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Research article

Dynamics analysis of dengue fever model with harmonic mean type under fractal-fractional derivative

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Abstract: Dengue is a viral illness transmitted by Aedes mosquitoes and is a significant global threat. In this study, we developed a model of the dengue epidemic that incorporates larvicide and adulticide, as well as the harmonic mean incidence rate under fractal-fractional derivatives. We examined various theoretical aspects of the model, including nonnegativity, boundedness, existence, uniqueness, and stability. We computed the basic reproduction number $\mathfrak{R}_0$ using the next-generation matrix. The model has two disease-free equilibriums, a trivial equilibrium, and a biologically realistic, along with one endemic equilibrium point. These findings enhanced our understanding of dengue transmission, providing valuable insights for awareness campaigns, control strategies, intervention approaches, decision support, guiding public health planning, and resource allocation to manage dengue effectively.

Keywords: dengue diseases; fractal-fractional model; stability; equilibrium points; basic reproduction numbers; simulation analysis
Mathematics Subject Classification: 92D30, 26A33, 34D20, 34Cxx, 92D30, 65C05

1. Introduction

Mosquitoes are among the most dangerous active vectors of several diseases such as dengue, Zika, and malaria [43]. It is endemic in more than 100 countries, particularly in tropical and subtropical regions, and manifests in various forms, ranging from asymptomatic cases to severe illnesses [33]. The majority of dengue cases, approximately 75%, present as dengue fever (DF), characterized by symptoms such as high fever, severe headache, joint and muscle pain, and a rash. In certain cases, dengue fever can advance to more severe conditions such as dengue hemorrhagic fever (DHF) and dengue shock syndrome (DSS), leading to manifestations of bleeding, organ failure, and a significant risk of mortality if immediate treatment is not provided [47]. Diagnosing dengue is challenging due to similarities in symptoms with other febrile illnesses and the need for specialized laboratory testing. Effective control strategies require integrated approaches that combine vector control measures, public awareness, early case detection, and a strengthened healthcare system for surveillance and response [15, 45]. Researchers have employed and developed mathematical models with various objectives to comprehend the intricacies of dengue fever, as documented in references [4, 16–18, 35, 37, 38, 40]. In the works [1, 2, 10, 46, 48], the authors focused on optimal control strategies for the spread of dengue fever to identify effective measures to mitigate the transmission of the disease. The aforementioned dengue fever models are expressed in the form of ordinary integer-order derivatives, which depict the dynamics of dengue infection. However, these models have limitations as they do not incorporate information about memory and learning mechanisms. Moreover, exploring the dynamics between two distinct points using models based on integer-order derivatives can be challenging. Researchers have proposed various alternative models in fractional orders to address these concerns. These alternative models aim to overcome the limitations of the traditional models by incorporating memory and learning mechanisms, and by facilitating the exploration of dynamics between different points.

Atangana and Baleanu have made significant contributions to fractional calculus by introducing operators based on fractional integrals and derivatives that utilize the Mittag-Leffler function [13]. The utilization of fractional order has proven valuable in studying complex phenomena and has been applied in various research areas, including disease modeling [6, 14, 22, 26, 28, 30, 34, 36, 41]. These studies collectively demonstrate the versatility and effectiveness of fractional calculus in elucidating the dynamics and mechanisms of infectious diseases, paving the way for improved disease understanding, control, and prevention strategies. Some studies have investigated recent mathematical models of dengue fever incorporating fractional order and contributed to the understanding of the dynamics and complexities of dengue infection [5, 19, 23, 27]. The transmission dynamics of dengue infection investigated by considering vaccination, treatment, and reinfection using the Atangana-Baleanu operator have been studied by Jan et al. [29]. Through their analysis, the researchers observed that the control parameters, including the index of memory, biting rate, transmission probability, and recruitment rate of mosquitoes, can be manipulated to reduce the infection intensity [24]. Moreover, in another study [25], the authors demonstrated that manipulating the index of memory within the system can effectively control the reproduction number and the endemic level of dengue infection.

Recently, Atangana [11] introduced novel fractal-fractional differential and integral operators, combining power law, exponential decay law, and generalized Mittag-Leffler law with fractal

derivatives to address nonlocal natural problems exhibiting fractal behavior [7–9]. In the literature, there are a few studies of the dengue model in the context of the fractal and fractional operator. For example, Khan et al. [31] studied the dynamics of dengue infection using fractal-fractional operators and real statistical data to gain a deeper understanding of the complex dynamics of dengue transmission by incorporating fractal and fractional approaches into the modeling framework. Fatmawati et al. [20] studied the dengue model with the fractal-fractional Caputo–Fabrizio operator. The harmonic mean type plays a significant role in the mathematical modeling of diseases by facilitating the calculation of incidence rates, capturing transmission dynamics, evaluating control measures, aiding parameter estimation, and enabling sensitivity analysis. Its inclusion enhances the accuracy and robustness of disease models, contributing to a better understanding of disease dynamics and informing public health interventions. There are some researchers studying the harmonic mean type incidence rate in modeling some diseases such as the smoking model [44], hepatitis B virus model [49], and the leishmania epidemic model [32].

Inspired by the studies above, our objective is to extend the works [1, 2, 21] and construct a deterministic compartmental model involving harmonic mean type incidence rate. Our focus is on examining the effects of personal protection measures and insecticide campaigns (including adulticide and larvicide) to prevent and control the transmission of dengue fever. This work presents several novel aspects that contribute to the understanding of dengue disease dynamics. To begin, it introduces an innovative eight-compartmental (φ_1, φ_2)-fractal-fractional model (FFM) that allows for a more detailed representation of the disease transmission process. Furthermore, the incorporation of a harmonic mean type incidence rate for both human and mosquito populations provides a realistic depiction of the reciprocal relationship between infected individuals, enhancing the accuracy of the model. Additionally, the utilization of generalized Mittag-Leffler fractal-fractional derivatives offer a comprehensive framework for describing the complex dynamics of dengue disease. Finally, the exploration of the impact of awareness campaigns on dengue transmission dynamics provides valuable insights into the effectiveness of such interventions in controlling the spread of the disease.

The remainder of this paper is structured as follows: Section 2 provides an introduction to our model, offering a description of its fractal fractional nature. In Section 3, we introduce the definitions and important lemmas related to the fractal fractional operator. Section 4 focuses on identifying an invariant region and analyzing the positivity and boundedness of the model. We also examine the equilibrium points and calculate the basic reproduction number, denoted as $\mathfrak{R}_0$. Additionally, the Lipschitz property and stability of the model are discussed in this section. Section 5 is dedicated to establishing sufficient conditions for the existence and uniqueness of solutions. In Section 6, we present a numerical scheme for solving the model, outlining the methodology and algorithms employed in the numerical analysis. The numerical results and corresponding discussions are presented in Section 7. Finally, the last section concludes the paper with some final remarks, summarizing the key contributions of our work and highlighting potential avenues for future research.

2. Description of the model

Here, we investigate a dengue model that incorporates the harmonic mean type incidence rate to accurately describe the evolutionary dynamics of the disease within the framework of fractal fractional derivatives. The model can be expressed as follows:

$$
\begin{aligned}
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{S}(\iota) &= \Theta - 2b_v\beta_H\left(\frac{\mathbb{I}_v(\iota)\,\mathbb{S}(\iota)}{\mathbb{I}_v(\iota)+\mathbb{S}(\iota)}\right) - \varpi\mathbb{S}(\iota),\\
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{E}(\iota) &= 2b_v\beta_H\left(\frac{\mathbb{I}_v(\iota)\,\mathbb{S}(\iota)}{\mathbb{I}_v(\iota)+\mathbb{S}(\iota)}\right) - (\mu+\varpi)\,\mathbb{E}(\iota),\\
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{I}(\iota) &= \mu\mathbb{E}(\iota) - (\gamma+\varpi)\,\mathbb{I}(\iota),\\
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{R}(\iota) &= \gamma\mathbb{I}(\iota) - \varpi\mathbb{R}(\iota),\\
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{A}(\iota) &= \varpi_B\left(1-\frac{\mathbb{A}(\iota)}{K}\right)(\mathbb{S}_v(\iota)+\mathbb{E}_v(\iota)+\mathbb{I}_v(\iota)) - (\mu_{\mathbb{A}}+\varpi_{\mathbb{A}})\,\mathbb{A}(\iota),\\
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{S}_v(\iota) &= \mu_{\mathbb{A}}\mathbb{A}(\iota) - b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\,\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right) - \varpi_v\mathbb{S}_v(\iota),\\
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{E}_v(\iota) &= b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\,\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right) - (\mu_v+\varpi_v)\,\mathbb{E}_v(\iota),\\
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{I}_v(\iota) &= \mu_v\mathbb{E}_v(\iota) - \varpi_v\mathbb{I}_v(\iota),
\end{aligned}
\tag{2.1}
$$

with the initial conditions

$$
\begin{aligned}
\mathbb{S}(0) &> 0,\ \mathbb{E}(0)>0,\ \mathbb{I}(0)>0,\ \mathbb{R}(0)>0,\\
\mathbb{A}(0) &> 0,\ \mathbb{S}_v(0)>0,\ \mathbb{E}_v(0)>0,\ \mathbb{I}_v(0)>0,
\end{aligned}
\tag{2.2}
$$

where $\Theta = \varpi\mathbb{P}_h(\iota)$ and ${}^{FFML}D_0^{\varphi_1,\varphi_2}$ is the (φ_1,φ_2)-fractal-fractional derivation operator equipped with the fractional order φ_1 and fractal order φ_2 such that $\varphi_1,\varphi_2 \in (0,1]$. At time ι, the total human population $\mathbb{P}_h(\iota)$, is divided into four distinct epidemiological classes, susceptible individuals $\mathbb{S}(\iota)$, exposed individuals $\mathbb{E}(\iota)$, infectious individuals $\mathbb{I}(\iota)$ and recovered individuals $\mathbb{R}(\iota)$, then, $\mathbb{P}_h(\iota) = \mathbb{S}(\iota)+\mathbb{E}(\iota)+\mathbb{I}(\iota)+\mathbb{R}(\iota)$. Similarly, at time ι, the total female mosquito population $\mathbb{P}_v(\iota)$ is divided into three compartments, susceptible mosquitoes $\mathbb{S}_v(\iota)$, exposed mosquitoes $\mathbb{E}_v(\iota)$ and infectious mosquitoes $\mathbb{I}_v(\iota)$, then, $\mathbb{P}_v(\iota) = \mathbb{S}_v(\iota)+\mathbb{E}_v(\iota)+\mathbb{I}_v(\iota)$. In model (2.1), the total size of the aquatic mosquito population represented by the epidemiological state $\mathbb{A}(\iota)$. It is assumed that $\mathbb{A}(\iota)$ is proportional to the total human population $\mathbb{P}_h(\iota)$, with a constant of proportionality denoted as k see [3, 39]. Therefore, the relationship between $\mathbb{A}(\iota)$ and $\mathbb{P}_h(\iota)$ can be expressed as $\mathbb{A}(\iota) = k\mathbb{P}_h(\iota)$. The constant k determines the scaling factor or the ratio between the aquatic mosquito population and the human population. It captures the relationship between the two populations within the model framework. By incorporating this relationship, the model considers the dependence of the mosquito population on the availability of suitable aquatic habitats provided by the human population. This assumption recognizes the role of human activities and environmental factors in creating breeding sites for mosquitoes and influencing their population dynamics. Similarly, $\mathbb{P}_v(\iota) = m\mathbb{P}_h(\iota)$ for some constant m see [3, 39]. The parameter Θ is the recruitment rate of humans. The mosquitoes bite the susceptible humans, and they are infected at the rate of $2b_v\beta_H\left(\frac{\mathbb{I}_v(\iota)\mathbb{S}(\iota)}{\mathbb{I}_v(\iota)+\mathbb{S}(\iota)}\right)$, where b_v is the rate at which mosquitoes bite humans and β_H is the probability of dengue fever transmission from infectious mosquitoes to susceptible humans. The infected humans after being bitten by the sand fly infect the sandflies at the rate $b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)$, where b_v is the rate at which mosquitoes bite humans and β_v is the probability of dengue fever transmission from infectious humans to susceptible mosquitoes. The remainder of the parameters in the model are defined as follows:

- γ: The rate at which humans recover from dengue fever.
- $\frac{1}{\varpi_v}$: The average lifespan of humans.
- μ: The rate at which exposed humans progress to the infectious stage of dengue fever.
- μ_v: The rate at which exposed mosquitoes progress to the infectious stage of dengue fever.
- $\frac{1}{\varpi_v}$: The average lifespan of female mosquitoes.
- $\varpi_{\mathbb{A}}$: The natural mortality rate of mosquito larvae.
- ϖ_B: The rate at which female mosquitoes lay eggs (per capita oviposition rate).
- $\mu_{\mathbb{A}}$: The rate at which mosquito larvae mature into female mosquitoes.
- k: The number of larvae per human.
- K: The maximum capacity of larvae per capita.
- m: The number of female mosquitoes per human.

The parameters used in the model play a crucial role in capturing the dynamics of disease transmission and population growth of both humans and mosquitoes. These parameters represent various biological and epidemiological factors that influence the spread and progression of dengue fever. By incorporating these factors into the model, we can thoroughly explore the interactions between human and mosquito populations, considering important factors such as mosquito feeding preferences, the effectiveness of control measures, and human mobility patterns.

3. Preliminaries

In this section, we present the necessary definitions and fundamental auxiliary results that are crucial for comprehending the fractal-fractional derivatives and integrals in the Caputo sense with the generalized Mittag-Leffler kernel. Let $\mathcal{J} = [0, T] \subset \mathbb{R}_+$ and $\mathcal{E} = C\left(\mathcal{J} \times \mathbb{R}_+^8, \mathbb{R}_+\right)$ be a Banach space with the following norm

$$\begin{aligned} \|\mathbb{U}\| &= \|\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{R}, \mathbb{A}, \mathbb{S}_v, \mathbb{E}_v, \mathbb{I}_v\| \\ &= \max_{\iota \in \mathcal{J}} \{|\mathbb{S}(\iota)| + |\mathbb{E}(\iota)| + |\mathbb{I}(\iota)| + |\mathbb{R}(\iota)| + |\mathbb{A}(\iota)| + |\mathbb{S}_v(\iota)| + |\mathbb{E}_v(\iota)| + |\mathbb{I}_v(\iota)|\}, \end{aligned}$$

where $(\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{R}, \mathbb{A}, \mathbb{S}_v, \mathbb{E}_v, \mathbb{I}_v) \in \mathcal{E}$.

