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Research article

On Schrödinger-Poisson equations with a critical nonlocal term

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Abstract: In this paper, we study the following non-autonomous Schrödinger-Poisson equation with a critical nonlocal term and a critical nonlinearity:

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi|u|^3 u = f(u) + (u^+)^5, & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & \text{in} \quad \mathbb{R}^3. \end{cases}$$

First, we consider the case that the nonlinearity satisfies the Berestycki-Lions type condition with critical growth. Second, we consider the case that $\text{int}V^{-1}(0)$ is contained in a spherical shell. By using variational methods, we obtain the existence and asymptotic behavior of positive solutions.

Keywords: Schrödinger-Poisson equation; critical nonlocal term; critical nonlinearity; variational method
Mathematics Subject Classification: 35A15, 35J60

1. Introduction

The Schrödinger-Poisson equation

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u), & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.1}$$

arises in a physical context. It is introduced while describing the interaction of a charged particle with an electrostatic field. More details can be found in [3]. Also, it appears in other fields like semiconductor theory, nonlinear optics, and plasma physics. The readers may refer to [18] and the references therein for further discussion. When $V \equiv 1$, $\lambda = 1$, and $f(x,u) = |u|^{p-2}u$, problem (1.1) has been studied sufficiently. We refer to [9] for $p \leq 2$ and $p \geq 6$, [7, 8, 10] for $4 \leq p < 6$, [2] for $3 < p < 6$, and [22] for $2 < p < 6$. In [31], the authors obtained an axially symmetric solution of the following

Schrödinger-Poisson equation in $\mathbb{R}^2$:

$$\begin{cases} -\Delta u + V(x)u + \phi u = K(x)f(u), & \text{in} \quad \mathbb{R}^2, \\ \Delta\phi = u^2, & \text{in} \quad \mathbb{R}^2, \end{cases}$$

where $f \in C(\mathbb{R}, \mathbb{R})$, and V and K are both axially symmetric functions. In [4, 5], the almost necessary and sufficient condition (Berestycki-Lions type condition) for the existence of ground state solutions of the problem

$$-\Delta u = g(u), \ u \in H^1(\mathbb{R}^N)$$

was given by [4] when $N = 2$ and [5] when $N \geq 3$. Precisely, they assumed g satisfies the following conditions:

(g_1) $g(s) \in C(\mathbb{R}, \mathbb{R})$ is continuous and odd.
(g_2) $-\infty < \liminf_{s\to 0} \frac{g(s)}{s} \leq \limsup_{s\to 0} \frac{g(s)}{s} = -a < 0$ for $N \geq 3$, and $\lim_{s\to 0} \frac{g(s)}{s} = -a < 0$ for $N = 2$.
(g_3) When $N \geq 3$, $\limsup_{s\to\infty} \frac{g(s)}{|s|^{\frac{N+2}{N-2}}} \leq 0$; when $N = 2$, for any $\alpha > 0$ there exists $C_\alpha > 0$ such that $g(s) \leq C_\alpha \exp(\alpha s^2)$ for all $s > 0$.
(g_4) There exists $\xi_0 > 0$ such that $G(\xi_0) > 0$, where $G(\xi_0) = \int_0^{\xi_0} g(s)\mathrm{d}s$.

When g satisfies the above Berestycki-Lions type condition, the authors in [19] studied the problem

$$\begin{cases} -\Delta u + q\phi u = g(u), & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = qu^2, & \text{in} \quad \mathbb{R}^3. \end{cases}$$

By using a truncation technique in [14], they proved that the problem admits a nontrivial positive radial solution for $q > 0$ small. For the critical case, the authors in [30] studied the existence of positive radial solutions of the problem

$$\begin{cases} -\Delta u + u + \phi u = \mu Q(x)|u|^{q-2}u + K(x)u^5, & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in} \quad \mathbb{R}^3, \end{cases}$$

where $q \in (2, 4)$, $\mu > 0$, and Q and K are radial functions satisfying the following conditions:

(h_1) $K \in C(\mathbb{R}^3, \mathbb{R})$, $\lim_{|x|\to\infty} K(x) = K_\infty \in (0, \infty)$ and $K(x) \geq K_\infty$ for $x \in \mathbb{R}^3$.
(h_2) $Q \in C(\mathbb{R}^3, \mathbb{R})$, $\lim_{|x|\to\infty} Q(x) = Q_\infty \in (0, \infty)$ and $Q(x) \geq Q_\infty$ for $x \in \mathbb{R}^3$.
(h_3) $|K(x) - K(x_0)| = o(|x - x_0|^\alpha)$, where $1 \leq \alpha < 3$ and $K(x_0) = \max_{\mathbb{R}^3} K(x)$.

In [25], we studied (1.1) with f satisfying the following Berestycki-Lions type condition with critical growth:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd, $\lim_{u\to 0+} \frac{f(u)}{u} = 0$ and $\lim_{u\to+\infty} \frac{f(u)}{u^5} = K > 0$.
(f_2) There exist $D > 0$ and $2 < q < 6$ such that $f(u) \geq Ku^5 + Du^{q-1}$ for $u \geq 0$.
(f_3) There exists $\theta > 2$ such that $\frac{1}{\theta} f(u)u - F(u) \geq 0$ for all $u \in \mathbb{R}^+$, where $F(u) = \int_0^u f(s)\mathrm{d}s$.

When $\lambda > 0$ is small, we obtained positive radial solutions for $q \in (4, 6)$, or $q \in (2, 4]$ with $D > 0$ large. In [29], the authors removed (f_3) by using a local deformation argument in [6]. It should be pointed out that, in [25, 29], the problems were considered in a radial setting.

When the nonlocal term is of critical growth, that is, u^2 is replaced by u^5, problem (1.1) is reduced to

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x, u), & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = u^5, & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{1.2}$$

These kind of equations are closely related with the Choquard-Pekar equation, which was proposed in [20] to study the quantum theory of a polaron at rest. Since the critical nonlocal term may cause the loss of compactness, problem (1.2) is quite different from the standard Schrödinger-Poisson equation. In [16], the authors considered the equation

$$\begin{cases} -\Delta u + bu + q\phi|u|^3u = f(u), & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $b \geq 0$, $q \in \mathbb{R}$, and the subcritical nonlinearity f satisfies the following conditions:

(H_1) $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\lim_{u\to 0+} \frac{f(u)}{bu+u^5} = 0$.
(H_2) $\lim_{u\to\infty} \frac{f(u)}{u^5} = 0$.
(H_3) There is a function $z \in H_r^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} F(z) > b\int_{\mathbb{R}^3} z^2$, where $F(z) = \int_0^z f(t)\mathrm{d}t$.
(H_4) There exist $r \in (4, 6)$, $A > 0$, $B > 0$ such that $F(t) \geq At^r - Bt^2$ for $t \geq 0$.

For $q \geq 0$, they proved that there exists $q_0 > 0$ such that for $q \in [0, q_0)$, and problem (1.3) has at least one positive radially symmetric solution if (H_1)–(H_3) hold. For $q = -1$, they proved that problem (1.3) has at least one positive radially symmetric solution if (H_1)–(H_2) and (H_4) hold. In [17], the authors studied the existence, nonexistence, and multiplicity of positive radially symmetric solutions of the equation

$$\begin{cases} -\Delta u + u + \lambda\phi|u|^3u = \mu|u|^{p-1}u, & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\lambda \in \mathbb{R}$, $\mu \geq 0$, and $p \in [1, 5]$. In [15], the author obtained positive solutions of the following equation with subcritical growth:

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi|u|^3u = f(x, u), & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = K(x)|u|^5, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.5}$$

where V, K, and f are asymptotically periodic functions of x. If the nonlinearity is of critical growth, the author in [12] studied ground state solutions of the equation

$$\begin{cases} -\Delta u + V(x)u - \phi|u|^3u = f(u) + u^5, & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.6}$$

where $V(x) = 1 + x_1^2 + x_2^2$ with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and f is an appropriate nonlinear function.

