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Research article

Analysis of a stochastic two-species Schoener's competitive model with Lévy jumps and Ornstein–Uhlenbeck process

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Abstract: This paper studies a stochastic two-species Schoener's competitive model with Lévy jumps by the mean-reverting Ornstein–Uhlenbeck process. First, the biological implication of introducing the Ornstein–Uhlenbeck process is illustrated. After that, we show the existence and uniqueness of the global solution. Moment estimates for the global solution of the stochastic model are then given. Moreover, by constructing the Lyapunov function and applying Itô's formula and Chebyshev's inequality, it is found that the model is stochastic and ultimately bounded. In addition, we give sufficient conditions for the extinction of species. Finally, numerical simulations are employed to demonstrate the analytical results.

Keywords: Schoener's competitive model; Ornstein–Uhlenbeck process; extinction
Mathematics Subject Classification: 60H10, 60H30, 92D25

1. Introduction

As a common form of ecosystem, competition relationship has been widely applied in many fields in recent years, and many scholars have conducted detailed research on the competition relationship between populations [1]. In 2005, Zhen et al. [2] studied a class of two-dimensional non-autonomous competitive Lotka-Volterra systems with pulses. In 2012, Liu et al. [3] investigated the quasi-periodic solution of a Lotka-Volterra competitive system with quasi-periodic perturbations. In 2014, Liu et al. [4] studied stochastic logistic models with Lévy noise and obtained necessary and sufficient conditions for stochastic persistence and extinction. The Lotka-Volterra competition model is widely employed to study a variety of systems, including prey-predator dynamics in ecological interaction [5–7]. However, its linearization poses limitations to modeling the dynamics of certain species [8, 9], for which more complex approaches are required. In 1974, a more practical competitive system was proposed and discussed by Schoener [10], as follows :

$$\begin{cases} dx(t) = x(t)\left[-r_1 - \beta_1 x(t) - \alpha_1 y(t) + \frac{\xi_1}{x(t)+\delta_1}\right]dt, \\ dy(t) = y(t)\left[-r_2 - \beta_2 x(t) - \alpha_2 y(t) + \frac{\xi_2}{y(t)+\delta_2}\right]dt, \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ reflect the population density at time t, r_1 and r_2 represent the death rates of the population, β_1 and α_2 represent the intra-specific competition rates, and β_2 and α_1 represent the inter-specific competition rates. All parameters in system (1.1) are assumed to be positive constants. In recent years, more and more scholars have studied Schoener's competition model and achieved many results. In 2008, Wu et al. [11] studied a class of discrete Schoener's competition models with time delay, and obtained sufficient conditions to ensure the durability of the model and the global attraction of the model's positive solution. In 2012, LV et al. [8] investigated the dynamics of Schoener's competition model for two stochastic populations. In 2016, Li et al. [10] obtained some sufficient conditions for the persistence of uniformly asymptotically stable positive almost periodic solutions for a class of pulsed Schoener competition model with pure-delays.

However, in the natural world, population systems are inevitably affected by environmental noise. Therefore, considering only deterministic models can be very flawed. Many parameters in ecological dynamics should fluctuate around some average values. Mao et al. [12] demonstrated that even small amounts of ambient noise can have a large impact on species populations, which means that stochastic population models can provide additional authenticity compared to deterministic population models. Therefore, it is generally assumed that environmental noise primarily affects the fundamental parameters of the model in order to study the dynamic properties of ecosystems in different environments [13]. Based on the fact that population death rates are easily affected by environmental fluctuations, we assume r_1 and r_2 in Schoener's competition model are two random variables. Currently, there are two common methods for simulating small disturbances in the environment. The most common method is to introduce Gaussian linear white noise into the deterministic model [14–17]. Another method is to incorporate the mean-reverting Ornstein-Uhlenbeck process to simulate environmental perturbations [18–20], which has been demonstrated to be a practical and biologically meaningful method.

First, we assume that the death rate is linearly correlated with Gaussian white noise in the random environment. That is to say,

$$r_i(t) = \bar{r}_i + \frac{\sigma_i dB_i(t)}{dt}, i = 1, 2,$$

where $\bar{r}_i$ represents the long-time average levels of $r_i(t)$. $B_i(t)$ denotes two independent standard Brownian motions defined on a complete probability space $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P\}$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions [12]. For any time interval $[0, t]$, we can obtain that

$$\langle r_i(t)\rangle := \frac{1}{t}\int_0^t r_i(s)\,ds = \bar{r}_i + \frac{\sigma_i B_i(t)}{t} \sim N\left(\bar{r}_i, \frac{\sigma_i^2}{t}\right), i = 1, 2,$$

where $\langle r_i(t)\rangle$ is the time average of $r_i(t)$, and $N(\cdot,\cdot)$ denotes the one-dimensional normal distribution. From the above formula, it is easy to see that the variance of $\langle r_i(t)\rangle$ is $\frac{\sigma_i^2}{t}$, which tends to infinity as $t \to 0^+$. This means that the mean value of the perturbation parameter will be more variable in a small amount of time. In practice, however, $r_i(t)$ should likewise fluctuate little over a small time interval. Thus, the use of Gaussian linear white noise to simulate small disturbances in the environment is unreasonable.

We now consider the second method, which is to introduce the Ornstein-Uhlenbeck process into the deterministic model. On account of this approach, one has

$$\begin{aligned} dr_1(t) &= \omega_1\left(\bar{r_1} - r_1\right)dt + \sigma_1 dB_1(t), \\ dr_2(t) &= \omega_2\left(\bar{r_2} - r_2\right)dt + \sigma_2 dB_2(t), \end{aligned} \tag{1.2}$$

where $\omega_i > 0$ and $\sigma_i > 0$ are the speed of reversion and the intensity of volatility, respectively. By performing a stochastic integral operation on Equation (1.2), we can get the following unique explicit solutions:

$$r_i(t) = \bar{r_i} + \left[r_i(0) - \bar{r_i}\right]e^{-\omega_i t} + \sigma_i \int_0^t e^{-\omega_i(t-s)} dB_i(s), i = 1, 2, \tag{1.3}$$

where $r_i(0)$ is the initial value of the Ornstein–Uhlenbeck process $r_i(t)$. For any time interval $[0, t]$, we have

$$\langle r_i(t) \rangle := \frac{1}{t}\int_0^t r_i(s)\, ds = \bar{r}_i + \frac{1}{t}\int_0^t \frac{\sigma_i}{\omega_i}\left(1 - e^{\omega_i(s-t)}\right)dB_i(s) \sim N\left(\bar{r}_i, \frac{\sigma_i^2 t}{3} + O\left(t^2\right)\right),$$

and as the time interval $t \to 0^+$, it is clear that $E[\langle r_i(t)\rangle] = \bar{r}_i$ and $Var[\langle r_i(t)\rangle] = 0$, which is consistent with the fact that the perturbation of the death rate in a small time interval is also small. Therefore, it is more reasonable to introduce the Ornstein-Uhlenbeck process to perturb the parameters $r_i(t), i = 1, 2$, than Gaussian white noise [20, 21].