Definition 3.1. *[11] Let $(0, T), T > 0$ be an open interval, and let $\mathbb{U}(\iota)$ be a differentiable function in $(0, T)$. If $\mathbb{U}$ is fractal differentiable on $(0, T)$ with order φ_2, then the fractal-fractional derivative of a function $\mathbb{U}$ of order φ_1 in Caputo sense with the generalized Mittag-Leffler kernel is given by*

$$^{FFML}\mathbb{D}_0^{\varphi_1,\varphi_2}\mathbb{U}(\iota) = \frac{M(\varphi_1)}{1-\varphi_1}\frac{d}{ds^{\varphi_2}}\int_0^{\iota}\mathbb{U}(s)E_{\varphi_1}\left(\frac{-\varphi_1}{1-\varphi_1}(\iota-s)^{\varphi_1}\right)ds,$$

where $M(\varphi_1) = 1 - \varphi_1 + \frac{\varphi_1}{\Gamma(\varphi_1)}$.

Definition 3.2. *[11] If $\mathbb{U}(\iota)$ is a continuous function in $(0, T)$, then the fractal-fractional integral of $\mathbb{U}$ with order φ_1 is defined by*

$$^{FFML}I_0^{\varphi_1,\varphi_2}\mathbb{U}(\iota) = \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}\mathbb{U}(\iota)}{M(\varphi_1)} + \frac{\varphi_2\varphi_1}{M(\varphi_1)}\int_0^{\iota}s^{\varphi_2-1}(\iota-s)^{\varphi_1-1}\mathbb{U}(s)ds.$$

4. Fundamental characteristics of the model (2.1)

Identifying the invariant region and analyzing solution positivity in the dengue model is essential for comprehending and capturing dengue transmission dynamics in populations. The invariant region represents a subset of the state space where the solutions remain confined, offering valuable information about the boundaries within which the variables can vary. This analysis helps determine the feasible range of values for the variables and provides insights into the system's long-term behavior. Furthermore, studying the positivity of solutions is crucial for ensuring the biological plausibility of the model. It ensures that variables such as the number of susceptible, infected, or recovered individuals remain positive over time, reflecting the realistic dynamics of dengue transmission.

4.1. Positivity and boundedness of the model

The analysis of positivity and boundedness ensures the model's validity, feasibility of solutions, and stability of disease dynamics. Our analysis will focus on the variables $\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{R}, \mathbb{S}_v, \mathbb{E}_v, \mathbb{I}_v$. These dynamics will be examined within a viable region denoted as G, which ensures the model's suitability for both human and mosquito populations. Through the application of theorems, we will establish the boundedness and positivity of solutions for the FFM (2.1) within the region G. This ensures that the model's predictions remain within realistic and biologically plausible ranges. We consider $G = G_h \times G_v \subset \mathbb{R}^4_+ \times \mathbb{R}^4_+$, where

$$\begin{aligned} G_h &= \left\{(\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{R}) \in \mathbb{R}^4_+; \mathbb{S} + \mathbb{E} + \mathbb{I} + \mathbb{R} \leq N\right\}, \\ G_v &= \left\{(\mathbb{A}, \mathbb{S}_v, \mathbb{E}_v, \mathbb{I}_v) \in \mathbb{R}^4_+, \mathbb{A} \leq kN;\ \mathbb{A} + \mathbb{S}_v + \mathbb{E}_v + \mathbb{I}_v \leq mN\right\}. \end{aligned} \tag{4.1}$$

Theorem 4.1. *The region G is positively invariant concerning the FFM (2.1).*

Proof. The fractal fractional derivative of the total human population in the model (2.1) at time ι is represented by the following term

$$\begin{aligned} {}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{P}_h(\iota) &= {}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{S}(\iota) + {}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{E}(\iota) + {}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{I}(\iota) + {}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{R}(\iota) \\ &= \Theta - \left[b_v\beta_H\left(\frac{2\mathbb{I}_v^2(\iota)}{(\mathbb{I}_v(\iota)+\mathbb{S}(\iota))^2}\right) + \varpi\right]\mathbb{S}(\iota) - b_v\beta_H\left(\frac{2\mathbb{S}^2(\iota)}{(\mathbb{I}_v(\iota)+\mathbb{S}(\iota))^2}\right)\mathbb{I}_v(\iota) \\ &\quad + b_v\beta_H\left(\frac{2\mathbb{I}_v^2(\iota)}{(\mathbb{I}_v(\iota)+\mathbb{S}(\iota))^2}\right)\mathbb{S}(\iota) + b_v\beta_H\left(\frac{2\mathbb{S}^2(\iota)}{(\mathbb{I}_v(\iota)+\mathbb{S}(\iota))^2}\right)\mathbb{I}_v(\iota) \\ &\quad - (\mu+\varpi)\mathbb{E}(\iota) + \mu\mathbb{E}(\iota) - (\gamma+\varpi)\mathbb{I}(\iota) + \gamma\mathbb{I}(\iota) - \varpi\mathbb{R}(\iota) \\ &= \Theta - \varpi(\mathbb{S}(\iota) - \mathbb{E}(\iota) - \mathbb{I}(\iota) - \mathbb{R}(\iota)) \\ &= \Theta - \varpi\mathbb{P}_h(\iota). \end{aligned} \tag{4.2}$$

Thus, we have

$${}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{P}_h(\iota) = \Theta - \varpi\mathbb{P}_h(\iota).$$

Further, it follows

$$\lim_{\iota\to\infty}\mathbb{P}_h(\iota) \leq \frac{\Theta}{\varpi} := N.$$

Hence, $\mathbb{P}_h(t)$ is bounded within the region G_h by $\frac{\Theta}{\varpi}$. Similarly, we prove that $\mathbb{P}_v(t)$ is bounded within the region G_v. Thus, the solution trajectory of the dengue model (2.1) remains bounded within a specific region over time, and it demonstrates the positive invariance of that region. This means that all solutions starting within that region will stay within it for all future time points. Hence, we gain confidence in the model's ability to accurately represent the real-world dynamics of dengue transmission. This information can be valuable for assessing the effectiveness of control strategies and predicting the impact of interventions designed to reduce the burden of dengue. □

Theorem 4.2. *Under the given assumption set of nonnegative initial conditions (2.2), the region G is invariant positively for the model (2.1).*

Proof. Let $\mathbb{U} = (\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{R}, \mathbb{A}, \mathbb{S}_v, \mathbb{E}_v, \mathbb{I}_v)^T$, where T is transposition. Let

$$\begin{aligned}
\Psi_{11} &= \left[b_v\beta_H\left(\frac{2\mathbb{I}_v^2(\iota)}{(\mathbb{I}_v(\iota)+\mathbb{S}(\iota))^2}\right)+\varpi\right],\\
\Psi_{18} &= b_v\beta_H\left(\frac{2\mathbb{S}^2(\iota)}{(\mathbb{I}_v(\iota)+\mathbb{S}(\iota))^2}\right),\\
\Psi_{21} &= b_v\beta_H\left(\frac{2\mathbb{I}_v^2(\iota)}{(\mathbb{I}_v(\iota)+\mathbb{S}(\iota))^2}\right),\\
\Psi_{63} &= b_v\beta_v\left(\frac{2\mathbb{S}_v^2(\iota)}{(\mathbb{I}(\iota)+\mathbb{S}_v(\iota))^2}\right),\\
\Psi_{66} &= \left[b_v\beta_v\left(\frac{2\mathbb{I}^2(\iota)}{(\mathbb{I}(\iota)+\mathbb{S}_v(\iota))^2}\right)+\varpi_v\right],\\
\Psi_{76} &= b_v\beta_v\left(\frac{2\mathbb{I}^2(\iota)}{(\mathbb{I}(\iota)+\mathbb{S}_v(\iota))^2}\right),\\
\Psi_{56} &= \Psi_{57}=\Psi_{58}=\varpi_B\left(1-\frac{\mathbb{A}(\iota)}{K}\right).
\end{aligned}$$

Now, we can write the model (2.1) in matrix form as follows

$$^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{U} = \mathcal{G}\mathbb{U}+\mathcal{G}_1,$$

where

$$\mathcal{G}=\begin{pmatrix}
-\Psi_{11} & 0 & 0 & 0 & 0 & 0 & 0 & -\Psi_{18}\\
\Psi_{21} & -(\mu+\varpi) & 0 & 0 & 0 & 0 & 0 & \Psi_{18}\\
0 & \mu & -(\gamma+\varpi) & -\varpi & 0 & 0 & 0 & 0\\
0 & 0 & \gamma & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -(\mu_{\mathbb{A}}+\varpi_{\mathbb{A}}) & \Psi_{56} & \Psi_{57} & \Psi_{58}\\
0 & 0 & -\Psi_{63} & 0 & \mu_{\mathbb{A}} & -\Psi_{66} & 0 & 0\\
0 & 0 & \Psi_{63} & 0 & 0 & \Psi_{76} & -(\mu_v+\varpi_v) & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \mu_v & -(\varpi_v)
\end{pmatrix},$$

and

$$\mathcal{G}_1 = \begin{pmatrix} \Theta \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Observing that $\mathcal{G}$ is the Metzler matrix and $\mathcal{G}_1 \geq 0$, we can conclude that the model (2.1) is positively invariant within the region G. This means that once the system enters the region G, the solutions of the model (2.1) will remain within G for all future time points. Additionally, the positivity of solutions is guaranteed, ensuring that all variables within the model maintain nonnegative values throughout the entire simulation. □

4.2. Equilibrium points

Equilibrium points and the basic reproduction number (BRN) are essential in studying and managing infectious diseases. Equilibrium points represent stable states that provide insights into the long-term behavior of the disease and its potential for persistence or elimination. The BRN, denoted as $\mathfrak{R}_0$, quantifies the transmission potential of a disease, helping assess its severity and spread. Incorporating these concepts into mathematical models enables researchers and public health officials to make informed decisions and develop effective strategies for controlling disease transmission. This includes interventions such as vaccination campaigns, quarantine measures, and behavior change programs aimed at minimizing the impact of infectious diseases on populations.

Theorem 4.3. *For the given G defined in (4.1), let*

$$\wp_m = \frac{\varpi_B \mu_{\mathbb{A}}}{\varpi_v \left(\mu_{\mathbb{A}} + \varpi_{\mathbb{A}}\right)},$$

where $\wp_m$ represents the net reproduction number of mosquitoes. In this context, it can be stated that the FFM (2.1) has, at most, two disease-free equilibrium points as follows:

(1) Trivial Equilibrium $\ell_1 = \left(\frac{\Theta}{\varpi}, 0, 0, 0, 0, 0, 0, 0\right)$.
(2) Biologically realistic disease-free equilibrium points

$$\ell_2 = \left(\frac{\Theta}{\varpi}, 0, 0, 0, \left(1 - \frac{1}{\wp_m}\right) K, \frac{\mu_{\mathbb{A}}}{\varpi_v}\left(1 - \frac{1}{\wp_m}\right) K, 0, 0\right).$$

Proof. To obtain the disease-free equilibrium point of the model (2.1), we set the variables representing the populations $\mathbb{P}h(\iota)$ and $\mathbb{P}v(\iota)$ equal to zero. This equilibrium point represents a state where the disease is not present, and all populations are in a healthy state. Thus, we have

$$\Theta - \left[b_v \beta_H \left(\frac{2\mathbb{I}_v^2(\iota)}{\left(\mathbb{I}_v(\iota) + \mathbb{S}(\iota)\right)^2}\right) + \varpi\right] \mathbb{S}(\iota) - b_v \beta_H \left(\frac{2\mathbb{S}^2(\iota)}{\left(\mathbb{I}_v(\iota) + \mathbb{S}(\iota)\right)^2}\right) \mathbb{I}_v(\iota) = 0, \tag{4.3}$$

$$b_v\beta_H\left(\frac{2\mathbb{I}_v^2(\iota)}{(\mathbb{I}_v(\iota)+\mathbb{S}(\iota))^2}\right)\mathbb{S}(\iota)+b_v\beta_H\left(\frac{2\mathbb{S}^2(\iota)}{(\mathbb{I}_v(\iota)+\mathbb{S}(\iota))^2}\right)\mathbb{I}_v(\iota)-(\mu+\varpi)\,\mathbb{E}(\iota)=0, \tag{4.4}$$

$$\mu\mathbb{E}(\iota)-(\gamma+\varpi)\,\mathbb{I}(\iota)=0, \tag{4.5}$$

$$\gamma\mathbb{I}(\iota)-\varpi\mathbb{R}(\iota)=0, \tag{4.6}$$

$$\varpi_B\left(1-\frac{\mathbb{A}(\iota)}{K}\right)(\mathbb{S}_v(\iota)+\mathbb{E}_v(\iota)+\mathbb{I}_v(\iota))-(\mu_{\mathbb{A}}+\varpi_{\mathbb{A}})\,\mathbb{A}(\iota)=0, \tag{4.7}$$

$$\mu_{\mathbb{A}}\mathbb{A}(\iota)-\left[b_v\beta_v\left(\frac{2\mathbb{I}^2(\iota)}{(\mathbb{I}(\iota)+\mathbb{S}_v(\iota))^2}\right)+\varpi_v\right]\mathbb{S}_v(\iota)-b_v\beta_v\left(\frac{2\mathbb{S}_v^2(\iota)}{(\mathbb{I}(\iota)+\mathbb{S}_v(\iota))^2}\right)\mathbb{I}(\iota)=0, \tag{4.8}$$

$$b_v\beta_v\left(\frac{2\mathbb{I}^2(\iota)}{(\mathbb{I}(\iota)+\mathbb{S}_v(\iota))^2}\right)\mathbb{S}_v(\iota)+b_v\beta_v\left(\frac{2\mathbb{S}_v^2(\iota)}{(\mathbb{I}(\iota)+\mathbb{S}_v(\iota))^2}\right)\mathbb{I}(\iota)-(\mu_v+\varpi_v)\,\mathbb{E}_v(\iota)=0, \tag{4.9}$$

and

$$\mu_v\mathbb{E}_v(\iota)-\varpi_v\mathbb{I}_v(\iota)=0. \tag{4.10}$$

From (4.6), we have

$$\mathbb{R}(\iota)=\frac{\gamma}{\varpi}\mathbb{I}(\iota). \tag{4.11}$$

Also, from (4.5), we have

$$\mathbb{E}(\iota)=\frac{(\gamma+\varpi)}{\mu}\mathbb{I}(\iota). \tag{4.12}$$

Now, by (4.3), we have

$$b_v\beta_H\left(\frac{2\mathbb{I}_v(\iota)\,\mathbb{S}(\iota)}{\mathbb{I}_v(\iota)+\mathbb{S}(\iota)}\right)=\Theta-\varpi\mathbb{S}(\iota). \tag{4.13}$$