In this paper, we study the following Schrödinger-Poisson equation with a critical nonlocal term:

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi|u|^3u = f(u) + (u^+)^5, & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.7}$$

where $(u^+)^5$ is a critical term with $u^+ := \max\{u, 0\}$ and $\lambda > 0$ is a parameter. When we study (1.7) for the case $\lambda < 0$, the boundedness of the Palais-Smale sequence can be derived directly. However, for the case $\lambda > 0$, the problem is quite different. Since the term $\int_{\mathbb{R}^3} \phi_u|u|^5 dx$ is homogeneous of degree 10, the corresponding Ambrosetti-Rabinowitz condition on f is the following:

(f') There exists $\theta \geq 10$ such that $tf(t) - \theta F(t) \geq 0$ for any $t \in \mathbb{R}$.

Obviously, this condition is not suitable for the problem in dimension three. To solve the problem, the authors in [16] used a truncation technique in [14]. However, the argument is invalid when we study non-autonomous problems in a non-radial setting. Motivated by the above considerations, we first study the non-autonomous problem (1.7) in a non-radial setting, where the nonlinearity satisfies the Berestycki-Lions type condition with critical growth. We assume V satisfies the following conditions:

(V_1) $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and $\inf_{\mathbb{R}^3} V := V_0 > 0$.
(V_2) $V(x) \leq \lim_{|x|\to\infty} V(x) := V_\infty$ for all $x \in \mathbb{R}^3$ and the inequality is strict in a set of positive Lebesgue measure.
(V_3) There exists $\theta \in (0, 1)$ such that $\frac{t^3}{2}V(tx) - \frac{t^3}{2}V(x) - \frac{t^3-1}{6}(\nabla V(x), x) \leq \frac{\theta(t-1)^2(t+2)}{24|x|^2}$ for $x \in \mathbb{R}^3 \setminus \{0\}$ and $t \in \mathbb{R}^+$.

The result is as follows.

Theorem 1.1. Assume that (V_1)–(V_3) and (f_1)–(f_2) hold. Then there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, problem (1.7) has a positive solution $(u_\lambda, \phi_\lambda)$. Moreover, as $\lambda \to 0$, $(u_\lambda, \phi_\lambda) \to (u, 0)$ in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, where u is a ground state solution of the following limiting equation:

$$-\Delta u + V(x)u = f(u) + (u^+)^5 \ \text{ in } \ \mathbb{R}^3. \tag{1.8}$$

When $V \equiv 1$, problem (1.7) is reduced to the following equation:

$$\begin{cases} -\Delta u + u + \lambda\phi|u|^3u = f(u) + (u^+)^5, & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{1.9}$$

Then we have the following result.

Corollary 1.1. Assume that (f_1)–(f_2) hold. Then there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, problem (1.9) has a positive solution $(u_\lambda, \phi_\lambda)$. Moreover, as $\lambda \to 0$, $(u_\lambda, \phi_\lambda) \to (u, 0)$ in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, where u is a ground state solution of the following limiting equation:

$$-\Delta u + u = f(u) + (u^+)^5 \ \text{ in } \ \mathbb{R}^3. \tag{1.10}$$

Remark 1.1. Corollary 1.1 is still valid if we replace (f_2) by (H_3). So, we generalize the result in [16] to the critical case.

In the next, we consider the case that $\text{int}V^{-1}(0)$ is contained in a spherical shell. We assume the following conditions.

(V_1') $V \in C(\mathbb{R}^3, \mathbb{R})$ and $V(x) = V(|x|)$ for all $x \in \mathbb{R}^3$.
(V_2') $V(x) = 0$ for $x \in \wedge_1$ and there exists $V_0 > 0$ such that $V(x) \geq V_0$ for $x \notin \wedge_2$, where $\wedge_1 := \{x \in \mathbb{R}^3 : r_1 < |x| < r_2\}$ and $\wedge_2 := \{x \in \mathbb{R}^3 : R_1 < |x| < R_2\}$ with $0 < R_1 < r_1 < r_2 < R_2$.
(f_3') There exists $\theta > 2$ such that $\frac{f(u)}{u^{\theta-1}}$ is increasing for all $u > 0$.

To the best of our knowledge, there are no related results even for the case $\lambda = 0$. We must face several difficulties. A main difficulty is how to get the compactness. In [11], del Pino and Felmer developed a penalization approach to deal with singularly perturbed problems. Motivated by [11], instead of studying (1.7) directly, we turn to consider a modified problem. By studying the influence of the potential on the compactness and the behavior of positive solutions at infinity, we solve the problem. When $\lambda > 0$, we have to prove the boundedness of the Palais-Smale sequence for the modified problem. This is another difficulty. Now we state the result.

Theorem 1.2. Assume that (V_1')–(V_2'), (f_1)–(f_2), and (f_3') hold. Then there exists $R' > 0$ such that for $R_1 > R'$, there exists $\lambda' > 0$ such that problem (1.7) has a positive solution $(u_\lambda, \phi_\lambda)$ for $\lambda \in (0, \lambda')$. Moreover, as $\lambda \to 0$, $(u_\lambda, \phi_\lambda) \to (u, 0)$ in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, where u is a positive solution of (1.8).

Notations.

- Denote $H^1 := H^1(\mathbb{R}^3)$ the Hilbert space with the norm $\|u\|_{H^1}^2 := \int_{\mathbb{R}^3}(|\nabla u|^2 + |u|^2)\mathrm{d}x$.
- Denote $D^{1,2} := D^{1,2}(\mathbb{R}^3) = \left\{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\right\}$ the Sobolev space with the norm $\|u\|_{D^{1,2}}^2 := \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x$.
- Denote the norm $\|u\|_s := \left(\int_{\mathbb{R}^3} |u|^s \mathrm{d}x\right)^{\frac{1}{s}}$, where $2 \leq s < \infty$.
- Denote C a universal positive constant (possibly different).

2. Proof of Theorem 1.1

Without loss of generality, we assume that $f(u) = 0$ for $u \leq 0$. Define the best Sobolev constant

$$S := \inf_{u \in D^{1,2}\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x\right)^{\frac{1}{3}}}. \tag{2.1}$$

By the Lax-Milgram theorem, for any $u \in D^{1,2}$ there exists a unique $\phi_u \in D^{1,2}$ such that $-\Delta\phi_u = |u|^5$. The function ϕ_u has the following properties.

Lemma 2.1. ([16])

(*i*) $\phi_u \geq 0$, $\phi_{tu} = |t|^5\phi_u$ and $\phi_{u(\frac{\cdot}{t})} = t^2\phi_u(\frac{\cdot}{t})$ for all $t > 0$.
(*ii*) $\|\phi_u\|_{D^{1,2}} \leq S^{-\frac{1}{2}}\|u\|_6^5$.
(*iii*) If $u_n \rightharpoonup u$ weakly in $L^6(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then $\phi_{u_n} \rightharpoonup \phi_u$ weakly in $D^{1,2}$ up to a subsequence.
(*iv*) Let $J(u) = \int_{\mathbb{R}^3} \phi_u |u|^5 \mathrm{d}x$, where $u \in D^{1,2}$. If $u_n \rightharpoonup u$ weakly in $L^6(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then

$$J(u_n) - J(u) - J(u_n - u) = o_n(1).$$

Define $X := \left\{u \in H^1 : \int_{\mathbb{R}^3} V(x)|u|^2 \mathrm{d}x < \infty\right\}$ as the Hilbert space with the norm $\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$. Define the functional on X by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{10}\int_{\mathbb{R}^3} \phi_u |u|^5 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |u^+|^6 \mathrm{d}x,$$

where $F(u) := \int_0^u f(s)\mathrm{d}s$. Obviously, the functional I_λ is of class C^1 and critical points of I_λ are weak solutions of (1.7). Let

$$m_0 := \inf\{I_0(u) : u \in X \setminus \{0\}, I_0'(u) = 0\}. \tag{2.2}$$

If $I_0'(u) = 0$, by the arguments in [16,21,24] we can derive the Pohozǎev type identity $J_0(u) = 0$, where

$$\begin{aligned} J_0(u) =&\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[3V(x) + (\nabla V(x), x)]|u|^2\mathrm{d}x - 3\int_{\mathbb{R}^3} F(u)\mathrm{d}x \\ &- \frac{1}{2}\int_{\mathbb{R}^3}|u^+|^6\mathrm{d}x. \end{aligned}$$

