary 3.1. *When there are no stochastic disturbances, i.e.,* $g(\mathbf{V}(\zeta), \mathbf{V}(\zeta - \tau(\zeta))) = 0$*, (3.2) reduces to the following form:*

$$\begin{cases} d\mathbf{V}(\zeta) \leq [-a(\zeta)\mathbf{V}(\zeta) + b(\zeta)\sup_{\zeta-\tau(\zeta)\leq s\leq\zeta}\mathbf{V}(s) - f(\mathbf{V}(\zeta)) - \mathbf{c}(\zeta)]d\zeta, \zeta \in [\zeta_0, +\infty), \\ \mathbf{V}(\zeta) = \mathbf{v}_0(\zeta), \zeta \in [\zeta_0 - \tau, \zeta_0), \end{cases} \tag{3.14}$$

Suppose the following conditions hold for the system (3.14).

(a). There exists an integrable function $\xi(\zeta)$ *that satisfies:*

$$\begin{cases} \lim\limits_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\xi(s)ds \to +\infty, \\ \sup\limits_{\zeta\geq\zeta_0}\left\{\int_{\zeta-\tau(\zeta)}^{\zeta}\xi(l)dl\right\} := \xi, \\ -a(\zeta) + b(\zeta)e^{\xi} + \xi(\zeta) \leq 0. \end{cases} \tag{3.15}$$

(b). For the Ω *type function* $f(\zeta)$*, there exists a constant* $\Im$ *that satisfies:* $f(\zeta_1) + f(\zeta_2) \geq \Im \cdot f(\zeta_1 + \zeta_2)$*,*$(\forall\zeta_1, \zeta_2 \in R^+)$*.*

(c). There exists a constant $c > 0$ *that satisfies:* $-\mathbf{c}(\zeta) \leq c[-a(\zeta) + b(\zeta)e^{\xi}] - hf(c)$*.*

Thus one has $\lim_{\zeta\to\zeta_0+T}\mathbf{V}(\zeta) = 0$*, and* $\mathbf{V}(\zeta) = 0$ *for* $\zeta \geq \zeta_0 + T$*, where* $T = \int_0^{+\infty}\frac{ds}{hf(s)}$*, and h is a positive constant,* $0 < h \leq \frac{1}{\tau\Im e^{\xi}b+1} \leq \frac{1}{\tau\Im e^{\xi}b(\zeta)+1}$*.*

Corollary 3.2. *In the case where* $a(\zeta)$ *and* $b(\zeta)$ *are constants and the function* $f(x)$ *is expressed as* $f(x) = e_1x + e_2x^{\alpha} + e_3x^{\beta}$*, with* $e_1, e_2, e_3 > 0$*,* $0 < \alpha < 1$*, and* $\beta > 1$*, the stochastic time-delay* *system* (3.2) *takes on the following reduced form:*

$$\begin{cases} d\mathbf{V}(\zeta) \leq [\,-a\mathbf{V}(\zeta) + b\sup_{\zeta-\tau(\zeta)\leq s\leq\zeta}\mathbf{V}(s) - (e_1\mathbf{V}(\zeta) + e_2\mathbf{V}^{\alpha}(\zeta) + e_3\mathbf{V}^{\beta}(\zeta)) - \mathbf{c}(\zeta)]d\zeta \\ \qquad\qquad +g(\mathbf{V}(\zeta), \mathbf{V}(\zeta-\tau(\zeta)))dw, \zeta \in [\zeta_0, +\infty), \\ \mathbf{V}(\zeta) = \mathbf{v}_0(\zeta), \zeta \in [\zeta_0 - \tau, \zeta_0), \end{cases}$$

Suppose there exists a positive value ξ such that $-a + be^{\xi} + \frac{\xi}{\tau} \leq 0$, and there exists a constant $c > 0$ that satisfies: $-\mathbf{c}(\zeta) \leq c[-a + be^{\xi}] - 2^{1-\beta}h \cdot f(c)$, then it can be concluded that $\lim_{\zeta\to\zeta_0+T}\mathcal{E}[\mathbf{V}(\zeta)] = 0$ and $\mathcal{E}[\mathbf{V}(\zeta)] = 0$ almost surely for $\zeta \geq \zeta_0 + T$, $T = \frac{\Psi}{h}$. Moreover, the system (3.1) *is fixed-time stability, where Ψ is given by*

$$\Psi = \left[\frac{\pi\csc(\pi y)}{\rho_3(\alpha+1-\beta)}\left(\frac{\rho_3}{\rho_2}\right)I\left(\frac{\rho_3}{\rho_3+\rho_2}, y, 1-y\right) + \frac{\pi\csc(\pi z)}{\rho_3(\alpha-1+\beta)}\left(\frac{\rho_3}{\rho_1}\right)I\left(\frac{\rho_3}{\rho_3+\rho_1}, z, 1-z\right)\right], \tag{3.16}$$

with $\rho_1 = he_2$, $\rho_2 = he_3$, $\rho_3 = \xi + he_1$, and $h = \frac{1}{2^{1-\beta}\tau e^{\xi}b+1}$. Here, $I(\lambda, x, y)$ represents the incomplete beta function.

Proof. According to Theorem 3.1, we just need to verify that $\Omega(x) = [\xi x + (e_1x + e_2x^{\alpha} + e_3x^{\beta})]$ is a Ω function, and $f(\zeta_1) + f(\zeta_2) \geq \Im \cdot f(\zeta_1 + \zeta_2)$,$(\forall\zeta_1, \zeta_2 \in R^+)$. We notice that $\Omega(x)$ is a monotonically increasing function. In addition, it is not difficult to obtain $\int_{\zeta_0}^{\infty}\frac{dx}{[\xi x+(e_1x+e_2x^{\alpha}+e_3x^{\beta})]} \leq \Psi < +\infty$, where the calculation of this integral can be found in [24].

Additionally, it is obvious that $(e_1\zeta_1 + e_2\zeta_1^{\alpha} + e_3\zeta_1^{\beta}) + (e_1\zeta_2 + e_2\zeta_2^{\alpha} + e_3\zeta_2^{\beta}) \geq 2^{1-\beta}(e_1(\zeta_1+\zeta_2) + e_2(\zeta_1+\zeta_2)^{\alpha} + e_3(\zeta_1+\zeta_2)^{\beta})$,$(\forall\zeta_1, \zeta_2 \in R^+)$. Thus, the conditions (a), (b), (c) of Theorem 3.1 are satisfied. According to the Theorem 3.1, we complete the proof.

Theorem 3.2. *Suppose that $V(\zeta) : R^n \to [0, +\infty)$ is a positive-definite, radially unbounded, and differentiable function. $x(\zeta)$ is the state of (3.1). Define $V(x(\zeta)) = \mathbf{V}(\zeta)$, $\mathbf{v}_0(\zeta) = V(\mathbf{x}_0(\zeta))$. If there exist integrable function $a(\zeta) : [\zeta_0, +\infty) \to R^+$, bounded functions $b_i(\zeta), (i = 1, 2, \cdots, m) : [\zeta_0, +\infty) \to [0, b]$, Ω type function $f(\zeta) : [\zeta_0, +\infty) \to R^n$, and $g(x, y) : [\zeta_0, +\infty) \times [\zeta_0, +\infty) \to R^n$, such that*

$$\begin{cases} d\mathbf{V}(\zeta) \leq \left[-a(\zeta)\mathbf{V}(\zeta) + \sum_{i=1}^{m} b_i(\zeta)\sup_{\zeta-\tau_i(\zeta)\leq s\leq\zeta}\mathbf{V}(s) - f(\mathbf{V}(\zeta)) - \mathbf{c}(\zeta)\right]d\zeta \\ \qquad\qquad +g(\mathbf{V}(\zeta), \mathbf{V}(\zeta-\tau(\zeta)))dw, \zeta \in [\zeta_0, +\infty), \\ \mathbf{V}(\zeta) = \mathbf{v}_0(\zeta), \zeta \in [\zeta_0 - \tau, \zeta_0). \end{cases} \tag{3.17}$$

Supposing the following conditions hold for (3.17).

(a). There exists an integrable function $\xi(\zeta)$ that satisfies:

$$\begin{cases} \lim\limits_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\xi(s)ds \to +\infty, \\ \sup\limits_{\zeta\geq\zeta_0}\left\{\int_{\zeta-\hat{\tau}(\zeta)}^{\zeta}\xi(l)dl\right\} := \xi, \\ -a(\zeta) + \sum_{i=1}^{m} b_i(\zeta)e^{\xi} + \xi(\zeta) \leq 0. \end{cases} \tag{3.18}$$

(b). For the Ω type function $f(\zeta)$, there exists a constant $\Im$ that satisfies: $f(\zeta_1) + f(\zeta_2) \geq \Im \cdot f(\zeta_1 + \zeta_2)$,$(\forall\zeta_1, \zeta_2 \in R^+)$.

