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Research article

The dual of a space of compact operators

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Abstract: Let X and Y be Banach spaces. We provide the representation of the dual space of compact operators $K(X, Y)$ as a subspace of bounded linear operators $\mathcal{L}(X, Y)$. The main results are: (1) If Y is separable, then the dual forms of $K(X, Y)$ can be represented by the integral operator and the elements of $C[0, 1]$. (2) If X^{**} has the weak Radon-Nikodym property, then the dual forms of $K(X, Y)$ can be represented by the trace of some tensor products.

Keywords: compact operators; Banach space; dual space; Radon-Nikodym property
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1. Introduction

Let X and Y be Banach spaces. The spaces of compact operators $K(X, Y)$ is the one of the most important spaces of Banach space theory. Many mathematicians have investigated the nature of $K(X, Y)$ and their duals [1–5]. In particular, Feder and Saphar proved the following theorem.

Theorem 1.1. *Suppose X^{**} or Y^* has the Radon-Nikodým property. For every $\phi \in \mathcal{K}(X, Y)^*$ and $\varepsilon > 0$, there are $(x_n^{**}) \subset X^{**}$ and $(y_n^*) \subset Y^*$ such that $\phi(T) = \sum_{n=1}^{\infty} x_n^{**} T^*(y_n^*)$ for all $T \in K(X, Y)$ where $\sum_{n=1}^{\infty} \|x_n^{**}\| \|y_n^*\| < \|\phi\| + \varepsilon$.*

Lima and Oja applied this theorem to solve the famous metric approximation problem [6]. The first author of this work provided the generalized representation of $\mathcal{K}(X, Y)^*$ concerning Feder and Saphar's theorem and the topological property of $\mathcal{K}(X, Y)^*$, in the case that X^{**} or Y^* has the weak Radon-Nikodym property (weak RNP) [7].

Theorem 1.2. *Let X and Y be Banach spaces such that X^{**} or Y^* has the weak Radon-Nikodým property. Then, for all $\phi \in \mathcal{K}(X, Y)^*$, there exist a sequence $(((x_i^n)^{**})_{i=1}^{m_n})_{n=1}^{\infty}$ in X^{**} and a* *sequence* $(((y_i^n)^*)_{i=1}^{m_n})_{n=1}^{\infty}$ *in* Y^* *such that*

$$\phi(T) = \lim_{n\to\infty} \sum_{i=1}^{m_n} (x_i^n)^{**}(T^*((y_i^n)^*))$$

for all $T \in \mathcal{K}(X, Y)$*. Moreover, we have*

$$\limsup_n \sum_{i=1}^{m_n} \|(x_i^n)^{**}\|\|(y_i^n)^*\| \le \|\phi\|.$$

Our aim in this paper is to provide the more general representation of $\mathcal{K}(X, Y)^*$ concerning Feder and Saphar's theorem and the topological property of $\mathcal{K}(X, Y)^*$. The main results are :

(1) If Y is separable, $\mathcal{K}(X, Y)^*$ can be represented by the integral operator and the elements of $C[0, 1]$.
(2) If X^{**} has the weak RNP, $\mathcal{K}(X, Y)^*$ can be represented by the trace of some tensor products.

To prove the main theorems, we shall use the technique of two-trunk tree in Banach space which Lima and Oja developed [8]. Our paper is organized as follows : In Section 2, we introduce the concepts of two-trunk tree in Banach space. Then, for given Banach spaces X and Y, we provide the representation of $\mathcal{K}(X, Y)^*$ in the case that Y is separable (see Theorem 2.4). In Section 3, we provide the new characterization of the weak RNP (see Theorem 3.2). By using this, we present the improved form of the representation of $\mathcal{K}(X, Y)^*$ compared to Theorem B. Our conclusion follows in Section 4.

2. Dual spaces of compact operator spaces: Separable case

In this section, we provide the representation of $\mathcal{K}(X, Y)^*$ when Y is separable. We need some lemmas and definitions. Let $F(X, Y)$ be the space of finite rank operators. For $S \in \mathcal{F}(X, Y)$, let

$$\|S\|_{\mathcal{N}^0} := \inf\Big\{ \sum_{n=1}^{l} \|x_n^*\|\|y_n\| : S = \sum_{n=1}^{l} x_n^* \otimes y_n \Big\}.$$

As usual, $\mathcal{L}(X, Y)$ is the space of bounded linear operators, and $\mathcal{I}$ is the space of integral operators. The space of Pietsch integral operators from X into Y with Pietsch integral norm $\|\cdot\|_{\mathcal{P}}$ is denoted by $\mathcal{P}(X, Y)$. Note that if T is a Pietsch integral operator, then T is an integral operator. Also, it is known that $\mathcal{I}(X, Y) = \mathcal{P}(X, Y)$ if Y is a dual space. We denote by $S_m \xrightarrow{\tau_c} R$ as $m \to \infty$ in $\mathcal{L}(X, Y)$ if for every compact subset K_0 of X and every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \geqslant N$, $\sup_{x\in K_0} \|S_m(x) - R(x)\| \le \varepsilon$.