Based on the above analysis, we decided to introduce the Ornstein-Uhlenbeck process into the studied model. By combining system (1.1) and system (1.2), we can obtain a stochastic model of the following form:

$$\begin{cases} dx(t) = x(t)\left[-r_1(t) - \beta_1 x(t) - \alpha_1 y(t) + \frac{\xi_1}{x(t)+\delta_1}\right]dt, \\ dy(t) = y(t)\left[-r_2(t) - \beta_2 x(t) - \alpha_2 y(t) + \frac{\xi_2}{y(t)+\delta_2}\right]dt, \\ dr_1(t) = \omega_1\left[\bar{r}_1 - r_1(t)\right]dt + \sigma_1 dB_1(t), \\ dr_2(t) = \omega_2\left[\bar{r}_2 - r_2(t)\right]dt + \sigma_2 dB_2(t). \end{cases} \tag{1.4}$$

However, in addition to small perturbations in the environment, population dynamics can be subject to sudden and violent environmental shocks, such as avalanches, earthquakes, and tsunamis. It is important to note that these environmental shocks may cause sharp jumps in the population size so that previously continuous solution trajectories are no longer continuous. However, this phenomenon cannot be described by Brownian motion, so stochastic differential equations with Lévy jumps are often used to simulate sudden random perturbations that occur in the environment [22, 23]. In 2014, Liu et al. [24] studied the Lotka-Volterra stochastic model disturbed by Lévy noise and established the necessary and sufficient conditions for persistence in the mean and extinction of each population. In 2018, Qiu and Deng [1] studied the optimal acquisition problem of randomly competing Lotka-Volterra models under the influence of time delay. Inspired by the above, this paper adds Lévy jumps to system

(1.4) to simulate sudden environmental disturbances in nature. It takes the form

$$\begin{cases} dx(t) = x(t^-)\left[-r_1(t^-) - \beta_1 x(t^-) - \alpha_1 y(t^-) + \frac{\xi_1}{x(t^-)+\delta_1}\right]dt + \int_Y x(t^-)\gamma_1(\mu)N(dt, d\mu), \\ dy(t) = y(t^-)\left[-r_2(t^-) - \beta_2 x(t^-) - \alpha_2 y(t^-) + \frac{\xi_2}{y(t^-)+\delta_2}\right]dt + \int_Y y(t^-)\gamma_2(\mu)N(dt, d\mu), \\ dr_1(t) = \omega_1[\bar{r}_1 - r_1(t)]dt + \sigma_1 dB_1(t), \\ dr_2(t) = \omega_2[\bar{r}_2 - r_2(t)]dt + \sigma_2 dB_2(t), \end{cases} \tag{1.5}$$

where $x(t^-)$, $y(t^-)$, and $r(t^-)$ represent the left limit at t, $N(\cdot,\cdot)$ is a Poisson counting measure with characteristic measure λ on a measurable subset Y of $(0, +\infty)$ with $\lambda(Y) < \infty$, and $N(\cdot,\cdot)$ is independent of Brownian motion $B(\cdot)$. Define the compensating random measure as $\tilde{N}$, then $N(dt, d\mu) = \tilde{N}(dt, d\mu) + \lambda(d\mu)dt$. $\gamma(\cdot)$ represents the disturbance intensity of Lévy jump noise to the population.

The model established in this paper is an improvement on the classical Schoener's competition model. First, system (1.5) uses the mean-reverting Ornstein-Uhlenbeck process to simulate small perturbations in the environment, which is more reasonable than assuming that population parameters are linearly distributed in Gaussian white noise. In addition, considering that species in the real world are often affected by sudden random perturbations, we also introduced Lévy jumps into the model. Therefore, this model takes into account both small environmental disturbances and sudden violent disturbances, which can better simulate some random phenomena in nature.

The rest of the paper is organized as follows. Section 2 gives some necessary lemmas and assumptions. Section 3 shows several dynamical properties of system (1.5), including the existence and uniqueness of global solutions, moment estimate, and stochastic ultimate boundness. In addition, we obtain sufficient conditions for species extinction. In Section 4, we carry out some numerical simulations to verify the theoretical results. Finally, we give some conclusions in Section 5.

Remark 1. The model studied in this paper perturbs the parameter r_i , where r_i represents the death rate of the population. Due to the characteristics of the Ornstein-Uhlenbeck process, the death rate can be taken as any real number in the model. However, it is well known that negative death rates are biologically implausible. For this phenomenon, we have the following explanation:

First, the positive or negative death rate does not affect the proof of the theorem, so even if the death rate may be negative, it does not affect the results obtained in this paper. Moreover, since the theorem is true for any value of death rate, it is also true for non-negative death rate. It is worth noting that the negative death rate is only a theoretical value, and we still assume that it is positive in practice and numerical simulation, that is, we mainly use the part of $r_i \geq 0, i = 1, 2$.

2. Materials and methods

Due to the need of subsequent proofs, several lemmas and assumptions will be given in this section.

Lemma 2.1. (Chebyshev inequality) If $c > 0$, $p > 0$, $X \in L^p$, we have

$$P\{\omega : |X(\omega)| \geq c|\} \leq c^{-p} E[|X|^p],$$

where L^p represents the set of random variables that take values on R^n and $E[|X|^p] < \infty$.

Lemma 2.2. Let $M(t)$ be a local martingale with initial value $M(0) = 0$. If $\lim_{t\to\infty} \rho_M(t) < \infty$, then

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0,$$

where

$$\rho_M(t) = \int_0^t \frac{d\langle M, M\rangle(s)}{(1+s)^2},\ t \geq 0.$$

Assumption 2.1. For any $i \in \{1, 2\}$, there is always a constant c that makes the following true:

$$\begin{aligned}
&(1)\ \int_Y \left(|\ln(1+\gamma_i(\mu))| \vee |\ln(1+\gamma_i(\mu))|^2\right)\lambda(\mu) < c. \\
&(2)\ \int_Y |\gamma_i(\mu)|\lambda(d\mu) < c. \\
&(3)\ \int_Y |(1+\gamma_i(\mu))^q - 1|\lambda(d\mu) < c.
\end{aligned} \tag{2.1}$$

3. Results

3.1. Existence and uniqueness of global solution

Theorem 3.1. For any initial value condition

$$(x(0), y(0), r_1(0), r_2(0)) \in R_+^2 \times R^2,$$

system (1.5) has a unique solution $(x(t), y(t), r_1(t), r_2(t))$ on $t \geq 0$, and it will remain in $R_+^2 \times R^2$ with probability one.

Proof. For $t \geq 0$, and for any initial value $(x(0), y(0), r_1(0), r_2(0)) \in R_+^2 \times R^2$, it is easy to prove that the equation coefficients in system (1.5) satisfy local Lipschitz conditions. Therefore, system (1.5) has a unique local solution $(x(t), y(t), r_1(t), r_2(t)) \in R_+^2 \times R^2$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [12].

To prove that the model has a global positive solution, we only need to show that $\tau_e = \infty$ a.s.. By defining a necessary set $\mathbb{H}_{n_0} = (-n_0, n_0) \times (-n_0, n_0) \times (-n_0, n_0) \times (-n_0, n_0)$, we can always determine a sufficiently large integer n_0 such that $(\ln x(0), \ln y(0), r_1(0), r_2(0)) \in \mathbb{H}_{n_0}$. For any integer $n \geq n_0$, we define a stopping time set τ_n by

$$\begin{aligned}
\tau_n = inf\{t \in [0, \tau_e) \,|\, \ln x(t) \notin (-n, n), or\ \ln y(t) \notin (-n, n), or\ r_1(t) \notin \\
(-n, n), or\ r_2(t) \notin (-n, n)\}.
\end{aligned} \tag{3.1}$$

Obviously, τ_n is monotonically increasing as n increases. For convenience, let $\tau_\infty = \lim_{n\to\infty} \tau_n$ and $inf\varnothing = \infty$, which implies $\tau_\infty \leq \tau_e$ a.s.. To prove Theorem 3.1, it suffices to verify $\tau_\infty = \infty$ a.s.. Consider the contradiction, i.e., $\tau_\infty < \infty$ a.s.. Then there are constants $T > 0$ and $\varepsilon \in (0, 1)$ to make $P\{\tau_\infty \leq T\} > \varepsilon$. Hence, there is a positive number $n_1 \geq n_0$ such that

$$P\{\tau_n \leq T\} \geq \varepsilon, \forall n \geq n_1. \tag{3.2}$$

For any $t \leq \tau_n$, by the inequality $x - 1 \geq \ln x \ (x > 0)$, a non-negative C^2-function $V(x(t), y(t), r_1(t), r_2(t))$ is constructed as follows:

$$V(x, y, r_1, r_2) = x(t) - 1 - \ln x(t) + y(t) - 1 - \ln y(t) + \frac{r_1^4(t)}{4} + \frac{r_2^4(t)}{4}. \tag{3.3}$$