When $V \equiv V_\infty$, problem (1.8) is reduced to the following equation:

$$-\Delta u + V_\infty u = f(u) + (u^+)^5 \ \text{ in } \ \mathbb{R}^3. \tag{2.3}$$

The functional associated with (2.3) is

$$I_0^\infty(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty|u|^2\mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3}|u^+|^6\mathrm{d}x, \quad u \in H^1.$$

Define

$$m_0^\infty := \inf\{I_0^\infty(u) : u \in H^1 \setminus \{0\}, (I_0^\infty)'(u) = 0\}. \tag{2.4}$$

Define

$$c_0^\infty := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_0^\infty(\gamma(t)), \tag{2.5}$$

where $\Gamma := \left\{\gamma \in C([0,1], H^1) : \gamma(0) = 0, I_0^\infty(\gamma(1)) < 0\right\}$.

Lemma 2.2. Assume that (V_1)–(V_3) hold. Then, for all $x \in \mathbb{R}^3 \setminus \{0\}$,

$$3V_\infty - 3V(x) - \frac{\theta}{4|x|^2} \le (\nabla V(x), x) \le \frac{\theta}{2|x|^2}. \tag{2.6}$$

Proof. Let

$$g(t) := \frac{t^3}{2}V(tx) - \frac{t^3}{2}V(x) - \frac{t^3-1}{6}(\nabla V(x), x) - \frac{\theta(t-1)^2(t+2)}{24|x|^2}.$$

By (V_3), we get $g(0) \le 0$. Then $(\nabla V(x), x) \le \frac{\theta}{2|x|^2}$ for all $x \in \mathbb{R}^3\setminus\{0\}$. By (V_2)–(V_3), we get $\lim_{t\to+\infty}\frac{g(t)}{t^3} \le 0$. Then $(\nabla V(x), x) \ge 3V_\infty - 3V(x) - \frac{\theta}{4|x|^2}$ for all $x \in \mathbb{R}^3 \setminus \{0\}$. □

Theorem 2.1. ([13]) Let X be a Banach space equipped with a norm $\|.\|_X$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $(I_\mu)_{\mu\in J}$ of C^1 -functionals on X of the form

$$I_\mu(u) = A(u) - \mu B(u), \quad \forall\, \mu \in J,$$

where $B(u) \geq 0$ for all $u \in X$, and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to \infty$. We assume there are two points v_1, v_2 in X such that

$$c_\mu := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)) > \max\{I_\mu(v_1), I_\mu(v_2)\}, \quad \forall\, \mu \in J,$$

where $\Gamma := \{\gamma \in C([0,1], X); \gamma(0) = v_1, \gamma(1) = v_2\}$. Then, for almost every $\mu \in J$, there is a sequence $\{v_n\} \subset X$ such that $\{v_n\}$ is bounded, $I_\mu(v_n) \to c_\mu$, and $I'_\mu(v_n) \to 0$ in X^{-1}. Moreover, the map $\mu \to c_\mu$ is continuous from the left-hand side.

Lemma 2.3. Assume that (V_1)–(V_3) and (f_1)–(f_2) hold. Then $m_0 \in (0, m_0^\infty)$ is attained by a positive function.

Proof. Let $\mu_0 \in (0,1)$. Define the functionals on X by

$$I_{0,\mu}(u) = \frac{1}{2}\|u\|^2 - \mu \int_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{\mu}{6}\int_{\mathbb{R}^3} |u^+|^6\mathrm{d}x,$$

where $\mu \in [\mu_0, 1]$. Similar to the argument in [27], we can use Theorem 2.1 to derive that for almost every $\mu \in [\mu_0, 1]$ there exists a positive function $u_\mu \in X$ such that $c_\mu = I_{0,\mu}(u_\mu)$ and $I'_{0,\mu}(u_\mu) = 0$.

Choose $\mu_n \uparrow 1$ such that $I_{0,\mu_n}(u_{\mu_n}) = c_{\mu_n}$ and $I'_{0,\mu_n}(u_{\mu_n}) = 0$. Then u_{μ_n} satisfies the following Pohozǎev type identity:

$$\begin{aligned}&\frac{1}{2}\|\nabla u_{\mu_n}\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[3V(x) + (\nabla V(x), x)]|u_{\mu_n}|^2\mathrm{d}x\\ &= 3\mu_n \int_{\mathbb{R}^3} F(u_{\mu_n})\mathrm{d}x + \frac{\mu_n}{2}\int_{\mathbb{R}^3}|u_{\mu_n}|^6\mathrm{d}x.\end{aligned} \tag{2.7}$$

By (2.7), Lemma 2.2, and the Hardy inequality,

$$c_{\mu_n} = \frac{1}{3}\|\nabla u_{\mu_n}\|_2^2 - \frac{1}{6}\int_{\mathbb{R}^3}(\nabla V(x), x)|u_{\mu_n}|^2\mathrm{d}x \geq \frac{1-\theta}{3}\|\nabla u_{\mu_n}\|_2^2, \tag{2.8}$$

and

$$\begin{aligned}&\frac{1}{2}\|\nabla u_{\mu_n}\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[3V(x) + (\nabla V(x), x)]|u_{\mu_n}|^2\mathrm{d}x\\ &\geq \frac{1-\theta}{2}\|\nabla u_{\mu_n}\|_2^2 + \frac{3}{2}\int_{\mathbb{R}^3} V_\infty |u_{\mu_n}|^2\mathrm{d}x.\end{aligned} \tag{2.9}$$

By (2.7)–(2.9) and (f_1), we get that $\|u_{\mu_n}\|$ is bounded. Then $I_0(u_{\mu_n}) \to c_1$ and $I'_0(u_{\mu_n}) \to 0$. Similar to the argument in [27], we get that there exists a positive function $u_0 \in X$ such that $u_{\mu_n} \to u_0$ in X, $I_0(u_0) = c_1$, and $I'_0(u_0) = 0$. Moreover, $0 < m_0 \leq c_1$ is attained. By [28], we get that $m_0^\infty = c_0^\infty$ is attained by a positive function u_0^∞. Then by (V_1)-(V_2) and a standard argument, we have $c_1 < c_0^\infty$. □

Let S_0 be the set of ground states of (1.8). By Lemma 2.3, we have $S_0 \neq \emptyset$.

Lemma 2.4. Assume that (V_1)–(V_3) and (f_1)–(f_2) hold. Then S_0 is compact in X.

Proof. By Lemma 2.3, for any $\{u_n\} \subset S_0$ we have $I_0(u_n) = m_0$, $I'_0(u_n) = 0$, and $J_0(u_n) = 0$. Moreover, $\|u_n\|$ is bounded. Assume that $u_n \rightharpoonup u_0$ weakly in X. Then $I'_0(u_0) = 0$. Let $v_n = u_n - u_0$. By (V_1), (f_1), and the Brezis-Lieb lemma in [24], we have

$$m_0 - I_0(u_0) + o_n(1) = I_0^\infty(v_n), \quad (I_0^\infty)'(v_n) = o_n(1). \tag{2.10}$$

Since $v_n \rightharpoonup 0$ weakly in X, by the Lions Lemma in [24], $v_n \to 0$ in $L^t(\mathbb{R}^3)$ for any $t \in (2,6)$, or there exists $\{y_n^1\} \subset \mathbb{R}^3$ with $|y_n^1| \to \infty$ such that $v_n^1 := v_n(. + y_n^1) \rightharpoonup v^1 \neq 0$ weakly in X. If $v_n \to 0$ in $L^t(\mathbb{R}^3)$ for any $t \in (2,6)$, by (f_1) we get $\int_{\mathbb{R}^3} F(v_n)\mathrm{d}x = o_n(1)$ and $\int_{\mathbb{R}^3} f(v_n)v_n\mathrm{d}x = o_n(1)$. Then

$$m_0 + o_n(1) = I_0(u_0) + \frac{1}{2}\|v_n\|^2 - \frac{1}{6}\|v_n\|_6^6, \quad \|v_n\|^2 = \|v_n\|_6^6 + o_n(1). \tag{2.11}$$