(c). There exists a constant $c > 0$ that satisfies: $-\mathbf{c}(\zeta) \leq c[-a(\zeta) + \sum_{i=1}^{m} b_i(\zeta)\lambda(\zeta)] - hf(c)$.

then it follows that $\lim_{\zeta\to\zeta_0+T}\mathcal{E}\mathbf{V}(\zeta)=0$. *For* $\zeta\geq\zeta_0+T$, *we have* $\mathcal{E}\mathbf{V}(\zeta)=0$ *almost surely, where* $T=\int_0^{+\infty}\frac{ds}{hf(s)}$, *and* h *is a positive constant,* $0<h\leq\frac{1}{\tau_i\Im e^{\xi}\sum_{i=1}^m b_i+1}\leq\frac{1}{\tau_i\Im e^{\xi}\sum_{i=1}^m b_i(\zeta)+1}$. *Moreover, the system (3.1) is fixed-time stable.*

Proof. At first, we construct the following stochastic differential equation

$$\begin{cases} d\mathbf{W}(\zeta)=\{[-a(\zeta)+\sum_{i=1}^m b_i(\zeta)\lambda(\zeta)]\,\mathbf{W}(\zeta)-hf(\mathbf{W}(\zeta))-\mathbf{c}(\zeta)\}\,d\zeta \\ \qquad\qquad +g(\mathbf{W}(\zeta),\mathbf{W}(\zeta-\tau(\zeta)))dw,\zeta\in[\zeta_0,+\infty), \\ \mathbf{W}(\zeta)=\sup_{s\leq\zeta_0}|\mathcal{E}\mathbf{v}_0(s)|,\zeta\in[\zeta_0-\tau,\zeta_0), \end{cases}$$

where h is a positive constant, $\lambda(\zeta)=e^{\sup_{\zeta-\hat{\tau}(\zeta)\leq s\leq\zeta}\int_s^{\zeta}\xi(s)ds}$, $(\hat{\tau}(\zeta)=\min_{1\leq i\leq m}\tau_i(\zeta))$, and $f(\zeta)>0$ is a monotone increasing function defined on $\zeta\in[\zeta_0,+\infty)$. It is obvious that

$$\mathcal{L}\mathbf{W}(\zeta)=\left[-a(\zeta)+\sum_{i=1}^m b_i(\zeta)\lambda(\zeta)\right]\mathbf{W}(\zeta)-hf(\mathbf{W}(\zeta))-\mathbf{c}(\zeta), \tag{3.19}$$

then based on Itô's formula, we have

$$\mathcal{E}\mathbf{W}(\zeta)-\mathcal{E}\mathbf{W}(\zeta_0)=\mathcal{E}\int_{\zeta_0}^{\zeta}\left[-a(\zeta)+\sum_{i=1}^m b_i(\zeta)\lambda(\zeta)\right]\mathbf{W}(\zeta)-f(\mathbf{W}(\zeta))-\mathbf{c}(\zeta)ds. \tag{3.20}$$

Let's define $\hat{\mathbf{W}}(\zeta)=\mathbf{W}(\zeta)+c$. Take the derivative of (3.20), and use Condition (a) of Theorem 3.2, which yields

$$\begin{aligned}\frac{d\mathcal{E}\hat{\mathbf{W}}(\zeta)}{d\zeta}&=\left[-a(\zeta)+\sum_{i=1}^m b_i(\zeta)\lambda(\zeta)\right]\mathcal{E}\mathbf{W}(\zeta)-h\cdot\mathcal{E}f(\mathbf{W}(\zeta))-\mathbf{c}(\zeta)\\ &\leq-\xi(\zeta)[\mathcal{E}\mathbf{W}(\zeta)+c]-h\Im\cdot\mathcal{E}f(\mathbf{W}(\zeta)+c).\end{aligned} \tag{3.21}$$

Thus, from (3.21) we have

$$\begin{aligned}\mathcal{E}\hat{\mathbf{W}}(\zeta-\tau_i(\zeta))\leq&\mathcal{E}\hat{\mathbf{W}}(\zeta)e^{\int_{\zeta}^{\zeta-\tau(\zeta)}-\xi(s)ds}-\int_{\zeta}^{\zeta-\tau_i(\zeta)}e^{\int_s^{\zeta-\tau_i(\zeta)}-\xi(\zeta)du}h\Im\cdot\mathcal{E}f(\hat{\mathbf{W}}(s))ds\\ =&\mathcal{E}\hat{\mathbf{W}}(\zeta)e^{\int_{\zeta-\tau_i(\zeta)}^{\zeta}\xi(s)ds}+\int_{\zeta-\tau_i(\zeta)}^{\zeta}e^{\int_{\zeta-\tau_i(\zeta)}^{s}\xi(u)du}h\Im\cdot\mathcal{E}f(\hat{\mathbf{W}}(s))ds.\end{aligned} \tag{3.22}$$

According to $\int_{\zeta-\tau_i(\zeta)}^{s}\xi(u)du\leq\underset{\zeta\geq\zeta_0}{sup}\left\{\int_{\zeta-\tau_i(\zeta)}^{\zeta}\xi^*(l)dl\right\}:=\xi$ and in the light of the integral mean value theorem, $\exists\theta\in(\zeta-\hat{\tau}(\zeta),\zeta)$, such that

$$\begin{aligned}&\int_{\zeta-\tau_i(\zeta)}^{\zeta}e^{\int_{\zeta-\tau_i(\zeta)}^{s}\xi(u)du}h\Im\cdot\mathcal{E}f(\hat{\mathbf{W}}(s))ds=h\Im\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\int_{\theta}^{\zeta}e^{\int_{\zeta-\tau_i(\zeta)}^{s}\xi(u)du}ds\\ \leq&h\Im\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\int_{\theta}^{\zeta}e^{\xi}ds\leq h\Im\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta))(\zeta-\theta)e^{\xi}\\ \leq&h\Im\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\tau_i(\zeta)e^{\xi}\leq h\Im\cdot\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\tau e^{\xi}.\end{aligned} \tag{3.23}$$

Thus, we derive $\mathcal{E}\hat{\mathbf{W}}(\zeta-\tau_i(\zeta)) \leq \mathcal{E}\hat{\mathbf{W}}(\zeta)e^{\int_{\zeta-\tau_i(\zeta)}^{\zeta}\xi(u)du} + h\mathfrak{I}\cdot f(\hat{\mathbf{W}}(\zeta))\tau_i e^{\xi}$. That is to say,

$$\begin{aligned}\sup_{\zeta-\tau_i(\zeta)\leq s\leq\zeta}\mathcal{E}\hat{\mathbf{W}}(s) \leq & \mathcal{E}\hat{\mathbf{W}}(\zeta)e^{\int_{\zeta-\tau_i(\zeta)}^{\zeta}\xi(u)du} + h\tau_i\mathfrak{I}\cdot e^{\xi}\mathcal{E}f(\hat{\mathbf{W}}(\zeta))\\ \leq & \mathcal{E}\hat{\mathbf{W}}(\zeta)\lambda(\zeta) + h\tau_i\mathfrak{I}\cdot e^{\xi}\mathcal{E}f(\hat{\mathbf{W}}(\zeta)), \zeta\in[\zeta_0,+\infty).\end{aligned}$$

then we obtain

$$-\mathcal{E}\hat{\mathbf{W}}(\zeta)\lambda(\zeta) \leq -\sup_{\zeta-\tau_i(\zeta)\leq s\leq\zeta}\mathcal{E}\hat{\mathbf{W}}(s) + h\tau_i\mathfrak{I}\cdot e^{\xi}\mathcal{E}f(\hat{\mathbf{W}}(\zeta)), \zeta\in[\zeta_0,+\infty). \tag{3.24}$$

Next, we aim to prove

$$\mathcal{E}\mathbf{V}(\zeta) + c \leq \mathcal{E}\hat{\mathbf{W}}(\zeta), \zeta\in[\zeta_0,+\infty).$$

We notice that

$$\mathcal{E}\mathbf{V}(\zeta) + c \leq \sup_{s\leq\zeta_0}|\mathcal{E}\mathbf{v}_0(s)| + c = \mathcal{E}\hat{\mathbf{W}}(\zeta), \zeta\in[\zeta_0-\hat{\tau},\zeta_0). \tag{3.25}$$