Applying the *Itô* formula yields,

$$\begin{aligned} dV =& \mathcal{L}V dt + r_1^3(t)\sigma_1 dB_1(t) + r_2^3(t)\sigma_2 dB_2(t) \\ &+ \int_Y \Big(x(t)\gamma_1(\mu) - \ln(1+\gamma_1(\mu))\Big)\widetilde{N}(dt, d\mu) \\ &+ \int_Y \Big(y(t)\gamma_2(\mu) - \ln(1+\gamma_2(\mu))\Big)\widetilde{N}(dt, d\mu), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \mathcal{L}V =& - x(t) r_1(t) - \beta_1 x^2(t) - \alpha_1 x(t) y(t) + \frac{\xi_1 x(t)}{x(t)+\delta_1} + r_1(t) + \beta_1 x(t) \\ &+ \alpha_1 y(t) - \frac{\xi_1}{x(t)+\delta_1} - y(t) r_2(t) - \beta_2 y(t) x(t) - \alpha_2 y^2(t) + \frac{\xi_2 y(t)}{y(t)+\delta_2} \\ &+ r_2(t) + \beta_2 x(t) + \alpha_2 y(t) - \frac{\xi_2}{y(t)+\delta_2} + \omega_1 r_1^3(t)(\bar{r}_1 - r_1(t)) \\ &+ \omega_2 r_2^3(t)(\bar{r}_2 - r_2(t)) + \frac{3}{2}\sigma_1^2 r_1^2(t) + \frac{3}{2}\sigma_2^2 r_2^2(t) \\ &+ \int_Y (x(t)\gamma_1(\mu) - \ln(1+\gamma_1(\mu)))\lambda(d\mu) \\ &+ \int_Y (y(t)\gamma_2(\mu) - \ln(1+\gamma_2(\mu)))\lambda(d\mu). \end{aligned} \tag{3.5}$$

Combined with Assumption 2.1, it is easy to know that there exists an upper bound K such that

$$\begin{aligned} \mathcal{L}V \leq& |r_1(t)|(1 + x(t)) + \frac{\xi_1 x(t)}{x(t)+\delta_1} + (\beta_1+\beta_2) x(t) + |r_2(t)|(1 + y(t)) \\ &+ \frac{\xi_2 y(t)}{y(t)+\delta_2} + (\alpha_1+\alpha_2) y(t) + \omega_1 \bar{r}_1 r_1^3(t) + \omega_2 \bar{r}_2 r_2^3(t) + \frac{3}{2}\sigma_1^2 r_1^2(t) \\ &+ \frac{3}{2}\sigma_2^2 r_2^2(t) - \beta_1 x^2(t) - \alpha_2 y^2(t) - (\alpha_1+\beta_2) x(t) y(t) + x(t)\int_Y \gamma_1(\mu)\lambda(d\mu) \\ &+ y(t)\int_Y \gamma_2(\mu)\lambda(d\mu) - \frac{\xi_1}{x(t)+\delta_1} - \frac{\xi_2}{y(t)+\delta_2} - \omega_1 r_1^4(t) - \omega_2 r_2^4(t) + 2c \\ \leq& K, \end{aligned}$$

thus

$$\begin{aligned} dV \leq& K dt + r_1^3(t)\sigma_1 dB_1(t) + r_2^3(t)\sigma_2 dB_2(t) \\ &+ \int_Y \Big(x(t)\gamma_1(\mu) - \ln(1+\gamma_1(\mu))\Big)\widetilde{N}(dt, d\mu) \\ &+ \int_Y \Big(y(t)\gamma_2(\mu) - \ln(1+\gamma_2(\mu))\Big)\widetilde{N}(dt, d\mu). \end{aligned} \tag{3.6}$$

Integrating the inequality from 0 to $\tau_n \wedge T$ and then taking the expectation on both sides of inequality (3.7), we have

$$E\left[V\left(x\left(\tau_n \wedge T\right), y\left(\tau_n \wedge T\right), r_1\left(\tau_n \wedge T\right), r_2\left(\tau_n \wedge T\right)\right)\right] \le V\left(x(0), y(0), r_1(0), r_2(0)\right) + KT. \tag{3.7}$$

For all $n_1 \ge n_0$, let $\Omega_n = \{\tau_n \le T\}$. Then we have $P\left(\Omega_n\right) \ge \varepsilon$. Note that, for any $\omega \in \Omega_n$, x, y, r_1, and r_2 equals either $-n$ or n, so there is

$$V\left(x\left(\tau_n, \omega\right), y\left(\tau_n, \omega\right), r_1\left(\tau_n, \omega\right), r_2\left(\tau_n, \omega\right)\right) \ge \min\left\{e^{-n} - 1 + n, e^{n} - 1 - n\right\}.$$

According to the inequality 3.7, it can be derived that

$$\begin{aligned} V\left(x(0), y(0), r_1(0), r_2(0)\right) + KT &\ge E\left[1_{\Omega_n(\omega)} V\left(x\left(\tau_n\right), y\left(\tau_n\right), r_1\left(\tau_n\right), r_2\left(\tau_n\right)\right)\right] \\ &\ge \varepsilon \min\left\{e^{-n} - 1 + n, e^{n} - 1 - n\right\}, \end{aligned} \tag{3.8}$$

where $1_{\Omega_n(\omega)}$ represents the index function. As $n \to \infty$, we have $\infty > V\left(x(0), y(0), r_1(0), r_2(0)\right) + KT \ge \infty$, which leads to a contradiction. Therefore, we have $\tau_\infty = \infty$ a.s.. This completes the proof of Theorem 3.1.

3.2. Moment estimation

In this section, we will provide a moment estimate for the global solution mentioned above.

Theorem 3.2. For any initial value $\left(x(0), y(0), r_1(0), r_2(0)\right) \in R_+^2 \times R^2$, the solution $\left(x(t), y(t), r_1(t), r_2(t)\right)$ of system (1.5) has the property

$$E\left[\left|x(t), y(t)\right|^q\right] \le K(q) \tag{3.9}$$

for any $q > 0$, where $K(q)$ is a continuous function with respect to q. That is to say, the *qth* moment of the solution $x(t), y(t)$ is bounded.

Proof. For any $q \ge 2$, we define a non-negative C^2-function

$$V_1\left(x(t), y(t), r_1(t), r_2(t)\right) : R_+^2 \times R^2$$

by

$$V_1\left(x(t), y(t), r_1(t), r_2(t)\right) = \frac{x^q(t)}{q} + \frac{y^q(t)}{q} + \frac{r_1^{2q}(t)}{2q} + \frac{r_2^{2q}(t)}{2q}.$$

Applying the generalized *Itô* formula, we obtain

$$\begin{aligned} dV_1 =& \mathcal{L}V_1 dt + r_1^{2q-1}(t)\sigma_1 dB_1(t) + r_2^{2q-1}(t)\sigma_2 dB_2(t) \\ &+ \frac{x^q(t)}{q}\int_Y \left(\left(1 + \gamma_1(\mu)\right)^q - 1\right)\widetilde{N}(dt, d\mu) \\ &+ \frac{y^q(t)}{q}\int_Y \left(\left(1 + \gamma_2(\mu)\right)^q - 1\right)\widetilde{N}(dt, d\mu). \end{aligned} \tag{3.10}$$

To simplify the notation, the subsequent proof process replaces $x(t)$, $y(t)$, $r_i(t)$, and $\gamma_i(\mu)$ with x, y, r_i, and γ_i, respectively. So, we have

$$\begin{aligned}\mathcal{L}V_1 =& - x^q r_1 - \beta_1 x^{q+1} - \alpha_1 x^q y + \frac{\xi_1 x^q}{x+\delta_1} - y^q r_2 - \beta_2 x y^q - \alpha_2 y^{q+1} \\ &+ \frac{\xi_2 y^q}{y+\delta_2} + \omega_1 r_1^{2q-1}(\bar{r}_1 - r_1) + \omega_2 r_2^{2q-1}(\bar{r}_2 - r_2) + \frac{2q-1}{2}\sigma_1^2 r_1^{2q-2} \\ &+ \frac{2q-1}{2}\sigma_2^2 r_2^{2q-2} + \frac{x^q}{q}\int_Y ((1+\gamma_1)^q - 1)\lambda(d\mu) \\ &+ \frac{y^q}{q}\int_Y ((1+\gamma_2)^q - 1)\lambda(d\mu).\end{aligned} \tag{3.11}$$