Similarly, from (4.4), we have

$$b_v\beta_H\left(\frac{2\mathbb{I}_v(\iota)\,\mathbb{S}(\iota)}{\mathbb{I}_v(\iota)+\mathbb{S}(\iota)}\right)=(\mu+\varpi)\,\mathbb{E}(\iota). \tag{4.14}$$

By (4.13) and (4.14), we get

$$\mathbb{S}(\iota)=\frac{\Theta}{\varpi}-\frac{(\mu+\varpi)}{\varpi}\mathbb{E}(\iota). \tag{4.15}$$

Put (4.12) in (4.15), we have

$$\mathbb{S}(\iota)=\frac{\Theta}{\varpi}-\frac{(\mu+\varpi)(\gamma+\varpi)}{\varpi\mu}\mathbb{I}(\iota). \tag{4.16}$$

Therefore, we obtain

$$\mathbb{R}^*(\iota) = \frac{\gamma}{\varpi}\mathbb{I}^*(\iota), \tag{4.17}$$

$$\mathbb{S}^*(\iota) = \frac{\Theta}{\varpi}-\frac{(\mu+\varpi)(\gamma+\varpi)}{\varpi\mu}\mathbb{I}^*(\iota), \tag{4.18}$$

$$\mathbb{E}^*(\iota) = \frac{(\gamma+\varpi)}{\mu}\mathbb{I}^*(\iota). \tag{4.19}$$

Similarly, by (4.7), we get

$$\varpi_B\left(1-\frac{\mathbb{A}(\iota)}{K}\right)(\mathbb{S}_v(\iota)+\mathbb{E}_v(\iota)+\mathbb{I}_v(\iota))=(\mu_{\mathbb{A}}+\varpi_{\mathbb{A}})\,\mathbb{A}(\iota).$$

From (4.8), we get

$$\mathbb{S}_v(\iota)=\frac{\mu_{\mathbb{A}}}{\left[b_v\beta_v\left(\frac{2\mathbb{I}(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)+\varpi_v\right]}\mathbb{A}(\iota). \tag{4.20}$$

From (4.9), we get

$$\mathbb{E}_v(\iota)=\frac{b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)}{(\mu_v+\varpi_v)}\mathbb{S}_v(\iota). \tag{4.21}$$

Put (4.20) in (4.21), and we obtain

$$\mathbb{E}_v(\iota)=\frac{b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)\mu_{\mathbb{A}}}{(\mu_v+\varpi_v)\left[b_v\beta_v\left(\frac{2\mathbb{I}(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)+\varpi_v\right]}\mathbb{A}(\iota). \tag{4.22}$$

From (4.10), we get

$$\mathbb{I}_v(\iota)=\frac{\mu_v}{\varpi_v}\mathbb{E}_v(\iota). \tag{4.23}$$

Put (4.22) in (4.23), and we obtain

$$\mathbb{I}_v(\iota)=\frac{\mu_v b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)\mu_{\mathbb{A}}}{\varpi_v(\mu_v+\varpi_v)\left[b_v\beta_v\left(\frac{2\mathbb{I}(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)+\varpi_v\right]}\mathbb{A}(\iota).$$

Consequently, we have

$$\varpi_B\left(1-\frac{\mathbb{A}(\iota)}{K}\right)(\mathbb{S}_v(\iota)+\mathbb{E}_v(\iota)+\mathbb{I}_v(\iota))=(\mu_{\mathbb{A}}+\varpi_{\mathbb{A}})\,\mathbb{A}(\iota), \tag{4.24}$$

$$\mathbb{S}_v(\iota)=\frac{\mu_{\mathbb{A}}}{\left[b_v\beta_v\left(\frac{2\mathbb{I}(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)+\varpi_v\right]}\mathbb{A}(\iota), \tag{4.25}$$

$$\mathbb{E}_v(\iota)=\frac{b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)\mu_{\mathbb{A}}}{(\mu_v+\varpi_v)\left[b_v\beta_v\left(\frac{2\mathbb{I}(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)+\varpi_v\right]}\mathbb{A}(\iota), \tag{4.26}$$

and

$$\mathbb{I}_v(\iota)=\frac{\mu_v b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)\mu_{\mathbb{A}}}{\varpi_v(\mu_v+\varpi_v)\left[b_v\beta_v\left(\frac{2\mathbb{I}(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)+\varpi_v\right]}\mathbb{A}(\iota). \tag{4.27}$$

To compute the disease-free equilibrium points without disease, we put $\mathbb{I}=\mathbb{I}_v=0$ in Eqs (4.24)–(4.27), then Eq (4.24) becomes

$$\varpi_B\left(1-\frac{\mathbb{A}(\iota)}{K}\right)\left(\frac{\mu_{\mathbb{A}}}{\varpi_v}\right)\mathbb{A}(\iota)=(\mu_{\mathbb{A}}+\varpi_{\mathbb{A}})\,\mathbb{A}(\iota). \tag{4.28}$$

By solving Eq (4.28), we get $\mathbb{A}(\iota) = 0$ or

$$\varpi_B\left(1-\frac{\mathbb{A}(\iota)}{K}\right)\left(\frac{\mu_{\mathbb{A}}}{\varpi_v}\right)-(\mu_{\mathbb{A}}+\varpi_{\mathbb{A}})=0. \tag{4.29}$$

The solution of Eq (4.29) in terms of $\mathbb{A}$ is given as follows

$$\left(1-\frac{\varpi_v(\mu_{\mathbb{A}}+\varpi_{\mathbb{A}})}{\varpi_B\mu_{\mathbb{A}}}\right)K=\mathbb{A}(\iota).$$

Therefore, we have

$$\mathbb{A}(\iota)=\left(1-\frac{1}{\wp_m}\right)K,\ if\ \wp_m>1.$$

The crucial aspect is to utilize the threshold $\wp_m$ as a means of controlling mosquito presence and ensuring that the model (2.1) precisely accommodates two disease-free equilibrium points. Now, if $\mathbb{A}=\mathbb{I}_v=\mathbb{I}=0$, we deduced from Eqs (4.17)–(4.19) and (4.24)–(4.27) that $\mathbb{S}_v=\mathbb{E}_v=\mathbb{R}=0$ and $\mathbb{S}(\iota)=\frac{\Theta}{\varpi}$. Hence, trivial equilibrium ℓ_1 is obtained as

$$\ell_1=\left(\frac{\Theta}{\varpi},0,0,0,0,0,0,0\right).$$

Since $\wp_m>1$ and $\mathbb{A}=\mathbb{A}^*=\left(1-\frac{1}{\wp_m}\right)K$, then from Eqs (4.17)–(4.19) and (4.24)–(4.27), we get

$$\begin{aligned}
\mathbb{S}^*(\iota) &= \frac{\Theta}{\varpi},\\
\mathbb{E}^*(\iota) &= 0,\\
\mathbb{R}^*(\iota) &= 0,\\
\mathbb{I}^*(\iota) &= 0,\\
\mathbb{A}^*(\iota) &= \left(1-\frac{1}{\wp_m}\right)K\\
\mathbb{S}_v^*(\iota) &= \frac{\mu_{\mathbb{A}}}{\varpi_v}\left(1-\frac{1}{\wp_m}\right)K,\\
\mathbb{E}_v^*(\iota) &= 0,\\
\mathbb{I}_v^*(\iota) &= 0.
\end{aligned}$$

Thus, the biologically realistic disease-free equilibrium points ℓ_2, presented by

$$\ell_2=\left(\frac{\Theta}{\varpi},0,0,0,\left(1-\frac{1}{\wp_m}\right)K,\frac{\mu_{\mathbb{A}}}{\varpi_v}\left(1-\frac{1}{\wp_m}\right)K,0,0\right).$$

□

4.3. Basic reproduction number $\mathfrak{R}_0$.

To obtain $\mathfrak{R}_0$, we employ the next-generation matrix method. We define $\mathbb{U}=(\mathbb{E},\mathbb{I},\mathbb{E}_v,\mathbb{I}_v)$ as the set of compartments representing the infected individuals. Thus, we have

$$\begin{cases}
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{E}(\iota)=b_v\beta_H\left(\frac{2\mathbb{I}_v(\iota)\mathbb{S}(\iota)}{\mathbb{I}_v(\iota)+\mathbb{S}(\iota)}\right)-(\mu+\varpi)\mathbb{E}(\iota),\\
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{I}(\iota)=\mu\mathbb{E}(\iota)-(\gamma+\varpi)\mathbb{I}(\iota),\\
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{E}_v(\iota)=b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)-(\mu_v+\varpi_v)\mathbb{E}_v(\iota),\\
{}^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{I}_v(\iota)=\mu_v\mathbb{E}_v(\iota)-(\varpi_v)\mathbb{I}_v(\iota).
\end{cases}$$

The above model can be written as

$$^{FFML}D_0^{\varphi_1,\varphi_2}\mathbb{U} = \mathcal{F} - \mathcal{V},$$

where $\mathcal{F}$ and $\mathcal{V}$ are the infection matrix and the transition matrix, respectively, defined as follows:

$$\mathcal{F} = \begin{pmatrix} b_v\beta_H\left(\frac{2\mathbb{I}_v(\iota)\mathbb{S}(\iota)}{\mathbb{I}_v(\iota)+\mathbb{S}(\iota)}\right) \\ 0 \\ b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right) \\ 0 \end{pmatrix},$$

and

$$\mathcal{V} = \begin{pmatrix} (\mu+\varpi)\,\mathbb{E}(\iota) \\ (\gamma+\varpi)\,\mathbb{I}(\iota) - \mu\mathbb{E}(\iota) \\ (\mu_v+\varpi_v)\,\mathbb{E}_v(\iota) \\ (\varpi_v)\,\mathbb{I}_v(\iota) - \mu_v\mathbb{E}_v(\iota) \end{pmatrix}.$$

The expressions for the Jacobian matrices F and V, which correspond to $\mathcal{F}$ and $\mathcal{V}$, respectively, are as follows:

$$F = \begin{pmatrix} 0 & 0 & 0 & 2b_v\beta_H \\ 0 & 0 & 0 & 0 \\ 0 & 2b_v\beta_v & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$V = \begin{pmatrix} (\mu+\varpi) & 0 & 0 & 0 \\ -\mu & (\gamma+\varpi) & 0 & 0 \\ 0 & 0 & (\mu_v+\varpi_v) & 0 \\ 0 & 0 & -\mu_v & (\varpi_v) \end{pmatrix}.$$

Thus

$$V^{-1} = \begin{pmatrix} \frac{1}{\mu+\varpi} & 0 & 0 & 0 \\ \frac{\mu}{(\mu+\varpi)(\gamma+\varpi)} & \frac{1}{\gamma+\varpi} & 0 & 0 \\ 0 & 0 & \frac{1}{\mu_v+\varpi_v} & 0 \\ 0 & 0 & \frac{\mu_v}{(\mu_v+\varpi_v)(\varpi_v)} & \frac{1}{\varpi_v} \end{pmatrix}.$$

Therefore, using the fact that $\mathfrak{R}_0 = \alpha\left(FV^{-1}\right)$, we obtain $\mathfrak{R}_0$ for the model (2.1)

$$\mathfrak{R}_0 = \sqrt{\frac{4b_v\beta_H b_v\beta_v\mu\mu_v}{(\mu+\varpi)(\gamma+\varpi)(\mu_v+\varpi_v)(\varpi_v)}}.$$

The behavior of $\mathfrak{R}_0$ against of its various parameters is shown in Figures 1 and 2. We have used the numerical value of Table 1.

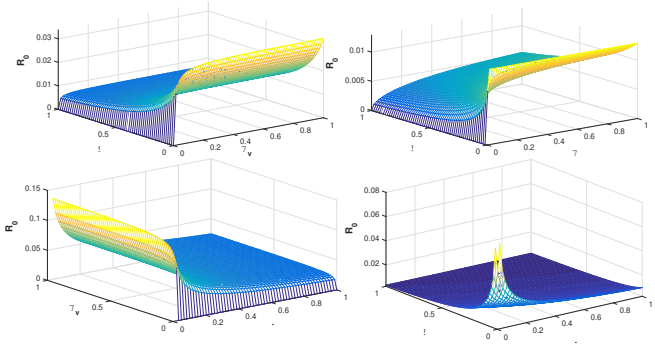

Figure 1. Behavior of R_0 using 3D profile.

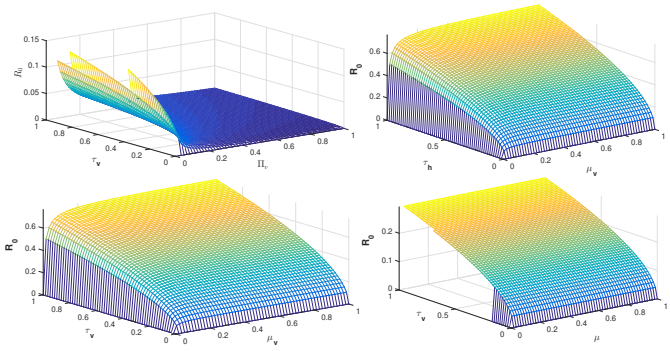

Figure 2. Behavior of R_0 using 3D profile.

Table 1. Numerical values of parameters.