Assume that there exists $\zeta^* > \zeta_0$ such that $\mathcal{E}\mathbf{V}(\zeta) + c < \mathcal{E}\hat{\mathbf{W}}(\zeta)$ for $\zeta\in[\zeta_0-\hat{\tau},\zeta^*)$, and $\mathcal{E}\mathbf{V}(\zeta^*) + c = \mathcal{E}\hat{\mathbf{W}}(\zeta^*)$, then we have $\left(\frac{d(\mathcal{E}\mathbf{V}(\zeta)+c)}{d\zeta} - \frac{d\mathcal{E}\hat{\mathbf{W}}(\zeta)}{d\zeta}\right)\Big|_{\zeta=\zeta^*} > 0$. On the other hand, combine (3.17) and (3.21), (3.24) to yield

$$\begin{aligned}&\left(\frac{d(\mathcal{E}\mathbf{V}(\zeta)+c)}{d\zeta} - \frac{d\mathcal{E}\mathbf{W}(\zeta)}{d\zeta}\right)\Big|_{\zeta=\zeta^*}\\ \leq & -a(\zeta^*)(\mathbf{V}(\zeta^*)-\mathbf{W}(\zeta^*)) - f(\mathbf{V}(\zeta^*)) + hf(\hat{\mathbf{W}}(\zeta^*))\\ &+\sum_{i=1}^m b_i(\zeta^*)\left[\sup_{\zeta^*-\tau_i(\zeta^*)\leq s\leq\zeta^*}\mathbf{V}(s) - \lambda(\zeta)\mathbf{W}(\zeta)\right]\\ \leq & -a(\zeta^*)(\mathbf{V}(\zeta^*)-\mathbf{W}(\zeta^*)) + \left[h + h\tau_i\mathfrak{I}\cdot e^{\xi}\sum_{i=1}^m b_i(\zeta^*)\right]f(\mathbf{W}(\zeta^*))\\ &- f(\mathbf{V}(\zeta^*)) + \sum_{i=1}^m b_i(\zeta^*)\sup_{\zeta^*-\tau_i(\zeta^*)\leq s\leq\zeta^*}[\mathbf{V}(s)-\mathbf{W}(s)]\\ \leq & -a(\zeta^*)(\mathbf{V}(\zeta^*)-\mathbf{W}(\zeta^*)) + [f(\mathbf{W}(\zeta^*)) - f(\mathbf{V}(\zeta^*))]\\ &+\sum_{i=1}^m b_i(\zeta^*)\sup_{\zeta^*-\tau_i(\zeta^*)\leq s\leq\zeta^*}[\mathbf{V}(s)-\mathbf{W}(s)]\\ = & \sum_{i=1}^m b_i(\zeta^*)\sup_{\zeta^*-\tau_i(\zeta^*)\leq s\leq\zeta^*}[\mathbf{V}(s)-\mathbf{W}(s)] \leq 0,\end{aligned}$$

which is a contradiction. Thus, we have $\mathcal{E}\mathbf{V}(\zeta) + c \leq \mathcal{E}\hat{\mathbf{W}}(\zeta)$, $\zeta\in[\zeta_0,+\infty)$. Based on (3.19) and condition (c), one obtains $\mathcal{L}\hat{\mathbf{W}}(\zeta) = [-a(\zeta) + \sum_{i=1}^m b_i(\zeta)\lambda(\zeta)]\hat{\mathbf{W}}(\zeta) - hf(\hat{\mathbf{W}}(\zeta)) \leq -\xi(\zeta)\hat{\mathbf{W}}(\zeta) - hf(\hat{\mathbf{W}}(\zeta)) \leq -hf(\hat{\mathbf{W}}(\zeta))$, and we notice that $f(\zeta)$ is a Ω type function. In accordance with Lemma 2.1, we deduce that $\lim_{\zeta\to T}\mathcal{E}\hat{\mathbf{W}}(\zeta) = 0$ and $\mathcal{E}\hat{\mathbf{W}}(\zeta) = 0$, almost surely, for $\zeta \geq \zeta_0 + T$, where $T = \int_0^{+\infty}\frac{ds}{hf(s)}$.

However, considering the inequalities $0 \leq \lim_{\zeta\to\zeta_0+T} \mathcal{E}[V(\zeta, x(\zeta))] + c \leq \lim_{\zeta\to\zeta_0+T} \mathcal{E}[\hat{\mathbf{W}}(x(\zeta))] \leq 0$, we arrive at the conclusion that there must be a $\vartheta < \zeta_0 + T$ such that $\mathcal{E}[V(\zeta, x(\zeta))] = 0$ for $\zeta \in [\vartheta, +\infty)$. If not, one derives $0 \leq \mathcal{E}[V(\zeta_0 + T, x(\zeta_0 + T))] < -c < 0$, which is a contradiction, then we have $\mathcal{E}[x(\zeta)] = 0$, almost surely, for $\zeta \geq \vartheta$.

3.2. Prescribed-time stability of stochastic time-delay system

In the forthcoming theorem, we will introduce two prescribed-time stability criteria for stochastic delay systems with the help of a new time-varying scaling function. To begin with, we give the following Theorem:

Theorem 3.3. *Suppose that $V(\zeta) : R^n \to [0, +\infty)$ is a positive-definite, radially unbounded, and differentiable function. $x(\zeta)$ is the state of (3.1), and $\Gamma(\zeta)$ is the time-varying scaling function of Definition 2.2. Define $V(x(\zeta)) = \mathbf{V}(\zeta)$, $\mathbf{v}_0(\zeta) = V(\mathbf{x}_0(\zeta))$. If there exist integrable function $a(\zeta), : [\zeta_0, +\infty) \to R^+$, bounded function $b(\zeta) : [\zeta_0, +\infty) \to [0, b]$, and a sufficiently small constant $\varepsilon > 0$, such that*

$$\begin{cases} \mathcal{L}\mathbf{V}(\zeta) \leq -a(\zeta)\mathbf{V}(\zeta) + b(\zeta)\sup_{\zeta-\tau(\zeta)\leq s\leq\zeta} \mathbf{V}(s) - \frac{1}{h}\frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}\mathbf{V}(\zeta) - \mathbf{c}(\zeta), \\ \mathbf{V}(\zeta) = \mathbf{v}_0(\zeta), \zeta \in [\zeta_0 - \tau, \zeta_0), \end{cases}$$

(a). There exists a constant $\xi > 0$ that satisfies: $-a(\zeta) + b(\zeta)e^{\tau\xi} \leq -\xi$, $(\xi > 0)$, and $0 < h \leq \frac{1}{\tau_i e^{\xi} b+1}$.

(b). There exists a constant $c > 0$ that satisfies: $-\mathbf{c}(\zeta) \leq c[-a(\zeta) + b(\zeta)\lambda(\zeta)] - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}c$.

then there exists a $R(\varepsilon) \in R^+$ such that the state $\lim\limits_{\zeta\to\zeta_0+T} \mathcal{E}x(\zeta)$ converges into the set $M = \{\mathcal{E}x(\zeta) \mid \|\mathcal{E}x(\zeta)\| \leq R(\varepsilon)\}$, and $\mathcal{E}x(\zeta) \in M$, a.s. when$\zeta \geq \zeta_0 + T$. (3.1) is prescribed-time quasi-stable in probability.

Proof. At first, we construct the following stochastic differential equation:

$$\begin{cases} \mathcal{L}\mathbf{W}(\zeta) = [-a(\zeta) + b(\zeta)e^{\tau\xi}]\mathbf{W}(\zeta) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}\mathbf{W}(\zeta) - \mathbf{c}(\zeta), \zeta \in [\zeta_0, +\infty), \\ \mathbf{W}(\zeta) = \sup_{s\leq\zeta_0} |\mathcal{E}\mathbf{v}_0(s)|, \zeta \in [\zeta_0 - \tau, \zeta_0), \end{cases}$$

Let $f(\mathbf{W}(\zeta)) = \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}\mathbf{W}(\zeta)$ and $\xi(\zeta) = \xi$, $\hat{\mathbf{W}}(\zeta) = \mathbf{W}(\zeta) + c$, $\hat{\mathbf{V}}(\zeta) = \mathbf{V}(\zeta) + c$ by repeating the procedure of Theorem 3.1. We obtain

$$\mathcal{E}\hat{\mathbf{V}}(\zeta) \leq \mathcal{E}\hat{\mathbf{W}}(\zeta). \tag{3.26}$$

then we derive

$$\begin{aligned} d\mathcal{E}\hat{\mathbf{W}}(\zeta) =& [-a(\zeta) + b(\zeta)e^{\tau\xi}]\mathcal{E}\hat{\mathbf{W}}(\zeta) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}\mathcal{E}\hat{\mathbf{W}}(\zeta) \\ \leq& -\xi\mathcal{E}\hat{\mathbf{W}}(\zeta) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}\mathcal{E}\hat{\mathbf{W}}(\zeta). \end{aligned} \tag{3.27}$$

