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Research article

Novel fixed-time synchronization results of fractional-order fuzzy cellular neural networks with delays and interactions

Jun Liu[1,2], **Wenjing Deng**[1], **Shuqin Sun**[3,*] **and Kaibo Shi**[4]

[1] School of Mathematics Sciences, University of Electronic Science and Technology of China, Chengdu Sichuan 611731, China
[2] Visual Computing and Virtual Reality Key Laboratory of Sichuan Province, Sichuan Normal University, Chengdu, Sichuan 610068, China
[3] School of Mathematics Education, China West Normal University, Nanchong Sichuan 637002, China
[4] School of Electronic Information and Electrical Engineering, Chengdu University, Chengdu, Sichuan 610106, China

* **Correspondence:** Email: sunshuqinsusan@163.com.

Abstract: This research investigated the fixed-time (FXT) synchronization of fractional-order fuzzy cellular neural networks (FCNNs) with delays and interactions based on an enhanced FXT stability theorem. By conceiving proper Lyapunov functions and applying inequality techniques, several sufficient conditions were obtained to vouch for the fixed-time synchronization (FXTS) of the discussed systems through two categories of control schemes. Moreover, in terms of another FXT stability theorem, different upper-bounding estimating formulas for settling time (ST) were given, and the distinctions between them were pointed out. Two examples were delivered at length to demonstrate the conclusions.

Keywords: synchronization; adaptive control; fractional-order; neural networks; interactions
Mathematics Subject Classification: 37N35, 93D15, 93D21, 93D40

1. Introduction

Cellular neural networks (CNN) were first presented in 1988 by Chua and Yang [1], which have gained widespread attention because of their many applications. Fuzzy mathematics was originally established by Zadeh, while Wu and Yang [2, 3] incorporated fuzzy operators into the structure of CNNs and introduced fuzzy cellular neural networks (FCNNs). Among conventional CNNs, FCNNs possess improved competence in pattern classification and image encryption [4–7]. In 1998, the authors in [8] first proposed fractional-order CNNs and investigated their dynamical behaviors in the case of chaos and bifurcation. Owing to the infinite memory and hereditary properties of fractional derivatives, various practical applications can be simulated with greater accuracy by fractional-order CNNs. Therefore, concerning the merits of FCNNs and fractional-order CNNs, the fractional-order FCNNs have been studied by many researchers, and several useful conclusions have been obtained [9–15].

The dynamic behavior of nonlinear systems [16, 17] is characterized by their stability. Similarly, the synchronization between systems is identical to the stabilization of their corresponding error systems. Among the previous literature discussed on the stability of nonlinear systems, there have been plenty of references to asymptotic stabilization and exponential stabilization [9, 10, 18–24]. Varying from infinite-time stabilization such as asymptotic stabilization, finite-time stability [11, 13, 14, 25, 26] is more appropriate for practical applications, as it can ensure that a system reaches a stable circumstance on a finite horizon. However, the settling time (ST) of a system depends on its initial values; therefore, the initial values must be known. But obtaining the initial values for any arbitrary system in engineering can be challenging, if not unattainable.

To overcome this drawback, the FXT stability theory was proposed [27]. The FXT stability indicates that the system is not only stable in a restricted time, but the ST estimation is foreign to the initial conditions. Based on these advantages, some scholars have studied this issue and obtained sufficient conditions to judge the FXT stability for dynamical systems [28–34]. However, the yielded estimating formula of the ST is not unique because of the variety of research methods and theoretical analyses. However, under the extant FXT stability theorems, several articles have been published on FXT stability or nonlinear system synchronization [35–37]. In [15], the FXTS of delayed fractional-order memristor-based FCNNs was studied by employing a feedback control. The authors in [38] enforced a sliding mode control scheme and discussed the FXTS of fractional-order memristive BAM neural networks. An innovative state-feedback control scheme was used in [39] to study the FXT stabilization of fractional-order memristive complex-valued BAM neural networks that included uncertain parameters and delays.

To achieve FXTS between drive-response systems, it is necessary to rely on the control schemes. Universal control schemes include state feedback control, adaptive control [38, 40–42], and impulsive control [43]. Among them, the application of adaptive control in the study of nonlinear systems is widespread, because it is robust and can automatically adjust parameters in terms of updated laws. The global asymptotical and exponential synchronization problems of chaotic fractional-order FCNNs were studied using a novel adaptive control scheme in [22]. The authors in [44] obtained the FXTS criteria of fuzzy stochastic CNNs with discrete and distributed delays using state feedback and adaptive control individually. In [45], two types of controllers were used to attain the FXTS of stochastic memristor-based neural networks involving state feedback and adaptive control.

Meanwhile, time delay is a factor that exists in actual systems and cannot be ignored. It can change the stable state of the system and culminate in intricate dynamical behaviors such as bifurcation and oscillation. Consequently, it is more reasonable to study nonlinear systems with time delays. In addition, interactions between two networks are inevitable. Authors in [46] introduced a model that included two coupled networks with interactions and investigated two synchronizations of the discussed model through adaptive control. Subsequently, a fuzzy neural network model with fractional-order involving interaction was proposed in [20], and its global asymptotic synchronization criteria were obtained.

To the best of our knowledge, few studies have considered the FXTS of fractional-order FCNNs with delays and interactions. Motivated by the aforementioned analysis, we have focused on the FXTS of fractional-order FCNNs with delays and interactions under state feedback control and adaptive control. The main contributions of this study are as follows:

(1) The model discussed in this study integrates fractional calculus, fuzzy operators, time delays, and interaction terms. Among the literature on FXTS, few of them involved the model used in this study. For example, [9–15] considered fractional calculus, fuzzy operators, and time delays, but didn't consider interaction terms or FXTS; [20, 46] considered fractional calculus, fuzzy operators, and interaction terms but didn't consider time delays and FXTS.

(2) Control strategies based on state feedback and adaptive control are presented to vouch for the FXTS of the proposed system. With an upgraded FXT stability theorem, by constructing Lyapunov functions, the criteria with the upper-bounding estimation of the ST are acquired to guarantee the FXTS of the discussed system.

(3) According to another FXT stability theorem, we infer the different upper-bounding estimating formulas for the ST and point out the distinctions between them. Essentially, the main conclusions can be generalized to the case of unbounded interaction functions between drive and response systems.

The remainder of this paper is organized in the following manner. In Section 2, the model of fraction-order FCNNs with delays and interactions is proposed, and mathematical preliminaries are presented which will be used to prove the main theorems. In Section 3, state feedback and adaptive control strategies are applied to obtain FXTS criteria. In Section 4, two examples of the simulated results explicate the effectiveness of these outcomes. The final section concludes the study and proposes future work.

Notation: Throughout this study, all fractional-order derivatives are based on the definition of the Caputo fractional differential operator ${}_C\mathscr{D}^{\alpha}_{t_0,t}$. $\mathbb{R}$ and $\mathbb{R}^n$ represent the set of real numbers and the n-dimensional Euclidean space, respectively. Let $\mathbb{R}^+ = [0, +\infty)$, and $\mathbb{Z}^+$ be the set of positive integers. $C^1(\mathbb{R}^+, \mathbb{R})$ is the space of continuous and differentiable functions from $\mathbb{R}^+$ into $\mathbb{R}$. The notation $sign(\cdot)$ denotes a sign function.