By $I_0'(u_0) = 0$, we have $J_0(u_0) = 0$. By Lemma 2.2 and the Hardy inequality, we get $I_0(u_0) \geq 0$. Assume that $\lim_{n\to\infty} \|v_n\|_6^6 = l$. If $l > 0$, by (2.11) and the definition of S, we get $l \geq S^{\frac{3}{2}}$. Then $m_0 \geq \frac{1}{3}S^{\frac{3}{2}}$, a contradiction. So, $l = 0$, from which we get $v_n \to 0$ in X. If there exists $\{y_n^1\} \subset \mathbb{R}^3$ with $|y_n^1| \to \infty$ such that $v_n^1 := v_n(. + y_n^1) \rightharpoonup v^1 \neq 0$ weakly in X, similar to the argument of Lemma 2.6 in [27] there exist $k \in \mathbb{N} \cup \{0\}$, $\{y_n^i\} \subset \mathbb{R}^3$ and $v^i \in X$ for $1 \leq i \leq k$ such that

$$\begin{aligned}
&|y_n^i| \to \infty \textit{ and } |y_n^i - y_n^j| \to \infty, \textit{ if } i \neq j,\ 1 \leq i, j \leq k,\\
&v_n(. + y_n^i) \rightharpoonup v^i \neq 0 \textit{ weakly in } X \textit{ and } (I_0^\infty)'(v^i) = 0,\ \forall\ 1 \leq i \leq k,\\
&\left\| v_n - \sum_{i=1}^k v^i(. - y_n^i) \right\| \to 0,\\
&m_0 = I_0(u_0) + \sum_{i=1}^k I_0^\infty(v^i).
\end{aligned} \tag{2.12}$$

Since $(I_0^\infty)'(v^i) = 0$, we have $I_0^\infty(v^i) \geq m_0^\infty$. If $k \geq 1$, by $I_0(u_0) \geq 0$ and (2.12) we get $m_0 \geq m_0^\infty$, a contradiction. So, $k = 0$, from which we get $u_n \to u_0$ in X. □

Lemma 2.5. Assume that (V_1)–(V_3) and (f_1) hold. If $u \in S_0$, then $m_0 = I_0(u) > I_0(u(\frac{\cdot}{t}))$ for all $t \in [0,1) \cup (1,+\infty)$. Also, there exists $t_0 > 1$ independent of $u \in S_0$ such that $I_0(u(\frac{\cdot}{t_0})) \leq -2$.

Proof. By $u \in S_0$, we have $J_0(u) = 0$. Then

$$\begin{aligned}
I_0\left(u\left(\frac{x}{t}\right)\right) - I_0(u) = &\int_{\mathbb{R}^3}\left[\frac{t^3}{2}V(tx) - \frac{t^3}{2}V(x) - \frac{t^3-1}{6}(\nabla V(x), x)\right]|u|^2\mathrm{d}x\\
&- \frac{(t-1)^2(t+2)}{6}\|\nabla u\|_2^2.
\end{aligned} \tag{2.13}$$

By (V_3) and the Hardy inequality, we get $I_0(u) > I_0(u(\frac{\cdot}{t}))$ for all $t \neq 1$. By Lemma 2.2 and the Hardy inequality,

$$\begin{aligned}
&\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[3V(x) + (\nabla V(x), x)]|u|^2\mathrm{d}x\\
\geq\ &\frac{1-\theta}{2}\|\nabla u\|_2^2 + \frac{3}{2}\int_{\mathbb{R}^3} V_\infty|u|^2\mathrm{d}x.
\end{aligned} \tag{2.14}$$

Since $J_0(u) = 0$, by (f_1) and (2.14) there exists $\varrho > 0$ independent of $u \in S_0$ such that $\|\nabla u\|_2^2 \geq \varrho$. So, by (V_3), the Hardy inequality, and (2.13) we get there exists $t_0 > 1$ independent of $u \in S_0$ such that $I_0(u(\frac{\cdot}{t_0})) \leq -2$. □

Lemma 2.6. Assume that (V_1)–(V_3) and (f_1) hold. Then there exist $\lambda_1, M_0 > 0$ independent of $u \in S_0$ such that $I_\lambda(u(\frac{\cdot}{t_0})) \leq -1$, $\max_{t\in[0,1]} \|u(\frac{\cdot}{tt_0})\| \leq M_0$ and $\|u\| \leq M_0$ for all $\lambda \in [0, \lambda_1]$ and $u \in S_0$.

Proof. If $u \in S_0$, then $m_0 = I_0(u)$ and $J_0(u) = 0$. By the Hardy inequality and Lemma 2.2, we have $m_0 \geq \frac{1-\theta}{3}\|\nabla u\|_2^2$. Together with (2.14), $J_0(u) = 0$, and (f_1), we derive that there exists $\sigma_1 > 0$ independent of $u \in S_0$ such that $\|u\|_{H^1} \leq \sigma_1$. We note that

$$\left\|u\left(\frac{\cdot}{tt_0}\right)\right\|^2 = tt_0\|\nabla u\|_2^2 + (tt_0)^3 \int_{\mathbb{R}^3} V(tt_0 x)|u|^2 \mathrm{d}x. \tag{2.15}$$

Together with (V_1) and $\|u\|_{H^1} \leq \sigma_1$, we get

$$\|u\|^2 \leq \left(1 + \max_{\mathbb{R}^3} V\right)\sigma_1^2, \quad \max_{t\in[0,1]} \left\|u\left(\frac{\cdot}{tt_0}\right)\right\|^2 \leq \left(t_0 + t_0^3 \max_{\mathbb{R}^3} V\right)\sigma_1^2. \tag{2.16}$$

By Lemma 2.1, we have

$$\begin{aligned} I_\lambda\left(u\left(\frac{\cdot}{tt_0}\right)\right) =& I_0\left(u\left(\frac{\cdot}{tt_0}\right)\right) + \frac{\lambda(tt_0)^5}{10}\int_{\mathbb{R}^3} \phi_u|u|^5 \mathrm{d}x \\ \leq& I_0\left(u\left(\frac{\cdot}{tt_0}\right)\right) + \frac{\lambda(tt_0)^5}{10S^6}\|\nabla u\|_2^{10}. \end{aligned} \tag{2.17}$$

By Lemma 2.5 and (2.17), we derive that there exists $\lambda_1 > 0$ independent of $u \in S_0$ such that $I_\lambda(u(\frac{\cdot}{t_0})) \leq -1$ for $\lambda \in (0, \lambda_1)$ and $u \in S_0$. □

Choose $U_0 \in S_0$. Define

$$b_\lambda := \inf_{g\in G_0} \max_{t\in[0,1]} I_\lambda(g(t)), \tag{2.18}$$

where $G_0 := \left\{g \in C([0,1], X) : g(0) = 0, g(1) = U_0\left(\frac{\cdot}{t_0}\right)\right\}$ and $\lambda \in (0, \lambda_1)$. Define

$$B_\lambda := \max_{t\in[0,1]} I_\lambda\left(U_0\left(\frac{\cdot}{tt_0}\right)\right). \tag{2.19}$$

Lemma 2.7. $\lim_{\lambda\to 0} b_\lambda = \lim_{\lambda\to 0} B_\lambda = m_0$.