When $\zeta \in [\zeta_0, \zeta_0 + T)$, multiplying $\Gamma^2(\zeta - \varepsilon)$ on both hands of (3.27),

$$\Gamma^2(\zeta-\varepsilon)d\mathcal{E}\hat{\mathbf{W}}(\zeta) \leq -\xi\Gamma^2(\zeta-\varepsilon)\mathcal{E}\hat{\mathbf{W}}(\zeta) - \dot{\Gamma}(\zeta-\varepsilon)\Gamma(\zeta-\varepsilon)\mathcal{E}\hat{\mathbf{W}}(\zeta) \tag{3.28}$$

which is equivalent to

$$\frac{d\left(\Gamma^2(\zeta-\varepsilon)\mathcal{E}\hat{\mathbf{W}}(\zeta)\right)}{d\zeta} \leq \xi\Gamma^2(\zeta-\varepsilon)\mathcal{E}\hat{\mathbf{W}}(\zeta)+\dot{\Gamma}(\zeta-\varepsilon)\Gamma(\zeta-\varepsilon)\mathcal{E}\hat{\mathbf{W}}(\zeta)=-\xi\left(\Gamma^2(\zeta-\varepsilon)\mathcal{E}\hat{\mathbf{W}}(\zeta)\right). \tag{3.29}$$

Thus,

$$\begin{aligned} &\Gamma^2(\zeta-\varepsilon)\mathcal{E}\hat{\mathbf{W}}(\zeta) \leq e^{-\xi(\zeta-\zeta_0)}\Gamma^2(\zeta_0-\varepsilon)\mathcal{E}\hat{\mathbf{W}}(\zeta_0)=e^{-\xi(\zeta-\zeta_0)}\mathcal{E}\hat{\mathbf{W}}(\zeta_0). \\ \Longrightarrow\quad &\mathcal{E}\hat{\mathbf{W}}(\zeta) \leq e^{-\xi(\zeta-\zeta_0)}\Gamma^{-2}(\zeta-\varepsilon)\mathcal{E}\hat{\mathbf{W}}(\zeta_0). \end{aligned} \tag{3.30}$$

When $\zeta \in [\zeta_0, \zeta_0+T)$, as $\underset{\varepsilon\to 0}{lim}\ \underset{\zeta\to\zeta_0+T}{lim}\Gamma^{-1}(\zeta-\varepsilon) = \underset{\varepsilon\to 0}{lim}\ \underset{\zeta\to\zeta_0+T}{lim}\cos^{\rho}\left(\frac{\pi(\zeta-\varepsilon)}{2(\zeta_0+T)}\right) = 0$, we can deduce that $0 \leq \lim_{\zeta\to\zeta_0+T}\mathcal{E}\mathbf{V}+c = \lim_{\zeta\to\zeta_0+T}\mathcal{E}\hat{\mathbf{V}}(\zeta) \leq \lim_{\zeta\to\zeta_0+T}\mathcal{E}\hat{\mathbf{W}}(\zeta) \leq R\varepsilon^{2\rho}$, $(R = e^{-\xi T}\frac{\pi}{\zeta_0+T}\mathcal{E}\hat{\mathbf{W}}(\zeta_0))$. We arrive at the conclusion that there must be a $\vartheta < \zeta_0+T$ such that $\mathcal{E}[V(\zeta, x(\zeta))] < R\varepsilon^{2\rho}$ for $\zeta \in [\vartheta, +\infty)$. If not, one derives $0 \leq \mathcal{E}[V(\zeta_0+T, x(\zeta_0+T))] < R\varepsilon^{2\rho} - c < 0$,(when ε is sufficiently small), which is a contradiction. Combining this with the initial condition $\mathbf{V}(0) = 0$, we arrive at $\underset{\varepsilon\to 0}{lim}\ \underset{\zeta\to\zeta_0+T}{lim}\mathcal{E}x(\zeta) = 0$, and there exists a $R(\varepsilon) \in R^+$ such that the state $\underset{\zeta\to\vartheta}{lim}\mathcal{E}x(\zeta)$ converges into the set $M = \{\mathcal{E}x(\zeta) \mid \|\mathcal{E}x(\zeta)\| \leq R(\varepsilon)$.

When $\zeta \in [\vartheta, +\infty)$, it is evident that $\frac{d((\zeta)\mathcal{E}\hat{\mathbf{W}}(\zeta))}{d\zeta} < 0$, causing $\mathcal{E}\hat{\mathbf{W}}(\zeta)$ to be monotonically decreasing. Despite this, due to $\lim_{\zeta\to\vartheta}\mathcal{E}\hat{\mathbf{W}}(\zeta) \leq R\varepsilon^{2\rho}$ and the nonnegativity of $\mathcal{E}\hat{\mathbf{W}}(\zeta)$, we conclude that $\mathcal{E}\hat{\mathbf{W}}(\zeta) \leq R\varepsilon^{2\rho}$ for $\zeta \in [\zeta_0+T, +\infty)$. Notice that the inequalities $0 \leq \mathcal{E}\hat{\mathbf{V}}(\zeta) \leq \mathcal{E}\hat{\mathbf{W}}(\zeta) \leq R\varepsilon^{2\rho}$, for $\zeta \in [\vartheta, +\infty)$. We arrive at the conclusion that $\mathcal{E}\hat{\mathbf{V}}(\zeta) \leq R\varepsilon^{2\rho}$ for $\zeta \in [\vartheta, +\infty)$, which further implies $\mathcal{E}x(\zeta) \in M$, almost surely, for $\zeta \in [\vartheta, +\infty)$. The proof is completed.

Corollary 3.3. *Let $V(\zeta) : R^n \to [0, +\infty)$ be a positive defined and radially unbounded function. $x(\zeta)$ is the state of (3.1), and $\Gamma(\zeta)$ is the time-varying scaling function. Define $V(x(\zeta)) = \mathbf{V}(\zeta)$, $\mathbf{v}_0(\zeta) = V(\mathbf{x}_0(\zeta))$. If there exist integrable function $a(\zeta) : [\zeta_0, +\infty) \to R^+$ and bounded functions $b_i(\zeta), (i = 1, 2, \cdots, m) : [\zeta_0, +\infty) \to [0, b]$, such that*

$$\begin{cases} \mathcal{L}\mathbf{V}(\zeta) \leq -a(\zeta)\mathbf{V}(\zeta) + \sum_i^m b_i(\zeta)\sup_{\zeta-\tau_i(\zeta)\leq s\leq\zeta}\mathbf{V}(s) + \frac{1}{h}\frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}\mathbf{V}(\zeta), \\ \mathbf{V}(\zeta) = \mathbf{v}_0(\zeta), \zeta \in [\zeta_0-\tau, \zeta_0), \end{cases}$$

(a). There exists a constant $\xi > 0$ that satisfies: $-a(\zeta) + \sum_i^m b_i(\zeta)e^{\tau\xi} \leq -\xi$, and $0 < h \leq \frac{1}{\tau_i e^{\xi}\sum_{i=1}^m b_i+1}$.

(b). There exists a constant $c > 0$ that satisfies: $-\mathbf{c}(\zeta) \leq c[-a(\zeta) + \sum_i^m b_i(\zeta)\lambda(\zeta)] - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}c$.

then there exists a $R(\varepsilon) \in R^+$ such that the state $\underset{\zeta\to\zeta_0+T}{lim}\mathcal{E}x(\zeta)$ converges into the set $M = \{\mathcal{E}x(\zeta) \mid \|\mathcal{E}x(\zeta)\| \leq R(\varepsilon)\}$, and $\mathcal{E}x(\zeta) \in M, a.s.$ when$\zeta \geq \zeta_0 + T$. (3.1) is prescribed-time quasi-stable in probability.

4. Applications to the stability analysis of stochastic time-delay systems

In this section, by the utilization of the novel fixed/prescribed stability criteria, we delve into the investigation of the fixed-time stabilization problem for a stochastic time-delay system, as well as the study of the prescribed-time quasi-stabilization issue for a multi-delay stochastic reaction-diffusion system.

4.1. Applications to a stochastic time-delay system

We consider a stochastic time-delay system

$$\begin{cases} dx(\zeta) = [ax(\zeta - \tau(\zeta)) + f(x(\zeta)) + u(\zeta)]d\zeta + g(x(\zeta), x(\zeta - \tau(\zeta))dw, \zeta \in [\zeta_0, \infty), \\ x(\zeta) = \mathbf{x}_0(\zeta), \zeta \in [\zeta_0 - \tau, \zeta_0), \end{cases} \tag{4.1}$$

where $x(\zeta) = (x_1(\zeta), x_2(\zeta), \cdots, x_n(\zeta)) \in R^n$ is the state vector, $\mathbf{x}_0(\zeta) \in PC^b_{\mathcal{F}_{\zeta_0}}([\zeta_0 - \tau, \zeta_0)$ stands for the initial function, and the vector field $f = (f_1, f_2, \cdots, f_n) : R^n \to R^n$, $g = (g_1, g_2, \cdots, g_n) : R^n \times R^+ \to R^n$ is a nonlinear function.