Taking the mathematical expectation of $e^{\eta t}V_1$, we obtain

$$\begin{aligned}E\left(e^{\eta t}V_1(x(t), y(t), r_1(t), r_2(t))\right) =& E\left(V_1(x(0), y(0), r_1(0), r_2(0))\right) \\ &+ \int_0^t E\left[\mathcal{L}\left(e^{\eta s}V_1(x(s), y(s), r_1(s), r_2(s))\right)\right] ds,\end{aligned} \tag{3.12}$$

where $\eta = q\min\{\omega_1, \omega_2\}$. Noting that

$$\begin{aligned}&\mathcal{L}\left[e^{\eta t}V_1(x, y, r_1, r_2)\right] \\ =&\eta e^{\eta t}V_1(x, y, r_1, r_2) + e^{\eta t}\mathcal{L}V_1(x, y, r_1, r_2) \\ =&e^{\eta t}\Big\{\frac{\eta}{q}x^q + \frac{\eta}{q}y^q - x^q r_1 - \beta_1 x^{q+1} - \alpha_1 x^q y + \frac{\xi_1 x^q}{x+\delta_1} - y^q r_2 - \beta_2 x y^q - \alpha_2 y^{q+1} \\ &+ \frac{\xi_2 y^q}{y+\delta_2} - \left(\omega_1 - \frac{\eta}{2q}\right) r_1^{2q} - \left(\omega_2 - \frac{\eta}{2q}\right) r_2^{2q} + \omega_1 r_1^{2q-1}\bar{r}_1 + \omega_2 r_2^{2q-1}\bar{r}_2 \\ &+ \frac{2q-1}{2}\sigma_1^2 r_1^{2q-2} + \frac{2q-1}{2}\sigma_2^2 r_2^{2q-2} + \frac{x^q}{q}\int_Y ((1+\gamma_1)^q - 1)\lambda(d\mu) \\ &+ \frac{y^q}{q}\int_Y ((1+\gamma_2)^q - 1)\lambda(d\mu)\Big\} \\ \leq& e^{\eta t}\Big\{\omega_1 x^q + \omega_2 y^q + |r_1|x^q + |r_2|y^q + \frac{\xi_1 x^q}{x+\delta_1} + \frac{\xi_2 y^q}{y+\delta_2} + \omega_1 r_1^{2q-1}\bar{r}_1 + \omega_2 r_2^{2q-1}\bar{r}_2 \\ &+ \frac{2q-1}{2}\sigma_1^2 r_1^{2q-2} + \frac{2q-1}{2}\sigma_2^2 r_2^{2q-2} - \beta_1 x^{q+1} - \alpha_2 y^{q+1} - \alpha_1 x^q y - \beta_2 x y^q \\ &- \frac{\omega_1}{2} r_1^{2q} - \frac{\omega_2}{2} r_2^{2q} + \frac{x^q}{q}\int_Y ((1+\gamma_1)^q - 1)\lambda(d\mu) + \frac{y^q}{q}\int_Y ((1+\gamma_2)^q - 1)\lambda(d\mu)\Big\} \\ \leq& k(q)\, e^{\eta t},\end{aligned} \tag{3.13}$$

where

$$k(q) = \sup_{(x,y,r_1,r_2)\in R_+^2\times R^2}\Big\{\omega_1 x^q + \omega_2 y^q + |r_1|x^q + |r_2|y^q + \frac{\xi_1 x^q}{x+\delta_1} + \frac{\xi_2 y^q}{y+\delta_2}$$
$$+ \omega_1 r_1^{2q-1}\bar{r}_1 + \omega_2 r_2^{2q-1}\bar{r}_2 + \frac{2q-1}{2}\sigma_1^2 r_1^{2q-2} + \frac{2q-1}{2}\sigma_2^2 r_2^{2q-2} - \beta_1 x^{q+1}$$
$$- \alpha_2 y^{q+1} - \alpha_1 x^q y - \beta_2 x y^q - \frac{\omega_1}{2} r_1^{2q} - \frac{\omega_2}{2} r_2^{2q}$$
$$+ \frac{x^q}{q}\int_Y \left((1+\gamma_1)^q - 1\right)\lambda(d\mu) + \frac{y^q}{q}\int_Y \left((1+\gamma_2)^q - 1\right)\lambda(d\mu)\Big\}$$
$$<\infty.$$

Combining formula (3.12) and formula (3.13), we obtain

$$\begin{aligned} &E\left(e^{\eta t}V_1\left(x(t), y(t), r_1(t), r_2(t)\right)\right) \\ &\le E\left(V_1\left(x(0), y(0), r_1(0), r_2(0)\right)\right) + \frac{k(q)\left(e^{\eta t}-1\right)}{\eta}, \end{aligned} \tag{3.14}$$

and then we have

$$\begin{aligned} \limsup_{t\to\infty} E\left[|x(t), y(t)|^q\right] &\le q \limsup_{t\to\infty} E\left[V_1\left(x(t), y(t), r_1(t), r_2(t)\right)\right] \\ &\le \lim_{t\to\infty}\frac{E\left[V_1\left(x(0), y(0), r_1(0), r_2(0)\right)\right]}{e^{\eta t}} + \lim_{t\to\infty}\frac{k(q)\left(e^{\eta t}-1\right)}{\eta e^{\eta t}} \\ &= 0 + \frac{k(q)}{\eta} = \frac{k(q)}{\eta} a.s.. \end{aligned} \tag{3.15}$$

The continuity of $x(t), y(t)$ on $t \in [0,\infty)$ along with this implies that there exists a constant $K(q)$ such that

$$E\left[|x(t), y(t)|^q\right] \le K(q), \quad \forall t \ge 0, q \ge 2.$$

By applying Hölder's inequality, we can conclude that, for any $q \in (0,2)$,

$$E\left(|x(t), y(t)|^q\right) \le \left(E\left(|x(t), y(t)|\right)^2\right)^{\frac{q}{2}} \le \left(K(2)\right)^{\frac{q}{2}}.$$

This completes the proof of Theorem 3.2.

3.3. Stochastic ultimate boundedness

First, the definition of stochastic ultimate boundedness is given.

Definition 3.1. The solution of system (1.5) is said to be stochastic and ultimately bounded, if for any $\varepsilon \in (0,1)$, there is a positive number $H(=H(\varepsilon))$ such that for any initial value $(x(0), y(0), r_1(0), r_2(0)) \in R_+^2 \times R^2$, the solution of the model satisfies

$$\limsup_{t\to\infty} P\left\{\sqrt{x^2(s) + y^2(s)} \le H\right\} \ge 1 - \varepsilon.$$

Theorem 3.3. The solutions of system (1.5) are stochastic and ultimately bounded.

Proof. According to Theorem 3.2, the solution of system (1.5) is moment bounded. Let $q = 0.5$, then there is a positive number K_0 such that for any initial value $(x(0), y(0), r_1(0), r_2(0)) \in R_+^2 \times R^2$, the solution of system (1.5) satisfies

$$\limsup_{t\to\infty} E\left(x^2(s) + y^2(s)\right)^{\frac{1}{2}} \le K_0.$$

According to the Lemma 2.1: $P\{|x(t)| > H\} \le \frac{E|x(t)|}{H}$, by setting $H = K_0/\varepsilon$, we can get

$$\limsup_{t\to\infty} P\left\{\sqrt{x^2(s) + y^2(s)} > H\right\} \le \varepsilon,$$

then

$$\limsup_{t\to\infty} P\left\{\sqrt{x^2(s) + y^2(s)} \le H\right\} \ge 1 - \varepsilon.$$

This completes the proof of Theorem 3.3.

3.4. Population extinction

Theorem 3.4. For any initial value $(x(0), y(0), r_1(0), r_2(0)) \in R_+^2 \times R^2$, the solution $(x(t), y(t), r_1(t), r_2(t))$ of system (1.5) has the property that

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(-r_1 + \frac{\xi_1}{\delta_1} + \int_Y \ln(1+\gamma_1)\lambda(d\mu)\right) ds,$$

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(-r_2 + \frac{\xi_2}{\delta_2} + \int_Y \ln(1+\gamma_2)\lambda(d\mu)\right) ds.$$

In particular, if

$$G_1 = \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(-r_1 + \frac{\xi_1}{\delta_1} + \int_Y \ln(1+\gamma_1)\lambda(d\mu)\right) ds < 0,$$

$$G_2 = \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(-r_2 + \frac{\xi_2}{\delta_2} + \int_Y \ln(1+\gamma_2)\lambda(d\mu)\right) ds < 0,$$

then $x(t)$, $y(t)$ are extinct.

