AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 12778–12799.
DOI: 10.3934/math.2024624
Received: 25 January 2024
Revised: 13 March 2024
Accepted: 19 March 2024
Published: 02 April 2024

Research article

Dynamic inequalities of Grüss, Ostrowski and Trapezoid type via diamond-α integrals and Montgomery identity

Marwa M. Tharwat[1], **Marwa M. Ahmed**[1], **Ammara Nosheen**[2], **Khuram Ali Khan**[2,*], **Iram Shahzadi**[2], **Dumitru Baleanu**[3,*] **and Ahmed A. El-Deeb**[4]

[1] Electrical Engineering Department, Faculty of Engineering, King Abdulaziz University, Jeddah 80204, Saudi Arabia

[2] Department of Mathematics, University of Sargodha, Sargodha, Pakistan

[3] Lebanese American University, Department of Computer Science and Mathematics, Beirut, Lebanon

[4] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt

* **Correspondence:** Email: khuram.ali@uos.edu.pk, dumitru.baleanu@lau.edu.lb.

Abstract: In this article, the Montgomery identity and Ostrowski inequality are established for univariate first-order diamond-alpha differentiable functions. We also investigate the generalization of Ostrowski-type inequalities for bivariate functions with bounded second-order diamond-alpha derivatives by applying integration by parts for $\diamond_\alpha$-integrals. Moreover, some extensions of dynamic trapezoid- and Grüss-type inequalities are also obtained by using the Montgomery identity.

Keywords: Gr*ü*ss-type inequality; diamond-α integral; Ostrowski inequality; trapezoid inequality; Montgomery identity
Mathematics Subject Classification: 39B62, 39B05, 26D15, 26D20, 26D10

1. Introduction

The Ostrowski inequality [1] is presented in 1938.

Theorem 1.1. *Suppose that $\breve{G} : [\eta_1, \eta_4] \to \mathbb{R}$ is a function which is continuous on $[\eta_1, \eta_4]$ and differentiable on (η_1, η_4); then, for all $\tau_3 \in [\eta_1, \eta_4]$, we have*

$$\left|\breve{G}(\tau_3) - \frac{1}{\eta_4 - \eta_1}\int_{\eta_1}^{\eta_4} \breve{G}(\tau_2)d\tau_2\right| \leq \sup_{\eta_1<\tau_2<\eta_4} |\breve{G}'(\tau_2)|(\eta_4 - \eta_1)\left[\frac{(\tau_3 - \frac{\eta_1+\eta_4}{2})^2}{(\eta_4 - \eta_1)^2} + \frac{1}{4}\right]. \tag{1.1}$$

Inequality (1.1) clearly indicates the absolute difference between the integral mean of $\breve{G}$ over $[\eta_1, \eta_4]$ and its value at a certain point in $[\eta_1, \eta_4]$. Many applications of Ostrowski's inequality have been explored in statistics, optimization and probability theory, numerical integration, and theory of the integral operators. Inequality (1.1) is also used to calculate error in the approximation of integrals. For more details, we refer the readers to [2–9].

In 1991, the trapezoid inequality [10] is estimated as follows:

Theorem 1.2. *Suppose that a function $\breve{G}$ is two times differentiable on $[\eta_1, \eta_4]$; then, we have*

$$\left|\frac{\breve{G}(\eta_1) + \breve{G}(\eta_4)}{2}(\eta_4 - \eta_1) - \int_{\eta_1}^{\eta_4} \breve{G}(\tau_2)d\tau_2\right| \leq \sup_{\eta_1<\tau_2<\eta_4} |\breve{G}''(\tau_2)|\frac{(\eta_4 - \eta_1)^3}{12}.$$

In 1935, *Grüss* [11] obtained the following inequality.

Theorem 1.3. *Suppose that $\breve{G}$ and $\breve{H}$ are continuous functions on $[\eta_1, \eta_4]$ such that*

$$\zeta_1 \leq \breve{G}(\tau_2) \leq \zeta_2 \quad and \quad \zeta_3 \leq \breve{H}(\tau_2) \leq \zeta_4$$

for all $\tau_2 \in [\eta_1, \eta_4]$ and $\zeta_i \in [\eta_1, \eta_4]$, where $i = 1, 2, 3, 4$. Then, we have

$$\begin{aligned}&\left|\frac{1}{\eta_4 - \eta_1}\int_{\eta_1}^{\eta_4} \breve{G}(\tau_2)\breve{H}(\tau_2)d\tau_2 - \frac{1}{(\eta_4 - \eta_1)^2}\int_{\eta_1}^{\eta_4} \breve{G}(\tau_2)d\tau_2 \int_{\eta_1}^{\eta_4} \breve{H}(\tau_2)d\tau_2\right| \\ &\leq \frac{1}{4}(\zeta_2 - \zeta_1)(\zeta_4 - \zeta_3).\end{aligned} \tag{1.2}$$

Certainly, (1.2) computes the absolute divergence of the integral means of two functions from the product of their integral means.

In 2003, Pachpatte [12] derived the Grüss- and trapezoid-type inequalities as follows:

Theorem 1.4. *Suppose that $\breve{G}, \breve{H} : [\eta_1, \eta_4] \to \mathbb{R}$ are differentiable functions on (η_1, η_4), whose first derivatives $\breve{G}', \breve{H}' : (\eta_1, \eta_4) \to \mathbb{R}$ are bounded on (η_1, η_4); then, we have*

$$\begin{aligned}&\left|\frac{1}{\eta_4 - \eta_1}\int_{\eta_1}^{\eta_4} \breve{G}(\tau_2)\breve{H}(\tau_2)d\tau_2 - \left(\frac{1}{\eta_4 - \eta_1}\int_{\eta_1}^{\eta_4} \breve{G}(\tau_2)d\tau_2\right)\left(\frac{1}{\eta_4 - \eta_1}\int_{\eta_1}^{\eta_4} \breve{H}(\tau_2)d\tau_2\right)\right| \\ &\leq \frac{1}{2(\eta_4 - \eta_1)^2}\int_{\eta_1}^{\eta_4} [M_1|\breve{G}(\tau_2)| + N_1|\breve{H}(\tau_2)|]\left[\frac{(\eta_4 - \eta_1)^2}{4} + \left(\tau_2 - \frac{\eta_1 + \eta_4}{2}\right)^2\right]d\tau_2,\end{aligned} \tag{1.3}$$

where $M_1 = \sup_{\eta_1<\tau_2<\eta_4} \breve{G}'(\tau_2)$ and $N_1 = \sup_{\eta_1<\tau_2<\eta_4} \breve{H}'(\tau_2)$.

Theorem 1.5. *Suppose that $\breve{G} : [\eta_1, \eta_4] \to \mathbb{R}$ is a differentiable function on (η_1, η_4), whose first derivative $\breve{G}' : (\eta_1, \eta_4) \to \mathbb{R}$ is bounded on (η_1, η_4); then,*

$$\left|\frac{\breve{G}^2(\eta_4) - \breve{G}^2(\eta_1)}{2} - \frac{\breve{G}(\eta_4) - \breve{G}(\eta_1)}{\eta_4 - \eta_1}\int_{\eta_1}^{\eta_4} \breve{G}(\tau_2)d\tau_2\right| \leq \frac{M_1^2(\eta_4 - \eta_1)^2}{3},$$

where $M_1 = \sup_{\eta_1<\tau_2<\eta_4} \breve{G}'(\tau_2)$.

The theory of time scales is a significant branch of mathematics because of its applications in a variety of fields. In 1988, calculus on measure chains was introduced by Stefan Hilger [13]. The valuable contributions of theory of unification, extension and discretization were identified by his Ph.D supervisor, Bernd Aulbach. The theory of time scales for integral inequalities has been explored by numerous researchers. When estimating the approximate error in integration, these inequalities facilitate the analysis of the consistency and steadiness of statistical calculations [14]. Its applications in engineering, optimization theory, functional spaces, mathematical biology and dynamic inequalities have also contributed to the literature. Time scale calculus is illustrated through the use of continuous, discrete, and quantum calculus. For convex functions, Ekinci [15] derived Ostrowski-type delta integral inequalities. Hu and Wang [16] investigated time scales inequalities and their applications to the persistence of a predator-prey system.

In 2006, Sheng et al. [17] developed a joint dynamic $\diamond_\alpha$-derivative as a linear combination of delta and nabla dynamic derivatives on time scales. For $\alpha = 1$ and $\alpha = 0$, the diamond-α derivative becomes the conventional delta and nabla derivative, respectively. On any discrete time scale, it gives a symmetric dynamic derivative for $\alpha = \frac{1}{2}$. Ahmad et al. [18] obtained a bivariate Montgomery identity by using α-diamond integrals. Liu and Tuna [19] established weighted Grüss-type and Ostrowski-type inequalities for $\diamond_\alpha$-integrals. Bohner et al. [20] derived diamond-alpha Grüss-type inequalities. Liu et al. [21] also presented weighted Grüss-type, Ostrowski-type, Ostrowski-Grüss-type and trapezoid-type inequalities. Du et al. [22] established the Y-function and L'Hospital-type monotonicity rules with nabla and diamond-alpha derivatives on time scales. Bilal et al. [23] obtained bounds of some divergence measures by applying Hermite polynomials in diamond integrals on time scales. Truong et al. [24] investigated the diamond-alpha differentiability of interval-valued functions and their applicability to interval differential equations on time scales.

Motivated by the work of Bohner and Mathews [25, 26] and El-Deeb [27], the objective of this manuscript is to obtain some Ostrowski-type, Grüss-type, and trapezoid-type inequalities via the Montgomery identity for diamond-alpha integrals on time scales. The proofs of these results rely on employing the properties of differentiation and integration on time scales and they not only provide the generalization of existing results, but also give some novel inequalities for diamond-alpha integrals through the choice of some special time scales.

This paper is organized as follows. Section 2 presents some early results on time scales that will be used later in this study. Section 3 proves the Montgomery identity and Ostrowski inequality for diamond-alpha differentiable functions. In addition, we derive Ostrowski-type, trapezoid-type and Grüss-type inequalities for twice diamond-alpha differentiable functions. Some classical and modern inequalities are derived. Section 4 gives the summary of the findings.

2. Preliminaries

We now go over some fundamental concepts and notations in time scales calculus.