2. Preliminaries

The fractional-order FCNNs with delays and interactions are portrayed by

$$\begin{aligned}{}_C\mathscr{D}^{\alpha}_{0,t}x_i(t) =& - c_i x_i(t) + \sum_{j\in\Xi} a_{ij} f_j(x_j(t)) + \sum_{j\in\Xi} b_{ij} f_j(x_j(t-\varsigma)) \\ &+ \bigwedge_{j\in\Xi} \alpha_{ij} f_j(x_j(t-\varsigma)) + \bigvee_{j\in\Xi} \beta_{ij} f_j(x_j(t-\varsigma)) + \varepsilon \sum_{j\in\Xi} d_{ij} h_j(y_j(t)) + I_i, \end{aligned} \tag{2.1}$$

$$\begin{aligned}{}_C\mathscr{D}^{\alpha}_{0,t}y_i(t) =& - c_i y_i(t) + \sum_{j\in\Xi} a_{ij} f_j(y_j(t)) + \sum_{j\in\Xi} b_{ij} f_j(y_j(t-\varsigma)) \\ &+ \bigwedge_{j\in\Xi} \alpha_{ij} f_j(y_j(t-\varsigma)) + \bigvee_{j\in\Xi} \beta_{ij} f_j(y_j(t-\varsigma)) + \varepsilon \sum_{j\in\Xi} \bar{d}_{ij} h_j(x_j(t)) + I_i + u_i(t). \end{aligned} \tag{2.2}$$

Models (2.1) and (2.2) are drive-response systems, where $0 < \alpha < 1$, $t \geq 0$, $\varsigma(\varsigma > 0)$ is the transmission delay, and ε is the strength of the outer interaction. In the drive and response systems, $x_i(t)$ and $y_i(t)$ represent the ith unit state variables at time t. The passive decay rate is indicated by

$c_i(c_i > 0)$. Feedback templates (fuzzy feedback templates) include elements a_{ij} and b_{ij} (α_{ij} and β_{ij}). The structures of these interactions are d_{ij} and $\bar{d}_{ij}$. Fuzzy AND and fuzzy OR are indicated by $\bigwedge$ and $\bigvee$. A bias is indicated by I_i and the control input is denoted by $u_i(t)$, of the ith neuron. In the jth neuron, $f_j(\cdot)$ ($f_j(0) = 0$) represents the activation function. The two networks interact with the function $h_j(\cdot)$ ($h_j(0) = 0$), $i, j \in \Xi = \{1, 2, \ldots, n\}$, n indicates the number of neurons.

Remark 2.1. *Systems (2.1) and (2.2) include interactions, fuzzy OR, and fuzzy AND operations. The complexity and uncertainty can be better described in mathematical modeling, and coupled strengths change dynamically.*

The initial values of (2.1) and (2.2) are given like this:

$$x_i(t) = \psi_i(t), \quad t \in [-\varsigma, 0], \tag{2.3}$$

$$y_i(t) = \varphi_i(t), \quad t \in [-\varsigma, 0]. \tag{2.4}$$

Assume $x_i(t)$ and $y_i(t)$ are arbitrary solutions of (2.1) with (2.3) and (2.2) with (2.4), respectively. Let $\varpi_i(t) = y_i(t) - x_i(t), i \in \Xi$. Then the error system is illustrated as follows:

$$\begin{aligned} {}_C\mathscr{D}^{\alpha}_{0,t}\varpi_i(t) = &- c_i\varpi_i(t) + \sum_{j\in\Xi} a_{ij} f_j(\varpi_j(t)) + \sum_{j\in\Xi} b_{ij} f_j(\varpi_j(t-\varsigma)) + \bigwedge_{j\in\Xi} \alpha_{ij} F_j(\varpi_j(t-\varsigma)) \\ &+ \bigvee_{j\in\Xi} \beta_{ij} F_j(\varpi_j(t-\varsigma)) + \varepsilon \sum_{j\in\Xi} \bar{d}_{ij} h_j(x_j(t)) - \varepsilon \sum_{j\in\Xi} d_{ij} h_j(y_j(t)) + u_i(t), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} f_j(\varpi_j(t)) =& f_j(y_j(t)) - f_j(x_j(t)), \\ \bigwedge_{j\in\Xi} \alpha_{ij} F_j(\varpi_j(t-\varsigma)) =& \bigwedge_{j\in\Xi} \alpha_{ij} f_j(y_j(t-\varsigma)) - \bigwedge_{j\in\Xi} \alpha_{ij} f_j(x_j(t-\varsigma)), \\ \bigvee_{j\in\Xi} \beta_{ij} F_j(\varpi_j(t-\varsigma)) =& \bigvee_{j\in\Xi} \beta_{ij} f_j(y_j(t-\varsigma)) - \bigvee_{j\in\Xi} \beta_{ij} f_j(x_j(t-\varsigma)). \end{aligned}$$

Definition 2.1. *[47] With fractional-order $\alpha > 0$, the Riemann-Liouville fractional integral of function $g(x)$ is defined as follows:*

$$ {}_{x_0}I^{\alpha}_x g(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-s)^{\alpha-1} g(s) ds,$$

where $\Gamma(\cdot)$ represents the Gamma function and is provided by $\Gamma(t) = \int_0^{\infty} \mu^{t-1} e^{-\mu} d\mu$.

Definition 2.2. *[47] With fractional-order $\alpha > 0$, the Caputo derivative of function $g(x)$ is defined as follows:*

$$ {}_C\mathscr{D}^{\alpha}_{x_0,x} g(x) = \frac{1}{\Gamma(n-\alpha)} \int_{x_0}^{x} (x-s)^{n-\alpha-1} g^{(n)}(s) ds,$$

where $n - 1 < \alpha < n \in \mathbb{Z}^+$.

In order to keep things simple, we refer to

$$ {}_{C}\mathscr{D}^{\alpha}_{t_0,t}g(t) = \mathscr{D}^{\alpha}_{t}g(t), $$

where t_0 is the initial time. In this paper, let $t_0 = 0$.

Lemma 2.1. *[48] Let $W(t) \in C^1(\mathbb{R}^+, \mathbb{R})$. Then*

$$ \mathscr{D}^{\alpha}_{t}|W(t)| \leq sign(W(t))\mathscr{D}^{\alpha}_{t}W(t), \quad 0 < \alpha \leq 1. $$

Lemma 2.2. *[2] Let $x_j(t)$ and $z_j(t)$ be two state variables of system (2.5). Then the following inequalities hold:*

$$ \left|\bigwedge_{j\in\Xi} a_{ij}g_j(x_j(t)) - \bigwedge_{j\in\Xi} a_{ij}g_j(z_j(t))\right| \leq \sum_{j\in\Xi} \left|a_{ij}\right| \left|g_j(x_j(t)) - g_j(z_j(t))\right|, $$

$$ \left|\bigvee_{j\in\Xi} b_{ij}g_j(x_j(t)) - \bigvee_{j\in\Xi} b_{ij}g_j(z_j(t))\right| \leq \sum_{j\in\Xi} \left|b_{ij}\right| \left|g_j(x_j(t)) - g_j(z_j(t))\right|. $$

Lemma 2.3. *[49] Let $a_i \geq 0, 0 < q < 1, p > 1$. Then we have*

$$ \begin{aligned} a_1^q + a_2^q + \cdots + a_K^q &\geq (a_1 + a_2 + \cdots + a_K))^q, \\ a_i^p + a_2^p + \cdots + a_K^p &\geq K^{1-p}(a_1 + a_2 + \cdots + a_K))^p. \end{aligned} $$

Lemma 2.4. *[29] Suppose $W(\cdot) : \mathbb{R}^n \to \mathbb{R}^+$ is a continuous radially unbounded function (CRUF) under the following conditions:*

(1) $W(\varpi(t)) = 0$ *if and only if* $\varpi(t) = 0$;

(2) *In system (2.5), any solution $\varpi(t)$ is satisfied by:*

$$ \dot{W}(\varpi(t)) \leq -aW^q((\varpi(t))) - bW^p((\varpi(t))) - cW(\varpi(t)), $$

where $a, b, c > 0, 0 < q < 1, p > 1$.