Proof. Applying the *Itô* formula to $\ln x(t)$ and $\ln y(t)$, it can be obtained that

$$\begin{aligned} d\ln x(t) = &\left(-r_1 - \beta_1 x - \alpha_1 y + \frac{\xi_1}{x+\delta_1} + \int_Y \ln(1+\gamma_1)\lambda(d\mu)\right) dt \\ &+ \int_Y \ln(1+\gamma_1)\tilde{N}(dt, d\mu), \end{aligned} \tag{3.16}$$

$$\begin{aligned} d\ln y(t) = &\left(-r_2 - \beta_2 x - \alpha_2 y + \frac{\xi_2}{y+\delta_2} + \int_Y \ln(1+\gamma_2)\lambda(d\mu)\right) dt \\ &+ \int_Y \ln(1+\gamma_2)\tilde{N}(dt, d\mu). \end{aligned} \tag{3.17}$$

Integrating from 0 to t, we have

$$\begin{aligned}\ln x(t) = \ln x(0) + \int_0^t \left(-r_1 - \beta_1 x - \alpha_1 y + \frac{\xi_1}{x+\delta_1} + \int_Y \ln(1+\gamma_1)\lambda(d\mu)\right)ds \\ + \int_0^t \int_Y \ln(1+\gamma_1)\tilde{N}(ds,d\mu),\end{aligned} \tag{3.18}$$

$$\begin{aligned}\ln y(t) = \ln y(0) + \int_0^t \left(-r_2 - \beta_2 x - \alpha_2 y + \frac{\xi_2}{y+\delta_2} + \int_Y \ln(1+\gamma_2)\lambda(d\mu)\right)ds \\ + \int_0^t \int_Y \ln(1+\gamma_2)\tilde{N}(ds,d\mu).\end{aligned} \tag{3.19}$$

According to equalities (3.18) and (3.19) , it can be obtained that

$$\ln x(t) \le \ln x(0) + \int_0^t \left(-r_1 + \frac{\xi_1}{\delta_1} + \int_Y \ln(1+\gamma_1)\lambda(d\mu)\right)ds + \int_0^t \int_Y \ln(1+\gamma_1)\tilde{N}(ds,d\mu), \tag{3.20}$$

$$\ln y(t) \le \ln y(0) + \int_0^t \left(-r_2 + \frac{\xi_2}{\delta_2} + \int_Y \ln(1+\gamma_2)\lambda(d\mu)\right)ds + \int_0^t \int_Y \ln(1+\gamma_2)\tilde{N}(ds,d\mu). \tag{3.21}$$

Let $\tilde{M}_1 = \int_0^t \int_Y \ln(1+\gamma_1)\tilde{N}(dt,d\mu)$, $\tilde{M}_2 = \int_0^t \int_Y \ln(1+\gamma_2)\tilde{N}(dt,d\mu)$. According to Lemma 2.2, we have

$$\lim_{t\to\infty}\frac{\tilde{M}_1(t)}{t} = 0, \lim_{t\to\infty}\frac{\tilde{M}_2(t)}{t} = 0.$$

Then,

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t} \le \limsup_{t\to\infty}\frac{1}{t}\int_0^t \left(-r_1 + \frac{\xi_1}{\delta_1} + \int_Y \ln(1+\gamma_1)\lambda(d\mu)\right)ds, \tag{3.22}$$

$$\limsup_{t\to\infty}\frac{\ln y(t)}{t} \le \limsup_{t\to\infty}\frac{1}{t}\int_0^t \left(-r_2 + \frac{\xi_2}{\delta_2} + \int_Y \ln(1+\gamma_2)\lambda(d\mu)\right)ds. \tag{3.23}$$

Therefore, when

$$G_1 = \limsup_{t\to\infty}\frac{1}{t}\int_0^t \left(-r_1 + \frac{\xi_1}{\delta_1} + \int_Y \ln(1+\gamma_1)\lambda(d\mu)\right)ds < 0,$$

$$G_2 = \limsup_{t\to\infty}\frac{1}{t}\int_0^t \left(-r_2 + \frac{\xi_2}{\delta_2} + \int_Y \ln(1+\gamma_2)\lambda(d\mu)\right)ds < 0,$$

it implies $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} y(t) = 0$, and then $x(t)$, $y(t)$ are extinct. This completes the proof of Theorem 3.4.

4. Discussion

In this section, we will verify the theoretical results by numerical simulation examples. First, the system (1.5) is discretized using the Milstein scheme for higher order discretization [25]. Consider the following stochastic Schoener's competitive model for two populations:

$$\begin{cases} dx(t) = x(t^-)\left[-r_1(t^-) - \beta_1 x(t^-) - \alpha_1 y(t^-) + \frac{\xi_1}{x(t^-)+\delta_1}\right]dt + \int_Y x(t^-)\gamma_1(\mu)N(dt,d\mu), \\ dy(t) = y(t^-)\left[-r_2(t^-) - \beta_2 x(t^-) - \alpha_2 y(t^-) + \frac{\xi_2}{y(t^-)+\delta_2}\right]dt + \int_Y y(t^-)\gamma_2(\mu)N(dt,d\mu), \\ dr_1(t) = \omega_1[\bar{r}_1 - r_1(t)]dt + \sigma_1 dB_1(t), \\ dr_2(t) = \omega_2[\bar{r}_2 - r_2(t)]dt + \sigma_2 dB_2(t). \end{cases} \tag{4.1}$$

Immediately afterwards, it is necessary to introduce the biological significance of the parameters related to the process being modeled. Computer simulations can then be performed to gain insights into the dynamics of the biological system (see Table 1).

Table 1. List of biological parameters.

Parameter	Explication
$\bar{r}_1$	Average mortality of species x
$\bar{r}_2$	Average mortality of species y
β_1	The intra-specific competition rates of species x
β_2	The inter-specific competition rates
α_1	The inter-specific competition rates
α_2	The intra-specific competition rates of species y
σ_1	The intensity of volatility of r_1
σ_2	The intensity of volatility of r_2

Based on the biological significance of the above parameters, and in combination with reference [8–10, 26], we choose the reasonable values as shown in Table 2.

Table 2. Several combinations of biological parameters of system (1.5) in Table 1.

Combinations	Value
$(\mathcal{A}_1)$ [8]	$\bar{r}_1 = 0.15, \bar{r}_2 = 0.2, \beta_1 = 0.4, \beta_2 = 0.1, \alpha_1 = 0.1, \alpha_2 = 0.1, \xi_1 = 0.3, \xi_2 = 0.2, \delta_1 = 0.6$
	$\delta_2 = 0.1, \gamma_1 = -0.5, \gamma_2 = -0.5, \omega_1 = 0.1, \omega_2 = 0.1, \sigma_1 = 0.04, \sigma_2 = 0.04$
$(\mathcal{A}_2)$ [9]	$\bar{r}_1 = 0.4, \bar{r}_2 = 0.2, \beta_1 = 0.3, \beta_2 = 0.4, \alpha_1 = 0.4, \alpha_2 = 0.2, \xi_1 = 0.65, \xi_2 = 0.5, \delta_1 = 0.65$
	$\delta_2 = 0.55, \gamma_1 = 0.2, \gamma_2 = 0.2, \omega_1 = 0.1, \omega_2 = 0.1, \sigma_1 = 0.05, \sigma_2 = 0.05$
$(\mathcal{A}_3)$ [10]	$\bar{r}_1 = 0.1, \bar{r}_2 = 0.3, \beta_1 = 0.3, \beta_2 = 0.0001, \alpha_1 = 0.0001, \alpha_2 = 0.3, \xi_1 = 1, \xi_2 = 1, \delta_1 = 2$
	$\delta_2 = 2, \gamma_1 = -0.2, \gamma_2 = -0.2, \omega_1 = 0.1, \omega_2 = 0.1, \sigma_1 = 0.05, \sigma_2 = 0.05$
$(\mathcal{A}_4)$	$\bar{r}_1 = 0.35, \bar{r}_2 = 0.15, \beta_1 = 0.3, \beta_2 = 0.15, \alpha_1 = 0.15, \alpha_2 = 0.2, \xi_1 = 0.2, \xi_2 = 0.1, \delta_1 = 0.55$
	$\delta_2 = 0.2, \gamma_1 = -0.3, \gamma_2 = -0.25, \omega_1 = 0.1, \omega_2 = 0.1, \sigma_1 = 0.05, \sigma_2 = 0.05$
$(\mathcal{A}_5)$	$\bar{r}_1 = 0.2, \bar{r}_2 = 0.4, \beta_1 = 0.1, \beta_2 = 0.2, \alpha_1 = 0.2, \alpha_2 = 0.3, \xi_1 = 0.2, \xi_2 = 0.37, \delta_1 = 0.3$
	$\delta_2 = 0.75, \gamma_1 = -0.2, \gamma_2 = -0.4, \omega_1 = 0.1, \omega_2 = 0.1, \sigma_1 = 0.05, \sigma_2 = 0.05$
$(\mathcal{A}_6)$ [26]	$\bar{r}_1 = 0.37, \bar{r}_2 = 0.37, \beta_1 = 1, \beta_2 = 0.1, \alpha_1 = 0.27, \alpha_2 = 2, \xi_1 = 0.2, \xi_2 = 0.1, \delta_1 = 0.55$
	$\delta_2 = 0.5, \gamma_1 = -0.3, \gamma_2 = -0.3, \omega_1 = 0.2, \omega_2 = 0.1, \sigma_1 = 0.001, \sigma_2 = 0.001$