An arbitrary nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$ is called a time scale. Consider the time scale $\mathbb{T}$ and $v_1 \in \mathbb{T}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined as follows: $\sigma(v_1) := inf\{w_1 \in \mathbb{T} : w_1 > v_1\}$ and $\rho(v_1) := sup\{w_1 \in \mathbb{T} : w_1 < v_1\}$, respectively. The Δ-derivative and Δ-integral of a mapping $\breve{G}$ are denoted by $\breve{G}^\Delta$ and $\int_{\mathbb{T}} \breve{G}(\eta)\Delta\eta$. Similarly, the ∇-derivative and ∇-integral of a mapping $\breve{G}$ are denoted respectively by $\breve{G}^\nabla$ and $\int_{\mathbb{T}} \breve{G}(\eta)\nabla\eta$.

Assume that a function $W_1 : \mathbb{T} \to \mathbb{R}$, $\tau_2 \in \mathbb{T}^{\kappa}_{\kappa}$, $\alpha \in [0,1]$, $\mu_{\tau_2\eta_1} = \sigma(\tau_2) - \eta_1$, and $\nu_{\tau_2\eta_1} = \rho(\tau_2) - \eta_1$. Suppose that $W_1^{\diamond_\alpha}(\tau_2) \in \mathbb{R}$ is a $\diamond_\alpha$-derivative of W_1 at τ_2 if for any $\epsilon > 0$, there exists a neighborhood W_0 of τ_2 such that, for all $\eta_1 \in W_0$, we have

$$\begin{aligned}&|\alpha[W_1(\sigma(\tau_2)) - W_1(\eta_1)]\nu_{\tau_2\eta_1} + (1-\alpha)[W_1(\rho(\tau_2)) - W_1(\eta_1)]\mu_{\tau_2\eta_1} - W_1^{\diamond_\alpha}(\tau_2)\mu_{\tau_2\eta_1}\nu_{\tau_2\eta_1}|\\ &\leq \epsilon|\mu_{\tau_2\eta_1}\nu_{\tau_2\eta_1}|.\end{aligned}$$

Moreover, W_1 is said to be $\diamond_\alpha$-differentiable if and only if it is delta- and nabla differentiable. For $\alpha = 1$ and $\alpha = 0$, the $\diamond_\alpha$-derivative reduces to the delta and nabla derivative, respectively [28].

In [29], the following results are given:
Assume that the functions $W_1, W_2 : \mathbb{T} \to \mathbb{R}$ are diamond-alpha differentiable at $\tau_2 \in \mathbb{T}$, and that $c_0 \in \mathbb{R}$. Then, we have

$$\begin{aligned}&\textbf{(a)}\ (W_1 + W_2)^{\diamond_\alpha}(\tau_2) = W_1^{\diamond_\alpha}(\tau_2) + W_2^{\diamond_\alpha}(\tau_2);\\ &\textbf{(b)}\ (c_0W_1)^{\diamond_\alpha}(\tau_2) = c_0W_1^{\diamond_\alpha}(\tau_2);\\ &\textbf{(c)}\ (W_1W_2)^{\diamond_\alpha}(\tau_2) = W_1^{\diamond_\alpha}(\tau_2)W_2(\tau_2) + \alpha W_1^{\sigma}(\tau_2)W_2^{\Delta}(\tau_2) + (1-\alpha)W_1^{\rho}(\tau_2)W_2^{\nabla}(\tau_2).\end{aligned} \tag{2.1}$$

If we take the integral on both sides of (2.1), we get the following formula for integration by parts. If $\eta_1, \eta_4 \in \mathbb{T}$ and W_1, W_2 are continuous functions, then

$$\begin{aligned}&\int_{\eta_1}^{\eta_4} W_1^{\diamond_\alpha}(\tau_2)W_2(\tau_2)\diamond_\alpha\tau_2 = (W_1W_2)(\eta_4) - (W_1W_2)(\eta_1)\\ &- \alpha\int_{\eta_1}^{\eta_4} W_1^{\sigma}(\tau_2)W_2^{\Delta}(\tau_2)\diamond_\alpha\tau_2 - (1-\alpha)\int_{\eta_1}^{\eta_4} W_1^{\rho}(\tau_2)W_2^{\nabla}(\tau_2)\diamond_\alpha\tau_2.\end{aligned} \tag{2.2}$$

Suppose that $W_1 : \mathbb{T} \to \mathbb{R}$ is a continuous function and $\eta_1, \eta_4 \in \mathbb{T}$. Then, the $\diamond_\alpha$-integral of W_1 over $[\eta_1, \eta_4]$ is described as follows:

$$\int_{\eta_1}^{\eta_4} W_1(\tau_2)\diamond_\alpha\tau_2 = \alpha\int_{\eta_1}^{\eta_4} W_1(\tau_2)\Delta\tau_2 + (1-\alpha)\int_{\eta_1}^{\eta_4} W_1(\tau_2)\nabla\tau_2, \quad 0 \leq \alpha \leq 1. \tag{2.3}$$

Assume that $\eta_1, \eta_4, \tau_2 \in \mathbb{T}$, $c_0 \in \mathbb{R}$ and W_1, W_2 are continuous functions on $[\eta_1, \eta_4]_{\mathbb{T}}$. Then,
(i) $\int_{\eta_1}^{\eta_4}[W_1(\tau_2) + W_2(\tau_2)]\diamond_\alpha\tau_2 = \int_{\eta_1}^{\eta_4} W_1(\tau_2)\diamond_\alpha\tau_2 + \int_{\eta_1}^{\eta_4} W_2(\tau_2)\diamond_\alpha\tau_2$;
(ii) $\int_{\eta_1}^{\eta_4} c_0W_1(\tau_2)\diamond_\alpha\tau_2 = c_0\int_{\eta_1}^{\eta_4} W_1(\tau_2)\diamond_\alpha\tau_2$;
(iii) $\int_{\eta_1}^{\eta_4} W_1(\tau_2)\diamond_\alpha\tau_2 = \int_{\eta_1}^{\eta} W_1(\tau_2)\diamond_\alpha\tau_2 + \int_{\eta}^{\eta_4} W_1(\tau_2)\diamond_\alpha\tau_2$;
(iv) $\int_{\eta_1}^{\eta_4} W_1(\tau_2)\diamond_\alpha\tau_2 = -\int_{\eta_4}^{\eta_1} W_1(\tau_2)\diamond_\alpha\tau_2$;
(v) $\left|\int_{\eta_1}^{\eta_4} W_1(\tau_2)\diamond_\alpha\tau_2\right| \leq \int_{\eta_1}^{\eta_4} |W_1(\tau_2)|\diamond_\alpha\tau_2$.

Let $\mathbb{T}$ be an arbitrary time scale. Suppose that the functions $h_k, \hat{h}_k : \mathbb{T}\times\mathbb{T} \to \mathbb{R}, k \in \mathbb{N}\cup\{0\}$, are defined recursively by

$$h_0(\tau_2, \eta_1) = 1, \quad h_{k+1}(\tau_2, \eta_1) = \int_{\eta_1}^{\tau_2} h_k(\tau_3, \eta_1)\Delta\tau_3,$$

and

$$\hat{h}_0(\tau_2, \eta_1) = 1, \quad \hat{h}_{k+1}(\tau_2, \eta_1) = \int_{\eta_1}^{\tau_2} \hat{h}_k(\tau_3, \eta_1)\nabla\tau_3.$$

Similarly, we define a function $\tilde{h}_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}, k \in \mathbb{N} \cup \{0\}$, as:

$$\tilde{h}_1(\tau_2, \eta_1) = 1, \quad \tilde{h}_{k+1}(\tau_2, \eta_1) = \alpha h_{k+1}(\tau_2, \eta_1) + (1-\alpha)\hat{h}_{k+1}(\tau_2, \eta_1), \quad \alpha \in [0, 1],$$

where h_k are right-dense continuous and $\hat{h}_k$ are left-dense continuous functions.
For further details, the readers are referred to [30–34].

3. Main results

In this section, the Montgomery identity is proved by utilizing the formula for integration by parts for diamond alpha integrals. Further, Ostrowski-, Grüss-, and trapezoid-type inequalities are established by using the Montgomery identity for second-order diamond-alpha-differentiable functions. Mathematical applications of this work are given in the form of examples and corollaries.

Theorem 3.1. *Assume that $\eta_1, \tau_2, \tau_3, \eta_4 \in \mathbb{T}$, with $\eta_1 < \eta_4$, $\alpha \in [0, 1]$ and $\breve{G} : [\eta_1, \eta_4]_{\mathbb{T}} \to \mathbb{R}$ as a diamond-alpha differentiable function. Then, for all $\tau_3 \in [\eta_1, \eta_4]_{\mathbb{T}}$*

$$\begin{aligned}\breve{G}(\tau_2) &= \frac{\alpha}{\eta_4 - \eta_1} \int_{\eta_1}^{\eta_4} \breve{G}^{\sigma}(\tau_3) \diamond_{\alpha} \tau_3 + \frac{1-\alpha}{\eta_4 - \eta_1} \int_{\eta_1}^{\eta_4} \breve{G}^{\rho}(\tau_3) \diamond_{\alpha} \tau_3 \\ &+ \frac{1}{\eta_4 - \eta_1} \int_{\eta_1}^{\eta_4} \breve{\Gamma}(\tau_2, \tau_3) \breve{G}^{\diamond_{\alpha}}(\tau_3) \diamond_{\alpha} \tau_3,\end{aligned} \tag{3.1}$$

where

$$\breve{\Gamma}(\tau_2, \tau_3) = \begin{cases} \tau_3 - \eta_1, & \tau_3 \in [\eta_1, \tau_2]_{\mathbb{T}}, \\ \tau_3 - \eta_4, & \tau_3 \in (\tau_2, \eta_4]_{\mathbb{T}}. \end{cases} \tag{3.2}$$