Thus, it is FXT stable at the origin of system (2.5), and the ST estimating formula T^1_{max} is

$$ T^1_{max} = \frac{p-q}{c(p-1)(1-q)} \ln\left(1 + \frac{c}{a}\left(\frac{a}{b}\right)^{\frac{1-q}{p-q}}\right). $$

Remark 2.2. *Lemma 2.4 is superior to the FXT stability theorem of [28] in the estimation of ST. In [28], ST $T^2_{max} = \frac{1}{c(1-q)}\ln(1 + \frac{c}{a}) + \frac{1}{c(p-1)}\ln(1 + \frac{c}{b})$. When $a = b$, $T^1_{max} = T^2_{max}$, otherwise $T^2_{max} > T^1_{max}$, (see [29]).*

Lemma 2.5. *[33] Let $W(\cdot) : \mathbb{R}^n \to \mathbb{R}^+$ be a CRUF, under the following conditions:*

(1) $W(\varpi(t)) = 0$ *if and only if* $\varpi(t) = 0$.

(2) *In system (2.5), any solution $\varpi(t)$ is satisfied by:*

$$ \dot{W}(\varpi(t)) \leq -aW^q((\varpi(t))) - bW^p((\varpi(t))) - c. $$

(3) *If $\varpi(t) = 0, \dot{W}(\varpi(t)) \leq 0$.*

Thus, it is FXT stable at the origin of system (2.5), and the ST estimating formula T^3_{max} is

$$ T^3_{max} = \frac{1}{a^{\frac{1}{q}}(1-q)}\left[(a^{\frac{1}{q}} + c^{\frac{1}{q}})^{1-q} - c^{\frac{1-q}{q}}\right] + \frac{2^{p-1}}{b^{\frac{1}{p}}(p-1)}(b^{\frac{1}{p}} + c^{\frac{1}{p}})^{1-p}. $$

Remark 2.3. *If* $c = 0$, $T^3_{max} = \frac{1}{a(1-q)} + \frac{1}{b(p-1)}$.

Assumption 2.1. *Let the activation functions* $f_i(x)$ *and interaction functions* $h_i(x)$ *be Lipschitz continuous, that is, there exist constants* $F_i, H_i > 0$, *which makes*

$$\begin{aligned} |f_i(x) - f_i(y)| \le & F_i \, |x - y| \, , \\ |h_i(x) - h_i(y)| \le & H_i \, |x - y| \, , \ \forall i \in \Xi. \end{aligned}$$

Assumption 2.2. *The interaction functions* $h_i(x)$ *are bounded. This means that there exists a positive constant* M_i, *which makes*

$$|h_i(x)| \le M_i, \quad \forall i \in \Xi.$$

3. Main results

In this section, we present some FXTS criteria for fractional-order FCNNs with delays and interactions based on the state feedback and adaptive control.

3.1. Feedback control

The state feedback controller $u_i(t)$ is designed as follows:

$$\begin{aligned} u_i(t) = & - \eta_i \varpi_i(t) - sign(\varpi_i(t))[\rho_i + \gamma_i |\varpi_i(t - \varsigma)| + k_1 (\mathscr{D}_t^{\alpha-1} |\varpi_i(t)|)^q \\ & + k_2 (\mathscr{D}_t^{\alpha-1} |\varpi_i(t)|)^p + k_3 \mathscr{D}_t^{\alpha-1} |\varpi_i(t)|], \end{aligned} \tag{3.1}$$

where $k_1, k_2, k_3 > 0, 0 < q < 1, p > 1$, and η_i, ρ_i, and γ_i are all positive constants.

Theorem 3.1. *Suppose Assumptions 2.1 and 2.2 hold. If the error system (2.5) is controlled by control law (3.1) with*

$$\begin{cases} c_i + \eta_i - \sum\limits_{j \in \Xi} \left(|a_{ji}| F_i + \epsilon t_{ji} H_i \right) \ge 0, \\ t_{ji} = \min \left\{ |d_{ji}|, \ |\bar{d}_{ji}| \right\}, \\ \gamma_i - \sum\limits_{j \in \Xi} F_i \left(|b_{ji}| + |\alpha_{ji}| + |\beta_{ji}| \right) \ge 0, \\ \sum\limits_{i \in \Xi} (\rho_i - \sum\limits_{j \in \Xi} \epsilon |\bar{d}_{ij} - d_{ij}| M_j) \ge 0, \ \forall i \in \Xi, \end{cases} \tag{3.2}$$

then systems (2.1) and (2.2) are FXTS. In addition, ST T^1_{max} *is estimated by*

$$T^1_{max} = \frac{p - q}{k_3 (p-1)(1-q)} \ln \left[1 + \frac{k_3}{k_1} \left(\frac{k_1}{k_2 n^{1-p}} \right)^{\frac{1-q}{p-q}} \right].$$

Proof. The chosen Lyapunov function is depicted by

$$V(\varpi(t)) = \sum_{i \in \Xi} \mathscr{D}_t^{\alpha-1} |\varpi_i(t)|.$$

This together with the error system (2.5) implies that

$$
\begin{aligned}
\dot{V}(\varpi(t)) &= \mathscr{D}_t^{\alpha}\left(\mathscr{D}_t^{1-\alpha}V(t)\right)\\
&= \mathscr{D}_t^{\alpha}\left(\mathscr{D}_t^{1-\alpha}\sum_{i\in\Xi}\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|\right)\\
&= \mathscr{D}_t^{\alpha}\left(\sum_{i\in\Xi}\mathscr{D}_t^{1-\alpha}\left(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|\right)\right)\\
&= \sum_{i\in\Xi}\mathscr{D}_t^{\alpha}|\varpi_i(t)|\\
&\le \sum_{i\in\Xi} sign\left(\varpi_i(t)\right)\mathscr{D}_t^{\alpha}\varpi_i(t).
\end{aligned}
\tag{3.3}
$$

Replacing $\mathscr{D}_t^{\alpha}\varpi_i(t)$ with Eqs (2.5) and (3.1), we have

$$
\begin{aligned}
\dot{V}(\varpi(t)) \le& \sum_{i\in\Xi} sign(\varpi_i(t))\Big[-c_i\varpi_i(t)+\sum_{j\in\Xi}a_{ij}f_j(\varpi_j(t))+\sum_{j\in\Xi}b_{ij}f_j(\varpi_j(t-\varsigma))+\bigwedge_{j\in\Xi}\alpha_{ij}F_j(\varpi_j(t-\varsigma))\\
&+\bigvee_{j\in\Xi}\beta_{ij}F_j(\varpi_j(t-\varsigma))+\epsilon\sum_{j\in\Xi}\bar{d}_{ij}h_j(x_j(t))-\epsilon\sum_{j\in\Xi}d_{ij}h_j(y_j(t))-\eta_i\varpi_i(t)-sign(\varpi_i(t))\rho_i\\
&-sign(\varpi_i(t))(\gamma_i|\varpi_i(t-\varsigma)|+k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q+k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p+k_3(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|))\Big]\\
=&\sum_{i\in\Xi}(-c_i-\eta_i)|\varpi_i(t)|+\sum_{i\in\Xi}sign(\varpi_i(t))\Big[\sum_{j\in\Xi}a_{ij}f_j(\varpi_j(t))+\sum_{j\in\Xi}b_{ij}f_j(\varpi_j(t-\varsigma))\\
&+\bigwedge_{j\in\Xi}\alpha_{ij}F_j(\varpi_j(t-\varsigma))+\bigvee_{j\in\Xi}\beta_{ij}F_j(\varpi_j(t-\varsigma))+\epsilon\sum_{j\in\Xi}\bar{d}_{ij}h_j(x_j(t))-\epsilon\sum_{j\in\Xi}d_{ij}h_j(y_j(t))\Big]\\
&-\sum_{i\in\Xi}[\rho_i+\gamma_i|\varpi_i(t-\varsigma)|+k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q+k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p+k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|]\\
\le&-\sum_{i\in\Xi}(c_i+\eta_i)|\varpi_i(t)|+\sum_{i,j\in\Xi}[|a_{ij}||f_j(\varpi_j(t))|+|b_{ij}||f_j(\varpi_j(t-\varsigma))|+\epsilon|\bar{d}_{ij}h_j(x_j(t))-d_{ij}h_j(y_j(t))|]\\
&+\sum_{i\in\Xi}[|\bigwedge_{j\in\Xi}\alpha_{ij}F_j(\varpi_j(t-\varsigma))|+|\bigvee_{j\in\Xi}\beta_{ij}F_j(\varpi_j(t-\varsigma))|]-\sum_{i\in\Xi}[\rho_i+\gamma_i|\varpi_i(t-\varsigma)|\\
&+k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q+k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p+k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|].
\end{aligned}
\tag{3.4}
$$