In order to investigate the fixed/prescribed-time stability of stochastic time-delay systems, the following assumption is made for nonlinear function g, f:

Assumption 1. *For any $x_1, x_2, y_1, y_2 \in R^n$, there exists a positive constant l such that the following inequality holds:*

$$\begin{aligned} & tr\left[g(x_1, y_1) - g(x_2, y_2)\right]^T \left[g(x_1, y_1) - g(x_2, y_2)\right] \\ \le & l\left[(x_1 - x_2)^T(x_1 - x_2) + (y_1 - y_2)^T(y_1 - y_2)\right]. \end{aligned}$$

Assumption 2. *If for $\forall x \in R^n$, there exists $m_i, b_i \in R$, $i = 1, 2, \cdots, n$ such that$f_i(x) \le m_i x + b_i$.*

The control mechanism is designed as follows.

$$u(\zeta) = -k_1 x(\zeta) - k_2 sign(x(\zeta)) \circ x^{\alpha}(\zeta) - k_3 sign(x(\zeta)) \circ x^{\beta}(\zeta) - \mathbf{c}(\zeta) \cdot sign(x(\zeta)) - B, \tag{4.2}$$

where $0 < \alpha < 1, \beta > 1$; $B = diag\{b_1, b_2, \cdots, b_n\}$; k_1, k_2, $k_3 > 0$ are constants and need to be designed later.

Theorem 4.1. *Under Assumption 2 and the control mechanism (4.2), if there exist constants k_1, k_2, k_3, c, which satisfy the following inequality*

$$\begin{gathered} k_1 \ge \max\{m_i\} + e^{\tau}|a| + 1, k_2 \ge 0, k_3 \ge 0, \\ -\mathbf{c}(\zeta) \le c[-a(\zeta) + b(\zeta)e^{\xi}] - h(k_2 c^{\alpha} + k_3 c^{\beta}). \end{gathered}$$

then the stochastic system (4.1) is fixed-time stable in probability, and the setting time $T(x_0) \le \frac{\Psi}{h}$, where $\Psi = \left(\frac{\pi csc(\pi y)}{\rho_3(\alpha+1-\beta)}\left(\frac{\rho_3}{\rho_2}\right)I\left(\frac{\rho_3}{\rho_3+\rho_2}, y, 1-y\right) + \frac{\pi csc(\pi z)}{\rho_3(\alpha-1+\beta)}\left(\frac{\rho_3}{\rho_1}\right)I\left(\frac{\rho_3}{\rho_3+\rho_1}, z, 1-z\right)\right)$, $\rho_1 = hk_2$, $\rho_2 = hk_3$, $\rho_3 = 1 + h$, and $h = \frac{1}{2^{1-\beta}\tau e+1}$.

Proof. Construct the Lyapunov function

$$V(\zeta) = \sum_{i=1}^{n} |x_i(\zeta)|. \tag{4.3}$$

Taking the derivative of V along the trajectory of system (4.1) gives $dV(\zeta) = \mathcal{L}V(x(\zeta))d\zeta + \mathcal{H}V(x(\zeta))dw$, and $\mathcal{H}V(x(\zeta))dw = x^T(\zeta)g(x(\zeta), x(\zeta - \tau(\zeta)))dw$, where

$$\mathcal{L}V(x(\zeta)) = \sum_{i=1}^{n} sign(x_i(\zeta))\Big[ax_i(\zeta - \tau(\zeta)) + f_i(x(\zeta)) + u_i(\zeta)\Big] \tag{4.4}$$

Additionally, we have $a \cdot sign(x_i(\zeta))x_i(\zeta-\tau(\zeta)) \le |ax_i(\zeta-\tau(\zeta))|$. According to Assumption 2, one derives

$$\begin{aligned}\mathcal{L}V(\zeta) \le& \sum_{i=1}^{n}(m_i sign(x_i(\zeta))x_i(\zeta) - k_1 sign(x_i(\zeta))x_i(\zeta)) + |a|\sum_{i=1}^{n}|x_i(\zeta-\tau(\zeta))| \\ &- \sum_{i=1}^{n}\Big(k_2 x_i^{\alpha}(\zeta) + k_3 x_i^{\beta}(\zeta)\Big) + \sum_{i=1}^{n} b_i sign(x_i(\zeta)) - \mathbf{c}(\zeta) \\ \le& - pV(\zeta) + |a|V(\zeta-\tau(\zeta)) - k_2 V^{\alpha}(\zeta) - k_3 V^{\beta}(\zeta) + \sum_{i=1}^{n} b_i sign(x_i(\zeta)) - \mathbf{c}(\zeta).\end{aligned}$$

where $p = (k_1 - \max\{m_i\})$. Denote $a(\zeta) = p$, $b(\zeta) = |a|$, $\xi = 1$, and we have $-a(\zeta) + b(\zeta)e^{\tau\cdot 1} \le -1$.

Thus, according to Corollary 3.2, the system (4.1) is fixed-time stable in probability, and the setting time $T(x_0) \le \frac{\Psi}{h}$, where $\Psi = \frac{\pi csc(\pi y)}{\rho_3(\alpha+1-\beta)}\Big(\frac{\rho_3}{\rho_2}\Big)I\Big(\frac{\rho_3}{\rho_3+\rho_2}, y, 1-y\Big) + \frac{\pi csc(\pi z)}{\rho_3(\alpha-1+\beta)}\Big(\frac{\rho_3}{\rho_1}\Big)I\Big(\frac{\rho_3}{\rho_3+\rho_1}, z, 1-z\Big)$, $\rho_1 = hk_2$, $\rho_2 = hk_3$,$\rho_3 = 1+h$, and $h = \frac{1}{2^{1-\beta}\tau e+1}$. The proof is completed.

4.2. Applications to multi-delay stochastic reaction-diffusion system

We consider the following multi-delay stochastic reaction-diffusion system:

$$\begin{cases} dv(x,\zeta) = \Big[a\frac{\partial^2 v(x,\zeta)}{\partial x^2} + \sum_{i=1}^{N} b_i v(x,\zeta-\tau_i(\zeta)) + u(x,\zeta)\Big]d\zeta \\ \qquad\qquad +g(v(x,\zeta), v(x,\zeta-\tau(\zeta)))dw, \\ v_x(0,\zeta) = 0, \quad v_x(l,\zeta) = 0, \quad x \in (0,l), \\ v(x,\zeta) = v_0(x,\zeta), \zeta \in [\zeta_0-\tau, \zeta_0). \end{cases} \tag{4.5}$$

where $v \in R^n$ denotes the state vector, $x \in [0,l]$ is the space variable, $\zeta \in [0,+\infty)$ is the time variable, $a > 0$ is the diffusivity parameter, and $0 \le \tau_i(\zeta)$ represents the unknown time-delay, and we just know its upper bound τ. $v_0(x,\zeta) \in PC^b_{\mathcal{F}_{\zeta_0}}(R, [\zeta_0-\tau, \zeta_0))$ stands for the initial function; $u(x,\zeta)$ stands for the control input. $w(\zeta)$ is the n-dimensional wiener process in complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\zeta\}_{\zeta\ge 0}, P)$. $g: R^n \times R^+ \to R^n$ is a nonlinear function, and $g(0,0) = 0$.

The control mechanism is designed as follows.

$$u(x,\zeta) = -k_1 v(x,\zeta) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}v(x,\zeta) - \mathbf{c}(\zeta)\cdot\frac{sign(v(x,\zeta))\circ v(x,\zeta)}{l\cdot\|v(x,\zeta)\|_2^2}, \tag{4.6}$$

where $\Gamma(\zeta) = \sec^2(\frac{\pi\zeta}{2(\zeta_0+T)})$, when $\zeta \in [\zeta_0, \zeta_0+T)$. $\Gamma(\zeta) = 0$, when $\zeta \in [\zeta_0+T, \infty]$, $k_1 > 0$, and $0 < \varepsilon \ll 1$.

Theorem 4.2. *Under Assumption 1 and the control mechanism (4.6), if there exist constants k_1, $c > 0$ that satisfy the following inequality:*

$$\begin{aligned} -\mathbf{c}(\zeta) &\le -c - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}c, \\ k_1 &\ge 2\sum_{i=1}^{N} b_i - \frac{\pi^2}{2l^2}a + (\sum_{i=1}^{N} b_i)e^{\tau} + 1. \end{aligned}$$

then the multi-delay stochastic reaction-diffusion system (4.5) is prescribed-time quasi-stable in probability.