Proof. By using (2.2), we have

$$\begin{aligned}\int_{\eta_1}^{\tau_2} (\tau_3 - \eta_1) \breve{G}^{\diamond_{\alpha}}(\tau_3) \diamond_{\alpha} \tau_3 &= \breve{G}(\tau_2)(\tau_2 - \eta_1) \\ - \alpha \int_{\eta_1}^{\tau_2} \breve{G}^{\sigma}(\tau_3) \diamond_{\alpha} \tau_3 &- (1-\alpha) \int_{\eta_1}^{\tau_2} \breve{G}^{\rho}(\tau_3) \diamond_{\alpha} \tau_3,\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\int_{\tau_2}^{\eta_4} (\tau_3 - \eta_4) \breve{G}^{\diamond_{\alpha}}(\tau_3) \diamond_{\alpha} \tau_3 &= \breve{G}(\tau_2)(\eta_4 - \tau_2) \\ - \alpha \int_{\tau_2}^{\eta_4} \breve{G}^{\sigma}(\tau_3) \diamond_{\alpha} \tau_3 &- (1-\alpha) \int_{\tau_2}^{\eta_4} \breve{G}^{\rho}(\tau_3) \diamond_{\alpha} \tau_3.\end{aligned} \tag{3.4}$$

Add (3.3) and (3.4) to obtain

$$\begin{aligned}\int_{\eta_1}^{\eta_4} \breve{\Gamma}(\tau_2, \tau_3) \breve{G}^{\diamond_{\alpha}}(\tau_3) \diamond_{\alpha} \tau_3 &= \breve{G}(\tau_2)(\eta_4 - \eta_1) \\ - \alpha \int_{\eta_1}^{\eta_4} \breve{G}^{\sigma}(\tau_3) \diamond_{\alpha} \tau_3 &- (1-\alpha) \int_{\eta_1}^{\eta_4} \breve{G}^{\rho}(\tau_3) \diamond_{\alpha} \tau_3.\end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned}\breve{G}(\tau_2) &= \frac{\alpha}{\eta_4-\eta_1}\int_{\eta_1}^{\eta_4}\breve{G}^{\sigma}(\tau_3)\diamond_\alpha\tau_3 + \frac{1-\alpha}{\eta_4-\eta_1}\int_{\eta_1}^{\eta_4}\breve{G}^{\rho}(\tau_3)\diamond_\alpha\tau_3\\ &+ \frac{1}{\eta_4-\eta_1}\int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_2,\tau_3)\breve{G}^{\diamond_\alpha}(\tau_3)\diamond_\alpha\tau_3\\ &= \frac{\alpha}{\eta_4-\eta_1}\int_{\eta_1}^{\eta_4}\breve{G}^{\sigma}(\tau_3)\diamond_\alpha\tau_3 + \frac{1-\alpha}{\eta_4-\eta_1}\int_{\eta_1}^{\eta_4}\breve{G}^{\rho}(\tau_3)\diamond_\alpha\tau_3\\ &+ \frac{1}{\eta_4-\eta_1}\Big[-\alpha\int_{\eta_1}^{\eta_4}\breve{G}^{\sigma}(\tau_3)\diamond_\alpha\tau_3 - (1-\alpha)\int_{\eta_1}^{\eta_4}\breve{G}^{\rho}(\tau_3)\diamond_\alpha\tau_3\\ &+ \breve{G}(\tau_2)(\eta_4-\eta_1)\Big].\end{aligned}$$

□

Remark 3.2. *(i) Put $\alpha = 1$ in Theorem 3.1 to get [26, Lemma 3.1];*
(ii) put $\alpha = 0$ in Theorem 3.1 to get [18, Remark 1.1];
(iii) put $\alpha = \frac{1}{2}$ in Theorem 3.1 to get the symmetric combination of the inequalities established in [26, Lemma 3.1] and [18, Remark 1.1].

Example 3.3. *Substitute $\mathbb{T} = \mathbb{Z}$ in Theorem 3.1 to get*

$$\begin{aligned}\breve{G}(\tau_2) &= \frac{\alpha}{\eta_4-\eta_1}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\breve{G}(\tau_3+1) + \alpha\breve{G}(\eta_1+1) + (1-\alpha)\breve{G}(\eta_4+1)\Big]\\ &+ \frac{1-\alpha}{\eta_4-\eta_1}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\breve{G}(\tau_3-1) + \alpha\breve{G}(\eta_1-1) + (1-\alpha)\breve{G}(\eta_4-1)\Big]\\ &+ \frac{1}{\eta_4-\eta_1}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\breve{\Gamma}(\tau_2,\tau_3)\breve{G}[\alpha\Delta + (1-\alpha)\nabla](\tau_3)\\ &+ \alpha\breve{\Gamma}(\tau_2,\tau_3)\breve{G}[\alpha\Delta + (1-\alpha)\nabla](\eta_1) + (1-\alpha)\breve{\Gamma}(\tau_2,\tau_3)\breve{G}[\alpha\Delta + (1-\alpha)\nabla](\eta_4)\Big].\end{aligned}$$

Theorem 3.4. *Let $\eta_1, \tau_2, \tau_3, \eta_4 \in \mathbb{T}$, with $\eta_1 < \eta_4$, $\alpha \in [0,1]$ and $\breve{G} : [\eta_1, \eta_4]_{\mathbb{T}}$ be diamond-alpha-differentiable. Then,*

$$\begin{aligned}&\left|\breve{G}(\tau_2) - \frac{\alpha}{\eta_4-\eta_1}\int_{\eta_1}^{\eta_4}\breve{G}^{\sigma}(\tau_3)\diamond_\alpha\tau_3 - \frac{1-\alpha}{\eta_4-\eta_1}\int_{\eta_1}^{\eta_4}\breve{G}^{\rho}(\tau_3)\diamond_\alpha\tau_3\right|\\ &\le \frac{Y_0}{\eta_4-\eta_1}(\tilde{h}_2(\tau_2,\eta_1) + \tilde{h}_2(\tau_2,\eta_4)),\end{aligned} \tag{3.6}$$

where

$$Y_0 = \sup_{\eta_1<\tau_2<\eta_4}|\breve{G}^{\diamond_\alpha}(\tau_3)|, \quad \tilde{h}_2(\tau_2,\eta_1) = \int_{\eta_1}^{\tau_2}(\tau_3-\eta_1)\diamond_\alpha\tau_3,$$

and

$$\tilde{h}_2(\tau_2,\eta_4) = \int_{\eta_4}^{\tau_2}(\tau_3-\eta_4)\diamond_\alpha\tau_3.$$

Proof. Using Theorem 3.1, we obtain

$$
\begin{aligned}
&\left|\breve{G}(\tau_2) - \frac{\alpha}{\eta_4-\eta_1}\int_{\eta_1}^{\eta_4}\breve{G}^{\sigma}(\tau_3)\diamond_\alpha\tau_3 - \frac{1-\alpha}{\eta_4-\eta_1}\int_{\eta_1}^{\eta_4}\breve{G}^{\rho}(\tau_3)\diamond_\alpha\tau_3\right| \\
&= \left|\frac{1}{\eta_4-\eta_1}\int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_2,\tau_3)\breve{G}^{\diamond_\alpha}(\tau_3)\diamond_\alpha\tau_3\right| \\
&\le \frac{Y_0}{\eta_4-\eta_1}\Big[\int_{\eta_1}^{\tau_2}(\tau_3-\eta_1)\diamond_\alpha\tau_3 + \int_{\tau_2}^{\eta_4}(\tau_3-\eta_4)\diamond_\alpha\tau_3\Big] \\
&= \frac{Y_0}{\eta_4-\eta_1}(\tilde{h}_2(\tau_2,\eta_1)+\tilde{h}_2(\tau_2,\eta_4)).
\end{aligned}
$$

□

Remark 3.5. *(i) Put $\alpha = 1$ in Theorem 3.4 to obtain [26, Theorem 3.5];*
(ii) set $\alpha = 1$ and $\mathbb{T} = \mathbb{R}$ to obtain [26, Corollary 3.7].

Example 3.6. *Substitute $\mathbb{T} = \mathbb{Z}$ in Theorem 3.4 to obtain*

$$
\begin{aligned}
&\Bigg|\breve{G}(\tau_2) - \frac{\alpha}{\eta_4-\eta_1}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\breve{G}(\tau_3+1) + \alpha\breve{G}(\eta_1+1) + (1-\alpha)\breve{G}(\eta_4+1)\Big] \\
&- \frac{1-\alpha}{\eta_4-\eta_1}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\breve{G}(\tau_3-1) + \alpha\breve{G}(\eta_1-1) + (1-\alpha)\breve{G}(\eta_4-1)\Big]\Bigg| \\
&\le \frac{Y_0}{\eta_4-\eta_1}[(\alpha h_2(\tau_2,\eta_1) + (1-\alpha)\hat{h}_2(\tau_2,\eta_1)) + (\alpha h_2(\tau_2,\eta_4) + (1-\alpha)\hat{h}_2(\tau_2,\eta_4))].
\end{aligned}
$$

Theorem 3.7. *Consider $\mathbb{T}$ to be a time scale with $\eta_1, \tau_2, \tau_3, \eta_4 \in \mathbb{T}$ and $\eta_1 < \eta_4$. Additionally, assume that a function $\breve{G} : [\eta_1, \eta_4]_{\mathbb{T}} \to \mathbb{T}$ is two times diamond-α-differentiable. Then, for all $\tau_3 \in [\eta_1, \eta_4]_{\mathbb{T}}$, $\tau, \nu \in \mathbb{R}$ and $\alpha \in [0, 1]$, we have*

$$
\begin{aligned}
&\Bigg|\breve{G}(\tau_3) - \frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2 + \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2 \\
&+ \int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_2}\frac{\tau}{\tau_2-\eta_1}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\rho(w_1^*))\diamond_\alpha w_1^*\diamond_\alpha\tau_2 \\
&+ \int_{\eta_1}^{\eta_4}\int_{\tau_2}^{\eta_4}\frac{\nu}{\eta_4-\tau_2}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\rho(w_1^*))\diamond_\alpha w_1^*\diamond_\alpha\tau_2\Big] \\
&- \frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2 \\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2 \\
&+ \int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_2}\frac{\tau}{\tau_2-\eta_1}\breve{\Gamma}(\tau_3,\tau_2)[\breve{G}^{\diamond_\alpha}(\sigma(w_1^*))-\breve{G}^{\diamond_\alpha}(\rho(w_1^*))]\diamond_\alpha w_1^*\diamond_\alpha\tau_2 \\
&+ \int_{\eta_1}^{\eta_4}\int_{\tau_2}^{\eta_4}\frac{\nu}{\eta_4-\tau_2}\breve{\Gamma}(\tau_3,\tau_2)[\breve{G}^{\diamond_\alpha}(\sigma(w_1^*))-\breve{G}^{\diamond_\alpha}(\rho(w_1^*))]\diamond_\alpha w_1^*\diamond_\alpha\tau_2\Big]\Bigg| \\
&\le \frac{Y_1}{(\tau+\nu)^2}\Big(\frac{\tau}{\tau_3-\eta_1}\tilde{h}_2(\tau_3,\eta_1) + \frac{\nu}{\eta_4-\tau_3}\tilde{h}_2(\tau_3,\eta_4)\Big)^2,
\end{aligned}
\tag{3.7}
$$

where

$$\breve{\Gamma}(\tau_3,\tau_2)=\begin{cases}\frac{\tau}{\tau+\nu}(\frac{\tau_2-\eta_1}{\tau_3-\eta_1}), & \eta_1\le\tau_2<\tau_3,\\ \frac{-\nu}{\tau+\nu}(\frac{\eta_4-\tau_2}{\eta_4-\tau_3}), & \tau_3\le\tau_2\le\eta_4,\end{cases}$$

and

$$Y_1=\sup_{\eta_1<\tau_2<\eta_4}|\breve{G}^{\diamond_\alpha\diamond_\alpha}(\tau_2)|<\infty.$$