Example 1. Theorem 3.1 is first tested by selecting the combination $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, and $(\mathcal{A}_3)$ in Table 2 as the coefficients of system (1.5). See Figure 1, Figure 2, and Figure 3.

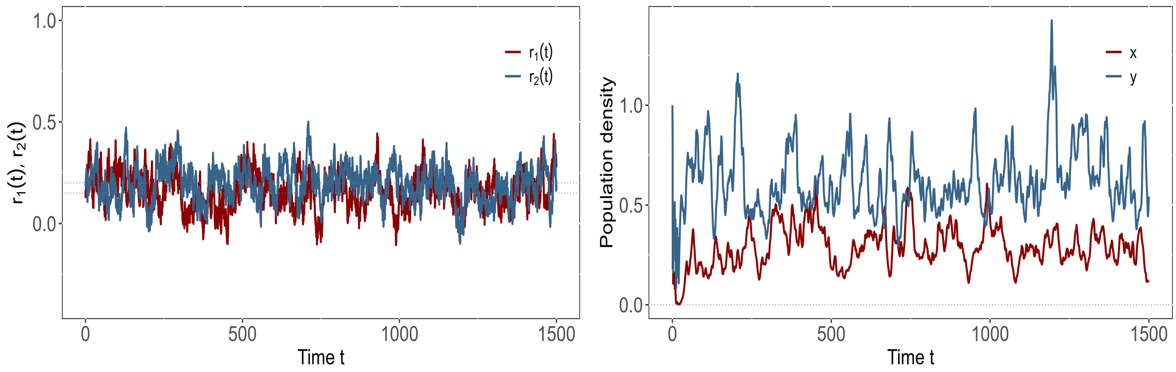


Figure 1. Coefficient combination ($\mathcal{A}_1$).

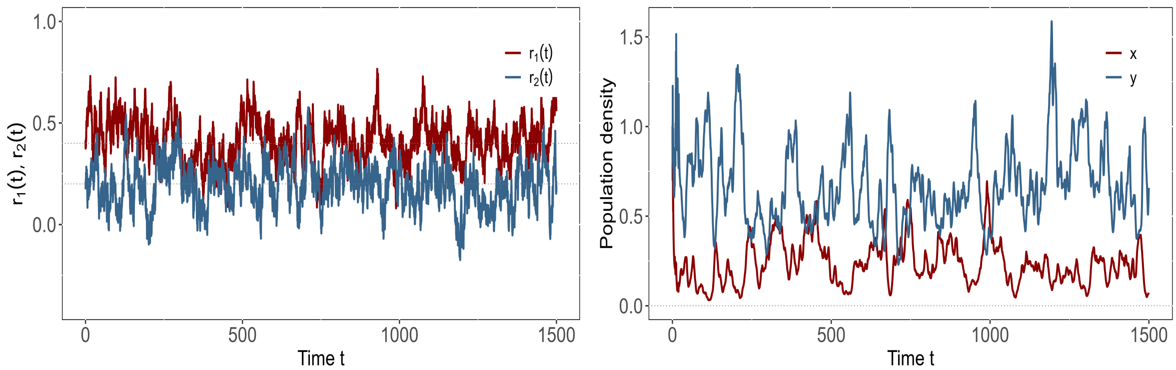


Figure 2. Coefficient combination ($\mathcal{A}_2$).

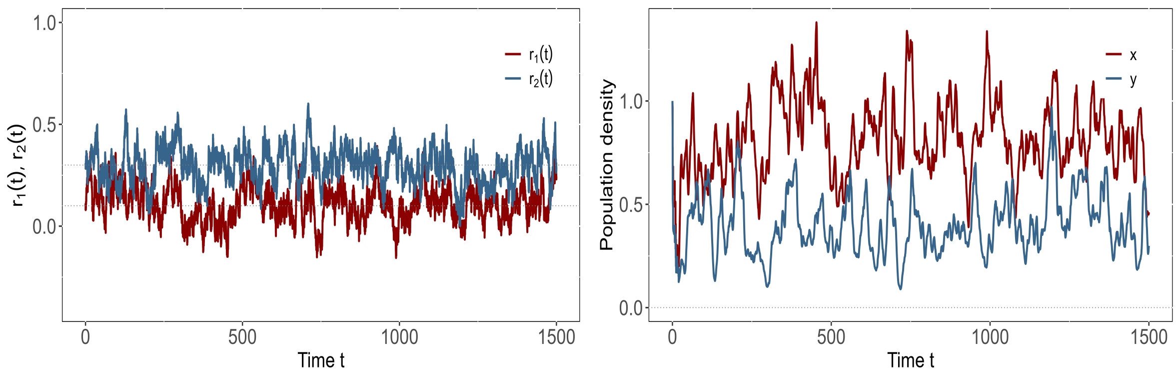
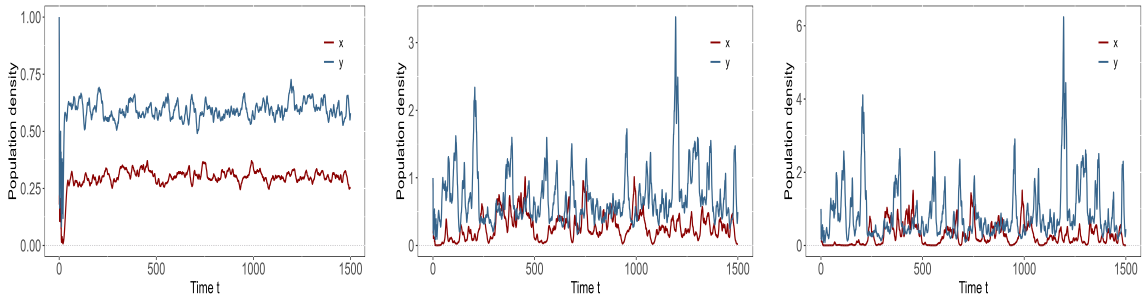


Figure 3. Coefficient combination ($\mathcal{A}_3$).

Remark 2. It can be seen that the death rate r_i is perturbed around the given mean value, which reflects the mean reversion characteristic of the Ornstein-Uhlenbeck process.

Comparing Figure 2 with Figure 1 and Figure 3, one can conclude that the death rate is not a determining factor in population density, and other factors such as competition intensity and environmental disturbance also have a great impact on the survival of the population.