Proof. Construct the Lyapunov function

$$V(\zeta) = \int_0^l \frac{1}{2} v^T(x,\zeta) v(x,\zeta) dx. \tag{4.7}$$

Taking the derivative of V along the trajectory of system (4.5) gives

$$\begin{aligned} dV(\zeta) =& \mathcal{L}V(v(x,\zeta))d\zeta + \mathcal{H}V(v(x,\zeta))dw \\ =& \int_0^l v^T(x,\zeta)\Big[a\frac{\partial^2 v(x,\zeta)}{\partial x^2} + \sum_{i=1}^N b_i v(x,\zeta-\tau_i(\zeta)) + u(x,\zeta)\Big]dxd\zeta \\ &+ \int_0^l \frac{1}{2} tr\Big\{g^T(v(x,\zeta), v(x,\zeta-\tau(\zeta)))g(v(x,\zeta), v(x,\zeta-\tau(\zeta)))\Big\}dxd\zeta \\ &+ \int_0^l v^T(x,\zeta) g(v(x,\zeta), v(x,\zeta-\tau(\zeta)))dxdw. \end{aligned} \tag{4.8}$$

Based on Assumption 1, we obtain

$$\begin{aligned} &tr[g(v(x,\zeta), v(x,\zeta-\tau(\zeta)))]^T[g(v(x,\zeta), v(x,\zeta-\tau(\zeta)))] \\ \leq& lv(x,\zeta)^T v(x,\zeta) + lv^T(x,\zeta-\tau(\zeta)))v(x,\zeta-\tau(\zeta)))). \end{aligned} \tag{4.9}$$

It is not difficult to obtain

$$\begin{aligned} &\int_0^l v^T(x,\zeta) a \frac{\partial^2 v(x,\zeta)}{\partial x^2} dx = a\int_0^l v^T(x,\zeta) d\Big(\frac{\partial v(x,\zeta)}{\partial x}\Big) \\ =& a\Big[v^T(x,\zeta)\frac{\partial v(x,\zeta)}{\partial x}\Big|_{x=l} - v^T(x,\zeta)\frac{\partial v(x,\zeta)}{\partial x}\Big|_{x=0}\Big] - a\int_0^l \Big(\frac{\partial v(x,\zeta)}{\partial x}\Big)^T \frac{\partial v(x,\zeta)}{\partial x} dx \\ =& -a\int_0^l \Big(\frac{\partial v(x,\zeta)}{\partial x}\Big)^T \frac{\partial v(x,\zeta)}{\partial x} dx. \end{aligned} \tag{4.10}$$

According to the Wirtinger's inequality [19], we have

$$\begin{aligned} a\int_0^l \Big(\frac{\partial v(x,\zeta)}{\partial x}\Big)^T \frac{\partial v(x,\zeta)}{\partial x} dx =& a\int_0^l (\frac{\partial (v(x,\zeta) - v(l,\zeta))}{\partial x})^T \frac{\partial (v(x,\zeta) - v(l,\zeta))}{\partial x} dx \\ \geq& \frac{\pi^2}{4l^2} a \int_0^l (v(x,\zeta) - v(l,\zeta))^T (v(x,\zeta) - v(l,\zeta)) dx \\ =& \frac{\pi^2}{4l^2} a \int_0^l v(x,\zeta)^T v(x,\zeta) dx. \end{aligned} \tag{4.11}$$

and

$$b_i v^T(x,\zeta) v(x,\zeta-\tau_i(\zeta)) \leq \frac{b_i}{2} v^T(x,\zeta)v(x,\zeta) + \frac{b_i}{2} v^T(x,\zeta-\tau_i(\zeta)))v(x,\zeta-\tau_i(\zeta)))). \tag{4.12}$$

Substitute (4.9)–(4.12) into (4.8), then

$$
\begin{aligned}
dV(\zeta) =& \mathcal{L}V(x(\zeta))d\zeta + \mathcal{H}V(x(\zeta))dw \\
=& \Big[-\frac{\pi^2}{4l^2} a \int_0^l v(x,\zeta)^T v(x,\zeta)dx + \frac{1}{2}\sum_{i=1}^N b_i \int_0^l v(x,\zeta)^T v(x,\zeta)dx \\
&+ \frac{1}{2}\sum_{i=1}^N b_i \int_0^l v^T(x,\zeta-\tau_i(\zeta)))v(x,\zeta-\tau_i(\zeta)))) + \int_0^l \frac{1}{2} v^T(x,\zeta)u(x,\zeta)dx \Big] d\zeta \\
&+ \Big[\int_0^l v^T(x,\zeta)g(v(x,\zeta),v(x,\zeta-\tau(\zeta)))dx \Big] dw \\
=& \left\{ \Big[-\frac{\pi^2}{2l^2} a + \sum_{i=1}^N b_i - k_1 \Big] V(\zeta) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)} V(\zeta) + \sum_{i=1}^N b_i \mathcal{E} V(\zeta-\tau_i(\zeta)) - \mathbf{c}(\zeta) \right\} d\zeta \\
&+ \Big[\int_0^l v^T(x,\zeta)g(v(x,\zeta),v(x,\zeta-\tau(\zeta)))dx \Big] dw \\
=& \left\{ \Big[\Big(\sum_{i=1}^N b_i \Big) e^{\tau} + 1 \Big] V(\zeta) + \Big(\sum_{i=1}^N b_i \Big) \sup_{\zeta-\tau\le s\le\zeta} V(s,x(s)) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)} V(\zeta) - \mathbf{c}(\zeta) \right\} d\zeta \\
&+ \Big[\int_0^l v^T(x,\zeta)g(v(x,\zeta),v(x,\zeta-\tau(\zeta)))dx \Big] dw.
\end{aligned}
$$

Thus, we have $\mathcal{L}V(x(\zeta)) \le \Big[\Big(\sum_{i=1}^N b_i\Big) e^{\tau} + 1 \Big] V(\zeta) + \Big(\sum_{i=1}^N b_i\Big) \sup_{\zeta-\tau\le s\le\zeta} V(s,x(s)) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)} V(\zeta)$. Denote $-a(\zeta) = \Big[\Big(\sum_{i=1}^N b_i\Big) e^{\tau} + 1 \Big]$, $b(\zeta) = \Big(\sum_{i=1}^N b_i\Big)$, $\xi = 1$, and we have $-a(\zeta) + b(\zeta)e^{\tau\cdot 1} \le -1$. According to Theorem 3.3, the multi-delay stochastic reaction-diffusion system (4.5) is prescribed-time quasi-stable in probability. The proof is completed.

Table 2. Different control mechanism of time-delay system.

Ref	control objective	control mechanism
[35]	prescribed-time stability	$u(\zeta) = -k_1 e(\zeta) - k_2 e^{\frac{\alpha}{\beta}}(\zeta) - k_2[\int_{\zeta-\tau(\zeta)}^{\zeta} e^T(s)e(s)ds]^{\frac{\alpha+\beta}{2\beta}} \frac{e(\zeta)}{\|e(\zeta)\|^2}$
[39]	prescribed-time stability	$u(\zeta) = -(k_1 + k_2 \frac{\dot{\Gamma}(\zeta)}{\Gamma(\zeta)})e(\zeta) - (k_2 + k_3 \frac{\dot{\Gamma}(\zeta)}{\Gamma(\zeta)}) \int_{\zeta-\tau(\zeta)}^{\zeta} e^T(s)e(s)ds$
this paper	prescribed-time stability	$u(x,\zeta) = -k_1 v(x,\zeta) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)} v(x,\zeta) - \mathbf{c}(\zeta) \cdot \frac{sign(v(x,\zeta))\circ v(x,\zeta)}{l\cdot\|v(x,\zeta)\|_2^2}$, k_1 satisfies (4.6)
[40]	fixed-time stability	$u(\zeta) = -\frac{k_1}{2} \frac{y(\zeta,1)}{\lvert y(\zeta,1)\rvert^2} (\int_0^1 y^T y dx + k_2 \int_0^1 \int_{\zeta-\tau(\zeta)}^{\zeta} y^T(s,x)y(s,x)dsdx)^{\delta}$
[41]	fixed-time stability	$u(\zeta) = -sign(e(\zeta))(k_1 e(\zeta) + k_2 e(\zeta-\tau(\zeta)) + k_2\|e(\zeta)\|^p + k_3\|e(\zeta)\|^q + k_2\|e(\zeta-\tau(\zeta))\|^p$ $+k_3\|e(\zeta-\tau(\zeta))\|^q)$
[33]	fixed-time stability	$u(\zeta) = -k_1 sign(e(\zeta))\lvert e(\zeta)\rvert - k_2 sign(e(\zeta))\lvert e(\zeta-\tau(\zeta))\rvert + k_2 sign(e(\zeta))\lvert e(\zeta)\rvert^p$
[34]	fixed-time stability	$u(\zeta) = -diag\{sign(e(\zeta))\}(sign(e^{\frac{1}{q}}(\zeta))\lvert e(\zeta)\rvert^{\frac{1}{q}} + sign(e(\zeta-\tau(\zeta)))\lvert e(\zeta-\tau(\zeta))\rvert)$
this paper	fixed-time stability	$u(\zeta) = -k_1 x(\zeta) - k_2 sign(x(\zeta)) \circ x^{\alpha}(\zeta) - k_3 sign(x(\zeta)) \circ x^{\beta}(\zeta) - \mathbf{c}(\zeta) \cdot sign(x(\zeta)) - B$

Table 2 gives the difference between the control mechanism in this paper and other literature.