Lemma 2.1. *([9], p. 102, Lemma 1) Let X, Y, and Z be Banach spaces. For every* $S \in \mathcal{F}(X, Y)$ *and every* $R \in \mathcal{I}(Y, Z)$*, we have*

$$\|RS\|_{\mathcal{N}^0} \le \|R\|_{\mathcal{I}}\|S\|.$$

Definition 2.1. *Let X be a Banach space. We say that a system* $((x_{k,2^n})_{k=0}^{2^n})_{n=0}^{\infty}$ *of elements of X is a two-trunk tree in X if for all* $n = 0, 1, \ldots$ *and* $k = 1, 2, \ldots, 2^n - 1$

$$x_{k,2^n} = \frac{1}{2}x_{2k-1,2^{n+1}} + x_{2k,2^{n+1}} + \frac{1}{2}x_{2k+1,2^{n+1}},$$

$$x_{0,2^n} = x_{0,2^{n+1}} + x_{1,2^{n+1}},$$

$$x_{2^n,2^n} = \frac{1}{2}x_{2^{n+1}-1,2^{n+1}} + x_{2^{n+1},2^{n+1}}.$$

The ℓ_1-tree space $\ell_1^{tree}(X)$ consists of all two-trunk tree $x = (x_{k,2^n}) = ((x_{k,2^n})_{k=0}^{2^n})_{n=0}^{\infty}$ in X such that

$$\|x\| := \sup_n \sum_{k=0}^{2^n} \|x_{k,2^n}\| < \infty.$$

By [8], $\ell_1^{tree}(X)$ is a Banach space.

The scaling function $\varphi : \mathbb{R} \to \mathbb{R}$ is defined by $\varphi(t) = 1 + t$ for $t \in [-1, 0]$, $\varphi(t) = 1 - t$ for $t \in [0, 1]$, and $\varphi(t) = 0$ for $t \notin [-1, 1]$. The functions $g_{k,2^n}, n = 0, 1, \ldots, k = 0, 1, \ldots, 2^n$ are defined by $\varphi(2^n t - k), t \in [0, 1]$. Then, $g_{k,2^n}$ satisfies the following:

$$g_{k,2^n}(\frac{k}{2^n}) = 1, \quad g_{k,2^n}(\frac{j}{2^n}) = 0, \quad j \neq k,$$

$$g_{k,2^n}(\frac{j}{2^{n+1}}) = 0, \quad j \notin \{2k-1, 2k, 2k+1\},$$

$$g_{k,2^n}(\frac{j}{2^{n+1}}) = 1, \quad j = 2k,$$

$$g_{k,2^n}(\frac{j}{2^{n+1}}) = \frac{1}{2}, \quad j \in \{2k-1, 2k+1\}.$$

Then, it is clear that $((g_{k,2^n})_{k=0}^{2^n})_{n=0}^{\infty}$ is a two-trunk tree in $C[0, 1]$.

Theorem 2.1. *([8], Theorem 3.2) Let X be a Banach space. Then, $\mathcal{P}(C[0, 1], X)$ is isometrically isomorphic to $\ell_1^{tree}(X)$, by mapping*

$$T \to ((Tg_{k,2^n})_{k=0}^{2^n})_{n=0}^{\infty}, T \in \mathcal{P}(C[0, 1], X).$$

The inverse mapping

$$((x_{k,2^n})_{k=0}^{2^n})_{n=0}^{\infty} \to T$$

is given by

$$Tf = \lim_n \sum_{k=0}^{2^n} f(\frac{k}{2^n}) x_{k,2^n}, f \in C[0, 1].$$

Now, we are in a position to state our main theorem.

Theorem 2.2. *Suppose that Y is separable. If $\phi \in (\mathcal{K}(X, Y), \|\cdot\|)^*$, then there exists $R \in \mathcal{I}(C[0, 1], X^{**})$ such that*

$$\phi(U) = \lim_m \sum_{k=0}^{2^m} R(g_{k,2^m})(U^* i^*(\delta_{k,2^m})), \quad \textit{for every } U \in \mathcal{K}(X, Y),$$

where $i : Y \to C[0, 1]$ is an isometry and $\sup_m \sum_{k=0}^{2^m} \|R(g_{k,2^m})\|_{X^{**}} = \|\phi\|$ and $\delta_{k,2^m} \in C[0, 1]^*$ is given by $\delta_{k,2^m}(f) = f(\frac{k}{2^m})$ for all $f \in C[0, 1]$.