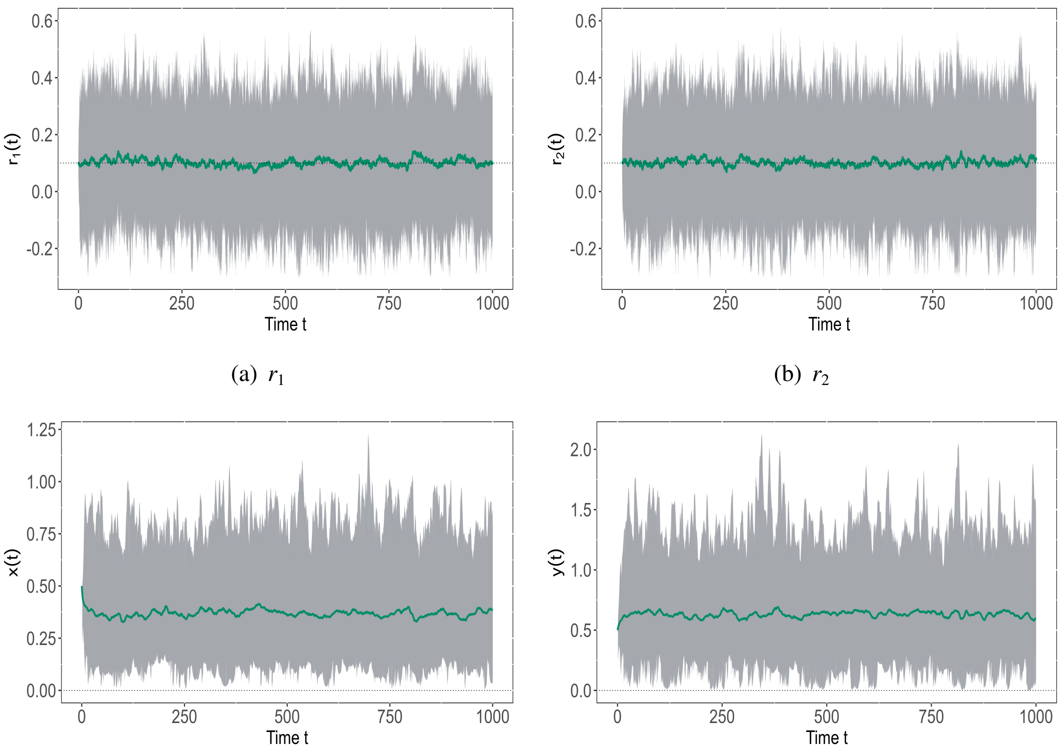
On the basis of Theorem 3.1, we next investigate the effect of environmental perturbations on the population by choosing the parameter combination ($\mathcal{A}_1$) and varying the value of the parameter $\sigma_i, i = 1, 2$, in it. Let σ_i =0.01, 0.08, 0.12, respectively. According to Figure 4, the population becomes more unstable as the environmental noise increases, which implies that the environmental perturbation has a great impact on the survival of the population.



(a) σ_i=0.01 (b) σ_i=0.08 (c) σ_i=0.12

Figure 4. The effect of different σ_i values on the populations.

In order to verify the existence and uniqueness of the solution of system (1.5) more comprehensively and clearly, we perform 100 simulations of Theorem 3.1. The solid green lines in (a), (b), (c), and (d) of Figure 5 represent the average of the 100 simulated paths, and all 100 paths are represented by gray lines. It can be seen that different coefficient combinations have different solutions, and the solutions are all existing and unique. Thus, the conclusion of Theorem 3.1 can be verified.

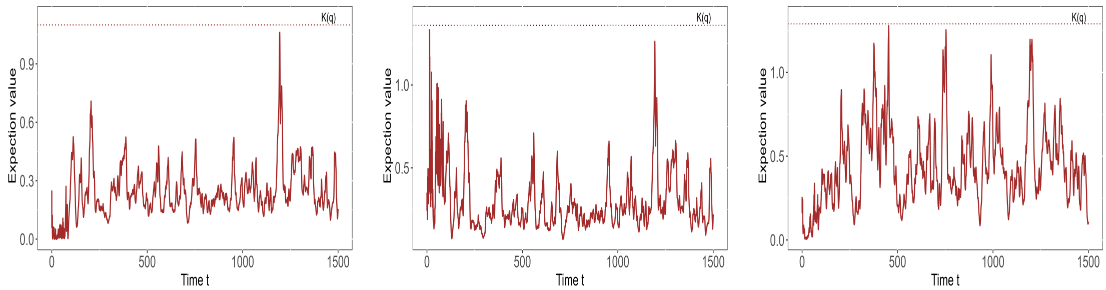


(a) r_1 (b) r_2

(c) x (d) y

Figure 5. 100 path simulation figures.

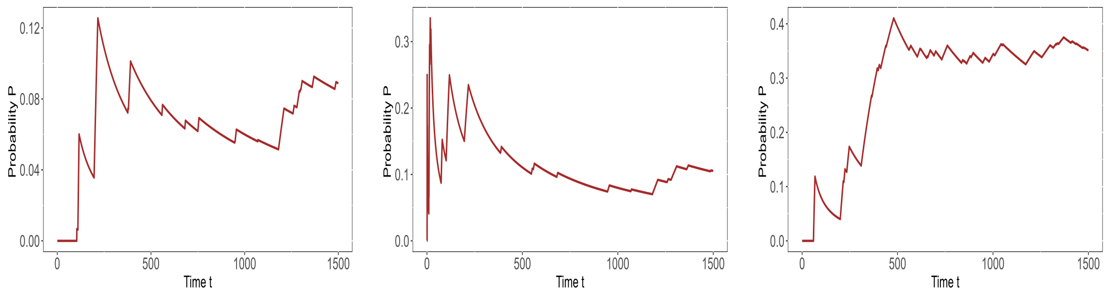
Example 2. Now we will verify the conclusion of **Theorem 3.2**. We still choose the combination $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, and $(\mathcal{A}_3)$ in Table 2 as the coefficients of system (1.5). Let q=2, then we have $E\left[|x(t), y(t)|^2\right] \leq K(2)$. It can be seen from Figure 6 that the expected value of the above three coefficient combinations is less than an upper bound $K(q)$ and this upper limit is not infinite, which indicates that the two-moment of the population is bounded. See Figure 6.



(a) Coefficient combination $(\mathcal{A}_1)$ (b) Coefficient combination $(\mathcal{A}_2)$ (c) Coefficient combination $(\mathcal{A}_3)$

Figure 6. Moment estimation.

Example 3. We will now verify the conclusion of **Theorem 3.3** by numerical simulation. By Table 2, We still use combination $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ for verification. See Figure 7



(a) Coefficient combination $(\mathcal{A}_1)$ (b) Coefficient combination $(\mathcal{A}_2)$ (c) Coefficient combination $(\mathcal{A}_3)$

Figure 7. Stochastic ultimate boundedness.

Remark 3. From the biological point of view, since the environmental resources are limited, no biological population can grow indefinitely, so we hope that the system solution is ultimately bounded. It can be seen from Figure 7 that, with the increase of time t, the probability $P\left\{\sqrt{x^2(s) + y^2(s)} \leq H\right\}$ is gradually stable and greater than a constant, which means $\limsup_{t\to\infty} P\left\{\sqrt{x^2(s) + y^2(s)} \leq H\right\} \geq 1 - \varepsilon$. Therefore, the above simulation verifies the stochastic ultimate boundedness.

In order to further verify the conclusions of Theorem 3.2 and Theorem 3.3, we choose 100 paths for simulation as in Theorem 3.1. The left figure is the simulation result of Theorem 3.2, and the right figure is the simulation result of Theorem 3.3. See Figure 8.