Proof. By (2.17) and Lemmas 2.5–2.6, we get

$$b_\lambda \leq B_\lambda \leq m_0 + \frac{\lambda(tt_0)^5 M_0^{10}}{10S^6}.$$

Then $\limsup_{\lambda\to 0} b_\lambda \leq \limsup_{\lambda\to 0} B_\lambda \leq m_0$. On the other hand, for any $g \in G_0$,

$$\max_{t\in[0,1]} I_\lambda(g(t)) \geq \max_{t\in[0,1]} I_0(g(t)) \geq b_0,$$

where $b_0 := \inf_{g\in G_0} \max_{t\in[0,1]} I_0(g(t))$. Then $b_\lambda \geq b_0$. By Lemma 2.6, there exists $\mu_0 \in (0, 1)$ such that $I_{0,\mu}(g(1)) \leq -\frac{1}{2}$ for $\mu \in (\mu_0, 1)$. Define

$$c_\mu := \inf_{g\in G_0} \max_{t\in[0,1]} I_{0,\mu}(g(t)).$$

By repeating the proof of Lemma 2.3, we get that c_μ is a critical value. Moreover, we can prove that b_0 is a critical value. Then $b_0 \geq m_0$. So, $\liminf_{\lambda\to 0} b_\lambda \geq m_0$. □

For $\eta, d > 0$, define $I_\lambda^\eta := \{u \in X : I_\lambda(u) \le \eta\}$ and $S_0^d := \{u \in X : \inf_{v \in S_0} \|u - v\| \le d\}$.
Lemma 2.8. Let $\{u_{\lambda_i}\} \subset S_0^d$ with $\lim_{i\to\infty} \lambda_i = 0$ be such that $\lim_{i\to\infty} I_{\lambda_i}(u_{\lambda_i}) \le m_0$ and $\lim_{i\to\infty} I'_{\lambda_i}(u_{\lambda_i}) = 0$. Then for $d > 0$ small, there exists $u_0 \in S_0$ such that $u_{\lambda_i} \to u_0$ in X up to a subsequence.
Proof. By the proof of Lemma 2.5, there exists $\varrho > 0$ independent of $u \in S_0$ such that $\|u\|^2 \ge \varrho$ for $u \in S_0$. Since $\{u_{\lambda_i}\} \subset S_0^d$, by choosing $d > 0$ small we get $\|u_{\lambda_i}\|^2 \ge \frac{\varrho}{2}$. By Lemma 2.4, we have that $\|u_{\lambda_i}\|$ is bounded. Then $\lim_{i\to\infty} I_0(u_{\lambda_i}) \le m_0$ and $\lim_{i\to\infty} I'_0(u_{\lambda_i}) = 0$. By the argument of Lemma 2.4, there exists $u_0 \in X$ such that $u_{\lambda_i} \to u_0$ in X up to a subsequence. So, $\|u_0\|^2 \ge \frac{\varrho}{2}$, $I_0(u_0) \le m_0$ and $I'_0(u_0) = 0$, which implies that $u_0 \in S_0$. □
Lemma 2.9. Let $d > 0$. Then there exists $\eta > 0$ such that for small $\lambda > 0$, $I_\lambda(\gamma(t)) \ge b_\lambda - \eta$ implies that $\gamma(t) \in S_0^{\frac{d}{2}}$, where $\gamma(0) = 0$ and $\gamma(t) = U_0(\frac{\cdot}{tt_0})$ for $t \in (0, 1]$.
Proof. By Lemma 2.5, if $\gamma(t) \notin S_0^{\frac{d}{2}}$, then there exists $\delta > 0$ such that $|tt_0 - 1| \ge \delta$. Moreover, there exists $\eta' > 0$ such that $I_0(\gamma(t)) \le m_0 - \eta'$. By Lemmas 2.1 and 2.6–2.7, there exists $\eta > 0$ such that for small $\lambda > 0$, it holds that $I_\lambda(\gamma(t)) < b_\lambda - \eta$. □
Proof of Theorem 1.1. Recall that if $u \in S_0$, then there exists $\varrho > 0$ independent of $u \in S_0$ such that $\|\nabla u\|_2^2 \ge \varrho$. So, we can choose $d > 0$ small such that $\|u\|^2 \ge \frac{\varrho}{2}$ for any $u \in S_0^d$. We use the idea in [6,29] to claim that for small $\lambda > 0$, there exists $\{u_n\} \subset S_0^d \cap I_\lambda^{B_\lambda}$ such that $I'_\lambda(u_n) \to 0$. Otherwise, there exists $a(\lambda) > 0$ such that $\|I'_\lambda(u)\| \ge a(\lambda)$ for $u \in S_0^d \cap I_\lambda^{B_\lambda}$. By Lemmas 2.7–2.8, there exists $\rho_0 > 0$ independent of $\lambda > 0$ small such that $\|I'_\lambda(u)\| \ge \rho_0$ for $u \in I_\lambda^{B_\lambda} \cap (S_0^d \setminus S_0^{\frac{d}{2}})$. We note that there exists a pseudo-gradient vector field Q_λ on a neighborhood Z_λ of $S_0^d \cap I_\lambda^{B_\lambda}$ for I_λ. Let η_λ be a Lipschitz continuous function on X such that $\eta_\lambda = 1$ on $S_0^d \cap I_\lambda^{B_\lambda}$, $\eta_\lambda = 0$ on $\mathrm{R}^3 \setminus Z_\lambda$, and $0 \le \eta_\lambda \le 1$ on R^3. Let ξ_λ be a Lipschitz continuous function such that $\xi_\lambda(t) = 1$ for $|t - b_\lambda| \le \frac{\eta}{2}$, $\xi_\lambda(t) = 0$ for $|t - b_\lambda| \ge \eta$, and $0 \le \xi_\lambda \le 1$ for $t \in \mathrm{R}^+$. Consider the initial value problem

$$\begin{cases} \dfrac{d\psi_\lambda(u,t)}{dt} = -\eta_\lambda(\psi_\lambda(u,t))\xi_\lambda(I_\lambda(\psi_\lambda(u,t)))Q_\lambda(\psi_\lambda(u,t)), \\ \psi_\lambda(u,0) = u. \end{cases} \tag{2.20}$$

Then (2.20) has a unique global solution $\psi_\lambda(u,t)$. Recall that $\lim_{\lambda\to0} b_\lambda = \lim_{\lambda\to0} B_\lambda = m_0$. Also, we have Lemma 2.9. By a standard argument, for any $t \in [0,1]$ there exists $s(t) \ge 0$ such that $\psi_\lambda(\gamma(t), s(t))$ is continuous in $t \in [0,1]$ and

$$\max_{t\in[0,1]} I_\lambda(\psi_\lambda(\gamma(t), s(t))) \le b_\lambda - \frac{\eta}{4},$$

where γ is given in Lemma 2.9. Let $\gamma_0(.) = \psi_\lambda(\gamma(.), s(.))$. Then $\gamma_0 \in G_0$, from which we get

$$\max_{t\in[0,1]} I_\lambda(\psi_\lambda(\gamma(t), s(t))) \ge b_\lambda,$$

a contradiction. Since for $\lambda > 0$ small there exists $\{u_n\} \subset I_\lambda^{B_\lambda} \cap S_0^d$ such that $I'_\lambda(u_n) \to 0$, by Lemma 2.4 we get that $\|u_n\|$ is bounded. Assume that $u_n \rightharpoonup u_\lambda$ weakly in X. By Lemma 2.1, we have $I'_\lambda(u_\lambda) = 0$. Let $u_n = v_n + w_n$, where $v_n \in S_0$ and $\|w_n\| \le d$. By Lemma 2.4, there exists $v_\lambda \in S_0$ such that $v_n \to v_\lambda$ in X. Assume that $w_n \rightharpoonup w_\lambda$ in X. Then $\|w_\lambda\| \le d$. So, $u_\lambda \in S_0^d$. Moreover, u_λ is positive. Together with Lemma 2.8, we get the result. □