Parameter	Value	Parameter	Value
Θ	3247700	β_H	0.31890
γ	0.54116	$\frac{1}{\varpi}$	74×52
μ	0.12899	b_v	0.66272
μ_v	0.00396	β_v	0.29294
ϖ_v	$\frac{1}{42}$	ϖ_A	0.20174
ϖ_B	3.01766	$\mu_{\mathbb{A}}$	0.08056
K	3	m	4

4.4. Lipschitz property

In this subsection, we shall establish the Lipschitz property of $\mathbb{F}_i(\iota, \mathbb{U}(\iota)), i = 1, 2, ..., 8$ where

$$\begin{aligned}
\mathbb{F}_1(\iota, \mathbb{U}(\iota)) &= \Theta - 2b_v\beta_H\left(\frac{\mathbb{I}_v(\iota)\,\mathbb{S}(\iota)}{\mathbb{I}_v(\iota) + \mathbb{S}(\iota)}\right) - \varpi\mathbb{S}(\iota),\\
\mathbb{F}_2(\iota, \mathbb{U}(\iota)) &= 2b_v\beta_H\left(\frac{\mathbb{I}_v(\iota)\,\mathbb{S}(\iota)}{\mathbb{I}_v(\iota) + \mathbb{S}(\iota)}\right) - (\mu + \varpi)\,\mathbb{E}(\iota),
\end{aligned}$$

$$\begin{aligned}
\mathbb{F}_3(\iota,\mathbb{U}(\iota)) &= \mu\mathbb{E}(\iota)-(\gamma+\varpi)\,\mathbb{I}(\iota),\\
\mathbb{F}_4(\iota,\mathbb{U}(\iota)) &= \gamma\mathbb{I}(\iota)-\varpi\mathbb{R}(\iota),\\
\mathbb{F}_5(\iota,\mathbb{U}(\iota)) &= \varpi_B\left(1-\frac{\mathbb{A}(\iota)}{K}\right)(\mathbb{S}_v(\iota)+\mathbb{E}_v(\iota)+\mathbb{I}_v(\iota))-(\mu_{\mathbb{A}}+\varpi_{\mathbb{A}})\,\mathbb{A}(\iota),\\
\mathbb{F}_6(\iota,\mathbb{U}(\iota)) &= \mu_{\mathbb{A}}\mathbb{A}(\iota)-b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\,\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)-\varpi_v\mathbb{S}_v(\iota),\\
\mathbb{F}_7(\iota,\mathbb{U}(\iota)) &= b_v\beta_v\left(\frac{2\mathbb{I}(\iota)\,\mathbb{S}_v(\iota)}{\mathbb{I}(\iota)+\mathbb{S}_v(\iota)}\right)-(\mu_v+\varpi_v)\,\mathbb{E}_v(\iota),\\
\mathbb{F}_8(\iota,\mathbb{U}(\iota)) &= \mu_v\mathbb{E}_v(\iota)-\varpi_v\mathbb{I}_v(\iota). \qquad (4.30)
\end{aligned}$$

Theorem 4.4. *Let* $\mathbb{S},\mathbb{E},\mathbb{I},\mathbb{R},\mathbb{A},\mathbb{S}_v,\mathbb{E}_v,\mathbb{I}_v,\widehat{\mathbb{S}},\widehat{\mathbb{E}},\widehat{\mathbb{I}},\widehat{\mathbb{R}},\widehat{\mathbb{A}},\widehat{\mathbb{S}}_v,\widehat{\mathbb{E}}_v,\widehat{\mathbb{I}}_v\in\mathcal{E}$, *and*

$$\begin{aligned}
\|\mathbb{S}\| &= \max_{\iota\in\mathcal{J}}|\mathbb{S}(\iota)|,\|\mathbb{E}\|=\max_{\iota\in\mathcal{J}}|\mathbb{E}(\iota)|,\\
\|\mathbb{I}\| &= \max_{\iota\in\mathcal{J}}|\mathbb{I}(\iota)|,\|\mathbb{R}\|=\max_{\iota\in\mathcal{J}}|\mathbb{R}(\iota)|,\\
\|\mathbb{A}\| &= \max_{\iota\in\mathcal{J}}|\mathbb{A}(\iota)|,\|\mathbb{S}_v\|=\max_{\iota\in\mathcal{J}}|\mathbb{S}_v(\iota)|,\\
\|\mathbb{E}_v\| &= \max_{\iota\in\mathcal{J}}|\mathbb{E}_v(\iota)|,\|\mathbb{I}_v\|=\max_{\iota\in\mathcal{J}}|\mathbb{I}_v(\iota)|.
\end{aligned}$$

Then, the functions $\mathbb{F}_i(\iota,\mathbb{U}(\iota)),i=1,2,....,8$ *defined by (4.30) are Lipschitz with Lipschitz constant* $\mathcal{L}_{\mathbb{F}}=\max_{i=1}^{8}\{\mathcal{L}_{\mathbb{F}_i}\}>0$, *such that*

$$\begin{aligned}
\mathcal{L}_{\mathbb{F}_1} &= \left(\frac{2b_v\beta_H\Theta}{\varpi}+\varpi\right),\mathcal{L}_{\mathbb{F}_2}=(\mu+\varpi),\\
\mathcal{L}_{\mathbb{F}_3} &= (\gamma+\varpi),\mathcal{L}_{\mathbb{F}_4}=\varpi,\\
\mathcal{L}_{\mathbb{F}_5} &= \frac{\varpi_B\Theta}{K\varpi}+(\mu_{\mathbb{A}}+\varpi_{\mathbb{A}}),\\
\mathcal{L}_{\mathbb{F}_6} &= \left(\frac{2b_v\beta_v\Theta}{\varpi}+\varpi_v\right),\mathcal{L}_{\mathbb{F}_7}=(\mu_v+\varpi_v),\\
\mathcal{L}_{\mathbb{F}_8} &= \varpi_v.
\end{aligned}$$

Proof. For $\mathbb{F}_1$, let $\mathbb{S},\widehat{\mathbb{S}}\in\mathcal{E}$, then

$$\begin{aligned}
&\left\|\mathbb{F}_1(\iota,\mathbb{S}(\iota))-\mathbb{F}_1\left(\iota,\widehat{\mathbb{S}}(\iota)\right)\right\|\\
=&\left\|\left[\Theta-2b_v\beta_H\left(\frac{\mathbb{I}_v(\iota)}{\mathbb{I}_v(\iota)+\mathbb{S}(\iota)}\right)\mathbb{S}(\iota)-\varpi\mathbb{S}(\iota)\right]-\left[\Theta-2b_v\beta_H\left(\frac{\mathbb{I}_v(\iota)}{\mathbb{I}_v(\iota)+\mathbb{S}(\iota)}\right)\widehat{\mathbb{S}}(\iota)-\varpi\widehat{\mathbb{S}}(\iota)\right]\right\|\\
\leq&\left\|-\left(2b_v\beta_H\frac{\|\mathbb{I}_v(\iota)\|}{\|\mathbb{I}_v(\iota)\|+\|\mathbb{S}(\iota)\|}+\varpi\right)\right\|\left\|\mathbb{S}-\widehat{\mathbb{S}}\right\|\\
\leq&\left(\frac{2b_v\beta_H\Theta}{\varpi}+\varpi\right)\left\|\mathbb{S}-\widehat{\mathbb{S}}\right\|.
\end{aligned}$$

Put $\mathcal{L}_{\mathbb{F}_1}=\left(\frac{2b_v\beta_H\Theta}{\varpi}+\varpi\right)>0$. Thus, we get

$$\left\|\mathbb{F}_1(\iota,\mathbb{S}(\iota))-\mathbb{F}_1\left(\iota,\widehat{\mathbb{S}}(\iota)\right)\right\|\leq\mathcal{L}_{\mathbb{F}_1}\left\|\mathbb{S}-\widehat{\mathbb{S}}\right\|.$$

Similarly, we can get the following

$$\begin{aligned}
\left\|\mathbb{F}_2(\iota,\mathbb{E}(\iota))-\mathbb{F}_2\left(\iota,\widehat{\mathbb{E}}(\iota)\right)\right\| &\leq \mathcal{L}_{\mathbb{F}_2}\left\|\mathbb{E}-\widehat{\mathbb{E}}\right\|,\\
\left\|\mathbb{F}_3(\iota,\mathbb{I}(\iota))-\mathbb{F}_3\left(\iota,\widehat{\mathbb{I}}(\iota)\right)\right\| &\leq \mathcal{L}_{\mathbb{F}_3}\left\|\mathbb{I}-\widehat{\mathbb{I}}\right\|,\\
\left\|\mathbb{F}_4(\iota,\mathbb{R}(\iota))-\mathbb{F}_4\left(\iota,\widehat{\mathbb{R}}(\iota)\right)\right\| &\leq \mathcal{L}_{\mathbb{F}_4}\left\|\mathbb{R}-\widehat{\mathbb{R}}\right\|,\\
\left\|\mathbb{F}_5(\iota,\mathbb{A}(\iota))-\mathbb{F}_5\left(\iota,\widehat{\mathbb{A}}(\iota)\right)\right\| &\leq \mathcal{L}_{\mathbb{F}_5}\left\|\mathbb{A}-\widehat{\mathbb{A}}\right\|,\\
\left\|\mathbb{F}_6(\iota,\mathbb{S}_v(\iota))-\mathbb{F}_6\left(\iota,\widehat{\mathbb{S}}_v(\iota)\right)\right\| &\leq \mathcal{L}_{\mathbb{F}_6}\left\|\mathbb{S}_v-\widehat{\mathbb{S}}_v\right\|,\\
\left\|\mathbb{F}_7(\iota,\mathbb{E}_v(\iota))-\mathbb{F}_7\left(\iota,\widehat{\mathbb{E}}_v(\iota)\right)\right\| &\leq \mathcal{L}_{\mathbb{F}_7}\left\|\mathbb{E}_v-\widehat{\mathbb{E}}_v\right\|,
\end{aligned}$$

and

$$\left\|\mathbb{F}_8(\iota,\mathbb{I}_v(\iota))-\mathbb{F}_8\left(\iota,\widehat{\mathbb{I}}_v(\iota)\right)\right\| \leq \mathcal{L}_{\mathbb{F}_8}\left\|\mathbb{I}_v-\widehat{\mathbb{I}}_v\right\|.$$

Let

$$\mathcal{L}_{\mathbb{F}}=\max_{i=1}^{8}\{\mathcal{L}_{\mathbb{F}_i}\}>0.$$

Thus, the function $\mathbb{F}(\iota,\mathbb{U}(\iota))$ satisfied the Lipschitz condition with Lipschitz constant $\mathcal{L}_{\mathbb{F}}>0$. □

4.5. Stability analysis

Understanding and managing the dynamics of dengue transmission requires an analysis of stability. This analysis provides insights for controlling epidemics, forecasting outbreaks, optimizing control measures, evaluating robustness, and validating models. Examining the stability properties can enhance our understanding of the disease's behavior and make informed choices to limit its spread. The model analysis revealed the presence of two disease-free equilibria, trivial equilibrium, and biologically realistic disease-free equilibrium, along with one endemic equilibrium point. The trivial equilibrium point represents a state where there is no active or dynamic interaction between the components of the system, resulting in a lack of interesting dynamics. In this section, we will discuss the stability of biologically realistic disease-free equilibrium. By the same technique used in works [1, 2, 21], we conclude that the biologically realistic disease-free equilibrium is locally and globally asymptotically stable. In the following, we discuss stability of the model in Ulam sense. Before that, we introduce the definitions and an auxiliary lemma that assists in the analysis. Here, we reformulated the (φ_1,φ_2)-FFM (2.1) of dengue as follows

$$\begin{cases}
{}^{ABR}D_0^{\varphi_1}\mathbb{S}(\iota)=\varphi_2\iota^{\varphi_2-1}\mathbb{F}_1(\iota,\mathbb{U}(\iota)),\\
{}^{ABR}D_0^{\varphi_1}\mathbb{E}(\iota)=\varphi_2\iota^{\varphi_2-1}\mathbb{F}_2(\iota,\mathbb{U}(\iota)),\\
{}^{ABR}D_0^{\varphi_1}\mathbb{I}(\iota)=\varphi_2\iota^{\varphi_2-1}\mathbb{F}_3(\iota,\mathbb{U}(\iota)),\\
{}^{ABR}D_0^{\varphi_1}\mathbb{R}(\iota)=\varphi_2\iota^{\varphi_2-1}\mathbb{F}_4(\iota,\mathbb{U}(\iota)),\\
{}^{ABR}D_0^{\varphi_1}A(\iota)=\varphi_2\iota^{\varphi_2-1}\mathbb{F}_5(\iota,\mathbb{U}(\iota)),\\
{}^{ABR}D_0^{\varphi_1}\mathbb{S}_v(\iota)=\varphi_2\iota^{\varphi_2-1}\mathbb{F}_6(\iota,\mathbb{U}(\iota)),\\
{}^{ABR}D_0^{\varphi_1}\mathbb{E}_v(\iota)=\varphi_2\iota^{\varphi_2-1}\mathbb{F}_7(\iota,\mathbb{U}(\iota)),\\
{}^{ABR}D_0^{\varphi_1}\mathbb{I}_v(\iota)=\varphi_2\iota^{\varphi_2-1}\mathbb{F}_8(\iota,\mathbb{U}(\iota)),
\end{cases} \tag{4.31}$$

where $\mathbb{F}_i(\iota, \mathbb{U}(\iota)), i = 1, 2, ..., 8$, as defined in (4.30). The model (4.31) is restructured as a compact initial value problem in the following form:

$$\begin{cases} {}^{ABR}D_0^{\varphi_1}\mathbb{U}(\iota) = \varphi_2 \iota^{\varphi_2 - 1}\mathbb{F}(\iota, \mathbb{U}(\iota)), \\ \mathbb{U}(0) = \mathbb{U}_0 > 0, \end{cases} \tag{4.32}$$

where

$$\begin{aligned} \mathbb{U}(\iota) &= (\mathbb{S}(\iota), \mathbb{E}(\iota), \mathbb{I}(\iota), \mathbb{R}(\iota), \mathbb{A}(\iota), \mathbb{S}_v(\iota), \mathbb{E}_v(\iota), \mathbb{I}_v(\iota))^T, \\ \mathbb{U}(0) &= \mathbb{U}_0 = (\mathbb{S}(0), \mathbb{E}(0), \mathbb{I}(0), \mathbb{R}(0), \mathbb{A}(0), \mathbb{S}_v(0), \mathbb{E}_v(0), \mathbb{I}_v(0))^T, \end{aligned}$$

and

$$\mathbb{F}(\iota, \mathbb{U}(\iota)) = \begin{pmatrix} \mathbb{F}_1(\iota, \mathbb{U}(\iota)) \\ \mathbb{F}_2(\iota, \mathbb{U}(\iota)) \\ \mathbb{F}_3(\iota, \mathbb{U}(\iota)) \\ \mathbb{F}_4(\iota, \mathbb{U}(\iota)) \\ \mathbb{F}_5(\iota, \mathbb{U}(\iota)) \\ \mathbb{F}_6(\iota, \mathbb{U}(\iota)) \\ \mathbb{F}_7(\iota, \mathbb{U}(\iota)) \\ \mathbb{F}_8(\iota, \mathbb{U}(\iota)) \end{pmatrix}.$$