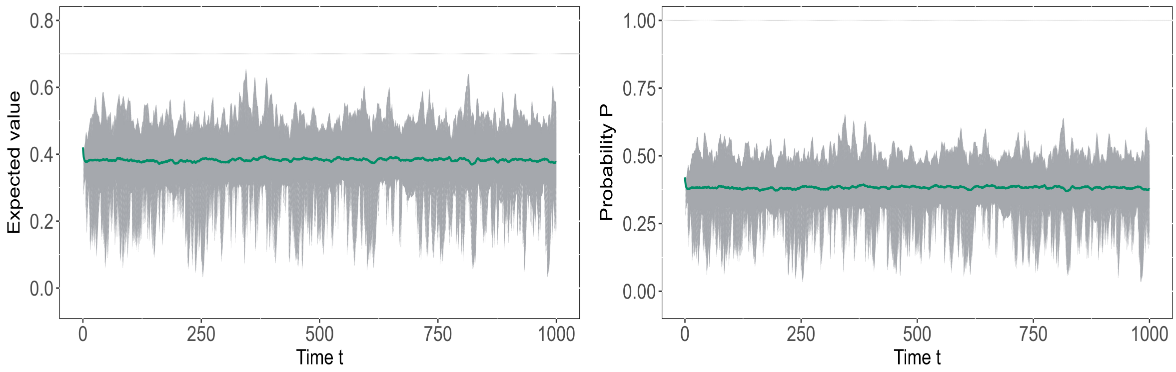


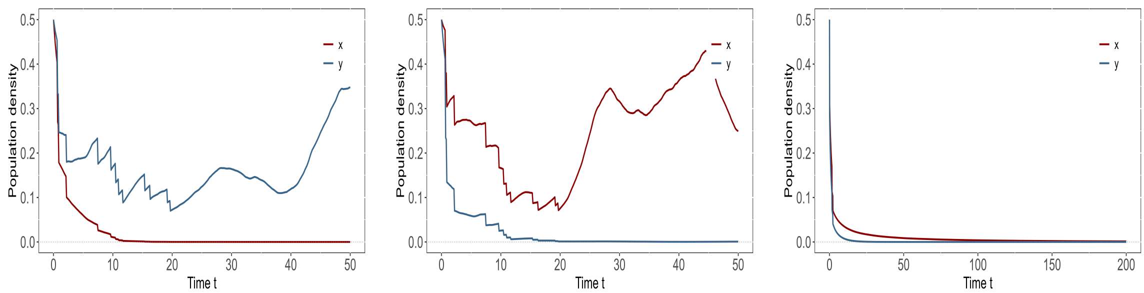
Figure 8. 100 path simulation figures.

Example 4. We now verify the conclusion of **Theorem 3.4** in terms of parameter combinations $(\mathcal{A}_4)$, $(\mathcal{A}_5)$, and $(\mathcal{A}_6)$.

In Figure 9 (a), we choose combination $(\mathcal{A}_4)$ as the biological parameter values of system (1.5). Direct calculations shows that $G_1 = -0.343 < 0$, it is easy to see that the parameters satisfy the condition of Theorem 3.4, and x is extinct. Figure 9 (a) confirms these.

In Figure 9 (b), we choose combination $(\mathcal{A}_5)$ as the biological parameter values of system (1.5). Direct calculations shows that $G_2 = -0.417 < 0$, it is easy to see that the parameters satisfy the condition of Theorem 3.4, and y is extinct. Figure 9 (b) confirms these.

In Figure 9 (c), we choose combination $(\mathcal{A}_6)$ as the biological parameter values of system (1.5). Direct calculations shows that $G_1 = -0.343 < 0$ and $G_2 = -0.527 < 0$, it is easy to see that the parameters satisfy the condition of Theorem 3.4, and all the species tend to be extinct. Figure 9 (c) confirms these.



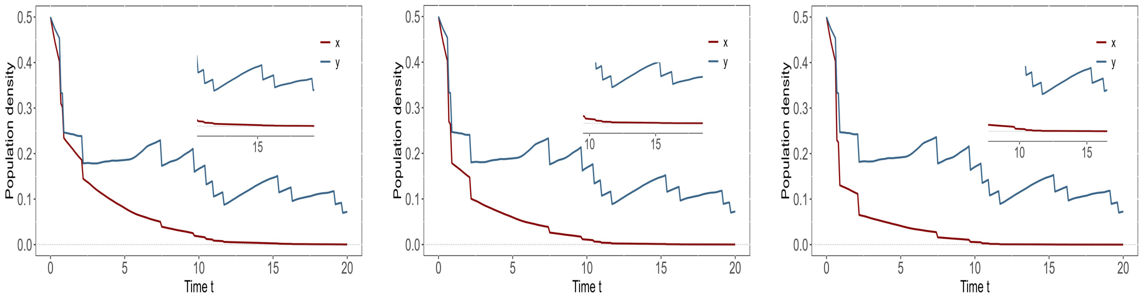
(a) x goes to extinction (b) y goes to extinction (c) x and y go to extinction

Figure 9. Population extinction.

Now explore the effect of different G values on the population.

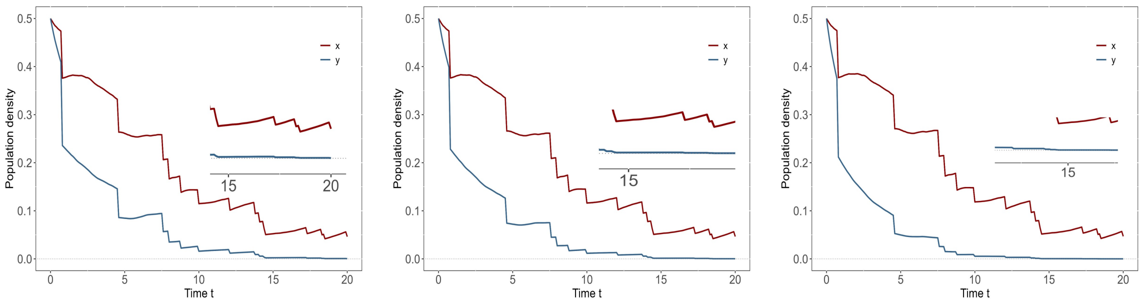
In Figure 10, we select parameter combination $(\mathcal{A}_4)$ and change the value of γ_1 in it. Let $\gamma_1 = -0.2, -0.3, -0.4$, respectively. It can be seen that, as the value of G_1 decreases, the extinction time of population x is advanced. It can also be seen that, when the jump noise coefficient $\gamma_1 < 0$, then the smaller γ_1 is, the greater the negative impact on the population.

In Figure 11, we select parameter combination $(\mathcal{A}_5)$ and change the value of $\bar{r}_2$ in it. Let $\bar{r}_2 = 0.36, 0.4, 0.46$, respectively. It can be seen that, as the value of G_2 decreases, the extinction time of population y is advanced. It can also be seen that the death rate accelerates population extinction, which is consistent with the phenomenon in nature.



(a) $\gamma_1 = -0.2, G_1 = -0.210$ (b) $\gamma_1 = -0.3, G_1 = -0.343$ (c) $\gamma_1 = -0.4, G_1 = -0.497$

Figure 10. Effect of G_1 on population x.



(a) $\bar{r}_2 = 0.36, G_2 = -0.377$ (b) $\bar{r}_2 = 0.4, G_2 = -0.417$ (c) $\bar{r}_2 = 0.46, G_2 = -0.477$

Figure 11. Effect of G_2 on population y.

5. Conclusions

In this paper, we first introduce a stochastic two-species Schoener's competitive model. Moreover, since populations in nature are often subject by sudden random perturbations, we introduced Lévy jumps to model this phenomenon. Previous research endeavors have traditionally employed white noise or telegraph noise to simulate environmental perturbations. Nonetheless, E.Allen [27] has identified certain conceptual and practical limitations of linear functions of Gaussian white noise. In addition, the role of the Ornstein-Uhlenbeck process in population dynamics models has received relatively little attention. It has been theoretically demonstrated that the mean-reverting Ornstein-Uhlenbeck process offers more stable environmental variability than linear and nonlinear perturbations. Therefore, we incorporate the Ornstein-Uhlenbeck process into the deterministic system, thereby obtaining a stochastic two-species Schoener's competitive model with Ornstein-Uhlenbeck process. It should be noted that the relevant dynamical properties of system (1.5) with the introduction of the Ornstein-Uhlenbeck process are changed. For example, the existence and uniqueness of the global positive solution studied in the traditional model becomes the existence and uniqueness of the global solution after adding the Ornstein-Uhlenbeck process. The Lyapunov functions used in the proofs of other properties (moment boundedness, extinction) are also different.

To the best of our ability, we prove the existence and uniqueness of the global solution, moment estimation, stochastic ultimate boundedness, and give sufficient conditions for population extinction. At present, few papers add Lévy jump or the Ornstein-Uhlenbeck process to Schoener's competitive model, and research on Schoener's competitive model mainly focus on periodic solution, stationarity, and persistence. Therefore, the study of other properties of system (1.5) has certain value.

In fact, our model still has areas for improvement. First, since the model we studied is two-dimensional, this leads to limitations in simulating the dynamical behaviour of populations in nature. Therefore, in the future, we will expand the model from two-dimensional to n-dimensional. In addition, it is well known that, in many natural ecosystems, there is a time lag in the interaction between populations, but we have not considered the effect of time lag on the model. In the future, we will utilize the relevant theories and methods of time-lagged generalized differential equations and impulse differential equations to build the corresponding dynamical models.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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