3. Proof of Theorem 1.2

Define $X_r := \left\{u \in H_r^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)|u|^2 dx < \infty\right\}$ as the Hilbert space with the norm $\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2 dx\right)^{\frac{1}{2}}$. By (V_2'), we derive that for all $u \in X_r$,

$$\begin{aligned}\|u\|_{H^1}^2 \leq & \int_{\wedge_2} \left(|\nabla u|^2 + u^2\right) dx + \int_{\mathbb{R}^3\setminus\wedge_2} \left(|\nabla u|^2 + \frac{V(x)}{V_0} u^2\right) dx \\ \leq & \int_{\wedge_2} |\nabla u|^2 dx + \left(\int_{\wedge_2} |u|^6 dx\right)^{\frac{1}{3}} |\wedge|^{\frac{2}{3}} \\ & + \max\left\{1, \frac{1}{V_0}\right\} \int_{\mathbb{R}^3\setminus\wedge_2} \left(|\nabla u|^2 + V(x)u^2\right) dx \\ \leq & \max\left\{1 + \frac{|\wedge_2|^{\frac{2}{3}}}{S}, \frac{1}{V_0}\right\} \|u\|^2. \end{aligned} \tag{3.1}$$

Then the imbedding $X_r \hookrightarrow H_r^1(\mathbb{R}^3)$ is continuous. Define $g(u) = 0$ for $u \leq 0$ and $g(u) = \min\left\{f(u) + (u^+)^5, \frac{V_0 u}{\kappa}\right\}$ for $u > 0$, where $\kappa > 2$. Let χ be the characteristic function such that $\chi(x) = 1$ for $x \in \wedge_2$ and $\chi(x) = 0$ for $x \in \mathbb{R}^3 \setminus \wedge_2$. Consider the truncated problem of (1.8) as

$$-\Delta u + V(x)u = h(x, u) \text{ in } \mathbb{R}^3, \tag{3.2}$$

where $h(x, u) = \chi(x)\left[f(u) + (u^+)^5\right] + (1 - \chi(x))g(u)$. The functional associated with (3.2) is

$$\hat{I}_0(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} H(x, u) dx, \quad u \in X_r,$$

where $H(x, u) = \int_0^u h(x, s) ds = \chi(x)\left[F(u) + \frac{1}{6}(u^+)^6\right] + (1 - \chi(x))G(u)$ with $G(u) = \int_0^u g(s) ds$. In what follows, we look for critical points of $\hat{I}_0$. Define

$$\hat{c}_0 := \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} \hat{I}_0(\gamma(t)), \tag{3.3}$$

where $\Gamma_0 := \left\{\gamma \in C([0, 1], X_r) : \gamma(0) = 0, \hat{I}_0(\gamma(1)) < 0\right\}$.

Lemma 3.1. There exists a bounded sequence $\{u_n\} \subset X_r$ such that $\hat{I}_0(u_n) \to \hat{c}_0 \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$ and $\hat{I}_0'(u_n) \to 0$.

Proof. By (f_1), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\max\{|h(x, u)u|, |H(x, u)|\} \leq \varepsilon|u|^2 + C_\varepsilon|u|^6, \quad \forall\, u \in \mathbb{R}. \tag{3.4}$$

Then there exist $\rho, \varrho > 0$ such that $\hat{I}_0(u) \geq \varrho$ for $\|u\| = \rho$, in view of the definition of S. Also, $\hat{I}_0(0) = 0$ and $\lim_{t\to+\infty} \hat{I}_0(t\varphi) = -\infty$ for any $\varphi \in C_0^\infty(\wedge_2) \setminus \{0\}$. By the mountain pass theorem in [1], there exists a sequence $\{u_n\} \subset X_r$ such that $\hat{I}_0(u_n) \to \hat{c}_0 \geq \varrho$ and $\hat{I}_0'(u_n) \to 0$. By (f_3'), we get $\frac{1}{\theta} f(u)u - F(u) \geq 0$ for all $u \in \mathbb{R}$. Then

$$\hat{c}_0 + o_n(1) + o_n(1)\|u_n\| = \hat{I}_0(u_n) - \frac{1}{\theta}\left(\hat{I}_0'(u_n), u_n\right)$$

$$\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2+\int_{\mathbb{R}^3\setminus\wedge_2}\left[\frac{1}{\theta}g(u_n)u_n-G(u_n)\right]\mathrm{d}x$$
$$\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\left(1-\frac{1}{\kappa}\right)\|u_n\|^2. \tag{3.5}$$

So, $\|u_n\|$ is bounded. By [24], the function $U(x) := \frac{3^{\frac{1}{4}}}{\left(1+|x|^2\right)^{\frac{1}{2}}}$ is a minimizer for S. Define $U_\varepsilon(x) := \varepsilon^{-\frac{1}{2}}U(\frac{x}{\varepsilon})$. Let $x_0 \in \wedge_1$. Choose $r > 0$ such that $B_{2r}(x_0) \subset \wedge_1$. Define $u_\varepsilon(x) := \psi(x)U_\varepsilon(x)$, where $\psi \in C_0^\infty(B_{2r}(x_0))$ such that $\psi(x) = 1$ for $x \in B_r(x_0)$, $\psi(x) = 0$ for $x \in \mathbb{R}^3 \setminus B_{2r}(x_0)$, $0 \leq \psi(x) \leq 1$, and $|\nabla\psi(x)| \leq C$. By the definition of $\hat{c}_0$, we get $\hat{c}_0 \leq \sup_{t\geq 0}\hat{I}_0(tu_\varepsilon)$. Moreover, by Lemma 2.1 in [28], we get $\hat{c}_0 < \frac{1}{3}S^{\frac{3}{2}}$. □

Lemma 3.2. $\hat{I}_0$ admits a positive critical point u_0 with $\hat{I}_0(u_0) = \hat{c}_0$.

Proof. By Lemma 3.1, there exists a bounded sequence $\{u_n\} \subset X_r$ such that $\hat{I}_0(u_n) \to \hat{c}_0 \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$ and $\hat{I}_0'(u_n) \to 0$. Assume that $u_n \rightharpoonup u_0$ weakly in X_r. Then $\hat{I}_0'(u_0) = 0$. For $R > R_2$, define $\psi_R \in C_0^\infty(\mathbb{R}^3)$ such that $\psi_R(x) = 0$ for $|x| \leq R$, $\psi_R(x) = 1$ for $|x| \geq 2R$, and $0 \leq \psi_R \leq 1$ and $|\nabla\psi_R| \leq \frac{C}{R}$. By $\left(\hat{I}_0'(u_n), \psi_R u_n\right) = o_n(1)$,

$$\int_{\mathbb{R}^3}\left(|\nabla u_n|^2\psi_R + V(x)u_n^2\psi_R\right)\mathrm{d}x + o_n(1)$$
$$\leq \int_{\mathbb{R}^3} g(u_n)u_n\psi_R\mathrm{d}x + \int_{\mathbb{R}^3}|\nabla u_n||\nabla\psi_R||u_n|\mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{R}^3}V(x)u_n^2\psi_R\mathrm{d}x + \frac{C}{R}.$$

Then, for any $\delta > 0$, there exists $R_\delta > 0$ such that for $R > R_\delta$,

$$\lim_{n\to+\infty}\int_{|x|\geq 2R}\left(|\nabla u_n|^2 + V(x)u_n^2\right)\mathrm{d}x \leq \delta. \tag{3.6}$$

Since $h(x,u)u \leq \frac{V_0}{\kappa}u^2$ for $x \in \mathbb{R}^3 \setminus \wedge_2$, by the Lebesgue dominated convergence theorem

$$\lim_{n\to+\infty}\int_{B_{2R}\setminus\wedge_2} h(x,u_n)u_n\mathrm{d}x = \int_{B_{2R}\setminus\wedge_2} h(x,u_0)u_0\mathrm{d}x. \tag{3.7}$$

By the argument of Lemma 2.1 in [26], we obtain that

$$\lim_{n\to+\infty}\int_{\wedge_2} h(x,u_n)u_n\mathrm{d}x = \int_{\wedge_2} h(x,u_0)u_0\mathrm{d}x. \tag{3.8}$$

Combining (3.6)–(3.8), we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} h(x,u_n)u_n\mathrm{d}x = \int_{\mathbb{R}^3} h(x,u_0)u_0\mathrm{d}x. \tag{3.9}$$

Let $v_n = u_n - u_0$. Then

$$o_n(1) = \left(\hat{I}_0'(u_n), u_n\right) - \left(\hat{I}_0'(u_0), u_0\right) = \|v_n\|^2 + o_n(1),$$

from which we derive that $u_n \to u_0$ in X_r, $\hat{I}_0(u_0) = \hat{c}_0$ and $\hat{I}_0'(u_0) = 0$. By $\left(\hat{I}_0'(u_0), u_0^-\right) = 0$, we get $u_0 \geq 0$. The maximum principle implies that u_0 is positive. □

Let $\hat{m}_0 := \inf\{\hat{I}_0(u) : u \in X_r, \hat{I}_0'(u) = 0\}$.