The equivalent integral equation of (4.32) is given as

$$\begin{aligned} \mathbb{U}(\iota) &= \mathbb{U}_0 + \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}(\iota, \mathbb{U}(\iota)) \\ &\quad + \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_0^\iota s^{\varphi_2-1}(\iota - s)^{\varphi_1-1}\mathbb{F}(s, \mathbb{U}(s))\, ds. \end{aligned}$$

Definition 4.5. *The (φ_1, φ_2)-FFM (2.1) of dengue is considered Ulam-Hyers stable if there exists a positive constant $\mathfrak{M}$ such that for any $\varepsilon > 0$ and any solution $\widehat{\mathbb{U}} \in \mathcal{E}$ of the (φ_1, φ_2)-FFM (2.1) of dengue satisfying the inequality*

$$\left| {}^{FFML}\mathbb{D}_0^{\varphi_1,\varphi_2}\widehat{\mathbb{U}}(\iota) - \mathbb{F}(\iota, \widehat{\mathbb{U}}(\iota)) \right| \leq \varepsilon, \tag{4.33}$$

there exists a unique solution $\mathbb{U} \in \mathcal{E}$ of the (φ_1, φ_2)-FFM (2.1) of dengue such that

$$\left\| \widehat{\mathbb{U}} - \mathbb{U} \right\| \leq \mathfrak{M}\varepsilon,$$

where

$$\widehat{\mathbb{U}}(\iota) = \begin{pmatrix} \widehat{\mathbb{S}}(\iota) \\ \widehat{\mathbb{E}}(\iota) \\ \widehat{\mathbb{I}}(\iota) \\ \widehat{\mathbb{R}}(\iota) \\ \widehat{\mathbb{A}}(\iota) \\ \widehat{\mathbb{S}}_v(\iota) \\ \widehat{\mathbb{E}}_v(\iota) \\ \widehat{\mathbb{I}}_v(\iota) \end{pmatrix}, \mathfrak{M} = \begin{pmatrix} \mathfrak{M}_1 \\ \mathfrak{M}_2 \\ \mathfrak{M}_3 \\ \mathfrak{M}_4 \\ \mathfrak{M}_5 \\ \mathfrak{M}_6 \\ \mathfrak{M}_7 \\ \mathfrak{M}_8 \end{pmatrix}, \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \\ \varepsilon_7 \\ \varepsilon_8 \end{pmatrix}.$$

In other words, if the difference between the fractal-fractional derivative of $\widehat{\mathbb{U}}(\iota)$ and the corresponding function $\mathbb{F}(\iota, \widehat{\mathbb{U}}(\iota))$ is small (within ε), then there exists another solution $\widehat{\mathbb{U}}$ that is close to $\mathbb{U}$ (within $\mathfrak{M}\varepsilon$) and satisfies the same model.

Remark 4.6. *The function $\widehat{\mathbb{U}} \in \mathcal{E}$ is a solution of the inequality*

$$\left| {}^{FFML}\mathbb{D}_0^{\varphi_1,\varphi_2}\widehat{\mathbb{U}}(\iota) - \mathbb{F}(\iota, \widehat{\mathbb{U}}(\iota)) \right| \leq \varepsilon,$$

if and only if there exist a small perturbation $w \in \mathcal{E}$ and $w(\iota) = (w_1(\iota), w_2(\iota), w_3(\iota), w_4(\iota), w_5(\iota), w_6(\iota))^T$ such that for each $\iota \in \mathcal{J}$, we have $|w(\iota)| \leq \varepsilon$ and

$${}^{FFML}\mathbb{D}_0^{\varphi_1,\varphi_2}\widehat{\mathbb{U}}(\iota) = \mathbb{F}(\iota, \widehat{\mathbb{U}}(\iota)) + w(\iota).$$

Lemma 4.7. *Let $\varphi_1 \in (0, 1)$. For any $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\} > 0$, let $\widehat{\mathbb{U}} \in \mathcal{E}$ be a solution that satisfies (4.33), then $\widehat{\mathbb{U}}$ satisfies the following integral inequalities*

$$\begin{aligned}&\left| \widehat{\mathbb{U}}(\iota) - \left(\mathbb{U}_0 + \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}\mathbb{F}(\iota, \widehat{\mathbb{U}}(\iota))}{M(\varphi_1)} + \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)} \int_0^\iota s^{\varphi_2-1}(\iota - s)^{\varphi_1-1}\mathbb{F}(s, \widehat{\mathbb{U}}(s))ds \right) \right| \\ &\leq \varepsilon \left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1, \varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)} \right].\end{aligned}$$

Proof. Let $\widehat{\mathbb{U}}$ be a solution that satisfies (4.33), then by Remark 4.6, $\widehat{\mathbb{U}}$ satisfies the following system

$${}^{FFML}\mathbb{D}_0^{\varphi_1,\varphi_2}\widehat{\mathbb{U}}(\iota) = \mathbb{F}(\iota, \widehat{\mathbb{U}}(\iota)) + w(\iota).$$

Thus, we have

$$\begin{aligned}\widehat{\mathbb{U}}(\iota) &= \mathbb{U}_0 + \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}\mathbb{F}(\iota, \widehat{\mathbb{U}}(\iota))}{M(\varphi_1)} + \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}w(\iota)}{M(\varphi_1)} \\ &\quad + \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)} \int_0^\iota s^{\varphi_2-1}(\iota - s)^{\varphi_1-1}\mathbb{F}(s, \widehat{\mathbb{U}}(s))ds \\ &\quad + \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)} \int_0^\iota s^{\varphi_2-1}(\iota - s)^{\varphi_1-1}w(s)ds.\end{aligned}$$

By Remark 4.6 and $|w(\iota)| \leq \varepsilon$, we get

$$\begin{aligned}&\left| \widehat{\mathbb{U}}(\iota) - \left(\mathbb{U}_0 + \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}\mathbb{F}(\iota, \widehat{\mathbb{U}}(\iota))}{M(\varphi_1)} + \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)} \int_0^\iota s^{\varphi_2-1}(\iota - s)^{\varphi_1-1}\mathbb{F}(s, \widehat{\mathbb{U}}(s))ds \right) \right| \\ &\leq \varepsilon \left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1, \varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)} \right].\end{aligned}$$

□

Theorem 4.8. *Let $\mathbb{F}(\iota, \mathbb{U}(\iota))$ be a function satisfying the Lipschitz condition with Lipschitz constant $\mathcal{L}_\mathbb{F} > 0$, then the (φ_1, φ_2)-FFM (2.1) of dengue will be Ulam-Hyers stable provided that*

$$\Lambda = \left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1, \varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)} \right] \mathcal{L}_\mathbb{F} < 1.$$

Proof. Let $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\} > 0$ and $\widehat{\mathbb{U}} \in \mathcal{E}$ be a solution of the (φ_1, φ_2)-FFM (2.1) of dengue satisfying (4.33) and let $\mathbb{U} \in \mathcal{E}$ be a unique solution of (φ_1, φ_2)-FFM (2.1) of dengue. Thus, we have

$$\mathbb{U}(\iota) = \mathbb{U}_0 + \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}(\iota, \mathbb{U}(\iota)) + \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_0^\iota s^{\varphi_2-1}(\iota-s)^{\varphi_1-1}\mathbb{F}(s, \mathbb{U}(s))ds.$$

Via the triangle inequality, and by Lemma 4.7, we have

$$\begin{aligned}
\left|\widehat{\mathbb{U}}(\iota) - \mathbb{U}(\iota)\right| &\leq \left|\widehat{\mathbb{U}}(\iota) - \left(\mathbb{U}_0 + \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}(\iota, \widehat{\mathbb{U}}(\iota))\right.\right. \\
&\quad \left.\left. + \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_0^\iota s^{\varphi_2-1}(\iota-s)^{\varphi_1-1}\mathbb{F}(s, \widehat{\mathbb{U}}(s))ds\right)\right| \\
&\quad + \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}}{M(\varphi_1)}\left|\mathbb{F}(\iota, \widehat{\mathbb{U}}(\iota)) - \mathbb{F}(\iota, \mathbb{U}(\iota))\right| \\
&\quad + \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_0^\iota s^{\varphi_2-1}(\iota-s)^{\varphi_1-1}\left|\mathbb{F}(s, \widehat{\mathbb{U}}(s)) - \mathbb{F}(s, \mathbb{U}(s))\right|ds \\
&\leq \varepsilon\left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right] \\
&\quad + \left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right]\mathcal{L}_{\mathbb{F}}\left\|\widehat{\mathbb{U}} - \mathbb{U}\right\|. \qquad (4.34)
\end{aligned}$$

It implies

$$\left\|\widehat{\mathbb{U}} - \mathbb{U}\right\| \leq \frac{\left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right]\varepsilon}{1 - \left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right]\mathcal{L}_{\mathbb{F}}}.$$

Let

$$\mathfrak{M} = \frac{\left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right]}{1 - \left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right]\mathcal{L}_{\mathbb{F}}} > 0.$$

Thus, we have

$$\left\|\widehat{\mathbb{U}} - \mathbb{U}\right\| \leq \mathfrak{M}\varepsilon.$$

Hence, the (φ_1, φ_2)-FFM (2.1) of dengue is Ulam-Hyers stable. □

5. Properties of the solution

This section focuses on investigating the existence and uniqueness of the solution for the (φ_1, φ_2)-FFM (2.1) of dengue through the application of the fixed-point technique.

5.1. Existence of the solution

The existence of solutions for (φ_1, φ_2)-FFM (2.1) of dengue fever is vital for ensuring mathematical consistency, making accurate predictions, estimating model parameters, validating the model against data, obtaining robust results, and enhancing our understanding of disease dynamics. They provide a solid foundation for reliable modeling and decision-making processes in the context of dengue fever control and management.

Theorem 5.1. *[42] (Schauder's fixed-point theorem) Suppose we have a Banach space X and a nonempty, compact, convex subset $\Omega \subset X$. If $\Delta : \Omega \to \Omega$ is a continuous mapping, and $\Delta(\Omega)$ is a relatively compact subset of X, then, we can conclude that the mapping Δ has at least one fixed point in Ω.*

For analysis of the existence and uniqueness, the following assumptions must be satisfied:
(H_1) : The function $\mathbb{F} : \mathcal{J} \times \mathcal{E} \to \mathbb{R}$ is continuous such that

$$|\mathbb{F}(\iota, \mathbb{U}(\iota))| \leq \theta_{\mathbb{F}} + |\mathbb{U}(\iota)|\, \eta_{\mathbb{F}},\ \theta_{\mathbb{F}}, \eta_{\mathbb{F}} > 0.$$

Theorem 5.2. *Under the hypotheses (H_1), the (φ_1, φ_2)-FFM (2.1) of dengue has a solution, provided that*

$$|\mathbb{U}_0| + \left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right]\theta_{\mathbb{F}} < 1.$$

Proof. Define an operator $\Delta : \mathcal{E} \to \mathcal{E}$ by

$$(\Delta\mathbb{U})(\iota) = \mathbb{U}_0 + \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}\mathbb{F}(\iota, \mathbb{U}(\iota))}{M(\varphi_1)} + \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_0^{\iota} s^{\varphi_2-1}(\iota - s)^{\varphi_1-1}\mathbb{F}(s, \mathbb{U}(s))ds.$$

Let $\Bbbk_r = \{\mathbb{U} \in \mathcal{E} : \|\mathbb{U}\| \leq r\}$ be a closed ball with

$$r \geq \frac{\left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right]\eta_{\mathbb{F}}}{1 - |\mathbb{U}_0| + \left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right]\theta_{\mathbb{F}}}.$$

To apply the Schauder's fixed-point theorem, we need proof of the following steps:
Step (1): $\Delta\mathbb{U}(\iota) \in \Bbbk_r$. For $\mathbb{U} \in \Bbbk_r$ with hypotheses (H_1), we have

$$\begin{aligned}
\|\Delta\mathbb{U}\| &= \max_{\iota\in\mathcal{J}}\left\{|\mathbb{U}_0| + \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}}{M(\varphi_1)}|\mathbb{F}(\iota, \mathbb{U}(\iota))|\right.\\
&\quad \left. + \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_0^{\iota} s^{\varphi_2-1}(\iota - s)^{\varphi_1-1}|\mathbb{F}(s, \mathbb{U}(s))|\, ds\right\}\\
&\leq |\mathbb{U}_0| + \left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right]\theta_{\mathbb{F}}\\
&\quad + \left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right]\eta_{\mathbb{F}} r\\
&\leq r.
\end{aligned}$$

This proves that $\Delta\mathbb{U}(\iota) \in \Bbbk_r$.

Step (2): Δ is relatively compact (i.e., continuous, uniform bounded, and equicontinuous).

Case 1: Δ is continuous. Since $\mathbb{U}(\iota)$ is continuous, then $\Delta(\mathbb{U}(\iota))$ is continuous.

Case 2: Δ is uniformly bounded on $\Bbbk_r$. Let $\mathbb{U}(\iota) \in \Bbbk_r$, then, we have

$$\begin{aligned}
\|\Delta\mathbb{U}\| &= \max_{\iota\in\mathcal{J}}\frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_0^{\iota} s^{\varphi_2-1}(\iota - s)^{\varphi_1-1}|\mathbb{F}(s, \mathbb{U}(s))|\, ds\\
&\leq \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\left[\theta_{\mathbb{F}} + \|\mathbb{U}\|\,\eta_{\mathbb{F}}\right]
\end{aligned}$$

$$\leq \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\left[\theta_{\mathbb{F}}+r\eta_{\mathbb{F}}\right].$$

Hence, Δ is uniformly bounded on $\mathbb{k}_r$.