Proof. Since Y is separable, there exists an isometry $i : Y \to C[0, 1]$. Thus, the map $J_1 : \mathcal{K}(X, Y) \to \mathcal{K}(X, C[0, 1])$ defined by $J_1(T) = iT$ is an isometry. Let $J_2 : \mathcal{K}(X, C[0, 1]) \to \mathcal{K}_{adj}(C[0, 1]^*, X^*) := \{T^* : T \in \mathcal{K}(X, C[0, 1])\}$ be the isometry via $J_2(T) = T^*$. Since $C[0, 1]$ has the metric approximation property, $\mathcal{K}(X, C[0, 1]) = \overline{\mathcal{F}(X, C[0, 1])}^{\|\cdot\|}$, and so

$\mathcal{K}_{adj}(X, C[0,1]) = \overline{\mathcal{F}_{adj}(X, C[0,1])}^{\|\cdot\|}$, which is isometrically isomorphic to the injective tensor product $C[0,1]\hat{\otimes}_\varepsilon X^*$ via the canonical isometry J_3.

$$\mathcal{K}(X,Y) \xrightarrow{J_1} \mathcal{K}(X, C[0,1]) \xrightarrow{J_2} \mathcal{K}_{adj}(C[0,1]^*, X^*) \xrightarrow{J_3} C[0,1]\hat{\otimes}_\varepsilon X^*.$$

Let $J := J_3J_2J_1$. Now, suppose that $\phi \in (\mathcal{K}(X,Y), \|\cdot\|)^*$. Then, $\phi J^{-1} \in (J(\mathcal{K}(X,Y)), \|\cdot\|_\varepsilon)^*$. Choose a Hahn-Banach extension $\widehat{\phi J^{-1}} \in (C[0,1]\hat{\otimes}_\varepsilon X^*)^*$ of ϕJ^{-1}. Let

$$\psi : \mathcal{I}(C[0,1], X^{**}) \to (C[0,1]\hat{\otimes}_\varepsilon X^*)^*$$

be the canonical isometry ([10], Section 3). Let $R := \psi^{-1}(\widehat{\phi J^{-1}}) \in \mathcal{I}(C[0,1], X^{**})$. By the well-known results of Grothendieck ([2], p.99), we have $\mathcal{P}(C[0,1], X^{**}) = \mathcal{I}(C[0,1], X^{**})$ as Banach spaces. Then, by Theorem 2.3, we have

$$R(f) = \lim_m \sum_{k=0}^{2^m} \delta_{k,2^m}(f) R(g_{k,2^m}), f \in C[0,1],$$

and $\|\phi\| = \|R\|_{\mathcal{I}} = \|R\|_{\mathcal{P}} = \sup_m \sum_{k=0}^{2^m} \|R(g_{k,2^m})\|_{X^{**}}$. For each $m \in \mathbb{N}$, let $S_m = \sum_{k=0}^{2^m} \delta_{k,2^m} \otimes R(g_{k,2^m})$. From Theorem 2.3, it is clear that S_m converges to R pointwisely in $\mathcal{I}(C[0,1], X^{**})$. Moreover, by ([8], proof of Theorem 3.2), we have

$$\|S_m\|_{\mathcal{I}} = \|S_m\|_{\mathcal{P}} = \sum_{k=0}^{2^m} \|R(g_{k,2^m})\|_{X^{**}} \le \|R\|_{\mathcal{I}}.$$

Or, put $P_m = \sum_{k=0}^{2^m} \delta_{k,2^m} \otimes g_{k,2^m}$. Then, P_m is a projection from $C[0,1]$ into $C[0,1]$ with $\|P_m\| = 1$. Since $S_m = RP_m$ for all m, by Lemma 2.1 we have $\|S_m\|_{\mathcal{I}} \leqslant \|S_m\|_{\mathcal{N}^0}\|R\|_I$.