Proof. By using (2.2), we obtain

$$\begin{aligned}&\int_{\eta_1}^{\tau_3}\frac{\tau}{\tau+\nu}\Big(\frac{\tau_2-\eta_1}{\tau_3-\eta_1}\Big)\breve{G}^{\diamond_\alpha}(\tau_2)\diamond_\alpha\tau_2\\&=\frac{\tau}{\tau+\nu}\breve{G}(\tau_3)-\frac{\tau}{(\tau+\nu)(\tau_3-\eta_1)}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2\\&-\frac{\tau\alpha}{(\tau+\nu)(\tau_3-\eta_1)}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2,\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}&\int_{\tau_3}^{\eta_4}-\frac{\nu}{\tau+\nu}\Big(\frac{\eta_4-\tau_2}{\eta_4-\tau_3}\Big)\breve{G}^{\diamond_\alpha}(\tau_2)\diamond_\alpha\tau_2\\&=\frac{\nu}{\tau+\nu}\breve{G}(\tau_3)-\frac{\nu}{(\tau+\nu)(\eta_4-\tau_3)}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2\\&-\frac{\alpha\nu}{(\tau+\nu)(\eta_4-\tau_3)}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2.\end{aligned}\tag{3.9}$$

By adding (3.8) and (3.9), we get

$$\begin{aligned}&\int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\tau_2)\diamond_\alpha\tau_2=\breve{G}(\tau_3)\\&-\frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2+\frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2\Big]\\&-\frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2+\frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2\Big].\end{aligned}\tag{3.10}$$

Likewise, we have

$$\begin{aligned}&\int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_2,w_1^*)\breve{G}^{\diamond_\alpha\diamond_\alpha}(w_1^*)\diamond_\alpha w_1^*=\breve{G}^{\diamond_\alpha}(\tau_2)\\&-\frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_2-\eta_1}\int_{\eta_1}^{\tau_2}\breve{G}^{\diamond_\alpha}(\rho(w_1^*))\diamond_\alpha w_1^*+\frac{\nu}{\eta_4-\tau_2}\int_{\tau_2}^{\eta_4}\breve{G}^{\diamond_\alpha}(\rho(w_1^*))\diamond_\alpha w_1^*\Big]\\&-\frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_2-\eta_1}\int_{\eta_1}^{\tau_2}[\breve{G}^{\diamond_\alpha}(\sigma(w_1^*))-\breve{G}^{\diamond_\alpha}(\rho(w_1^*))]\diamond_\alpha w_1^*\\&+\frac{\nu}{\eta_4-\tau_2}\int_{\tau_2}^{\eta_4}[\breve{G}^{\diamond_\alpha}(\sigma(w_1^*))-\breve{G}^{\diamond_\alpha}(\rho(w_1^*))]\diamond_\alpha w_1^*\Big].\end{aligned}\tag{3.11}$$

By substituting (3.11) into (3.10), we obtain

$$\begin{aligned}
&\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_3,\tau_2)\breve{\Gamma}(\tau_2,w_1^*)\breve{G}^{\diamond_\alpha\diamond_\alpha}(w_1^*)\diamond_\alpha w_1^*\diamond_\alpha\tau_2\\
&+\frac{1}{\tau+\nu}\Big[\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_2}\frac{\tau}{\tau_2-\eta_1}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\rho(w_1^*))\diamond_\alpha w_1^*\diamond_\alpha\tau_2\\
&+\int_{\eta_1}^{\eta_4}\int_{\tau_2}^{\eta_4}\frac{\nu}{\eta_4-\tau_2}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\rho(w_1^*))\diamond_\alpha w_1^*\diamond_\alpha\tau_2\Big]\\
&+\frac{\alpha}{\tau+\nu}\Big[\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_2}\frac{\tau}{\tau_2-\eta_1}\breve{\Gamma}(\tau_3,\tau_2)[\breve{G}^{\diamond_\alpha}(\sigma(w_1^*))-\breve{G}^{\diamond_\alpha}(\rho(w_1^*))]\diamond_\alpha w_1^*\diamond_\alpha\tau_2\\
&+\int_{\eta_1}^{\eta_4}\int_{\tau_2}^{\eta_4}\frac{\nu}{\eta_4-\tau_2}\breve{\Gamma}(\tau_3,\tau_2)[\breve{G}^{\diamond_\alpha}(\sigma(w_1^*))-\breve{G}^{\diamond_\alpha}(\rho(w_1^*))]\diamond_\alpha w_1^*\diamond_\alpha\tau_2\Big]\\
&=\breve{G}(\tau_3)-\frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2+\frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2\Big]\\
&-\frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2\\
&+\frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2\Big].
\end{aligned}\tag{3.12}$$

Using the properties of the modulus and the definition of $\tilde{h}_2(.,.)$, inequality (3.7) follows directly from (3.12). This concludes the theorem. □

Remark 3.8. *(i) Put $\alpha = 1$ in Theorem 3.7 to obtain [27, Theorem 3.1];*
(ii) set $\alpha = 1$ and $\mathbb{T} = \mathbb{R}$ to obtain [27, Corollary 3.2].

Corollary 3.9. *Substitute $\tau = \nu = 1$ in* (3.7) *to get*

$$\begin{aligned}
&\Big|\breve{G}(\tau_3)-\frac{1}{2}\Big[\frac{1}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2+\frac{1}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2\\
&+\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_2}\frac{1}{\tau_2-\eta_1}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\rho(w_1^*))\diamond_\alpha w_1^*\diamond_\alpha\tau_2\\
&+\int_{\eta_1}^{\eta_4}\int_{\tau_2}^{\eta_4}\frac{1}{\eta_4-\tau_2}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\rho(w_1^*))\diamond_\alpha w_1^*\diamond_\alpha\tau_2\Big]\\
&-\frac{\alpha}{2}\Big[\frac{1}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2\\
&+\frac{1}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2\\
&+\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_2}\frac{1}{\tau_2-\eta_1}\breve{\Gamma}(\tau_3,\tau_2)[\breve{G}^{\diamond_\alpha}(\sigma(w_1^*))-\breve{G}^{\diamond_\alpha}(\rho(w_1^*))]\diamond_\alpha w_1^*\diamond_\alpha\tau_2\\
&+\int_{\eta_1}^{\eta_4}\int_{\tau_2}^{\eta_4}\frac{1}{\eta_4-\tau_2}\breve{\Gamma}(\tau_3,\tau_2)[\breve{G}^{\diamond_\alpha}(\sigma(w_1^*))-\breve{G}^{\diamond_\alpha}(\rho(w_1^*))]\diamond_\alpha w_1^*\diamond_\alpha\tau_2\Big]\Big|\\
&\leq\frac{Y_1}{4}\Big(\frac{1}{\tau_3-\eta_1}\tilde{h}_2(\tau_3,\eta_1)+\frac{1}{\eta_4-\tau_3}\tilde{h}_2(\tau_3,\eta_4)\Big)^2.
\end{aligned}\tag{3.13}$$

Example 3.10. *If we substitute* $\mathbb{T}=\mathbb{Z}$ *in* (3.7)*, then we obtain*