Based on Assumption 2.1 and Lemma 2.2, one has

$$
\begin{gathered}
|f_j(\varpi_j(t))| \le F_j|\varpi_j(t)|,\\
|f_j(\varpi_j(t-\varsigma))| \le F_j|\varpi_j(t-\varsigma)|,\\
|h_j(x_j(t))-h_j(y_j(t))| \le H_j|\varpi_j(t)|,\\
|\bigwedge_{j\in\Xi}\alpha_{ij}F_j(\varpi_j(t-\varsigma))| \le \sum_{j\in\Xi}|\alpha_{ij}||f_j(y_j(t-\varsigma))-f_j(x_j(t-\varsigma))| \le \sum_{j\in\Xi}|\alpha_{ij}|F_j|\varpi_j(t-\varsigma)|,\\
|\bigvee_{j\in\Xi}\beta_{ij}F_j(\varpi_j(t-\varsigma))| \le \sum_{j\in\Xi}|\beta_{ij}||f_j(y_j(t-\varsigma))-f_j(x_j(t-\varsigma))| \le \sum_{j\in\Xi}|\beta_{ij}|F_j|\varpi_j(t-\varsigma)|.
\end{gathered}
$$

Therefore,

$$\begin{aligned}\dot{V}(\varpi(t)) \leq & \sum_{i\in\Xi}(-c_i-\eta_i)|\varpi_i(t)| + \sum_{i,j\in\Xi}[|a_{ij}|F_j|\varpi_j(t)| + |b_{ij}|F_j|\varpi_j(t-\varsigma)|] + \sum_{i,j\in\Xi}[|\alpha_{ij}|F_j|\varpi_j(t-\varsigma)| \\ & + |\beta_{ij}|F_j|\varpi_j(t-\varsigma)| + \epsilon|\bar{d}_{ij}h_j(x_j(t)) - d_{ij}h_j(y_j(t))|] - \sum_{i\in\Xi}[\rho_i + \gamma_i|\varpi_i(t-\varsigma)| \\ & + k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q + k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p + k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|]. \end{aligned} \tag{3.5}$$

Since

$$\begin{aligned}\sum_{i,j\in\Xi}|\bar{d}_{ij}h_j(x_j(t)) - d_{ij}h_j(y_j(t))| &\leq \sum_{i,j\in\Xi}[|\bar{d}_{ij}h_j(x_j(t)) - \bar{d}_{ij}h_j(y_j(t))| + |\bar{d}_{ij}h_j(y_j(t)) - d_{ij}h_j(y_j(t))|] \\ &\leq \sum_{i,j\in\Xi}[|\bar{d}_{ij}|H_j|\varpi_j(t)| + |\bar{d}_{ij} - d_{ij}|M_j], \end{aligned} \tag{3.6}$$

$$\begin{aligned}\sum_{i,j\in\Xi}|\bar{d}_{ij}h_j(x_j(t)) - d_{ij}h_j(y_j(t))| &\leq \sum_{i,j\in\Xi}[|\bar{d}_{ij}h_j(x_j(t)) - d_{ij}h_j(x_j(t))| + |d_{ij}h_j(x_j(t)) - d_{ij}h_j(y_j(t))|] \\ &\leq \sum_{i,j\in\Xi}[|\bar{d}_{ij} - d_{ij}|M_j + |d_{ij}|H_j|\varpi_j(t)|]. \end{aligned} \tag{3.7}$$

According to Eqs (3.6) and (3.7), we have

$$\sum_{i,j\in\Xi}|\bar{d}_{ij}h_j(x_j(t)) - d_{ij}h_j(y_j(t))| \leq \sum_{i,j\in\Xi}[|\bar{d}_{ij} - d_{ij}|M_j + t_{ij}H_j|\varpi_j(t)|], \tag{3.8}$$

where $t_{ij} = \min\left\{|d_{ij}|, |\bar{d}_{ij}|\right\}$.

Placing Eq (3.8) into (3.5), one obtains

$$\begin{aligned}\dot{V}(\varpi(t)) \leq & \sum_{i\in\Xi}(-c_i-\eta_i)|\varpi_i(t)| + \sum_{i,j\in\Xi}[|a_{ij}|F_j|\varpi_j(t)| + |b_{ij}|F_j|\varpi_j(t-\varsigma)|] + \sum_{i,j\in\Xi}[|\alpha_{ij}|F_j|\varpi_j(t-\varsigma)| \\ & + |\beta_{ij}|F_j|\varpi_j(t-\varsigma)| + \epsilon t_{ij}H_j|\varpi_j(t)|] + \epsilon\sum_{i,j\in\Xi}|\bar{d}_{ij} - d_{ij}|M_j - \sum_{i\in\Xi}[\rho_i + \gamma_i|\varpi_i(t-\varsigma)| \\ & + k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q] - k_2\sum_{i\in\Xi}[(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p + k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|] \\ = & -\sum_{i\in\Xi}[c_i + \eta_i - \sum_{j\in\Xi}\left(|a_{ji}|F_i + \epsilon t_{ji}H_i\right)]|\varpi_i(t)| - \sum_{i\in\Xi}\gamma_i|\varpi_i(t-\varsigma)| + \sum_{i,j\in\Xi}F_i(|b_{ji}| + |\alpha_{ji}| \\ & + |\beta_{ji}|)|\varpi_i(t-\varsigma)| - \sum_{i\in\Xi}(\rho_i - \epsilon\sum_{j\in\Xi}|\bar{d}_{ij} - d_{ij}|M_j) \\ & - \sum_{i\in\Xi}[k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q + k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p + k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|] \\ \leq & -\sum_{i\in\Xi}[k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q + k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p + k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|]. \end{aligned} \tag{3.9}$$

From Lemma 2.3, we derive that

$$\sum_{i\in\Xi}(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q \geq (\sum_{i\in\Xi}\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q,$$

$$\sum_{i\in\Xi}(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p \geq n^{1-p}(\sum_{i\in\Xi}\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p.$$

Then

$$\begin{aligned}\dot{V}(\varpi(t)) &\leq -k_1(\sum_{i\in\Xi}\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q - k_2n^{1-p}(\sum_{i\in\Xi}\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p - k_3\sum_{i\in\Xi}\mathscr{D}_t^{\alpha-1}|\varpi_i(t)| \\ &= -k_1V^q(\varpi(t)) - k_2n^{1-p}V^p(\varpi(t)) - k_3V(\varpi(t)). \end{aligned} \tag{3.10}$$

By Lemma 2.4, systems (2.1) and (2.2) are FXTS within

$$T_{max}^1 = \frac{p-q}{k_3(p-1)(1-q)}\ln\left[1+\frac{k_3}{k_1}\left(\frac{k_1}{k_2n^{1-p}}\right)^{\frac{1-q}{p-q}}\right].$$

From Lemma 2.5, we yield the next corollary.