For every $V \in \mathcal{I}(C[0,1], X^{**})$, we denote by $[\psi(V), \cdot\,]$ the dual action on $C[0,1]\hat{\otimes}_\varepsilon X^*$. Now, let $U \in \mathcal{K}(X,Y)$. Since ψ is an isometric isomorphism and S_m converges to R pointwisely, we have

$$\lim_m [\psi(S_m), J(U)] = [\psi(R), J(U)].$$

For every $m \in \mathbb{N}$, by the definition of J and the dual action $[\psi(\cdot), \cdot\,]$ on $C[0,1]\hat{\otimes}_\varepsilon X^*$, we have

$$[\psi(S_m), J(U)] = \sum_{k=0}^{2^m} R(g_{k,2^n})((iU)^*\delta_{k,2^m}).$$

By the above arguments, we have the desired equation that

$$\begin{aligned}
\phi(U) &= \phi J^{-1}(J(U)) \\
&= \psi\psi^{-1}(\widehat{\phi J^{-1}})(J(U)) \\
&= [\psi(R), J(U)] \\
&= \lim_m [\psi(S_m), J(U)] \\
&= \lim_m \sum_{k=0}^{2^m} R(g_{k,2^n})((iU)^*\delta_{k,2^m}) \\
&= \lim_m \sum_{k=0}^{2^m} R(g_{k,2^n})(U^*i^*\delta_{k,2^m}).
\end{aligned}$$

Let $\mathcal{K}_{w^*}(X^*, Y)$ be the space of weak*-to-norm continuous compact operators from X^* to Y. We provide the similar theorem in $\mathcal{K}_{w^*}(X^*, Y)$.

Theorem 2.3. *Suppose that Y is separable. If $\phi \in (\mathcal{K}_{w^*}(X^*, Y), \|\cdot\|)^*$, then there exists $R \in \mathcal{I}(C[0,1], X^*)$ such that*

$$\phi(U) = \lim_m \sum_{k=0}^{2^m} i^*(\delta_{k,2^m})(UR(g_{k,2^m})), \quad \forall U \in \mathcal{K}_{w^*}(X^*, Y),$$

where $i : Y \to C[0,1]$ is an isometry and $\sup_m \sum_{k=0}^{2^m} \|R(g_{k,2^m})\|_{X^*} = \|\phi\|$ and $\delta_{k,2^m} \in C[0,1]^*$ is given by $\delta_{k,2^m}(f) = f(\frac{k}{2^m})$ for all $f \in C[0,1]$.

Proof. Since Y is separable, there exists an isometry $i : Y \to C[0,1]$. Thus, the map $J_1 : \mathcal{K}(X, Y) \to \mathcal{K}(X, C[0,1])$ defined by $J_1(T) = iT$ is an isometry. Recall that if a Banach space B has the *approximation property* (AP), then for every Banach space Z, we have $\mathcal{K}_{w^*}(Z^*, B) = \overline{\mathcal{F}_{w^*}(Z^*, B)}^{\|\cdot\|}$. Since $C[0,1]$ has AP, we have $\mathcal{K}_{w^*}(X^*, C[0,1]) = \overline{\mathcal{F}_{w^*}(X^*, C[0,1])}^{\|\cdot\|}$, which is isometric to the injective tensor product $C[0,1]\hat{\otimes}_\varepsilon X$ via the isometry J_2.

$$\mathcal{K}_{w^*}(X^*, Y) \xrightarrow{J_1} \mathcal{K}_{w^*}(X^*, C[0,1]) \xrightarrow{J_2} C[0,1]\hat{\otimes}_\varepsilon X.$$

Let $J := J_2 J_1$. Then, $\phi J^{-1} \in (J(\mathcal{K}_{w^*}(X^*, Y)), \|\cdot\|_\varepsilon)^*$. Choose a Hahn-Banach extension $\widehat{\phi J^{-1}} \in (C[0,1]\hat{\otimes}_\varepsilon X)^*$ of ϕJ^{-1}. Let $\psi : \mathcal{I}(C[0,1], X^*) \to (C[0,1]\hat{\otimes}_\varepsilon X)^*$ be the canonical isometry ([10], Section 3). Let $R := \psi^{-1}(\widehat{\phi J^{-1}}) \in \mathcal{I}(C[0,1], X^*)$. By the well-known results of Grothendieck ([2], p. 99), we have $\mathcal{P}(C[0,1], X^*) = \mathcal{I}(C[0,1], X^*)$ as Banach spaces. Then, by Theorem 2.3, we have

$$R(f) = \lim_m \sum_{k=0}^{2^m} \delta_{k,2^m}(f) R(g_{k,2^m}), \quad f \in C[0,1],$$