$$
\begin{aligned}
&\Big|\breve{G}(\tau_3)-\frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\Big[\sum_{\tau_2=\eta_1+1}^{\tau_3-1}\breve{G}(\tau_2-1)+\alpha\breve{G}(\eta_1-1)+(1-\alpha)\breve{G}(\tau_3-1)\Big]\\
&+\frac{\nu}{\eta_4-\tau_3}\Big[\sum_{\tau_2=\tau_3+1}^{\eta_4-1}\breve{G}(\tau_2-1)+\alpha\breve{G}(\tau_3-1)+(1-\alpha)\breve{G}(\eta_4-1)\Big]\\
&+\sum_{\tau_2=\eta_1+1}^{\eta_4-1}\sum_{w_1^*=\eta_1+1}^{\tau_2-1}\frac{\tau}{\tau_2-\eta_1}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(w_1^*-1)\\
&+\frac{\tau}{\tau_2-\eta_1}\breve{\Gamma}(\tau_3,\tau_2)[\alpha\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\eta_1-1)\\
&+(1-\alpha)\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\tau_2-1)]\\
&+\sum_{\tau_2=\eta_1+1}^{\eta_4-1}\sum_{w_1^*=\tau_2+1}^{\eta_4-1}\frac{\nu}{\eta_4-\tau_2}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(w_1^*-1)\\
&+\frac{\nu}{\eta_4-\tau_2}\breve{\Gamma}(\tau_3,\tau_2)[\alpha\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\tau_2-1)+(1-\alpha)\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\eta_4-1)]\Big]\\
&-\frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\Big[\sum_{\tau_2=\eta_1+1}^{\tau_3-1}[\breve{G}(\tau_2+1)-\breve{G}(\tau_2-1)]+\alpha[\breve{G}(\eta_1+1)\\
&-\breve{G}(\eta_1-1)]+(1-\alpha)[\breve{G}(\tau_3+1)-\breve{G}(\tau_3-1)]\Big]+\frac{\nu}{\eta_4-\tau_3}\Big[\sum_{\tau_2=\tau_3+1}^{\eta_4-1}[\breve{G}(\tau_2+1)\\
&-\breve{G}(\tau_2-1)]+\alpha[\breve{G}(\tau_3+1)-\breve{G}(\tau_3-1)]+(1-\alpha)[\breve{G}(\eta_4+1)-\breve{G}(\eta_4-1)]\Big]\\
&+\sum_{\tau_2=\eta_1+1}^{\eta_4-1}\sum_{w_1^*=\eta_1+1}^{\tau_2-1}\frac{\tau}{\tau_2-\eta_1}\breve{\Gamma}(\tau_3,\tau_2)[\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(w_1^*+1)\\
&-\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(w_1^*-1)+\frac{\tau}{\tau_2-\eta_1}\breve{\Gamma}(\tau_3,\tau_2)[\alpha(\breve{G}(\alpha\Delta\\
&+(1-\alpha)\nabla)(\eta_1+1)-\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\eta_1-1))\\
&+(1-\alpha)(\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\tau_2+1)-\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\tau_2-1))]\\
&+\sum_{\tau_2=\eta_1+1}^{\eta_4-1}\sum_{w_1^*=\tau_2+1}^{\eta_4-1}\frac{\nu}{\eta_4-\tau_2}\breve{\Gamma}(\tau_3,\tau_2)[\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(w_1^*+1)\\
&-\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(w_1^*-1)]+\frac{\nu}{\eta_4-\tau_2}\breve{\Gamma}(\tau_3,\tau_2)[\alpha[\breve{G}(\alpha\Delta\\
&+(1-\alpha)\nabla)(\tau_2+1)-\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\tau_2-1)]\\
&+(1-\alpha)[\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\eta_4+1)-\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\eta_4-1)]]\Big|\\
&\leq\frac{Y_1}{(\tau+\nu)^2}\Big(\frac{\tau}{\tau_3-\eta_1}[\alpha h_2(\tau_3,\eta_1)+(1-\alpha)\hat{h_2}(\tau_3,\eta_1)]\\
&+\frac{\nu}{\eta_4-\tau_3}[\alpha h_2(\tau_3,\eta_4)+(1-\alpha)\hat{h_2}(\tau_3,\eta_4)]\Big)^2,
\end{aligned}
$$

where

$$Y_1 = \sup_{\eta_1<\tau_2<\eta_4} |\breve{G}(\alpha\Delta^2 + (1-\alpha)\nabla^2)(\tau_2)| < \infty.$$

Theorem 3.11. *Using the assumptions given in Theorem 3.7, we have*

$$\begin{aligned}
&\Bigg|\breve{G}^2(\eta_4) - \breve{G}^2(\eta_1) - \frac{1}{\tau+\nu}\int_{\eta_1}^{\eta_4} \breve{G}^{\diamond_\alpha}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3} \breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2 \\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4} \breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3 \\
&- \frac{\alpha}{\tau+\nu}\int_{\eta_1}^{\eta_4} \breve{G}^{\diamond_\alpha}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3} [\breve{G}^{\sigma}(\tau_2) - \breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2 \\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4} [\breve{G}^{\sigma}(\tau_2) - \breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3 \\
&- \frac{\alpha}{\tau+\nu}\int_{\eta_1}^{\eta_4} \breve{G}^{\Delta}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3} \breve{G}^{\rho}(\sigma(\tau_2))\diamond_\alpha\tau_2 \\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4} \breve{G}^{\rho}(\sigma(\tau_2))\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3 \\
&- \frac{\alpha^2}{\tau+\nu}\int_{\eta_1}^{\eta_4} \breve{G}^{\Delta}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3} [\breve{G}^{\sigma^2}(\tau_2) - \breve{G}^{\rho}(\sigma(\tau_2))]\diamond_\alpha\tau_2 \\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4} [\breve{G}^{\sigma^2}(\tau_2) - \breve{G}^{\rho}(\sigma(\tau_2))]\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3 \\
&- \frac{1-\alpha}{\tau+\nu}\int_{\eta_1}^{\eta_4} \breve{G}^{\nabla}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3} \breve{G}^{\rho^2}(\tau_2)\diamond_\alpha\tau_2 \\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4} \breve{G}^{\rho^2}(\tau_2)\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3 \\
&- \frac{\alpha(1-\alpha)}{\tau+\nu}\int_{\eta_1}^{\eta_4} \breve{G}^{\nabla}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3} [\breve{G}^{\sigma}(\rho(\tau_2)) - \breve{G}^{\rho^2}(\tau_2)]\diamond_\alpha\tau_2 \\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4} [\breve{G}^{\sigma}(\rho(\tau_2)) - \breve{G}^{\rho^2}(\tau_2)]\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\Bigg| \\
&\leq [Y_0^2 + \alpha Y_2 Z_0 + (1-\alpha)Y_3 X_1]\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4} |\breve{\Gamma}(\tau_3,\tau_2)|\diamond_\alpha\tau_2\diamond_\alpha\tau_3,
\end{aligned} \tag{3.14}$$

where

$$Y_0 = \sup_{\eta_1<\tau_2<\eta_4} |\breve{G}^{\diamond_\alpha}(\tau_2)|, \quad Y_2 = \sup_{\eta_1<\tau_2<\eta_4} |\breve{G}^{\diamond_\alpha}(\sigma(\tau_2))|, \quad Y_3 = \sup_{\eta_1<\tau_2<\eta_4} |\breve{G}^{\diamond_\alpha}(\rho(\tau_2))|,$$

$$Z_0 = \sup_{\eta_1<\tau_2<\eta_4} |\breve{G}^{\Delta}(\tau_2)| \quad \textit{and} \quad X_1 = \sup_{\eta_1<\tau_2<\eta_4} |\breve{G}^{\nabla}(\tau_2)|.$$

Proof. Rewrite (3.10) for $\breve{G}^{\sigma}(\tau_3)$ and $\breve{G}^{\rho}(\tau_3)$ as follows:

$$\begin{aligned}\breve{G}^{\sigma}(\tau_3) &= \int_{\eta_1}^{\eta_4} \breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\sigma(\tau_2))\diamond_\alpha\tau_2\\ &+ \frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho}(\sigma(\tau_2))\diamond_\alpha\tau_2 + \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho}(\sigma(\tau_2))\diamond_\alpha\tau_2\Big]\\ &+ \frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma^2}(\tau_2)-\breve{G}^{\rho}(\sigma(\tau_2))]\diamond_\alpha\tau_2\\ &+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma^2}(\tau_2)-\breve{G}^{\rho}(\sigma(\tau_2))]\diamond_\alpha\tau_2\Big],\end{aligned} \tag{3.15}$$

$$\begin{aligned}\breve{G}^{\rho}(\tau_3) &= \int_{\eta_1}^{\eta_4} \breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\rho(\tau_2))\diamond_\alpha\tau_2 + \frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\\ &\int_{\eta_1}^{\tau_3}\breve{G}^{\rho^2}(\tau_2)\diamond_\alpha\tau_2 + \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho^2}(\tau_2)\diamond_\alpha\tau_2\Big] + \frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\\ &\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma}(\rho(\tau_2))-\breve{G}^{\rho^2}(\tau_2)]\diamond_\alpha\tau_2 + \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma}(\rho(\tau_2))-\breve{G}^{\rho^2}(\tau_2)]\diamond_\alpha\tau_2\Big].\end{aligned} \tag{3.16}$$

Multiply (3.10) by $\breve{G}^{\diamond_\alpha}(\tau_3)$, (3.15) by $\alpha\breve{G}^{\Delta}(\tau_3)$, and (3.16) by $(1-\alpha)\breve{G}^{\nabla}(\tau_3)$, add them, use the product formula and integrate the obtained identity with respect to τ_3 over $[\eta_1,\eta_4]$ to obtain

$$\begin{aligned}\breve{G}^2(\eta_4)-\breve{G}^2(\eta_1) &= \int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\tau_3)\breve{G}^{\diamond_\alpha}(\tau_2)\diamond_\alpha\tau_2\diamond_\alpha\tau_3\\ &+ \alpha\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\sigma(\tau_2))\breve{G}^{\Delta}(\tau_3)\diamond_\alpha\tau_2\diamond_\alpha\tau_3\\ &+ (1-\alpha)\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\rho(\tau_2))\breve{G}^{\nabla}(\tau_3)\diamond_\alpha\tau_2\diamond_\alpha\tau_3\\ &+ \frac{1}{\tau+\nu}\int_{\eta_1}^{\eta_4}\breve{G}^{\diamond_\alpha}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2\\ &+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\\ &+ \frac{\alpha}{\tau+\nu}\int_{\eta_1}^{\eta_4}\breve{G}^{\diamond_\alpha}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2\\ &+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\\ &+ \frac{\alpha}{\tau+\nu}\int_{\eta_1}^{\eta_4}\breve{G}^{\Delta}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho}(\sigma(\tau_2))\diamond_\alpha\tau_2\\ &+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho}(\sigma(\tau_2))\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\alpha^2}{\tau+\nu}\int_{\eta_1}^{\eta_4} \breve{G}^{\Delta}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma^2}(\tau_2)-\breve{G}^{\rho}(\sigma(\tau_2))]\diamond_\alpha\tau_2\\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma^2}(\tau_2)-\breve{G}^{\rho}(\sigma(\tau_2))]\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\\
&+ \frac{1-\alpha}{\tau+\nu}\int_{\eta_1}^{\eta_4} \breve{G}^{\nabla}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho^2}(\tau_2)\diamond_\alpha\tau_2\\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\eta_4{}^2}(\tau_2)\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\\
&+ \frac{\alpha(1-\alpha)}{\tau+\nu}\int_{\eta_1}^{\eta_4} \breve{G}^{\nabla}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma}(\rho(\tau_2))-\breve{G}^{\rho^2}(\tau_2)]\diamond_\alpha\tau_2\\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma}(\rho(\tau_2))-\breve{G}^{\rho^2}(\tau_2)]\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3.
\end{aligned}
$$