Remark 6. *In most fixed/prescribed-time control methods for time-delay systems, the common approach is to directly incorporate the time-delayed states $e(\zeta-\tau(\zeta))$ into the controller to mitigate the influence of system delay. However, this approach is only suitable when the delay is known. When the time-delay is unknown, these control designs become ineffective. In contrast, the fixed/prescribed-time controller (4.2), (4.6) is independent from the time-delayed states $e(\zeta-\tau(\zeta))$. This means that even in the presence of unknown delay, the time-delayed states $e(\zeta-\tau(\zeta))$ are unavailable, and the controller (4.2), (4.6) can still achieve fixed/prescribed-time control objectives.*

Remark 7. *It is worth noting that the control mechanisms proposed in this paper (4.2), (4.6) have the following limitations: 1) Both controllers contain sign functions, which can lead to chattering phenomena. 2) Controller (4.6) is a full-state controller, requiring actuators to be deployed across the entire two-dimensional space, potentially reducing its applicability compared to boundary controllers.*

5. Numerical simulation

In this section, two examples are presented to verify the theoretical analysis and to test the effectiveness of the controller. *Example 1.* Consider the following stochastic delay system:

$$\begin{cases} dx(\zeta) = [-\zeta x(\zeta) + 2\zeta x(\zeta-\tau(\zeta)) + sin(x(\zeta)) + u(\zeta)]d\zeta \\ \qquad\quad +x(\zeta)cos(x(\zeta))dw, \zeta \in [\zeta_0, \infty), \\ x(\zeta) = 9, \zeta \in [-1, \zeta_0), \end{cases} \tag{5.1}$$

where $x \in R$ is the state vector, $\tau(\zeta) = \frac{1}{1+\zeta}$. The control protocol is designed as

$$u(\zeta) = -(e\zeta + \frac{3}{2})x(\zeta) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}x(\zeta) - \frac{\mathbf{c}(\zeta)sign(x(\zeta))}{\|x(\zeta)\|} - sign(x(\zeta)), \tag{5.2}$$

where $\rho = 2$. $-\mathbf{c}(\zeta) \leq -c - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}c$, $c = 0.01$, $\varepsilon = 0.01$

$$\Gamma(\zeta) = \begin{cases} \sec^{\rho}(\frac{\pi\zeta}{2(\zeta_0+T_C)}), & \zeta \in [\zeta_0, \zeta_0 + T_C), \\ 0, & \zeta \in [\zeta_0 + T_C, \infty]. \end{cases}$$

Let $V(x(\zeta)) = x^T(\zeta)x(\zeta)$, for $\zeta \geq \zeta_0$, and we obtain

$$\begin{aligned} \mathcal{L}V(x(\zeta)) =& 2x^T(\zeta)\big[-\zeta x(\zeta) + 2\zeta x(\zeta-\tau(\zeta)) + sin(x(\zeta)) + u(\zeta)\big] \\ \leq& 2\zeta x^T(\zeta-\tau(\zeta))x(\zeta-\tau(\zeta)) + 2x^T(\zeta)x(\zeta) + 2x^T(\zeta)u(\zeta) \\ \leq& -4e\zeta\mathcal{E}V(x(\zeta)) + 4\zeta\mathcal{E}\sup_{\zeta-\tau(\zeta)\leq s\leq\zeta} V(s) - \mathcal{E}V(x(\zeta)) - 2\frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)}V(x(\zeta)) - \mathbf{c}(\zeta). \end{aligned} \tag{5.3}$$

Obviously, $a(\zeta) = 4e\zeta + 1$, $b(\zeta) = 4\zeta$, $0 < \tau(\zeta) = \frac{1}{1+\zeta} < 1$, $f(x(\zeta)) = sin(x(\zeta))$. We choose $\xi(\zeta) = 1$, then $\xi = \sup_{\zeta>0}\int_{\zeta-\frac{1}{1+\zeta}}^{\zeta} 1ds \leq \int_{\zeta_0}^{1} 1ds = 1$. Therefore, $-a(\zeta) + b(\zeta)e^{\tau\xi} \leq -\xi$ is satisfied.

Set $T_C = 14$. Figure 1 records the trajectory of states $x(\zeta)$ with $u(\zeta) = 0$. Figure 2 records the trajectory of states $x(\zeta)$ with controller (5.2). From Figure 2, we can see the $x(\zeta)$ tends to zero before the prescribed-time $T_C = 14$. This shows that system (5.1) is prescribed-time quasi-stable in probability under the controller (5.2).

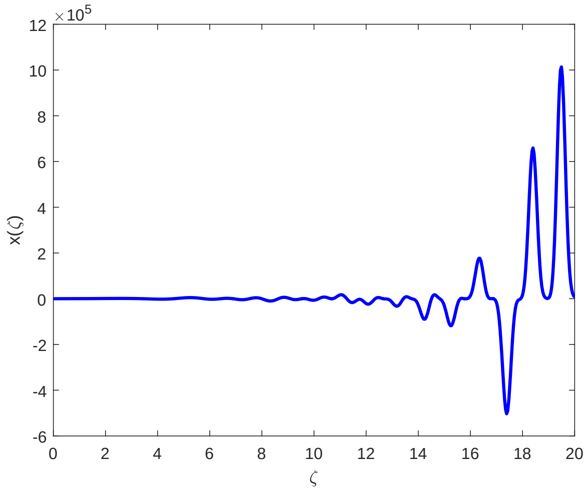

Figure 1. The states response $x(\zeta)$ with $u(\zeta) = 0$.

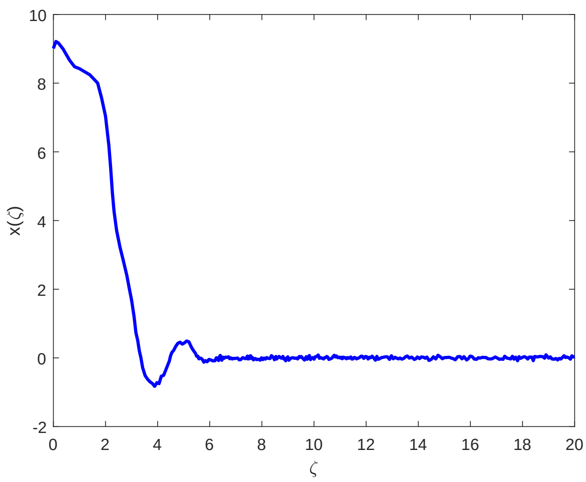

Figure 2. The states response $x(\zeta)$ with controller (5.2).

Example 2. We consider the following multi-delay stochastic reaction-diffusion system:

$$\begin{cases} dv(x,\zeta) = \left[\frac{\partial^2 v(x,\zeta)}{\partial x^2} + \sum_{i=1}^{2} b_i v(x,\zeta-\tau_i(\zeta)) + u(x,\zeta)\right] d\zeta + v(x,\zeta)dw, \\ v_x(0,\zeta) = 0, \quad v_x(10,\zeta) = 0, \quad x \in (0,10), \\ v(x,0) = sin(3x) - cos(x), \zeta \in [-0.3,\zeta_0). \end{cases} \tag{5.4}$$

where $v \in R^n$ denotes the state vector, $x \in [0,10]$ is the space variable, and $\zeta \in [0,+\infty)$ is the time variable.

Take $\zeta_0 = 0$, $b_1 = 0.03$, $b_2 = 0.02$, $\tau_1(\zeta) = 0.2$, $\tau_2(\zeta) = 0.3$, $T = 3$. The control mechanism is designed as follows.

$$u(x,\zeta) = -k_1 v(x,\zeta) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)} v(x,\zeta) - \frac{\mathbf{c}(\zeta) sign(v(x,\zeta))}{\|v(x,\zeta)\|}, \tag{5.5}$$

where $k_1 \geq 1 - \frac{\pi^2}{200} + (0.05)e^{0.3}$. Construct the Lyapunov function

$$V(\zeta) = \int_{\zeta_0}^{l} \frac{1}{2} v^T(x,\zeta) v(x,\zeta) dx. \tag{5.6}$$

It's not difficult to verify that

$$\mathcal{L}V(\zeta)d\zeta \leq \left[(0.05)\, e^{0.3} + 1\right] V(\zeta) + 0.05 \sup_{\zeta - 0.3 \leq s \leq \zeta} V(s) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)} V(\zeta) - \mathbf{c}(\zeta). \tag{5.7}$$

where $-\mathbf{c}(\zeta) \leq -c - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{\Gamma(\zeta-\varepsilon)} c$, $c = 0.01$, $\varepsilon = 0.01$. Denote $-a(\zeta) = \left[0.05e^{0.3} + 1\right]$, $b(\zeta) = 0.05$, $\xi = 1$, and we have $-a(\zeta) + b(\zeta)e^{\tau \cdot 1} \leq -1$. Figure 3 records the trajectory of states $v(x,\zeta)$ of system (5.4) without the controller. Figure 4 records the trajectory of states $v(x,\zeta)$ of system (5.4) under the controller (5.5). Figure 5 displays the evolution of $u(x,\zeta)$. From Figure 4, we can see the states $v(x,\zeta)$ converge to the neighborhood of 0 in prescribed time $T = 3$.