Corollary 3.1. *Suppose Assumptions 2.1 and 2.2 hold. If*

$$\begin{cases} c_i + \eta_i - \sum_{j\in\Xi}\left(|a_{ji}|F_i + \epsilon t_{ji}H_i\right) \geq 0, \\ t_{ji} = \min\left\{|d_{ji}|,\ |\bar{d}_{ji}|\right\}, \\ \gamma_i - \sum_{j\in\Xi}F_i\left(|b_{ji}| + |\alpha_{ji}| + |\beta_{ji}|\right) \geq 0, \\ \lambda_1 = \sum_{i\in\Xi}(\rho_i - \sum_{j\in\Xi}\epsilon|\bar{d}_{ij} - d_{ij}|M_j) \geq 0,\ \forall i\in\Xi, \end{cases} \tag{3.11}$$

then the control scheme is as follows:

$$u_i(t) = -\eta_i\varpi_i(t) - sign(\varpi_i(t))[\rho_i + \gamma_i|\varpi_i(t-\varsigma)| + k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q + k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p], \tag{3.12}$$

and the systems (2.1) and (2.2) are FXTS. Moreover, the ST T_{max}^3 is

$$T_{max}^3 = \begin{cases} \frac{1}{k_1^{\frac{1}{q}}(1-q)}\left[(k_1^{\frac{1}{q}} + \lambda_1^{\frac{1}{q}})^{1-q} - \lambda_1^{\frac{1-q}{q}}\right] + \frac{2^{p-1}}{k_2^{\frac{1}{p}}n^{\frac{1-p}{p}}(p-1)}\left(k_2^{\frac{1}{p}}n^{\frac{1-p}{p}} + \lambda_1^{\frac{1}{p}}\right)^{1-p},\ \lambda_1 > 0, \\ \frac{1}{k_1(1-q)} + \frac{1}{k_2n^{1-p}(p-1)},\ \lambda_1 = 0. \end{cases}$$

Remark 3.1. *From Theorem 3.1 and Corollary 3.1, we obtain that the FXTS between drive-response systems (2.1) and (2.2) can be reached under the control of (3.1) or (3.12). However, there are still some differences. Compared with the control scheme (3.1), fewer parameters are needed in the controller (3.12), which can help strengthen its enforceability. However, the ST of Theorem 3.1 is irrelevant to the system parameters, whereas the ST is related to the system parameters in Corollary 3.1. From this perspective, control strategy (3.1) is more practical.*

If the interaction functions h_i are unbounded, the following is the design of the state feedback controller:

$$\begin{aligned} u_i(t) = &-\eta_i\varpi_i(t) - \epsilon\sum_{j\in\Xi}(\bar{d}_{ij} - d_{ij})h_j(x_j(t)) - sign(\varpi_i(t))[\gamma_i|\varpi_i(t-\varsigma)| + k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^q \\ &+ k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^p + k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|]. \end{aligned} \tag{3.13}$$

Corollary 3.2. *Suppose Assumption 2.1 holds. If*

$$\begin{cases} c_i + \eta_i - \sum\limits_{j\in\Xi} \left(|a_{ji}|F_i + \epsilon|d_{ji}|H_i\right) \geq 0, \\ \gamma_i - \sum\limits_{j\in\Xi} F_i \left(|b_{ji}| + |\alpha_{ji}| + |\beta_{ji}|\right) \geq 0, \ \forall i \in \Xi. \end{cases} \tag{3.14}$$

Then, under control scheme (3.13), systems (2.1) and (2.2) are FXTS. Moreover, the ST is equal to T^1_{max} *in Theorem 3.1.*

3.2. Adaptive control

However, the control gain η_i of the state feedback control (3.1) is not easy to determine, so we carry out an adaptive control strategy to tackle the FXTS between systems (2.1) and (2.2). The adaptive control scheme $u_i(t)$ is expressed as

$$\begin{aligned} u_i(t) = & -\eta_i(t)\varpi_i(t) - sign(\varpi_i(t))[\rho_i + \gamma_i|\varpi_i(t-\varsigma)| + k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{q+1}{2}} \\ & + k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{p+1}{2}} + k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|]. \end{aligned} \tag{3.15}$$

The updated law is given as

$$\dot{\eta}_i(t) = \frac{1}{2}|\varpi_i(t)| - \frac{1}{2}sign(\eta_i(t)-\eta_1)[k_1|\eta_i(t)-\eta_1|^q + k_2|\eta_i(t)-\eta_1|^p + k_3|\eta_i(t)-\eta_1|], \tag{3.16}$$

where η_1 is a constant to be determined, and the other parameters have the same meanings as those in controller (3.1).

Theorem 3.2. *Suppose Assumptions 2.1 and 2.2 hold. If the error system (2.5) is controlled by control law (3.15) with*

$$\begin{cases} c_i + \eta_1 - \sum\limits_{j\in\Xi} \left(|a_{ji}|F_i + \epsilon t_{ji}H_i\right) \geq 0, \\ t_{ji} = \min\left\{|d_{ji}|,\ |\bar{d}_{ji}|\right\}, \\ \gamma_i - \sum\limits_{j\in\Xi} F_i \left(|b_{ji}| + |\alpha_{ji}| + |\beta_{ji}|\right) \geq 0, \\ \sum\limits_{i\in\Xi} (\rho_i - \sum\limits_{j\in\Xi} \epsilon|\bar{d}_{ij} - d_{ij}|M_j) \geq 0, \ \forall i \in \Xi, \end{cases} \tag{3.17}$$

thus, systems (2.1) and (2.2) are FXTS. In addition, ST $\bar{T}^1_{max}$ *is estimated by*

$$\bar{T}^1_{max} = \frac{2(p-q)}{k_3(p-1)(1-q)} \ln\left[1 + \frac{k_3}{k_1}\Big(\frac{k_1}{k_2(2n)^{\frac{1-p}{2}}}\Big)^{\frac{1-q}{p-q}}\right].$$

Proof. The chosen Lyapunov function is depicted by

$$V(\varpi(t)) = \sum_{i\in\Xi} [\mathscr{D}_t^{\alpha-1}|\varpi_i(t)| + (\eta_i(t)-\eta_1)^2].$$

According to the analysis of Theorem 3.1,

$$\begin{aligned}\dot{V}(\varpi(t)) \leq &\sum_{i\in\Xi}[sign(\varpi_i(t))\mathscr{D}_t^{\alpha}\varpi_i(t)+2(\eta_i(t)-\eta_1)\dot{\eta}_i(t)]\\ \leq &-\sum_{i\in\Xi}[c_i+\eta_1-\sum_{j\in\Xi}\left(|a_{ji}|F_i+\epsilon t_{ji}H_i\right)]|\varpi_i(t)|\\ &+\sum_{i\in\Xi}[(\eta_i(t)-\eta_1)|\varpi_i(t)|-2(\eta_i(t)-\eta_1)\dot{\eta}_i(t)]\\ &-\sum_{i\in\Xi}[\gamma_i-\sum_{j\in\Xi}F_i(|b_{ji}|+|\alpha_{ji}|+|\beta_{ji}|)]|\varpi_i(t-\varsigma)|\\ &-\sum_{i\in\Xi}[(\rho_i-\epsilon\sum_{j\in\Xi}|\bar{d}_{ij}-d_{ij}|M_j)+k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{q+1}{2}}\\ &+k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{p+1}{2}}+k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|].\end{aligned}\tag{3.18}$$

In terms of Eq (3.17), one obtains

$$\begin{aligned}\dot{V}(\varpi(t)) \leq &-\sum_{i\in\Xi}[(\eta_i(t)-\eta_1)|\varpi_i(t)|-2\eta_i(t)\dot{\eta}_i(t)]-\sum_{i\in\Xi}[k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{q+1}{2}}+k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|\\ &-\sum_{i\in\Xi}k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{p+1}{2}}-2\eta_1\dot{\eta}_i(t)]\\ =&-\sum_{i\in\Xi}[k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{q+1}{2}}+k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|]-\sum_{i\in\Xi}[k_1|\eta_i(t)-\eta_1|^{q+1}+k_3|\eta_i(t)-\eta_1|^2]\\ &-\sum_{i\in\Xi}[k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{p+1}{2}}+k_2|\eta_i(t)-\eta_1|^{p+1}]\\ =&-\sum_{i\in\Xi}\{k_1[(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{q+1}{2}}+|\eta_i(t)-\eta_1|^{q+1}]+k_2[(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{p+1}{2}}+|\eta_i(t)-\eta_1|^{p+1}]\\ &-k_3\sum_{i\in\Xi}[\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|+(\eta_i(t)-\eta_1)^2]\}.\end{aligned}\tag{3.19}$$