By using the properties of the modulus, we get

$$
\begin{aligned}
&\Big|\breve{G}^2(\eta_4)-\breve{G}^2(\eta_1)-\frac{1}{\tau+\nu}\int_{\eta_1}^{\eta_4}\breve{G}^{\diamond_\alpha}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2\\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho}(\tau_2)\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\\
&- \frac{\alpha}{\tau+\nu}\int_{\eta_1}^{\eta_4}\breve{G}^{\diamond_\alpha}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2\\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2)]\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\\
&- \frac{\alpha}{\tau+\nu}\int_{\eta_1}^{\eta_4}\breve{G}^{\Delta}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho}(\sigma(\tau_2))\diamond_\alpha\tau_2\\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho}(\sigma(\tau_2))\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\\
&- \frac{\alpha^2}{\tau+\nu}\int_{\eta_1}^{\eta_4}\breve{G}^{\Delta}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma^2}(\tau_2)-\breve{G}^{\rho}(\sigma(\tau_2))]\diamond_\alpha\tau_2\\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma^2}(\tau_2)-\breve{G}^{\rho}(\sigma(\tau_2))]\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\\
&- \frac{1-\alpha}{\tau+\nu}\int_{\eta_1}^{\eta_4}\breve{G}^{\nabla}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{G}^{\rho^2}(\tau_2)\diamond_\alpha\tau_2\\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{G}^{\rho^2}(\tau_2)\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\\
&- \frac{\alpha(1-\alpha)}{\tau+\nu}\int_{\eta_1}^{\eta_4}\breve{G}^{\nabla}(\tau_3)\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{G}^{\sigma}(\rho(\tau_2))-\breve{G}^{\rho^2}(\tau_2)]\diamond_\alpha\tau_2\\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma}(\rho(\tau_2))-\breve{G}^{\rho^2}(\tau_2)]\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\Big|\\
&= \Big|\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_3,\tau_2)\breve{G}^{\diamond_\alpha}(\tau_2)\breve{G}^{\diamond_\alpha}(\tau_3)\diamond_\alpha\tau_2\diamond_\alpha\tau_3
\end{aligned}
$$

$$+ \alpha \int_{\eta_1}^{\eta_4} \int_{\eta_1}^{\eta_4} \breve{\Gamma}(\tau_3, \tau_2) \breve{G}^{\diamond_\alpha}(\sigma(\tau_2)) \breve{G}^{\Delta}(\tau_3) \diamond_\alpha \tau_2 \diamond_\alpha \tau_3$$

$$+ (1-\alpha) \int_{\eta_1}^{\eta_4} \int_{\eta_1}^{\eta_4} \breve{\Gamma}(\tau_3, \tau_2) \breve{G}^{\diamond_\alpha}(\rho(\tau_2)) \breve{G}^{\nabla}(\tau_3) \diamond_\alpha \tau_2 \diamond_\alpha \tau_3 \Big|$$

$$\leq \int_{\eta_1}^{\eta_4} \int_{\eta_1}^{\eta_4} |\breve{\Gamma}(\tau_3, \tau_2)||\breve{G}^{\diamond_\alpha}(\tau_2)||\breve{G}^{\diamond_\alpha}(\tau_3)| \diamond_\alpha \tau_2 \diamond_\alpha \tau_3$$

$$+ \alpha \int_{\eta_1}^{\eta_4} \int_{\eta_1}^{\eta_4} |\breve{\Gamma}(\tau_3, \tau_2)||\breve{G}^{\diamond_\alpha}(\sigma(\tau_2))||\breve{G}^{\Delta}(\tau_3)| \diamond_\alpha \tau_2 \diamond_\alpha \tau_3$$

$$+ (1-\alpha) \int_{\eta_1}^{\eta_4} \int_{\eta_1}^{\eta_4} |\breve{\Gamma}(\tau_3, \tau_2)||\breve{G}^{\diamond_\alpha}(\rho(\tau_2))||\breve{G}^{\nabla}(\tau_3)| \diamond_\alpha \tau_2 \diamond_\alpha \tau_3$$

$$\leq [Y_0^2 + \alpha Y_2 Z_0 + (1-\alpha) Y_3 X_1] \int_{\eta_1}^{\eta_4} \int_{\eta_1}^{\eta_4} |\breve{\Gamma}(\tau_3, \tau_2)| \diamond_\alpha \tau_2 \diamond_\alpha \tau_3.$$

□

Remark 3.12. *(i) Put $\alpha = 1$ in Theorem 3.11 to obtain [27, Theorem 3.4];*
(ii) apply $\alpha = 1$ and $\mathbb{T} = \mathbb{R}$ to obtain [27, Corollary 3.5].

Corollary 3.13. *Put $\tau = \nu = 1$ in* (3.14) *to obtain*

$$\Big| \breve{G}^2(\eta_4) - \breve{G}^2(\eta_1) - \frac{1}{2} \int_{\eta_1}^{\eta_4} \breve{G}^{\diamond_\alpha}(\tau_3) \Big[\frac{1}{\tau_3 - \eta_1} \int_{\eta_1}^{\tau_3} \breve{G}^{\rho}(\tau_2) \diamond_\alpha \tau_2$$

$$+ \frac{1}{\eta_4 - \tau_3} \int_{\tau_3}^{\eta_4} \breve{G}^{\rho}(\tau_2) \diamond_\alpha \tau_2 \Big] \diamond_\alpha \tau_3$$

$$- \frac{\alpha}{2} \int_{\eta_1}^{\eta_4} \breve{G}^{\diamond_\alpha}(\tau_3) \Big[\frac{1}{\tau_3 - \eta_1} \int_{\eta_1}^{\tau_3} [\breve{G}^{\sigma}(\tau_2) - \breve{G}^{\rho}(\tau_2)] \diamond_\alpha \tau_2$$

$$+ \frac{1}{\eta_4 - \tau_3} \int_{\tau_3}^{\eta_4} [\breve{G}^{\sigma}(\tau_2) - \breve{G}^{\rho}(\tau_2)] \diamond_\alpha \tau_2 \Big] \diamond_\alpha \tau_3$$

$$- \frac{\alpha}{2} \int_{\eta_1}^{\eta_4} \breve{G}^{\Delta}(\tau_3) \Big[\frac{1}{\tau_3 - \eta_1} \int_{\eta_1}^{\tau_3} \breve{G}^{\rho}(\sigma(\tau_2)) \diamond_\alpha \tau_2$$

$$+ \frac{1}{\eta_4 - \tau_3} \int_{\tau_3}^{\eta_4} \breve{G}^{\rho}(\sigma(\tau_2)) \diamond_\alpha \tau_2 \Big] \diamond_\alpha \tau_3$$

$$- \frac{\alpha^2}{2} \int_{\eta_1}^{\eta_4} \breve{G}^{\Delta}(\tau_3) \Big[\frac{1}{\tau_3 - \eta_1} \int_{\eta_1}^{\tau_3} [\breve{G}^{\sigma^2}(\tau_2) - \breve{G}^{\rho}(\sigma(\tau_2))] \diamond_\alpha \tau_2$$

$$+ \frac{1}{\eta_4 - \tau_3} \int_{\tau_3}^{\eta_4} [\breve{G}^{\sigma^2}(\tau_2) - \breve{G}^{\rho}(\sigma(\tau_2))] \diamond_\alpha \tau_2 \Big] \diamond_\alpha \tau_3$$

$$- \frac{1-\alpha}{2} \int_{\eta_1}^{\eta_4} \breve{G}^{\nabla}(\tau_3) \Big[\frac{1}{\tau_3 - \eta_1} \int_{\eta_1}^{\tau_3} \breve{G}^{\rho^2}(\tau_2) \diamond_\alpha \tau_2$$

$$+ \frac{1}{\eta_4 - \tau_3} \int_{\tau_3}^{\eta_4} \breve{G}^{\rho^2}(\tau_2) \diamond_\alpha \tau_2 \Big] \diamond_\alpha \tau_3$$

$$- \frac{\alpha(1-\alpha)}{2} \int_{\eta_1}^{\eta_4} \breve{G}^{\nabla}(\tau_3) \Big[\frac{1}{\tau_3 - \eta_1} \int_{\eta_1}^{\tau_3} [\breve{G}^{\sigma}(\rho(\tau_2)) - \breve{G}^{\rho^2}(\tau_2)] \diamond_\alpha \tau_2$$

$$+\frac{1}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{G}^{\sigma}(\rho(\tau_2))-\breve{G}^{\rho^2}(\tau_2)]\diamond_\alpha\tau_2\Big]\diamond_\alpha\tau_3\Big|$$
$$\leq [Y_0^2+\alpha Y_2Z_0+(1-\alpha)Y_3X_1]\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}|\breve{\Gamma}(\tau_3,\tau_2)|\diamond_\alpha\tau_2\diamond_\alpha\tau_3.$$