Case 3: Δ equicontinuous. Let $\mathbb{U}(\iota)\in\mathbb{k}_r$ and $0<\iota_1<\iota_2<T$, then, we have

$$\begin{aligned}
\|\Delta\mathbb{U}(\iota_2)-\Delta\mathbb{U}(\iota_1)\| &= \max_{\iota\in\mathcal{J}}\left|\frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_0^{\iota_2} s^{\varphi_2-1}(\iota_2-s)^{\varphi_1-1}\mathbb{F}(s,\mathbb{U}(s))ds\right.\\
&\quad\left.-\frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_0^{\iota_1} s^{\varphi_2-1}(\iota_1-s)^{\varphi_1-1}\mathbb{F}(s,\mathbb{U}(s))ds\right|\\
&\leq \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_{\iota_1}^{\iota_2} s^{\varphi_2-1}(\iota_2-s)^{\varphi_1-1}\left|\mathbb{F}(s,\mathbb{U}(s))\right|ds\\
&\quad+\frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_0^{\iota_1} s^{\varphi_2-1}\left[(\iota_2-s)^{\varphi_1-1}-(\iota_1-s)^{\varphi_1-1}\right]\left|\mathbb{F}(s,\mathbb{U}(s))\right|ds\\
&\leq \frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)\left[\theta_{\mathbb{F}}+r\eta_{\mathbb{F}}\right]}{M(\varphi_1)}(\iota_2-\iota_1)^{\varphi_2+\varphi_1-1}.
\end{aligned}$$

It follows that

$$\|\Delta\mathbb{U}(\iota_2)-\Delta\mathbb{U}(\iota_1)\|\to 0,\ \text{As}\ \iota_1\to\iota_2.$$

By the Arzelá–Ascoli theorem, Δ is relatively compact and, hence, completely continuous. Thus, Theorem 5.1 implies that (φ_1,φ_2)-FFM (2.1) of dengue has at least one solution. □

5.2. Uniqueness of the solution

The uniqueness of solutions allows us to make precise predictions about the behavior and evolution of dengue fever. Having a unique solution ensures that there is a single trajectory that the disease dynamics will follow, providing clear insights into the spread of the disease and the effectiveness of control measures. This is crucial for decision-making and planning interventions to mitigate the impact of dengue fever. In this subsection, we shall establish the uniqueness of the solution for (φ_1,φ_2)-FFM (2.1) of dengue. Before that, we investigate the Lipschitz property for the functions $\mathbb{F}(\iota,\mathbb{U}(\iota))$ defined in (4.30).

Theorem 5.3. *Assume that the function $\mathbb{F}(\iota,\mathbb{U}(\iota))$ satisfies the Lipschitz condition with Lipschitz constant $\mathcal{L}_{\mathbb{F}}>0$. If*

$$\left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)}+\frac{\varphi_2\varphi_1 B(\varphi_1,\varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right]\mathcal{L}_{\mathbb{F}}<1, \tag{5.1}$$

then the (φ_1,φ_2)-FFM (2.1) of dengue has a unique solution.

Proof. Define the operator $\Delta:\mathcal{E}\to\mathcal{E}$ by

$$\begin{aligned}
(\Delta\mathbb{U})(\iota) &= \mathbb{U}_0+\frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}\mathbb{F}(\iota,\mathbb{U}(\iota))}{M(\varphi_1)}\\
&\quad+\frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\int_0^{\iota} s^{\varphi_2-1}(\iota-s)^{\varphi_1-1}\mathbb{F}(s,\mathbb{U}(s))ds.
\end{aligned}$$

Let $\mathbb{U}_1, \mathbb{U}_2 \in \mathcal{E}$ and $\iota \in \mathcal{J}$, then

$$\begin{aligned}
\|\Delta\mathbb{U}_1 + \Delta\mathbb{U}_2\| &\leq \max_{\iota\in J} \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}}{M(\varphi_1)} |\mathbb{F}(\iota, \mathbb{U}_1(\iota)) - \mathbb{F}(\iota, \mathbb{U}_2(\iota))| \\
&\quad + \max_{\iota\in J} \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)} \int_0^\iota s^{\varphi_2-1}(\iota - s)^{\varphi_1-1} |\mathbb{F}(s, \mathbb{U}_1(s)) - \mathbb{F}(s, \mathbb{U}_2(s))| \, ds \\
&\leq \frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} \mathcal{L}_{\mathbb{F}} \|\mathbb{U}_1 - \mathbb{U}_2\| \\
&\quad + \frac{\varphi_2\varphi_1 B(\varphi_1, \varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)} \mathcal{L}_{\mathbb{F}} \|\mathbb{U}_1 - \mathbb{U}_2\| \\
&\leq \left[\frac{\varphi_2(1-\varphi_1)T^{\varphi_2-1}}{M(\varphi_1)} + \frac{\varphi_2\varphi_1 B(\varphi_1, \varphi_2)T^{\varphi_2+\varphi_1-1}}{M(\varphi_1)}\right] \mathcal{L}_{\mathbb{F}} \|\mathbb{U}_1 - \mathbb{U}_2\| .
\end{aligned}$$

Due to (5.1), Δ is contraction. Thus, (φ_1, φ_2)-FFM (2.1) of dengue has a unique solution. □

6. Numerical scheme via the Newton polynomial method

In this section, we present a numerical scheme for the solutions of the (φ_1, φ_2)-FFM (2.1) of dengue, as introduced by Atangana in [11, 12]. To facilitate this, we will obtain the compact form of the initial value problem (4.32). Thus, we have

$$\mathbb{U}(\iota) - \mathbb{U}_0 = \frac{\varphi_2(1-\varphi_1)\iota^{\varphi_2-1}}{M(\varphi_1)} \mathbb{F}(\iota, \mathbb{U}(\iota)) + \frac{\varphi_2\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)} \int_0^\iota s^{\varphi_2-1}(\iota - s)^{\varphi_1-1} \mathbb{F}(s, \mathbb{U}(s)) \, ds.$$

Let $\overline{\mathbb{F}}(\iota, \mathbb{U}(\iota)) = \varphi_2\iota^{\varphi_2-1}\mathbb{F}(\iota, \mathbb{U}(\iota))$, then, we get

$$\mathbb{U}(\iota) - \mathbb{U}_0 = \frac{(1-\varphi_1)}{M(\varphi_1)} \overline{\mathbb{F}}(\iota, \mathbb{U}(\iota)) + \frac{\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)} \int_0^\iota (\iota - s)^{\varphi_1-1} \overline{\mathbb{F}}(s, \mathbb{U}(s)) \, ds. \tag{6.1}$$

By discretizing the Eq (6.1) at $\iota = \iota_{m+1} = (m+1)h$, where h represents the time step size, we obtain the following discrete equations:

$$\mathbb{U}(\iota_{m+1}) - \mathbb{U}_0 = \frac{(1-\varphi_1)}{M(\varphi_1)} \overline{\mathbb{F}}(\iota_m, \mathbb{U}(\iota_m)) + \frac{\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)} \int_0^{\iota_{m+1}} (\iota_{m+1} - s)^{\varphi_1-1} \overline{\mathbb{F}}(s, \mathbb{U}(s)) \, ds.$$

If we approximate the above integral, it can be expressed as follows:

$$\mathbb{U}(\iota_{m+1}) - \mathbb{U}_0 = \frac{(1-\varphi_1)}{M(\varphi_1)} \overline{\mathbb{F}}(\iota_m, \mathbb{U}(\iota_m)) + \frac{\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)} \sum_{n=1}^{m} \int_{\iota_n}^{\iota_{n+1}} (\iota_{m+1} - s)^{\varphi_1-1} \overline{\mathbb{F}}(s, \mathbb{U}(s)) \, ds. \tag{6.2}$$

Now, we approximate the functions $\overline{\mathbb{F}}(\iota, \mathbb{U}(\iota))$ on $[\iota_n, \iota_{n+1}]$ through the interpolation polynomial with $h = \iota_{n+1} - \iota_n$ as follows:

$$\begin{aligned}
\mathcal{Z}_m(s) &= \overline{\mathbb{F}}(\iota_{m-2}, \mathbb{U}(\iota_{m-2})) + \frac{\overline{\mathbb{F}}(\iota_{m-1}, \mathbb{U}(\iota_{m-1})) - \overline{\mathbb{F}}(\iota_{m-2}, \mathbb{U}(\iota_{m-2}))}{h} (s - \iota_{m-2}) \\
&\quad + \frac{\overline{\mathbb{F}}(\iota_m, \mathbb{U}(\iota_m)) - 2\overline{\mathbb{F}}(\iota_{m-1}, \mathbb{U}(\iota_{m-1})) + \overline{\mathbb{F}}(\iota_{m-2}, \mathbb{U}(\iota_{m-2}))}{2h^2} (s - \iota_{m-2})(s - \iota_{m-1}) .
\end{aligned} \tag{6.3}$$

Put (6.3) in (6.2), and we get

$$\begin{aligned}\mathbb{U}(\iota_{m+1})-\mathbb{U}_0 &= \frac{(1-\varphi_1)}{M(\varphi_1)}\overline{\mathbb{F}}(\iota_m,\mathbb{U}(\iota_m))\\ &+\frac{\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\sum_{n=1}^{m}\int_{\iota_n}^{\iota_{n+1}}(\iota_{m+1}-s)^{\varphi_1-1}\Big[\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\\ &+\frac{\overline{\mathbb{F}}(\iota_{n-1},\mathbb{U}(\iota_{n-1}))-\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))}{h}(s-\iota_{n-2})\\ &+\frac{\overline{\mathbb{F}}(\iota_n,\mathbb{U}(\iota_n))-2\overline{\mathbb{F}}(\iota_{n-1},\mathbb{U}(\iota_{n-1}))+\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))}{2h^2}\\ &\times(s-\iota_{n-2})(s-\iota_{n-1})\Big]ds.\end{aligned}$$

It implies that

$$\begin{aligned}\mathbb{U}(\iota_{m+1})-\mathbb{U}_0 &= \frac{(1-\varphi_1)}{M(\varphi_1)}\overline{\mathbb{F}}(\iota_m,\mathbb{U}(\iota_m))\\ &+\frac{\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\sum_{n=1}^{m}\Bigg[\int_{\iota_n}^{\iota_{n+1}}(\iota_{m+1}-s)^{\varphi_1-1}\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\,ds\\ &+\int_{\iota_n}^{\iota_{n+1}}\frac{\overline{\mathbb{F}}(\iota_{n-1},\mathbb{U}(\iota_{n-1}))-\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))}{h}(s-\iota_{n-2})(\iota_{m+1}-s)^{\varphi_1-1}ds\\ &+\int_{\iota_n}^{\iota_{n+1}}\frac{\overline{\mathbb{F}}(\iota_n,\mathbb{U}(\iota_n))-2\overline{\mathbb{F}}(\iota_{n-1},\mathbb{U}(\iota_{n-1}))+\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))}{2h^2}\\ &\times(s-\iota_{n-2})(s-\iota_{n-1})(\iota_{m+1}-s)^{\varphi_1-1}ds\Bigg].\end{aligned}$$

Consequently, we have

$$\begin{aligned}\mathbb{U}(\iota_{m+1})-\mathbb{U}_0 &= \frac{(1-\varphi_1)}{M(\varphi_1)}\overline{\mathbb{F}}(\iota_m,\mathbb{U}(\iota_m))\\ &+\frac{\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\sum_{n=1}^{m}\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\int_{\iota_n}^{\iota_{n+1}}(\iota_{m+1}-s)^{\varphi_1-1}ds\\ &+\frac{\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\sum_{n=1}^{m}\frac{\overline{\mathbb{F}}(\iota_{n-1},\mathbb{U}(\iota_{n-1}))-\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))}{h}\int_{\iota_n}^{\iota_{n+1}}(s-\iota_{n-2})(\iota_{m+1}-s)^{\varphi_1-1}ds\\ &+\frac{\varphi_1}{M(\varphi_1)\Gamma(\varphi_1)}\sum_{n=1}^{m}\frac{\overline{\mathbb{F}}(\iota_n,\mathbb{U}(\iota_n))-2\overline{\mathbb{F}}(\iota_{n-1},\mathbb{U}(\iota_{n-1}))+\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))}{2h^2}\\ &\times\int_{\iota_n}^{\iota_{n+1}}(s-\iota_{n-2})(s-\iota_{n-1})(\iota_{m+1}-s)^{\varphi_1-1}ds.\end{aligned}\tag{6.4}$$

Now, we compute the above three integrals separately, and we get

$$\int_{\iota_n}^{\iota_{n+1}}(\iota_{m+1}-s)^{\varphi_1-1}ds=\frac{h^{\varphi_1}}{\varphi_1}\left[(m-n+1)^{\varphi_1}-(m-n)^{\varphi_1}\right],\tag{6.5}$$

$$\int_{\iota_n}^{\iota_{n+1}} (s-\iota_{n-2})(\iota_{m+1}-s)^{\varphi_1-1}ds = \frac{h^{\varphi_1+1}}{\varphi_1(\varphi_1+1)}\left[(m+1-n)^{\varphi_1}(m-n+3+2\varphi_1)\right. \\ \left.-(m-n+1)^{\varphi_1}(m-n+3+3\varphi_1)\right], \tag{6.6}$$

and

$$\int_{\iota_n}^{\iota_{n+1}} (s-\iota_{n-2})(s-\iota_{n-1})(\iota_{m+1}-s)^{\varphi_1-1}ds = \frac{h^{\varphi_1+2}}{\varphi_1(\varphi_1+1)(\varphi_1+2)}\left\{(m+1-n)^{\varphi_1}\left[2(m-n)^2\right.\right. \\ \left.+(3\varphi_1+10)(m-n)+2\varphi_1^2+9\varphi_1+12\right] \\ -(m-n)^{\varphi_1}\left[2(m-n)^2+(5\varphi_1+10)(m-n)\right. \\ \left.\left.+6\varphi_1^2+18\varphi_1+12\right]\right\}. \tag{6.7}$$