From Lemma 2.3, we derive that

$$\begin{aligned}\sum_{i\in\Xi}[(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{q+1}{2}}+|\eta_i(t)-\eta_1|^{q+1}] \geq &\sum_{i\in\Xi}[\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|+(\eta_i(t)-\eta_1)^2]^{\frac{q+1}{2}}\\ \geq &[\sum_{i\in\Xi}\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|+(\eta_i(t)-\eta_1)^2]^{\frac{q+1}{2}},\\ \sum_{i\in\Xi}[(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{p+1}{2}}+|\eta_i(t)-\eta_1|^{p+1}] \geq &2^{\frac{1-p}{2}}\sum_{i\in\Xi}[\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|+(\eta_i(t)-\eta_1)^2]^{\frac{p+1}{2}}\\ \geq &(2n)^{\frac{1-p}{2}}[\sum_{i\in\Xi}\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|+(\eta_i(t)-\eta_1)^2]^{\frac{p+1}{2}}.\end{aligned}$$

Therefore,

$$\dot{V}(\varpi(t)) \leq -k_1[\sum_{i\in\Xi}\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|+(\eta_i(t)-\eta_1)^2]^{\frac{q+1}{2}}-k_2(2n)^{\frac{1-p}{2}}[\sum_{i\in\Xi}\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|+(\eta_i(t)-\eta_1)^2]^{\frac{p+1}{2}}$$

$$-k_3\Big[\sum_{i\in\Xi}\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|+(\eta_i(t)-\eta_1)^2\Big]$$
$$=-k_1V^{\frac{q+1}{2}}(\varpi(t))-k_2(2n)^{\frac{1-p}{2}}V^{\frac{p+1}{2}}(\varpi(t))-k_3V(\varpi(t)). \tag{3.20}$$

By Lemma 2.4, systems (2.1) and (2.2) are FXT synchronizations within

$$\bar{T}_{max}^1=\frac{2(p-q)}{k_3(p-1)(1-q)}\ln\left[1+\frac{k_3}{k_1}\Big(\frac{k_1}{k_2(2n)^{\frac{1-p}{2}}}\Big)^{\frac{1-q}{p-q}}\right]. \tag{3.21}$$

From Lemma 2.5, we yield the next corollary.

Corollary 3.3. *Suppose Assumptions 2.1 and 2.2 hold. If*

$$\begin{cases} c_i+\eta_1-\sum\limits_{j\in\Xi}\Big(|a_{ji}|F_i+\epsilon t_{ji}H_i\Big)\geq 0,\\ t_{ji}=\min\Big\{|d_{ji}|,\ |\bar{d}_{ji}|\Big\},\\ \gamma_i-\sum\limits_{j\in\Xi}F_i\Big(|b_{ji}|+|\alpha_{ji}|+|\beta_{ji}|\Big)\geq 0,\\ \lambda_1=\sum\limits_{i\in\Xi}(\rho_i-\sum\limits_{j\in\Xi}\epsilon|\bar{d}_{ij}-d_{ij}|M_j)\geq 0,\ \forall i\in\Xi, \end{cases} \tag{3.22}$$

then the design of the control scheme is

$$u_i(t)=-\eta_i(t)\varpi_i(t)-sign(\varpi_i(t))[\rho_i+\gamma_i|\varpi_i(t-\varsigma)|+k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{q+1}{2}}+k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{p+1}{2}}], \tag{3.23}$$

where $\dot{\eta}_i(t)=\frac{1}{2}|\varpi_i(t)|-\frac{1}{2}sign(\eta_i(t)-\eta_1)[k_1|\eta_i(t)-\eta_1|^q+k_2|\eta_i(t)-\eta_1|^p]$*, and systems (2.1) and (2.2) are FXTS. Moreover, ST* $\bar{T}_{max}^3$ *is*

$$\bar{T}_{max}^3=\begin{cases}\dfrac{2\triangle_1}{k_1^{\frac{2}{q+1}}(1-q)}+\dfrac{2^{\frac{p+1}{2}}\triangle_2}{k_2^{\frac{2}{p+1}}(2n)^{\frac{1-p}{p+1}}(p-1)},\ \lambda_1>0,\\ \dfrac{2}{k_1(1-q)}+\dfrac{2}{k_2(2n)^{\frac{1-p}{2}}(p-1)},\ \lambda_1=0,\end{cases}$$

where $\triangle_1=(k_1^{\frac{2}{q+1}}+\lambda_1^{\frac{2}{q+1}})^{\frac{1-q}{2}}-\lambda_1^{\frac{1-q}{q+1}}$, $\triangle_2=\left[k_2^{\frac{2}{p+1}}(2n)^{\frac{1-p}{p+1}}+\lambda_1^{\frac{2}{p+1}}\right]^{\frac{1-p}{2}}$.

If the interaction functions h_i are unbounded, the design of the adaptive controller follows:

$$u_i(t)=-\eta_i(t)\varpi_i(t)-\epsilon\sum_{j\in\Xi}(\bar{d}_{ij}-d_{ij})h_j(x_j(t))-sign(\varpi_i(t))[\gamma_i|\varpi_i(t-\varsigma)|$$
$$+k_1(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{q+1}{2}}+k_2(\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|)^{\frac{p+1}{2}}+k_3\mathscr{D}_t^{\alpha-1}|\varpi_i(t)|], \tag{3.24}$$

where $\dot{\eta}_i(t)=\frac{1}{2}|\varpi_i(t)|-\frac{1}{2}sign(\eta_i(t)-\eta_1)[k_1|\eta_i(t)-\eta_1|^q+k_2|\eta_i(t)-\eta_1|^p+k_3|\eta_i(t)-\eta_1|]$.

Corollary 3.4. *Suppose Assumption 2.1 holds. If*

$$\begin{cases} c_i+\eta_1-\sum\limits_{j\in\Xi}\Big(|a_{ji}|F_i+\epsilon|d_{ji}|H_i\Big)\geq 0,\\ \gamma_i-\sum\limits_{j\in\Xi}F_i\Big(|b_{ji}|+|\alpha_{ji}|+|\beta_{ji}|\Big)\geq 0,\ \forall i\in\Xi,\end{cases} \tag{3.25}$$

then, under control scheme (3.24), systems (2.1) and (2.2) are FXTS. Moreover, ST is equal to $\bar{T}_{max}^1$ *in Theorem 3.2.*

4. Numerical examples

The following section illustrates the derived theorems and corollaries with two examples.