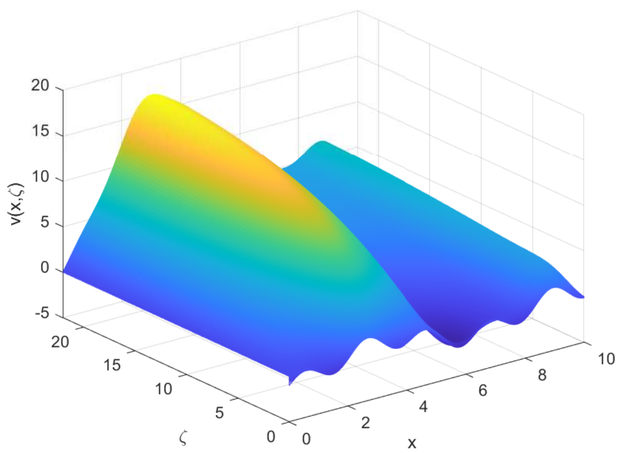

Figure 3. The states response $v(x,\zeta)$ with $u(x,\zeta) = 0$.

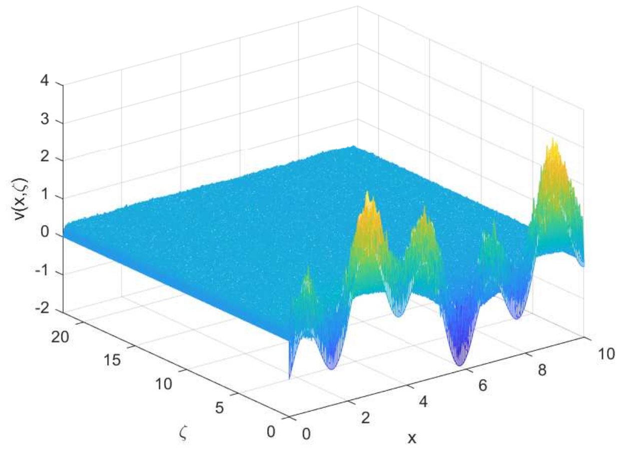

Figure 4. The states response $v(x,\zeta)$ with controller (5.5).

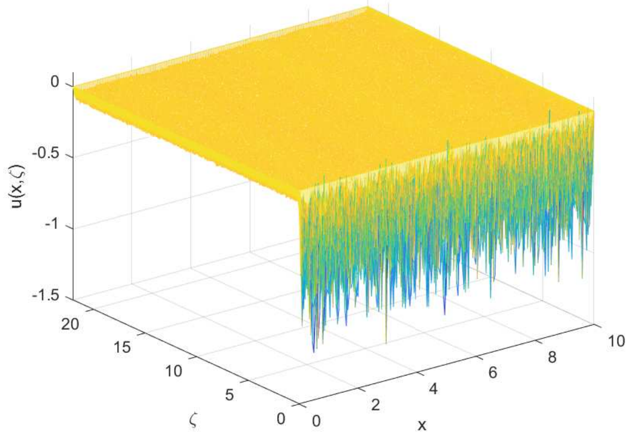

Figure 5. The control input $u(x,\zeta)$.

Example 3. Consider a second-order stochastic strict feedback as follows:

$$\begin{cases} dx_1(\zeta) = (x_2(\zeta) + sin(x_1(\zeta)))d\zeta + x_1(\zeta)dw \\ dx_2(\zeta) = (x_1(\zeta)\cdot x_2(\zeta) + u(\zeta))d\zeta, \end{cases} \tag{5.8}$$

Denote $e_1(\zeta) = x_1(\zeta)$, $\xi(\zeta) = -sin(x_1(\zeta)) - x_1(\zeta)\frac{\dot{\Gamma}(\zeta-\varepsilon)}{2\Gamma(\zeta-\varepsilon)}e_1(\zeta)$, $e_2(\zeta) = x_2(\zeta) - \xi(\zeta)$. The designed controller is $u(\zeta) = -x_1(\zeta)\cdot x_2(\zeta) - \dot{\xi}(\zeta) + \frac{\dot{\Gamma}(\zeta-\varepsilon)}{2\Gamma(\zeta-\varepsilon)}(x_2(\zeta) - \xi(\zeta))$, where $\rho = 2$, $\varepsilon = 0.001$.

$$\Gamma(\zeta) = \begin{cases} \sec^{\rho}(\frac{\pi\zeta}{2(\zeta_0+T_C)}), & \zeta \in [\zeta_0, \zeta_0 + T_C), \\ 0, & \zeta \in [\zeta_0 + T_C, \infty], \end{cases}$$

Let $V(\zeta) = \frac{1}{2}e_1^2(\zeta) + \frac{1}{2}e_2^2(\zeta)$. One obtains

$$\begin{aligned}\mathcal{L}V(x(\zeta)) =& e_1(\zeta)(x_2(\zeta) + sin(x_1(\zeta))) + x_1^2(\zeta) + e_2(\zeta)(x_1(\zeta)\cdot x_2(\zeta) + u(\zeta) + \dot{\xi}(\zeta))\\ \leq& e_1(\zeta)[\xi(\zeta) + sin(x_1(\zeta)) + x_1(\zeta)] + e_2(\zeta)(x_1(\zeta)\cdot x_2(\zeta) + u(\zeta) + \dot{\xi}(\zeta))\\ \leq& -\frac{\dot{\Gamma}(\zeta-\varepsilon)}{2\Gamma(\zeta-\varepsilon)}e_1^2(\zeta) - \frac{\dot{\Gamma}(\zeta-\varepsilon)}{2\Gamma(\zeta-\varepsilon)}e_2^2(\zeta) = -\frac{\dot{\Gamma}(\zeta-\varepsilon)}{2\Gamma(\zeta-\varepsilon)}V(\zeta).\end{aligned} \tag{5.9}$$

Set the preset time as $T = 1$. The trajectories of states of different initial values with respect system (5.8) is shown in Figure 6. The trajectories of control input is shown in Figure 7. From Figure 6, we can see that $\mathcal{E}x_i, (i = 1, 2)$ with different initial values, and $u(t)$ tends to zero before the preset time $T = 1$.

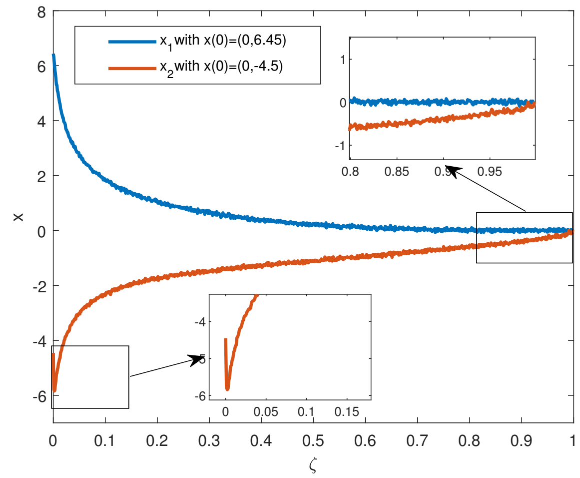

Figure 6. The states response $x_1(\zeta)$, $x_2(\zeta)$.

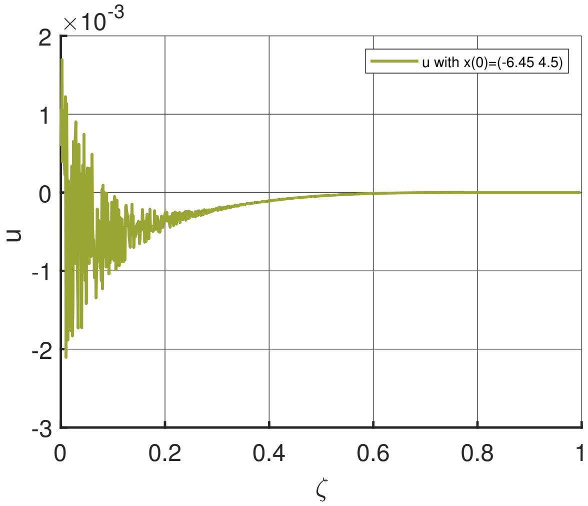

Figure 7. The control input u.

6. Conclusions

This paper has examined the fixed/prescribed-time stability issues in stochastic delay systems. It has established novel fixed-time stability and prescribed-time stability criteria for both stochastic delay systems and multi-delay systems. Additionally, the fixed-time stabilization of a stochastic time-delay system and the prescribed-time stabilization of a multi-delay stochastic reaction-diffusion system have been investigated. Two new delay-independent control mechanisms have been designed. By utilizing the newly established fixed/prescribed-time stability criteria, the conditions for determining the control gain of the delay-independent controller have been obtained. In the future, the focus will be on the study of prescribed-time stability for complex networks with delay impulses.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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