Put (6.5)–(6.7) in (6.4), and we get

$$\begin{aligned}\mathbb{U}(\iota_{m+1})-\mathbb{U}_0 &= \frac{(1-\varphi_1)}{M(\varphi_1)}\overline{\mathbb{F}}(\iota_m,\mathbb{U}(\iota_m)) \\ &+\frac{\varphi_1 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+1)}\sum_{n=1}^{m}\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\left[(m-n+1)^{\varphi_1}-(m-n)^{\varphi_1}\right] \\ &+\frac{\varphi_1 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+2)}\sum_{n=1}^{m}\overline{\mathbb{F}}(\iota_{n-1},\mathbb{U}(\iota_{n-1}))-\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2})) \\ &\left[(m+1-n)^{\varphi_1}(m-n+3+2\varphi_1)-(m-n+1)^{\varphi_1}(m-n+3+3\varphi_1)\right] \\ &+\frac{\varphi_1 h^{\varphi_1}}{2M(\varphi_1)\Gamma(\varphi_1+3)}\sum_{n=1}^{m}\overline{\mathbb{F}}(\iota_n,\mathbb{U}(\iota_n))-2\overline{\mathbb{F}}(\iota_{n-1},\mathbb{U}(\iota_{n-1}))+\overline{\mathbb{F}}(\iota_{n-2},\mathbb{U}(\iota_{n-2})) \\ &\left\{(m+1-n)^{\varphi_1}\left[2(m-n)^2+(3\varphi_1+10)(m-n)+2\varphi_1^2+9\varphi_1+12\right]\right. \\ &\left.-(m-n)^{\varphi_1}\left[2(m-n)^2+(5\varphi_1+10)(m-n)+6\varphi_1^2+18\varphi_1+12\right]\right\}. \end{aligned} \tag{6.8}$$

Now, replace $\overline{\mathbb{F}}(\iota,\mathbb{U}(\iota))=\varphi_2\iota^{\varphi_2-1}\mathbb{F}(\iota,\mathbb{U}(\iota))$ in the above equation, and we have

$$\begin{aligned}\mathbb{U}(\iota_{m+1})-\mathbb{U}_0 &= \frac{(1-\varphi_1)\varphi_2\iota_m^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}(\iota_m,\mathbb{U}(\iota_m)) \\ &+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+1)}\sum_{n=1}^{m}\iota_{n-2}^{\varphi_2-1}\mathbb{F}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\,\mathcal{N}_1(m,n,\varphi_1) \\ &+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+2)}\sum_{n=1}^{m}\left[\iota_{n-1}^{\varphi_2-1}\mathbb{F}(\iota_{n-1},\mathbb{U}(\iota_{n-1}))-\iota_{n-2}^{\varphi_2-1}\mathbb{F}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\right]\mathcal{N}_2(m,n,\varphi_1) \\ &+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{2M(\varphi_1)\Gamma(\varphi_1+3)}\sum_{n=1}^{m}\left[\iota_n^{\varphi_2-1}\mathbb{F}(\iota_n,\mathbb{U}(\iota_n))-2\iota_{n-1}^{\varphi_2-1}\mathbb{F}(\iota_{n-1},\mathbb{U}(\iota_{n-1}))\right. \\ &\left.+\iota_{n-2}^{\varphi_2-1}\mathbb{F}(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\right]\mathcal{N}_3(m,n,\varphi_1), \end{aligned} \tag{6.9}$$

where

$$\mathcal{N}_1(m,n,\varphi_1)=(m-n+1)^{\varphi_1}-(m-n)^{\varphi_1},$$

$$\mathcal{N}_2(m,n,\varphi_1) = (m+1-n)^{\varphi_1}(m-n+3+2\varphi_1) - (m-n+1)^{\varphi_1}(m-n+3+3\varphi_1),$$

and

$$\begin{aligned}\mathcal{N}_3(m,n,\varphi_1) &= (m+1-n)^{\varphi_1}\left[2(m-n)^2 + (3\varphi_1+10)(m-n) + 2\varphi_1^2 + 9\varphi_1 + 12\right] \\ &\quad - (m-n)^{\varphi_1}\left[2(m-n)^2 + (5\varphi_1+10)(m-n) + 6\varphi_1^2 + 18\varphi_1 + 12\right].\end{aligned}$$

Based on (4.30) and the numerical scheme obtained in (6.9), the numerical solutions of the (φ_1,φ_2)-FFM (2.1) of dengue are given by

$$\begin{aligned}\mathbb{S}(\iota_{m+1}) - \mathbb{S}_0 &= \frac{(1-\varphi_1)\varphi_2\iota_m^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}_1(\iota_m,\mathbb{U}(\iota_m)) \\ &\quad + \frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+1)}\sum_{n=1}^{m}\iota_{n-2}^{\varphi_2-1}\mathbb{F}_1(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\,\mathcal{N}_1(m,n,\varphi_1) \\ &\quad + \frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+2)}\sum_{n=1}^{m}\left[\iota_{n-1}^{\varphi_2-1}\mathbb{F}_1(\iota_{n-1},\mathbb{U}(\iota_{n-1})) - \iota_{n-2}^{\varphi_2-1}\mathbb{F}_1(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\right]\mathcal{N}_2(m,n,\varphi_1) \\ &\quad + \frac{\varphi_1\varphi_2 h^{\varphi_1}}{2M(\varphi_1)\Gamma(\varphi_1+3)}\sum_{n=1}^{m}\Big[\iota_n^{\varphi_2-1}\mathbb{F}_1(\iota_n,\mathbb{U}(\iota_n)) - 2\iota_{n-1}^{\varphi_2-1}\mathbb{F}_1(\iota_{n-1},\mathbb{U}(\iota_{n-1})) \\ &\quad + \iota_{n-2}^{\varphi_2-1}\mathbb{F}_1(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\Big]\mathcal{N}_3(m,n,\varphi_1), \qquad (6.10)\end{aligned}$$

$$\begin{aligned}\mathbb{E}(\iota_{m+1}) - \mathbb{E}_0 &= \frac{(1-\varphi_1)\varphi_2\iota_m^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}_2(\iota_m,\mathbb{U}(\iota_m)) \\ &\quad + \frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+1)}\sum_{n=1}^{m}\iota_{n-2}^{\varphi_2-1}\mathbb{F}_2(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\,\mathcal{N}_1(m,n,\varphi_1) \\ &\quad + \frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+2)}\sum_{n=1}^{m}\left[\iota_{n-1}^{\varphi_2-1}\mathbb{F}_2(\iota_{n-1},\mathbb{U}(\iota_{n-1})) - \iota_{n-2}^{\varphi_2-1}\mathbb{F}_2(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\right]\mathcal{N}_2(m,n,\varphi_1) \\ &\quad + \frac{\varphi_1\varphi_2 h^{\varphi_1}}{2M(\varphi_1)\Gamma(\varphi_1+3)}\sum_{n=1}^{m}\Big[\iota_n^{\varphi_2-1}\mathbb{F}_2(\iota_n,\mathbb{U}(\iota_n)) - 2\iota_{n-1}^{\varphi_2-1}\mathbb{F}_2(\iota_{n-1},\mathbb{U}(\iota_{n-1})) \\ &\quad + \iota_{n-2}^{\varphi_2-1}\mathbb{F}_2(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\Big]\mathcal{N}_3(m,n,\varphi_1),\end{aligned}$$

$$\begin{aligned}\mathbb{I}(\iota_{m+1}) - \mathbb{I}_0 &= \frac{(1-\varphi_1)\varphi_2\iota_m^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}_3(\iota_m,\mathbb{U}(\iota_m)) \\ &\quad + \frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+1)}\sum_{n=1}^{m}\iota_{n-2}^{\varphi_2-1}\mathbb{F}_3(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\,\mathcal{N}_1(m,n,\varphi_1) \\ &\quad + \frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+2)}\sum_{n=1}^{m}\left[\iota_{n-1}^{\varphi_2-1}\mathbb{F}_3(\iota_{n-1},\mathbb{U}(\iota_{n-1})) - \iota_{n-2}^{\varphi_2-1}\mathbb{F}_3(\iota_{n-2},\mathbb{U}(\iota_{n-2}))\right]\mathcal{N}_2(m,n,\varphi_1) \\ &\quad + \frac{\varphi_1\varphi_2 h^{\varphi_1}}{2M(\varphi_1)\Gamma(\varphi_1+3)}\sum_{n=1}^{m}\Big[\iota_n^{\varphi_2-1}\mathbb{F}_3(\iota_n,\mathbb{U}(\iota_n)) - 2\iota_{n-1}^{\varphi_2-1}\mathbb{F}_3(\iota_{n-1},\mathbb{U}(\iota_{n-1}))\end{aligned}$$

$$+\iota_{n-2}^{\varphi_2-1}\mathbb{F}_3\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\Big]\mathcal{N}_3\left(m,n,\varphi_1\right),$$

$$\begin{aligned}
\mathbb{R}(\iota_{m+1})-\mathbb{R}_0 &= \frac{(1-\varphi_1)\varphi_2\iota_m^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}_4\left(\iota_m,\mathbb{U}(\iota_m)\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+1)}\sum_{n=1}^{m}\iota_{n-2}^{\varphi_2-1}\mathbb{F}_4\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\mathcal{N}_1\left(m,n,\varphi_1\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+2)}\sum_{n=1}^{m}\Big[\iota_{n-1}^{\varphi_2-1}\mathbb{F}_4\left(\iota_{n-1},\mathbb{U}(\iota_{n-1})\right)-\iota_{n-2}^{\varphi_2-1}\mathbb{F}_4\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\Big]\mathcal{N}_2\left(m,n,\varphi_1\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{2M(\varphi_1)\Gamma(\varphi_1+3)}\sum_{n=1}^{m}\Big[\iota_n^{\varphi_2-1}\mathbb{F}_4\left(\iota_n,\mathbb{U}(\iota_n)\right)-2\iota_{n-1}^{\varphi_2-1}\mathbb{F}_4\left(\iota_{n-1},\mathbb{U}(\iota_{n-1})\right)\\
&+\iota_{n-2}^{\varphi_2-1}\mathbb{F}_4\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\Big]\mathcal{N}_3\left(m,n,\varphi_1\right),
\end{aligned}$$

$$\begin{aligned}
\mathbb{A}(\iota_{m+1})-\mathbb{A}_0 &= \frac{(1-\varphi_1)\varphi_2\iota_m^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}_5\left(\iota_m,\mathbb{U}(\iota_m)\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+1)}\sum_{n=1}^{m}\iota_{n-2}^{\varphi_2-1}\mathbb{F}_5\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\mathcal{N}_1\left(m,n,\varphi_1\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+2)}\sum_{n=1}^{m}\Big[\iota_{n-1}^{\varphi_2-1}\mathbb{F}_5\left(\iota_{n-1},\mathbb{U}(\iota_{n-1})\right)-\iota_{n-2}^{\varphi_2-1}\mathbb{F}_5\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\Big]\mathcal{N}_2\left(m,n,\varphi_1\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{2M(\varphi_1)\Gamma(\varphi_1+3)}\sum_{n=1}^{m}\Big[\iota_n^{\varphi_2-1}\mathbb{F}_5\left(\iota_n,\mathbb{U}(\iota_n)\right)-2\iota_{n-1}^{\varphi_2-1}\mathbb{F}_5\left(\iota_{n-1},\mathbb{U}(\iota_{n-1})\right)\\
&+\iota_{n-2}^{\varphi_2-1}\mathbb{F}_5\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\Big]\mathcal{N}_3\left(m,n,\varphi_1\right),
\end{aligned}$$

$$\begin{aligned}
\mathbb{S}_v(\iota_{m+1})-\mathbb{S}_{v0} &= \frac{(1-\varphi_1)\varphi_2\iota_m^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}_6\left(\iota_m,\mathbb{U}(\iota_m)\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+1)}\sum_{n=1}^{m}\iota_{n-2}^{\varphi_2-1}\mathbb{F}_6\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\mathcal{N}_1\left(m,n,\varphi_1\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+2)}\sum_{n=1}^{m}\Big[\iota_{n-1}^{\varphi_2-1}\mathbb{F}_6\left(\iota_{n-1},\mathbb{U}(\iota_{n-1})\right)-\iota_{n-2}^{\varphi_2-1}\mathbb{F}_6\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\Big]\mathcal{N}_2\left(m,n,\varphi_1\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{2M(\varphi_1)\Gamma(\varphi_1+3)}\sum_{n=1}^{m}\Big[\iota_n^{\varphi_2-1}\mathbb{F}_6\left(\iota_n,\mathbb{U}(\iota_n)\right)-2\iota_{n-1}^{\varphi_2-1}\mathbb{F}_6\left(\iota_{n-1},\mathbb{U}(\iota_{n-1})\right)\\
&+\iota_{n-2}^{\varphi_2-1}\mathbb{F}_6\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\Big]\mathcal{N}_3\left(m,n,\varphi_1\right),
\end{aligned}$$

$$\mathbb{E}_v(\iota_{m+1})-\mathbb{E}_{v0} = \frac{(1-\varphi_1)\varphi_2\iota_m^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}_7\left(\iota_m,\mathbb{U}(\iota_m)\right)$$

$$
\begin{aligned}
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+1)}\sum_{n=1}^{m}\iota_{n-2}^{\varphi_2-1}\mathbb{F}_7\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\mathcal{N}_1\left(m,n,\varphi_1\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+2)}\sum_{n=1}^{m}\left[\iota_{n-1}^{\varphi_2-1}\mathbb{F}_7\left(\iota_{n-1},\mathbb{U}(\iota_{n-1})\right)-\iota_{n-2}^{\varphi_2-1}\mathbb{F}_7\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\right]\mathcal{N}_2\left(m,n,\varphi_1\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{2M(\varphi_1)\Gamma(\varphi_1+3)}\sum_{n=1}^{m}\left[\iota_{n}^{\varphi_2-1}\mathbb{F}_7\left(\iota_{n},\mathbb{U}(\iota_{n})\right)-2\iota_{n-1}^{\varphi_2-1}\mathbb{F}_7\left(\iota_{n-1},\mathbb{U}(\iota_{n-1})\right)\right.\\
&\left.+\iota_{n-2}^{\varphi_2-1}\mathbb{F}_7\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\right]\mathcal{N}_3\left(m,n,\varphi_1\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{I}_v(\iota_{m+1})-\mathbb{I}_{v0} &= \frac{(1-\varphi_1)\varphi_2\iota_m^{\varphi_2-1}}{M(\varphi_1)}\mathbb{F}_8\left(\iota_m,\mathbb{U}(\iota_m)\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+1)}\sum_{n=1}^{m}\iota_{n-2}^{\varphi_2-1}\mathbb{F}_8\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\mathcal{N}_1\left(m,n,\varphi_1\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{M(\varphi_1)\Gamma(\varphi_1+2)}\sum_{n=1}^{m}\left[\iota_{n-1}^{\varphi_2-1}\mathbb{F}_8\left(\iota_{n-1},\mathbb{U}(\iota_{n-1})\right)-\iota_{n-2}^{\varphi_2-1}\mathbb{F}_8\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\right]\mathcal{N}_2\left(m,n,\varphi_1\right)\\
&+\frac{\varphi_1\varphi_2 h^{\varphi_1}}{2M(\varphi_1)\Gamma(\varphi_1+3)}\sum_{n=1}^{m}\left[\iota_{n}^{\varphi_2-1}\mathbb{F}_8\left(\iota_{n},\mathbb{U}(\iota_{n})\right)-2\iota_{n-1}^{\varphi_2-1}\mathbb{F}_8\left(\iota_{n-1},\mathbb{U}(\iota_{n-1})\right)\right.\\
&\left.+\iota_{n-2}^{\varphi_2-1}\mathbb{F}_8\left(\iota_{n-2},\mathbb{U}(\iota_{n-2})\right)\right]\mathcal{N}_3\left(m,n,\varphi_1\right).
\end{aligned}
$$

7. Numerical presentations and discussion

Here we use the following initial values for compartments:

$$
(\mathbb{S}_0,\mathbb{E}_0,\mathbb{I}_0,\mathbb{R}_0,\mathbb{A}_0,\mathbb{S}_{v0},\mathbb{E}_{v0},\mathbb{I}_{v0})=(3247543,120,37,0,9743100,12990600,100,100),
$$

and the other nomenclature values are given in Table 1.