Example 4.1. *The 2-dimensional fractional-order FCNNs with delays and interactions are described by*

$$\begin{aligned}\mathscr{D}_t^{\alpha} x_i(t) =& - c_i x_i(t) + \sum_{j=1,2}[a_{ij} f_j(x_j(t)) + b_{ij} f_j(x_j(t-\varsigma)) + \varepsilon d_{ij} h_j(y_j(t))] \\ &+ \bigwedge_{j=1,2} \alpha_{ij} f_j(x_j(t-\varsigma)) + \bigvee_{j=1,2} \beta_{ij} f_j(x_j(t-\varsigma)) + I_i,\end{aligned} \tag{4.1}$$

$$\begin{aligned}\mathscr{D}_t^{\alpha} y_i(t) =& - c_i y_i(t) + \sum_{j=1,2}[a_{ij} f_j(y_j(t)) + b_{ij} f_j(y_j(t-\varsigma)) + \varepsilon \bar{d}_{ij} h_j(x_j(t))] \\ &+ \bigwedge_{j=1,2} \alpha_{ij} f_j(y_j(t-\varsigma)) + \bigvee_{j=1,2} \beta_{ij} f_j(y_j(t-\varsigma)) + I_i + u_i(t),\end{aligned} \tag{4.2}$$

where $\alpha = 0.95, \varepsilon = 1, \varsigma = 0.1, f(\cdot) = h(\cdot) = tanh(\cdot)$, *and* $t \geq 0$, $(c_1, c_2) = (2, 1), (I_1, I_2) = (1, -1), (a_{11}, a_{12}, a_{21}, a_{22}) = (0.5, 0.3, 0.6, 0.4)$, $(b_{11}, b_{12}, b_{21}, b_{22}) = (-0.2, 0.1, 0.1, -0.3)$, $(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}) = (0.6, 0.4, 0.5, 0.2)$, $(\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}) = (0.1, 0.3, 0.5, 0.3)$, $(d_{11}, d_{12}, d_{21}, d_{22}) = (1, 2, 2, 1)$, $\bar{d}_{ij} = 1, i, j = 1, 2$.

To guarantee FXTS between systems (4.1) and (4.2), we apply the state-feedback control strategy as follows

$$\begin{cases} u_1(t) = - 1.5\varpi_1(t) - sign(\varpi_1(t))[1 + 2|\varpi_1(t-\varsigma)| + (\mathscr{D}_t^{-0.05}|\varpi_1(t)|)^{0.2} \\ \qquad + (\mathscr{D}_t^{-0.05}|\varpi_1(t)|)^{1.8} + 2\mathscr{D}_t^{-0.05}|\varpi_1(t)|], \\ u_2(t) = - 2\varpi_2(t) - sign(\varpi_2(t))[1 + 1.8|\varpi_2(t-\varsigma)| + (\mathscr{D}_t^{-0.05}|\varpi_2(t)|)^{0.2} \\ \qquad + (\mathscr{D}_t^{-0.05}|\varpi_2(t)|)^{1.8} + 2\mathscr{D}_t^{-0.05}|\varpi_2(t)|]. \end{cases} \tag{4.3}$$

Obviously, Assumptions 2.1 and 2.2 hold with $F_i = H_i = M_i = 1, i = 1, 2$. *It is easily verified that condition (3.2) is satisfied. Therefore, according to Theorem 3.1, the FXTS between systems (4.1) and (4.2) can be reached under controller (4.3) with* $T^1_{max} \approx 1.61$. *Choosing the initial values* $x_1(t) = 20, x_2(t) = -12, y_1(t) = -16, y_2(t) = 9, t \in [-0.1, 0]$, *Figures 1 and 2 indicate the trajectories of systems (4.1) and (4.2) without external input and under control (4.3), respectively. Moreover, the error trajectories between systems (4.1) and (4.2) with multiple sets of initial values are exhibited in Figure 3.*

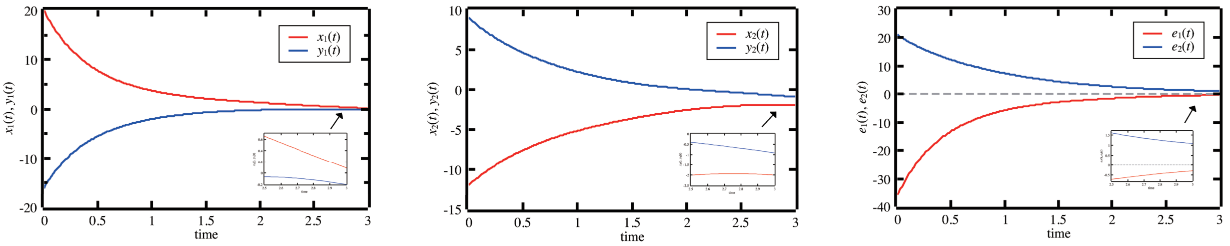

Figure 1. The trajectories of systems (4.1) and (4.2) without external input.

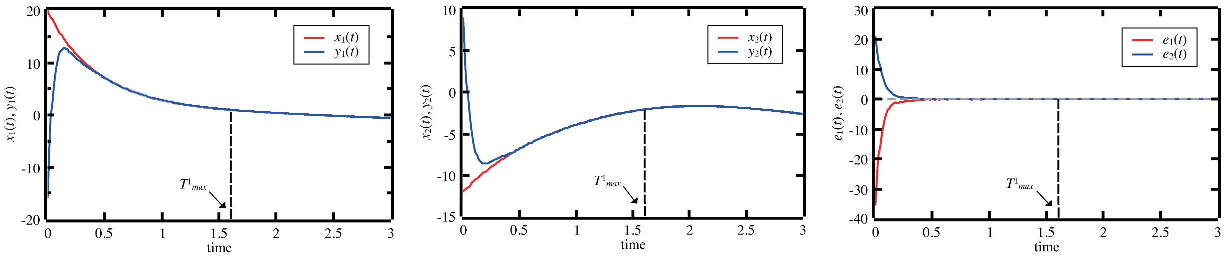

Figure 2. The trajectories of systems (4.1) and (4.2) under control scheme (4.3).

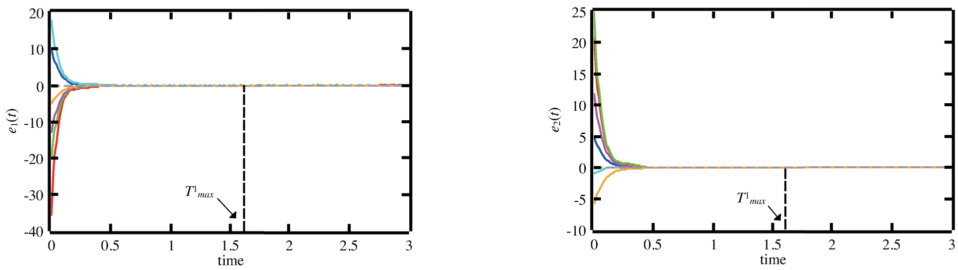

Figure 3. The error trajectories of systems (4.1) and (4.2) under control scheme (4.3) with multiple sets of initial values.

From Corollary 3.1, systems (4.1) and (4.2) can also reach FXTS using the following control scheme with $T^3_{max} \approx 3.43$ *(see Figure 4):*

$$\begin{cases} u_1(t) = -1.5\varpi_1(t) - sign(\varpi_1(t))[1 + 2|\varpi_1(t-\varsigma)| + (\mathscr{D}_t^{-0.05}|\varpi_1(t)|)^{0.2} + (\mathscr{D}_t^{-0.05}|\varpi_1(t)|)^{1.8}], \\ u_2(t) = -2\varpi_2(t) - sign(\varpi_2(t))[1 + 1.8|\varpi_2(t-\varsigma)| + (\mathscr{D}_t^{-0.05}|\varpi_2(t)|)^{0.2} + (\mathscr{D}_t^{-0.05}|\varpi_2(t)|)^{1.8}]. \end{cases} \tag{4.4}$$

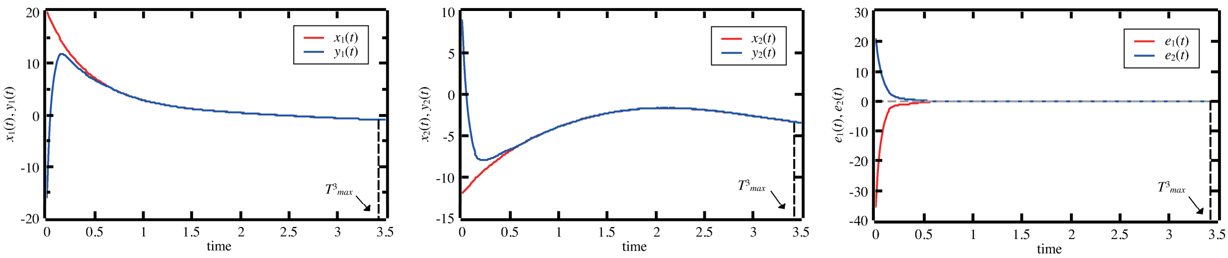

Figure 4. The trajectories of systems (4.1) and (4.2) under control scheme (4.4).