Example 3.14. *If we put* $\mathbb{T}=\mathbb{Z}$ *in* (3.14)*, then we get*

$$\Big|\breve{G}^2(\eta_4)-\breve{G}^2(\eta_1)-\frac{1}{\tau+\nu}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\tau_3)+\alpha\breve{G}(\alpha\Delta$$
$$+(1-\alpha)\nabla)(\eta_1)+(1-\alpha)\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\eta_4)\Big[\frac{\tau}{\tau_3-\eta_1}\Big[\sum_{\tau_2=\eta_1+1}^{\tau_3-1}\breve{G}(\tau_2-1)$$
$$+\alpha\breve{G}(\eta_1-1)+(1-\alpha)\breve{G}(\tau_3-1)\Big]+\frac{\nu}{\eta_4-\tau_3}\Big[\sum_{\tau_2=\tau_3+1}^{\eta_4-1}\breve{G}(\tau_2-1)+\alpha\breve{G}(\tau_3-1)$$
$$+(1-\alpha)\breve{G}(\eta_4-1)\Big]\Big]\Big]-\frac{\alpha}{\tau+\nu}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\tau_3)+\alpha\breve{G}(\alpha\Delta$$
$$+(1-\alpha)\nabla)(\eta_1)+(1-\alpha)\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\eta_4)\Big[\frac{\tau}{\tau_3-\eta_1}\Big[\sum_{\tau_2=\eta_1+1}^{\tau_3-1}[\breve{G}(\tau_2+1)$$
$$-\breve{G}(\tau_2-1)]+\alpha[\breve{G}(\eta_1+1)-\breve{G}(\eta_1-1)]+(1-\alpha)[\breve{G}(\tau_3+1)-\breve{G}(\tau_3-1)]\Big]$$
$$+\frac{\nu}{\eta_4-\tau_3}\Big[\sum_{\tau_2=\tau_3+1}^{\eta_4-1}[\breve{G}(\tau_2+1)-\breve{G}(\tau_2-1)]+\alpha[\breve{G}(\tau_3+1)-\breve{G}(\tau_3-1)]$$
$$+(1-\alpha)[\breve{G}(\eta_4+1)-\breve{G}(\eta_4-1)]\Big]\Big]\Big]-\frac{\alpha}{\tau+\nu}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\Delta\breve{G}(\tau_3)$$
$$+\alpha\Delta\breve{G}(\eta_1)+(1-\alpha)\Delta\breve{G}(\eta_4)\Big[\frac{\tau}{\tau_3-\eta_1}\Big[\sum_{\tau_2=\eta_1+1}^{\tau_3-1}\breve{G}(\tau_2)$$
$$+\alpha\breve{G}(\eta_1)+(1-\alpha)\breve{G}(\tau_3)\Big]+\frac{\nu}{\eta_4-\tau_3}\Big[\sum_{\tau_2=\tau_3+1}^{\eta_4-1}\breve{G}(\tau_2)+\alpha\breve{G}(\tau_3)+(1-\alpha)\breve{G}(\eta_4)\Big]\Big]\Big]$$
$$-\frac{\alpha^2}{\tau+\nu}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\Delta\breve{G}(\tau_3)+\alpha\Delta\breve{G}(\eta_1)+(1-\alpha)\Delta\breve{G}(\eta_4)\Big[\frac{\tau}{\tau_3-\eta_1}\Big[\sum_{\tau_2=\eta_1+1}^{\tau_3-1}[\breve{G}(\tau_2+2)$$
$$-\breve{G}(\tau_2)]+\alpha[\breve{G}(\eta_1+2)-\breve{G}(\eta_1)]+(1-\alpha)[\breve{G}(\tau_3+2)-\breve{G}(\tau_3)]\Big]$$
$$+\frac{\nu}{\eta_4-\tau_3}\Big[\sum_{\tau_2=\tau_3+1}^{\eta_4-1}[\breve{G}(\tau_2+2)-\breve{G}(\tau_2)]$$
$$+\alpha[\breve{G}(\tau_3+2)-\breve{G}(\tau_3)]+(1-\alpha)[\breve{G}(\eta_4+2)-\breve{G}(\eta_4)]\Big]\Big]\Big]-\frac{1-\alpha}{\tau+\nu}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\nabla\breve{G}(\tau_3)$$

$$+\alpha\nabla\breve{G}(\eta_1)+(1-\alpha)\nabla\breve{G}(\eta_4)\Big[\frac{\tau}{\tau_3-\eta_1}\Big[\sum_{\tau_2=\eta_1+1}^{\tau_3-1}\breve{G}(\tau_2-2)$$

$$+\alpha\breve{G}(\eta_1-2)+(1-\alpha)\breve{G}(\tau_3-2)\Big]$$

$$+\frac{\nu}{\eta_4-\tau_3}\Big[\sum_{\tau_2=\tau_3+1}^{\eta_4-1}\breve{G}(\tau_2-2)+\alpha\breve{G}(\tau_3-2)+(1-\alpha)\breve{G}(\eta_4-2)\Big]\Big]\Big]$$

$$-\frac{\alpha(1-\alpha)}{\tau+\nu}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\nabla\breve{G}(\tau_3)+\alpha\nabla\breve{G}(\eta_1)$$

$$+(1-\alpha)\breve{G}(\eta_4)\Big[\frac{\tau}{\tau_3-\eta_1}\Big[\sum_{\tau_2=\eta_1+1}^{\tau_3-1}[\breve{G}(\tau_2)-\breve{G}(\tau_2-2)]+\alpha[\breve{G}(\eta_1)-\breve{G}(\eta_1-2)]$$

$$+(1-\alpha)[\breve{G}(\tau_3)-\breve{G}(\tau_3-2)]\Big]$$

$$+\frac{\nu}{\eta_4-\tau_3}\Big[\sum_{\tau_2=\tau_3+1}^{\eta_4-1}[\breve{G}(\tau_2)-\breve{G}(\tau_2-2)]+\alpha[\breve{G}(\tau_3)-\breve{G}(\tau_3-2)]$$

$$+(1-\alpha)[\breve{G}(\eta_4)-\breve{G}(\eta_4-2)]\Big]\Big]\Big]\Big|$$

$$\leq[Y_0^2+\alpha Y_2Z_0+(1-\alpha)Y_3X_1]\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\sum_{\tau_2=\eta_1+1}^{\eta_4-1}|\breve{\Gamma}(\tau_3,\tau_2)|,$$

where

$$Y_0=\sup_{\eta_1<\tau_2<\eta_4}|\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\tau_2)|,$$
$$Y_2=\sup_{\eta_1<\tau_2<\eta_4}|\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\tau_2+1)|,$$
$$Y_3=\sup_{\eta_1<\tau_2<\eta_4}|\breve{G}(\alpha\Delta+(1-\alpha)\nabla)(\tau_2-1)|,$$
$$Z_0=\sup_{\eta_1<\tau_2<\eta_4}|\Delta\breve{G}(\tau_2)| \quad and \quad X_1=\sup_{\eta_1<\tau_2<\eta_4}|\nabla\breve{G}(\tau_2)|.$$

Theorem 3.15. *Consider $\mathbb{T}$ to be a time scale with $\eta_1,\tau_2,\tau_3,\eta_4\in\mathbb{T}$ and $\eta_1<\eta_4$. Further, suppose that the functions $\breve{G},\breve{H}:[\eta_1,\eta_4]_{\mathbb{T}}\to\mathbb{R}$ are diamond-alpha differentiable. Then, for all $\tau_3\in[\eta_1,\eta_4]_{\mathbb{T}}$, $\alpha\in[0,1]$, and $\tau,\nu\in\mathbb{R}$, we have*

$$\Big|2\int_{\eta_1}^{\eta_4}\breve{G}(\tau_3)\breve{H}(\tau_3)\diamond_\alpha\tau_3$$

$$-\frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_3}[\breve{G}^\rho(\tau_2)\breve{H}(\tau_3)+\breve{H}^\rho(\tau_2)\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3$$

$$+\frac{\nu}{\eta_4-\tau_3}\int_{\eta_1}^{\eta_4}\int_{\tau_3}^{\eta_4}[\breve{G}^\rho(\tau_2)\breve{H}(\tau_3)+\breve{H}^\rho(\tau_2)\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\Big]$$

$$-\frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_3}[(\breve{G}^\sigma(\tau_2)-\breve{G}^\rho(\tau_2))\breve{H}(\tau_3)$$

$$
\begin{aligned}
&+ (\breve{H}^{\sigma}(\tau_2) - \breve{H}^{\rho}(\tau_2))\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3 \\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\eta_1}^{\eta_4}\int_{\tau_3}^{\eta_4}[(\breve{G}^{\sigma}(\tau_2) - \breve{G}^{\rho}(\tau_2))\breve{H}(\tau_3) \\
&+ (\breve{H}^{\sigma}(\tau_2) - \breve{H}^{\rho}(\tau_2))\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\Big]\Big| \\
&\le \int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}|\breve{\Gamma}(\tau_3,\tau_2)|[Y_0|\breve{H}(\tau_3)| + S_0|\breve{G}(\tau_3)|]\diamond_\alpha\tau_2\diamond_\alpha\tau_3,
\end{aligned}
\tag{3.17}
$$

where

$$
Y_0 = \sup_{\eta_1<\tau_2<\eta_4}|\breve{G}^{\diamond_\alpha}(\tau_2)| < \infty \quad \textit{and} \quad S_0 = \sup_{\eta_1<\tau_2<\eta_4}|\breve{H}^{\diamond_\alpha}(\tau_2)| < \infty.
$$

Proof. Replace $\breve{H}$ with $\breve{G}$ in (3.10) to get

$$
\begin{aligned}
\breve{H}(\tau_3) &= \int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_3,\tau_2)\breve{H}^{\diamond_\alpha}(\tau_2)\diamond_\alpha\tau_2 \\
&+ \frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}\breve{H}^{\rho}(\tau_2)\diamond_\alpha\tau_2 + \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}\breve{H}^{\rho}(\tau_2)\diamond_\alpha\tau_2\Big] \\
&+ \frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\tau_3}[\breve{H}^{\sigma}(\tau_2) - \breve{H}^{\rho}(\tau_2)]\diamond_\alpha\tau_2 \\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\tau_3}^{\eta_4}[\breve{H}^{\sigma}(\tau_2) - \breve{H}^{\rho}(\tau_2)]\diamond_\alpha\tau_2\Big].
\end{aligned}
\tag{3.18}
$$

Multiply (3.10) by $\breve{H}(\tau_3)$ and (3.18) by $\breve{G}(\tau_3)$, add them and integrate the obtained identity with respect to τ_3 over $[\eta_1,\eta_4]$ to obtain

$$
\begin{aligned}
&2\int_{\eta_1}^{\eta_4}\breve{G}(\tau_3)\breve{H}(\tau_3)\diamond_\alpha\tau_3 \\
&= \int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_3,\tau_2)[\breve{G}^{\diamond_\alpha}(\tau_2)\breve{H}(\tau_3) + \breve{H}^{\diamond_\alpha}(\tau_2)\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3 \\
&+ \frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_3}[\breve{G}^{\rho}(\tau_2)\breve{H}(\tau_3) + \breve{H}^{\rho}(\tau_2)\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3 \\
&+ \frac{\nu}{\eta_4-\tau_3}\int_{\eta_1}^{\eta_4}\int_{\tau_3}^{\eta_4}[\breve{G}^{\rho}(\tau_2)\breve{H}(\tau_3) + \breve{H}^{\rho}(\tau_2)\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\Big] \\
&+ \frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_3}[(\breve{G}^{\sigma}(\tau_2) - \breve{G}^{\rho}(\tau_2))\breve{H}(\tau_3) \\
&+ (\breve{H}^{\sigma}(\tau_2) - \breve{H}^{\rho}(\tau_2))\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3 + \frac{\nu}{\eta_4-\tau_3}\int_{\eta_1}^{\eta_4}\int_{\tau_3}^{\eta_4}[(\breve{G}^{\sigma}(\tau_2) \\
&- \breve{G}^{\rho}(\tau_2))\breve{H}(\tau_3) + (\breve{H}^{\sigma}(\tau_2) - \breve{H}^{\rho}(\tau_2))\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\Big].
\end{aligned}
$$