Lemma 3.3. $\hat{m}_0 \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$ is attained.

Proof. By Lemmas 3.1–3.2, we get $\hat{m}_0 \le \hat{I}_0(u_0) = \hat{c}_0 < \frac{1}{3}S^{\frac{3}{2}}$. By the definition of $\hat{m}_0$, there exists $\{u_n\} \subset X_r$ such that $\hat{I}_0(u_n) \to \hat{m}_0$ and $\hat{I}_0'(u_n) = 0$. By $\left(\hat{I}_0'(u_n), u_n\right) = 0$, (3.4), and the definition of S, there exists $C_1 > 0$ such that $\|u_n\|^2 \ge C_1 S^{\frac{3}{2}}$. Similar to (3.5), we get $\hat{m}_0 > 0$. Also, there exists $C_2 > 0$ such that $\|u_n\|^2 \le C_2 S^{\frac{3}{2}}$. Assume that $u_n \rightharpoonup u_0$ weakly in X_r. Then $\hat{I}_0'(u_0) = 0$. Similar to the argument of Lemma 3.2, we get $u_n \to u_0$ in X_r. So $\hat{m}_0 = \hat{I}_0(u_0)$ and $\hat{I}_0'(u_0) = 0$, that is, $\hat{m}_0$ is attained. □

Define by $\hat{S}_0$ the set of ground states of (3.2). By Lemma 3.3, we get $\hat{S}_0 \ne \emptyset$.

Lemma 3.4. $\hat{S}_0$ is compact and there exist $C_1, C_2 > 0$ such that $C_1 S^{\frac{3}{2}} \le \|u\|^2 \le C_2 S^{\frac{3}{2}}$ for all $u \in \hat{S}_0$.

Proof. Similar to the argument of Lemma 3.3, we get $C_1 S^{\frac{3}{2}} \le \|u\|^2 \le C_2 S^{\frac{3}{2}}$ for all $u \in \hat{S}_0$. For any $\{u_n\} \subset \hat{S}_0$, since $\|u_n\|^2 \le C_2 S^{\frac{3}{2}}$, we assume that $u_n \rightharpoonup u$ weakly in X_r. By Lemma 3.3, we get $\hat{I}_0(u_n) = \hat{m}_0 \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$. Similar to the argument of Lemma 3.2, we obtain that $u_n \to u$ in X_r. So, $\hat{S}_0$ is compact. □

Lemma 3.5. ([23]) There exists a constant $C_0 > 0$ such that for all $u \in H_r^1(\mathbb{R}^3)$, there holds $|u(x)| \le \frac{C_0}{|x|^{\frac{1}{2}}}\|u\|_{H^1}$ for any $x \ne 0$.

By (f_1), there exists $C' > 0$ such that

$$|f(u) + (u^+)^5| \le \frac{V_0}{2\kappa}|u| + C'|u|^5, \quad \forall\, u \in \mathbb{R}. \tag{3.10}$$

Choose $R' > 0$ such that for $R_1 > R'$,

$$\frac{2C_2C_0^2S^{\frac{3}{2}}}{R_1}\max\left\{1 + \frac{|\wedge_2|^{\frac{2}{3}}}{S}, \frac{1}{V_0}\right\} \le \sqrt{\frac{V_0}{2\kappa C'}}. \tag{3.11}$$

Lemma 3.6. If $u \in \hat{S}_0$, then $\hat{m}_0 = \hat{I}_0(u) > \hat{I}_0(tu)$ for all $t \ne 1$. Also, there exists $t_0 > 1$ independent of $u \in \hat{S}_0$ such that $\hat{I}_0(t_0 u) \le -2$.

Proof. We claim that

$$\left|\mathrm{supp}u \cap \{x \in \mathbb{R}^3 : \chi(x) > 0\}\right| > 0, \quad \forall\, u \in \hat{S}_0. \tag{3.12}$$

Otherwise, there exists $u \in \hat{S}_0$ such that $\left|\mathrm{supp}u \cap \{x \in \mathbb{R}^3 : \chi(x) > 0\}\right| = 0$. By $(\hat{I}_0'(u), u) = 0$,

$$\|u\|^2 = \int_{\{x\in\mathbb{R}^3:\chi(x)=0\}} g(u)u\mathrm{d}x \le \frac{V_0}{\kappa}\int_{\{x\in\mathbb{R}^3:\chi(x)=0\}} u^2\mathrm{d}x \le \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2\mathrm{d}x,$$

a contradiction. Let $l(t) = \hat{I}_0(tu)$, where $t \ge 0$ and $u \in \hat{S}_0$. Then $l'(t) = ty(t)$, where

$$y(t) = \|u\|^2 - \int_{\mathbb{R}^3}\frac{(1-\chi(x))g(tu)u}{t}\mathrm{d}x - \int_{\mathbb{R}^3}\chi(x)\left(\frac{f(tu)u}{t} + t^4|u|^6\right)\mathrm{d}x.$$

Since $l'(1) = 0$, we have $y(1) = 0$. By (f_3'), we get that $y(t)$ is strictly decreasing on $t > 0$. Then $l'(t) > 0$ for $t \in (0,1)$ and $l'(t) < 0$ for $t > 1$, from which we get $\hat{I}_0(u) > \hat{I}_0(tu)$ for all $t \ne 1$. By $\left(\hat{I}_0'(u), u\right) = 0$, (3.4), and the definition of S, there exists $\delta_0 > 0$ independent of $u \in \hat{S}_0$ such that $\int_{\mathbb{R}^3}\chi(x)|u|^6\mathrm{d}x \ge \delta_0$. Together with Lemma 3.4, we derive that there exists $t_0 > 1$ independent of $u \in \hat{S}_0$ such that $\hat{I}_0(t_0u) \le -2$. □

We consider the following truncated problem of (1.7):

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi|u|^3 u = h(x,u), & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{3.13}$$

The functional associated with (3.13) is as follows:

$$\hat{I}_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{10}\int_{\mathbb{R}^3}\phi_u|u|^5\mathrm{d}x - \int_{\mathbb{R}^3}H(x,u)\mathrm{d}x, \quad u \in X_r.$$

Lemma 3.7. There exists $\lambda_1' > 0$ independent of $u \in \hat{S}_0$ such that $\hat{I}_\lambda(t_0 u) \leq -1$ for $\lambda \in (0, \lambda_1')$.
Proof. By Lemma 2.1, we have

$$\hat{I}_\lambda(t_0 u) = \hat{I}_0(t_0 u) + \frac{\lambda t_0^{10}}{10}\int_{\mathbb{R}^3}\phi_u|u|^5\mathrm{d}x \leq \hat{I}_0(t_0 u) + \frac{\lambda t_0^{10}}{10 S^6}\|\nabla u\|_2^{10}. \tag{3.14}$$

By Lemma 3.4, Lemma 3.6, and (3.14), we derive that there exists $\lambda_1' > 0$ independent of $u \in \hat{S}_0$ such that $\hat{I}_\lambda(t_0 u) \leq -1$. □