7.1. Case-I

Here we plot the numerical results for various compartments in Figure 3 using different fractals fractional orders. In the first case we present the numerical plots for various compartments using the fractals fractional order values in the interval [0.75, 1.0].

The population dynamics of susceptible individuals and susceptible mosquitoes decline at different rates. This suggests that the factors influencing the decline of susceptible individuals may differ from those affecting mosquito populations. It could be due to variations in factors such as vector control measures, population density, or environmental conditions. The population of exposed individuals and mosquitoes initially increases and then decreases, following a similar pattern. This means a relationship between the exposure of individuals and the corresponding exposure or maturation of mosquitoes. It could reflect the interplay between human-mosquito interactions and the disease's incubation period. The population of infected individuals shows an upward trend as the infection spreads within society. Similarly, the population of infectious mosquitoes also increases. This

indicates that the transmission of the disease is ongoing and the number of infected individuals and infectious mosquitoes is growing. It highlights the compulsory of implementing effective control measures to curtail disease transmission. The population of recovered individuals grows as they overcome the infection. This means that individuals who have been infected with the disease are gradually recovering and developing immunity. The increasing population of recovered individuals contributes to reducing the susceptible population and can have implications for herd immunity and disease control. The information indicates that variations in the fractal fractional order correspond to variations in the behavior of epidemiological states. This confirms the fractal fractional operator used in our work provides a better understanding of the dynamics and characteristics of the disease.

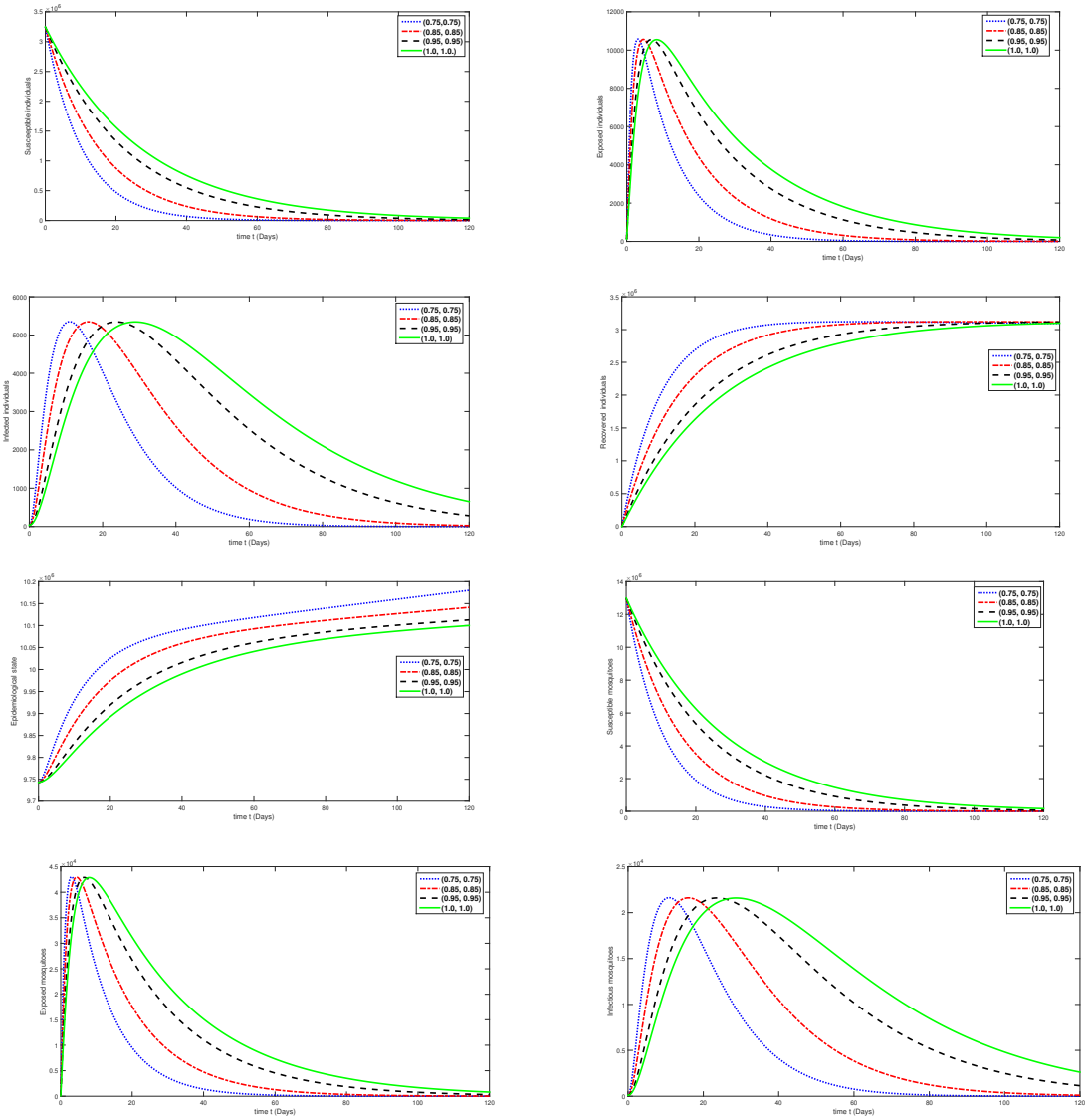

Figure 3. Numerical solutions of susceptible individuals, exposed individuals, infected individuals, recovered individuals, epidemiological state, susceptible mosquitoes, exposed mosquitoes, and infected mosquitoes at various fractional order φ_1 and fractal order φ_2, $(\varphi_1, \varphi_1) = (0.75, 0.75), (0.85, 0.85), (0.95, 0.95), (1.0, 1.0)$

7.2. *Case-II*

Here we plot the numerical results for various compartments in Figure 4 using small different fractals fractional orders. In first case we present the numerical plots for various compartments using the fractals fractional order values in the interval $(0, 0.65]$.

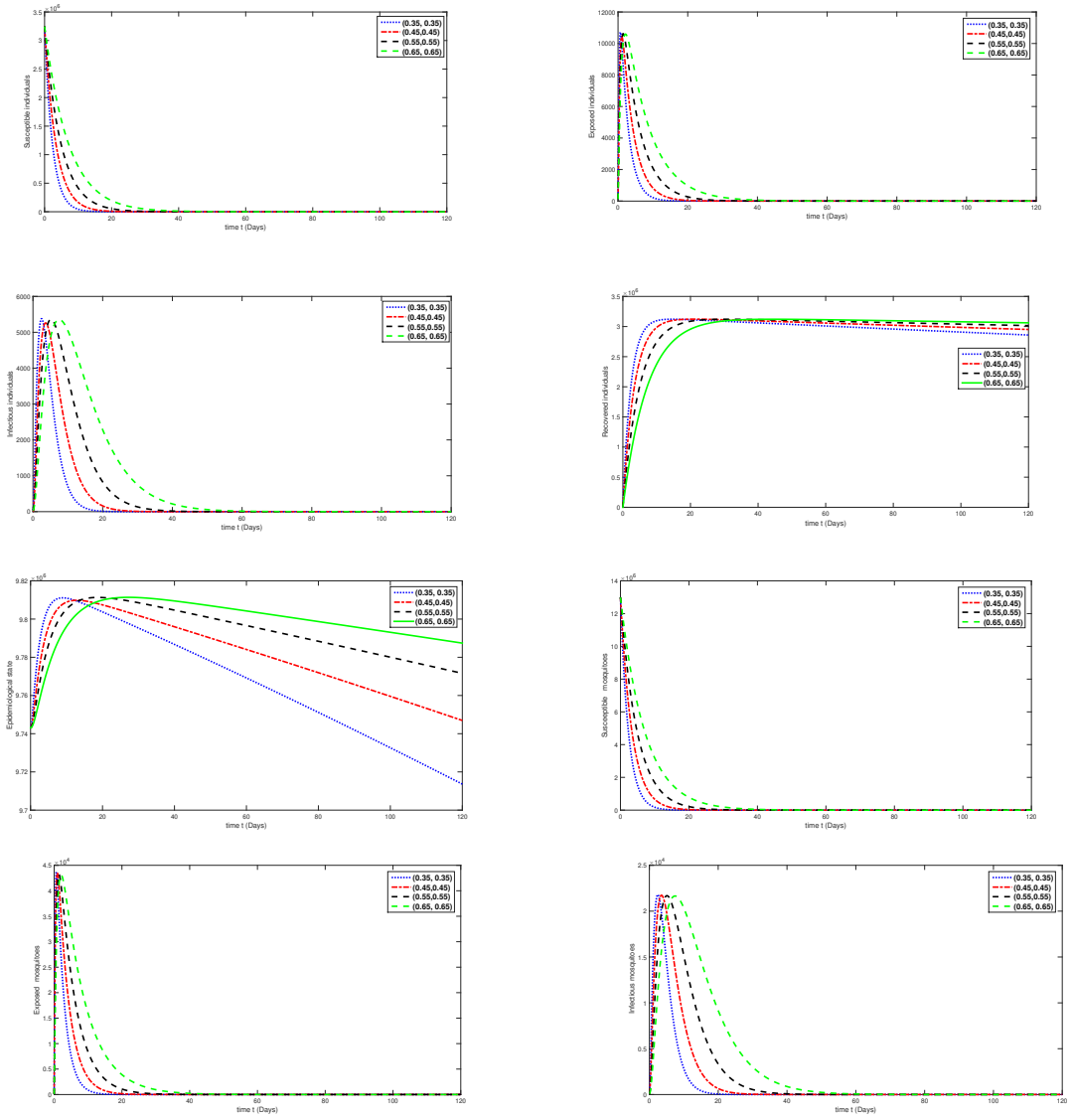

Figure 4. Numerical solutions of susceptible individuals, exposed individuals, infected individuals, recovered individuals, epidemiological state, susceptible mosquitoes, exposed mosquitoes, and infected mosquitoes at various fractional order φ_1 and fractal order φ_2, $(\varphi_1, \varphi_1) = (0.35, 0.35), (0.45, 0.45), (0.55, 0.55), (0.65, 0.65)$

Both sensitive humans and susceptible mosquito populations are declining in distinct proportions. Similarly, the number of exposed people and mosquitoes increases initially before declining in the same situation. As the sickness continues to spread throughout society, the number of affected people is initially rising. Similarly, the number of mosquitoes carrying infectious diseases is rising. After

becoming immune to the pathogen, the population that has recovered is growing. Additionally, the behavior of the epidemiological state varies on various fractional orders of fractals. Also, we noted that the basic reproduction number's most sensitive parameter is b_v, representing the rate of disease transmission from infected individuals to susceptible individuals. Conversely, the parameter ϖ_v, denoting the natural mortality rate of vectors, is the least sensitive.

8. Conclusions

This paper presents a compartmental model for dengue fever disease, utilizing a harmonic mean type incidence rate. The analysis employs nonsingular, fractal fractional differential operators. The fundamental analysis includes the equilibrium points and the basic reproductive number. Additionally, results related to the stability theory are derived. The qualitative analysis investigates the existence and uniqueness of the solution for the model using a fixed-point approach. Furthermore, a numerical scheme based on the Newton polynomial method is developed for the dengue model. By utilizing numerical values for initial data and model parameters, various graphical presentations are provided to investigate the transmission dynamics for different fractal fractional order values. The results obtained contribute to the understanding of dengue transmission dynamics and have broader implications for the application of fractal-fractional derivatives in various mathematical and practical contexts. The information mentioned in the numerical presentations and discussion section indicates that variations in the fractal fractional order correspond to variations in the behavior of epidemiological states. This confirms the fractal fractional operator used in our work provides a better understanding of the dynamics and characteristics of the disease. Furthermore, if $\mathfrak{R}_0 < 1$, we conclude that the biologically realistic disease-free equilibrium is locally and globally asymptotically stable. Simulations on the application of insecticides (larvicide and adulticide), were made. The adulticide was the most effective control, from the fact that with a low percentage of insecticide, the basic reproduction number is kept below unit and the infected number of humans was smaller. We aim in future work to present some studies in depth that describe the dengue model, with optimal control strategies of the outcomes.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that the research was conducted in the absence of any conflict of interest.

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