By using the modulus properties, we get

$$
\Big|2\int_{\eta_1}^{\eta_4}\breve{G}(\tau_3)\breve{H}(\tau_3)\diamond_\alpha\tau_3
$$

$$
\begin{aligned}
&-\frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_3}[\breve{G}^{\rho}(\tau_2)\breve{H}(\tau_3)+\breve{H}^{\rho}(\tau_2)\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\\
&+\frac{\nu}{\eta_4-\tau_3}\int_{\eta_1}^{\eta_4}\int_{\tau_3}^{\eta_4}[\breve{G}^{\rho}(\tau_2)\breve{H}(\tau_3)+\breve{H}^{\rho}(\tau_2)\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\Big]\\
&-\frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_3}[(\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2))\breve{H}(\tau_3)\\
&+(\breve{H}^{\sigma}(\tau_2)-\breve{H}^{\rho}(\tau_2))\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\\
&+\frac{\nu}{\eta_4-\tau_3}\int_{\eta_1}^{\eta_4}\int_{\tau_3}^{\eta_4}[(\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2))\breve{H}(\tau_3)\\
&+(\breve{H}^{\sigma}(\tau_2)-\breve{H}^{\rho}(\tau_2))\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\Big]\Big|\\
&=\Big|\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}\breve{\Gamma}(\tau_3,\tau_2)[\breve{G}^{\diamond_\alpha}(\tau_2)\breve{H}(\tau_3)+\breve{H}^{\diamond_\alpha}(\tau_2)\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\Big|\\
&\le\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}|\breve{\Gamma}(\tau_3,\tau_2)|[|\breve{G}^{\diamond_\alpha}(\tau_2)||\breve{H}(\tau_3)|+|\breve{H}^{\diamond_\alpha}(\tau_2)||\breve{G}(\tau_3)|]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\\
&\le\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}|\breve{\Gamma}(\tau_3,\tau_2)|[Y_0|\breve{H}(\tau_3)|+S_0|\breve{G}(\tau_3)|]\diamond_\alpha\tau_2\diamond_\alpha\tau_3.
\end{aligned}
$$

□

Remark 3.16. *(i) Put $\alpha = 1$ in Theorem 3.15 to obtain [27, Theorem 3.7];*
(ii) apply $\alpha = 1$ and $\mathbb{T} = \mathbb{R}$ to obtain [27, Corollary 3.8].

Corollary 3.17. *Substitute $\tau = \nu = 1$ in* (3.17) *to get*

$$
\begin{aligned}
&\Big|2\int_{\eta_1}^{\eta_4}\breve{G}(\tau_3)\breve{H}(\tau_3)\diamond_\alpha\tau_3\\
&-\frac{1}{2}\Big[\frac{1}{\tau_3-\eta_1}\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_3}[\breve{G}^{\rho}(\tau_2)\breve{H}(\tau_3)+\breve{H}^{\rho}(\tau_2)\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\\
&+\frac{1}{\eta_4-\tau_3}\int_{\eta_1}^{\eta_4}\int_{\tau_3}^{\eta_4}[\breve{G}^{\rho}(\tau_2)\breve{H}(\tau_3)+\breve{H}^{\rho}(\tau_2)\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\Big]\\
&-\frac{\alpha}{2}\Big[\frac{1}{\tau_3-\eta_1}\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\tau_3}[(\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2))\breve{H}(\tau_3)\\
&+(\breve{H}^{\sigma}(\tau_2)-\breve{H}^{\rho}(\tau_2))\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\\
&+\frac{1}{\eta_4-\tau_3}\int_{\eta_1}^{\eta_4}\int_{\tau_3}^{\eta_4}[(\breve{G}^{\sigma}(\tau_2)-\breve{G}^{\rho}(\tau_2))\breve{H}(\tau_3)\\
&+(\breve{H}^{\sigma}(\tau_2)-\breve{H}^{\rho}(\tau_2))\breve{G}(\tau_3)]\diamond_\alpha\tau_2\diamond_\alpha\tau_3\Big]\Big|\\
&\le\int_{\eta_1}^{\eta_4}\int_{\eta_1}^{\eta_4}|\breve{\Gamma}(\tau_3,\tau_2)|[Y_0|\breve{H}(\tau_3)|+S_0|\breve{G}(\tau_3)|]\diamond_\alpha\tau_2\diamond_\alpha\tau_3.
\end{aligned}
\tag{3.19}
$$

Example 3.18. *If we substitute* $\mathbb{T}=\mathbb{Z}$ *in* (3.17)*, then we obtain*

$$\begin{aligned}
&\Big|2\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\breve{G}(\tau_3)\breve{H}(\tau_3)+\alpha\breve{G}(\eta_1)\breve{H}(\eta_1)+(1-\alpha)\breve{G}(\eta_4)\breve{H}(\eta_4)\Big]\\
&-\frac{1}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\sum_{\tau_2=\eta_1+1}^{\tau_3-1}[\breve{G}(\tau_2-1)\breve{H}(\tau_3)+\breve{H}(\tau_2-1)\breve{G}(\tau_3)]\\
&+\alpha[\breve{G}(\eta_1-1)\breve{H}(\eta_1)+\breve{H}(\eta_1-1)\breve{G}(\eta_1)]+(1-\alpha)[\breve{G}(\tau_3-1)\breve{H}(\eta_4)+\breve{H}(\tau_3-1)\breve{G}(\eta_4)]\Big]\Big]\\
&+\frac{\nu}{\eta_4-\tau_3}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\sum_{\tau_2=\tau_3+1}^{\eta_4-1}[\breve{G}(\tau_2-1)\breve{H}(\tau_3)+\breve{H}(\tau_2-1)\breve{G}(\tau_3)]\\
&+\alpha[\breve{G}(\tau_3-1)\breve{H}(\eta_1)+\breve{H}(\eta_1-1)\breve{G}(\tau_3)]+(1-\alpha)[\breve{G}(\eta_4-1)\breve{H}(\eta_4)+\breve{H}(\eta_4-1)\breve{G}(\eta_4)]\Big]\Big]\\
&-\frac{\alpha}{\tau+\nu}\Big[\frac{\tau}{\tau_3-\eta_1}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\sum_{\tau_2=\eta_1+1}^{\tau_3-1}[[\breve{G}(\tau_2+1)-\breve{G}(\tau_2-1)]\breve{H}(\tau_3)\\
&+[\breve{H}(\tau_2+1)-\breve{H}(\tau_2-1)]\breve{G}(\tau_3)]+\alpha[[\breve{G}(\eta_1+1)-\breve{G}(\eta_1-1)]\breve{H}(\eta_1)\\
&+[\breve{H}(\eta_1+1)-\breve{H}(\eta_1-1)]\breve{G}(\eta_1)]+(1-\alpha)[[\breve{G}(\tau_3+1)-\breve{G}(\tau_3-1)]\breve{H}(\eta_4)\\
&+[\breve{H}(\eta_3+1)-\breve{H}(\tau_3-1)]\breve{G}(\eta_4)]\Big]\\
&+\frac{\nu}{\eta_4-\tau_3}\Big[\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\sum_{\tau_2=\tau_3+1}^{\eta_4-1}[[\breve{G}(\tau_2+1)-\breve{G}(\tau_2-1)]\breve{H}(\tau_3)\\
&+[\breve{H}(\tau_2+1)-\breve{H}(\tau_2-1)]\breve{G}(\tau_3)]+\alpha[[\breve{G}(\tau_3+1)-\breve{G}(\tau_3-1)]\breve{H}(\eta_1)\\
&+[\breve{H}(\tau_3+1)-\breve{H}(\tau_3-1)]\breve{G}(\eta_1)]+(1-\alpha)[[\breve{G}(\eta_4+1)-\breve{G}(\eta_4-1)]\breve{H}(\eta_4)\\
&+[\breve{H}(\eta_4+1)-\breve{H}(\eta_4-1)]\breve{G}(\eta_4)]\Big]\Big]\Big|\\
&\leq\sum_{\tau_3=\eta_1+1}^{\eta_4-1}\sum_{\tau_2=\eta_1+1}^{\eta_4-1}|\breve{\Gamma}(\tau_3,\tau_2)|[Y_0|\breve{H}(\tau_3)|+S_0|\breve{G}(\tau_3)|],
\end{aligned}$$

where

$$Y_0=\sup_{\eta_1<\tau_2<\eta_4}|\breve{G}^{\diamond_\alpha}(\tau_2)|<\infty,$$

and

$$S_0=\sup_{\eta_1<\tau_2<\eta_4}|\breve{H}^{\diamond_\alpha}(\tau_2)|<\infty.$$

4. Conclusions

In the present manuscript, the Ostrowski-type integral inequality has been established through the use of the Montgomery identity for diamond-alpha integrals. Moreover, some extensions of dynamic Grüss- and trapezoid-type inequalities have been investigated for bivariate functions which are two times diamond-α-differentiable. Special cases of our results not only produce the results of [18, 26, 27], they also give a symmetric combination of the results established in [18, 26, 27]. Truong et al. [24] presented the diamond-alpha differentiability of interval-valued functions and their applicability to interval differential equations on time scales which can help to extend the results of the present manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research work was funded by the Institutional Fund Project under grant no. (IFPIP: 1159-144-1443). The authors gratefully acknowledge the technical and financial support provided by the Ministry of Education and King Abdulaziz University, DSR, Jeddah, Saudi Arabia.

Conflict of interest

No potential conflict of interest is reported by the authors.

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