Choose $V_0 \in \hat{S}_0$. Define

$$d_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\hat{I}_\lambda(\gamma(t)), \tag{3.15}$$

where $\Gamma := \{\gamma \in C([0,1], X_r) : \gamma(0) = 0, \gamma(1) = t_0 V_0\}$ and $\lambda \in (0, \lambda_1')$. Define

$$D_\lambda := \max_{t\in[0,1]}\hat{I}_\lambda(t t_0 V_0). \tag{3.16}$$

Lemma 3.8. $\lim_{\lambda\to0} d_\lambda = \lim_{\lambda\to0} D_\lambda = \hat{m}_0$.
Proof. By (3.14), Lemma 3.4, and 3.6, we get

$$d_\lambda \leq D_\lambda \leq \hat{m}_0 + \frac{\lambda t_0^{10}}{10 S^6}\left(C_2 S^{\frac{3}{2}}\right)^5.$$

Then $\limsup_{\lambda\to0} d_\lambda \leq \limsup_{\lambda\to0} D_\lambda \leq \hat{m}_0$. By Lemma 3.6, for any $\gamma \in \Gamma$,

$$\max_{t\in[0,1]}\hat{I}_\lambda(\gamma(t)) \geq \max_{t\in[0,1]}\hat{I}_0(\gamma(t)) \geq \hat{c}_0,$$

from which we get $d_\lambda \geq \hat{c}_0$. By Lemma 3.2, we have $\hat{c}_0 \geq \hat{m}_0$, which implies that $\liminf_{\lambda\to0} d_\lambda \geq \hat{m}_0$. □

For η, $d > 0$, define $\hat{I}_\lambda^\eta := \{u \in X_r : \hat{I}_\lambda(u) \leq \eta\}$ and $\hat{S}_0^d := \{u \in X_r : \inf_{v\in S_0}\|u - v\| \leq d\}$. By Lemma 3.4, we can choose $d > 0$ small such that $\frac{C_1}{2}S^{\frac{3}{2}} \leq \|u\|^2 \leq 2C_2 S^{\frac{3}{2}}$ for all $u \in \hat{S}_0^d$.
Lemma 3.9. Let $\{u_{\lambda_i}\} \subset \hat{S}_0^d$ with $\lim_{i\to\infty}\lambda_i = 0$ be such that $\lim_{i\to\infty}\hat{I}_{\lambda_i}(u_{\lambda_i}) \leq \hat{m}_0$ and $\lim_{i\to\infty}\hat{I}_{\lambda_i}'(u_{\lambda_i}) = 0$. Then, for $d > 0$ small, there exists $u_0 \in \hat{S}_0$ such that $u_{\lambda_i} \to u_0$ in X_r up to a subsequence.
Proof. Since $\{u_{\lambda_i}\} \subset \hat{S}_0^d$, we have $\frac{C_1}{2}S^{\frac{3}{2}} \leq \|u_{\lambda_i}\|^2 \leq 2C_2 S^{\frac{3}{2}}$. Moreover, $\lim_{i\to\infty}\hat{I}_0(u_{\lambda_i}) \leq \hat{m}_0$ and $\lim_{i\to\infty}\hat{I}_0'(u_{\lambda_i}) = 0$. Similar to the argument of Lemma 3.2, we derive that there exists $u_0 \in X_r$ such that $u_{\lambda_i} \to u_0$ in X_r. So, $\|u_0\|^2 \geq \frac{C_1}{2}S^{\frac{3}{2}}$, $\hat{I}_0(u_0) \leq \hat{m}_0$, and $\hat{I}_0'(u_0) = 0$, from which we get $u_0 \in \hat{S}_0$. □

Lemma 3.10. Let $d > 0$. Then there exists $\eta > 0$ such that for small $\lambda > 0$, $\hat{I}_\lambda(\gamma(t)) \geq d_\lambda - \eta$ implies that $\gamma(t) \in \hat{S}_0^{\frac{d}{2}}$, where $\gamma(t) = tt_0 V_0$ for $t \in [0, 1]$.

Proof. By Lemma 3.6, if $\gamma(t) \notin \hat{S}_0^{\frac{d}{2}}$, then there exists $\delta > 0$ such that $|tt_0 - 1| \geq \delta$. Moreover, there exists $\eta' > 0$ such that $\hat{I}_0(\gamma(t)) \leq m_0 - \eta'$. By Lemma 2.1, Lemma 3.4, and Lemma 3.8, there exists $\eta > 0$ such that for small $\lambda > 0$, it holds that $\hat{I}_\lambda(\gamma(t)) < d_\lambda - \eta$. □

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, we can use Lemmas 3.8–3.10 to derive that, for small $\lambda > 0$, there exists $\{u_n\} \subset \hat{S}_0^d \cap \hat{I}_\lambda^{D_\lambda}$ such that $\hat{I}'_\lambda(u_n) \to 0$. Then $\frac{C_1}{2} S^{\frac{3}{2}} \leq \|u_n\|^2 \leq 2C_2 S^{\frac{3}{2}}$. Assume that $u_n \rightharpoonup u_\lambda$ weakly in X_r. Then $\hat{I}'_\lambda(u_\lambda) = 0$. Let $u_n = v_n + w_n$, where $v_n \in \hat{S}_0$ and $\|w_n\| \leq d$. By Lemma 3.4, there exists $v_\lambda \in \hat{S}_0$ such that $v_n \to v_\lambda$ in X_r. Assume that $w_n \rightharpoonup w_\lambda$ in X_r. Then $\|w_\lambda\| \leq d$. So, $u_\lambda \in \hat{S}_0^d$. Moreover, $\frac{C_1}{2} S^{\frac{3}{2}} \leq \|u_\lambda\|^2 \leq 2C_2 S^{\frac{3}{2}}$. Together with (3.1) and Lemma 3.5, we have

$$|u_\lambda(x)|^2 \leq 2C_2 C_0^2 S^{\frac{3}{2}} \max\left\{1 + \frac{|\wedge_2|^{\frac{2}{3}}}{S}, \frac{1}{V_0}\right\} \frac{1}{|x|}, \quad \forall x \neq 0. \tag{3.17}$$

By (3.11), we get $\max_{x \in \overline{\wedge_2}} u_\lambda(x) \leq \sqrt[4]{\frac{V_0}{2\kappa C'}}$. Let $\varphi = (u_\lambda - \sigma)^+$, where $\sigma = \sqrt[4]{\frac{V_0}{2\kappa C'}}$. By $\left(\hat{I}'_\lambda(u_\lambda), \varphi\right) = 0$,

$$\begin{aligned}
&\int_{(\mathrm{R}^3 \setminus \wedge_2) \cap \{x \in \mathrm{R}^3 : u_\lambda(x) > \sigma\}} |\nabla u_\lambda|^2 \mathrm{d}x + \int_{\mathrm{R}^3 \setminus \wedge_2} V(x) u_\lambda (u_\lambda - \sigma)^+ \mathrm{d}x \\
&\leq \int_{\mathrm{R}^3 \setminus \wedge_2} g(u_\lambda)(u_\lambda - \sigma)^+ \mathrm{d}x \leq \frac{1}{2} \int_{\mathrm{R}^3 \setminus \wedge_2} V(x) u_\lambda (u_\lambda - \sigma)^+ \mathrm{d}x.
\end{aligned} \tag{3.18}$$

Since $V(x) \geq V_0$ for $x \in \mathrm{R}^3 \setminus \wedge_2$, by (3.18), we get $u_\lambda(x) \leq \sigma$ for $x \in \mathrm{R}^3 \setminus \wedge_2$. Then $h(x, u_\lambda) = f(u_\lambda) + u_\lambda^5$, from which we get $I'_\lambda(u_\lambda) = 0$. Together with Lemma 3.9, we get the result. □

4. Conclusions

In this paper, we study the existence and asymptotic behavior of positive solutions of a non-autonomous Schrodinger-Poisson equation with critical growth. First, we consider the case that the nonlinearity satisfies the Berestycki-Lions type condition with critical growth. To the best of our knowledge, existing results on Schrodinger-Poisson equations are about radial solutions. However, the problem is quite different when we consider the problem in a non-radial setting. Second, we consider the case that the zero set of the potential is contained in a spherical shell. To the best of our knowledge, there are no results on this question. By developing some techniques in variational methods, we solve the problem successfully.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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