Example 4.2. *Consider the same systems (4.1) and (4.2) with the following adaptive controller*

$$\begin{cases} u_1(t) = -\eta_1(t)\varpi_1(t) - sign(\varpi_1(t))[1 + 2|\varpi_1(t-\varsigma)| + 2(\mathscr{D}_t^{-0.05}|\varpi_1(t)|)^{0.6} \\ \qquad + 2(\mathscr{D}_t^{-0.05}|\varpi_1(t)|)^{1.4} + 3\mathscr{D}_t^{-0.05}|\varpi_1(t)|], \\ u_2(t) = -\eta_2(t)\varpi_2(t) - sign(\varpi_2(t))[1 + 1.8|\varpi_2(t-\varsigma)| + 2(\mathscr{D}_t^{-0.05}|\varpi_2(t)|)^{0.6} \\ \qquad + 2(\mathscr{D}_t^{-0.05}|\varpi_2(t)|)^{1.4} + 3\mathscr{D}_t^{-0.05}|\varpi_2(t)|], \end{cases} \tag{4.5}$$

where $\eta_i(t)$ *is the adaptive regulated feedback gain, and the updated law is*

$$\dot{\eta}_i(t) = \frac{1}{2}|\varpi_i(t)| - \frac{1}{2}sign(\eta_i(t) - 1.8)[2|\eta_i(t) - 1.8|^{0.2} + 2|\eta_i(t) - 1.8|^{1.8} + 3|\eta_i(t) - 1.8|], \ i = 1, 2. \tag{4.6}$$

According to Theorem 3.2, systems (4.1) and (4.2) can realize FXTS under the adaptive control strategy (4.5) with $\bar{T}^1_{max} \approx 1.40$ *(see Figure 5). Figure 6 shows the error trajectories for different initial values.*

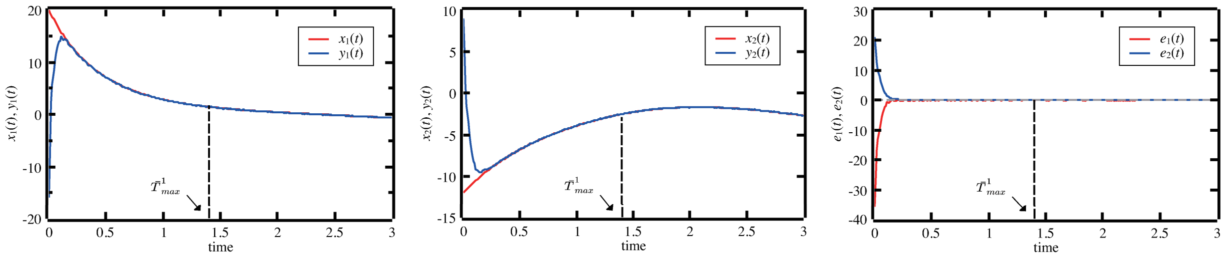

Figure 5. The trajectories of systems (4.1) and (4.2) under control scheme (4.5).

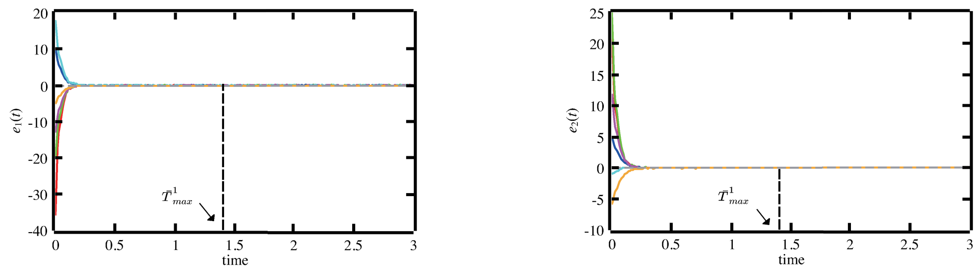

Figure 6. The error trajectories of systems (4.1) and (4.2) under control scheme (4.5) with multiple sets of initial values.

In Corollary 3.3, the FXTS of systems (4.1) and (4.2) can be realized through adaptive controller (4.7) with $\bar{T}^3_{max} \approx 3.43$ *(see Figure 7):*

$$\begin{cases} u_1(t) = -\eta_1(t)\varpi_1(t) - sign(\varpi_1(t))[1 + 2|\varpi_1(t-\varsigma)| + 2(\mathscr{D}_t^{-0.05}|\varpi_1(t)|)^{0.6} + 2(\mathscr{D}_t^{-0.05}|\varpi_1(t)|)^{1.4}], \\ u_2(t) = -\eta_2(t)\varpi_2(t) - sign(\varpi_2(t))[1 + 1.8|\varpi_2(t-\varsigma)| + 2(\mathscr{D}_t^{-0.05}|\varpi_2(t)|)^{0.6} + 2(\mathscr{D}_t^{-0.05}|\varpi_2(t)|)^{1.4}], \end{cases} \tag{4.7}$$

where the updated law is

$$\dot{\eta}_i(t) = \frac{1}{2}|\varpi_i(t)| - \frac{1}{2}sign(\eta_i(t) - 1.8)[2|\eta_i(t) - 1.8|^{0.2} + 2|\eta_i(t) - 1.8|^{1.8}],\ i = 1, 2. \tag{4.8}$$

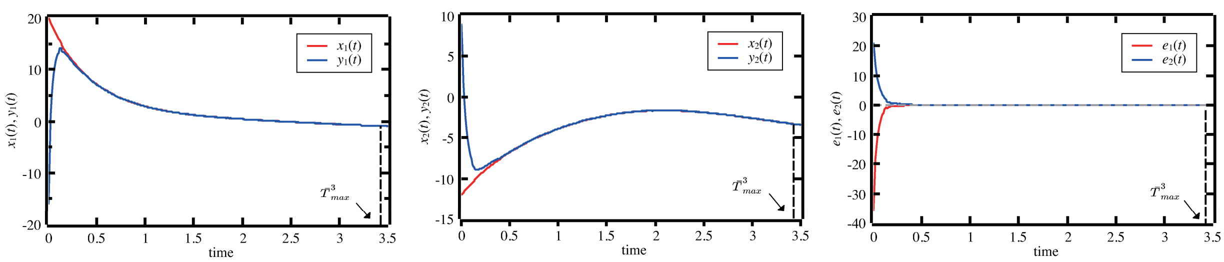

Figure 7. The trajectories of systems (4.1) and (4.2) under control scheme (4.6).

5. Conclusions

In this study, we investigated the FXTS of fractional-order FCNNs with delays and interactions. Two different control strategies, adaptive and state-feedback controllers, were devised to perform the FXTS. Based on the bounded interaction functions and Lipschitz continuous activation functions, some innovative and productive criteria with ST estimations were acquired to reach the FXTS of the discussed systems. In addition, according to another FXT stability theorem, we deduced the different upper bound estimation formulas for ST and pointed out the distinctions between them. Finally, two numerical examples were presented to corroborate the practicability of the aforementioned theorems and corollaries.

The Lipschitz continuity of the activation functions and interaction functions was essential for the results of this study. However, this condition was not always satisfied in practice. Therefore, our future work will pay attention to the FXTS of fractional-order FCNNs with delays, interactions, discontinuous activation functions, and discontinuous interaction functions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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