and $\|\phi\| = \|R\|_\mathcal{I} = \|R\|_\mathcal{P} = \sup_m \sum_{k=0}^{2^m} \|R(g_{k,2^m})\|_{X^*}$. For each $m \in \mathbb{N}$, let $S_m = \sum_{k=0}^{2^m} \delta_{k,2^m} \otimes R(g_{k,2^m})$. From Theorem 2.3, it is clear that S_m converges to R pointwisely in $\mathcal{I}(C[0,1], X^*)$ and $\|S_m\| \le \|R\|_I$. Then we obtain that $S_m \xrightarrow{\tau_c} R$ as $m \to \infty$. Moreover, by ([8], proof of Theorem 3.2), we have

$$\|S_m\|_\mathcal{I} = \|S_m\|_\mathcal{P} = \sum_{k=0}^{2^m} \|R(g_{k,2^m})\|_{X^*} \le \|R\|_\mathcal{I}.$$

For every $V \in \mathcal{I}(C[0,1], X^*)$, denote by $[\psi(V), \cdot\,]$ the dual action on $C[0,1]\hat{\otimes}_\varepsilon X$. Now, let $U \in \mathcal{K}(X, Y)$. Since ψ is isometric isomorphism and S_m converges to R pointwisely, we have

$$\lim_m [\psi(S_m), J(U)] = [\psi(R), J(U)].$$

For every m, by the definition of J and the dual action $[\psi(\cdot), \cdot\,]$ on $C[0,1]\hat{\otimes}_\varepsilon X^*$,

$$[\psi(S_m), J(U)] = \sum_{k=0}^{2^m} \delta_{k,2^m}(iUR(g_{k,2^m})).$$

By the above arguments, we have

$$\begin{aligned}\phi(U) &= \phi J^{-1}(J(U))\\ &= \psi\psi^{-1}(\widehat{\phi J^{-1}})(J(U))\\ &= [\psi(R), J(U)]\\ &= \lim_m[\psi(S_m), J(U)]\\ &= \lim_m \sum_{k=0}^{2^m} \delta_{k,2^m}(iUR(g_{k,2^m}))\\ &= \lim_m \sum_{k=0}^{2^m} i^*(\delta_{k,2^m})(UR(g_{k,2^m})).\end{aligned}$$

Now, suppose X is separable. Using the proof (b) of ([7], Corollary 4.4), we obtain the following.

Proposition 2.1. *Suppose that X is separable. If $\phi \in (\mathcal{K}_{w^*}(X^*, Y), \|\cdot\|)^*$, then there exists $R \in \mathcal{I}(C[0,1], Y^*)$ such that*

$$\phi(U) = \lim_m \sum_{k=0}^{2^m} R(g_{k,2^m})(Ui^*(\delta_{k,2^m})), \quad \forall U \in \mathcal{K}_{w^*}(X^*, Y),$$

where $i : X \to C[0,1]$ is an isometry, $\sup_m \sum_{k=0}^{2^m} \|R(g_{k,2^m})\|_{Y^*} = \|\phi\|$, and $\delta_{k,2^m} \in C[0,1]^*$ is given by $\delta_{k,2^m}(f) = f(\frac{k}{2^m})$ for all $f \in C[0,1]$.

3. Dual spaces of compact operator spaces: The weak Radon-Nikodym property case

In this section, we provide the representation of $\mathcal{K}(X,Y)^*$ when X^{**} has the weak RNP. We need some definitions. First, we introduce the definition of weak RNP. To do this, we have to define weakly integrable functions. Let (Ω, Σ, μ) be a finite complete measure space. We recall that a μ-measurable function $f : \Omega \to X$ is said to be weakly integrable with respect to μ if the function $x^* f$ is integrable for every $x^* \in X^*$. We say that the weakly integrable function f is Pettis integrable if the Dunford integral $\int_E f d\mu$ belongs to X for every $E \in \Sigma$. A Banach space X has the weak RNP if for each finite complete measure space (Ω, Σ, μ) and each μ-continuous X-valued countably additive vector measure $m : \Sigma \to X$ of bounded variation, there exists a Pettis integrable function $f : \Omega \to X$ such that

$$m(E) = \int_E f d\mu, \quad \forall E \in \Sigma.$$

Musial proved that a Banach space X contains no isomorphic copy of ℓ_1 if and only if X^* has the weak RNP [11].

Now, we provide the definition of p-integral and p-nuclear operators. Let K be a compact topological space and $\mathcal{B}_K$ be the σ-algebra of Borel sets on K. Recall that if $T : C(K) \to X$ is a bounded linear operator, then $\mu : \mathcal{B}_K \to X^{**}$ is called the representing measure for the operator T if

$\mu(E) = T^{**}(\chi_E)$. The vector sequence $\{x_n\}$ in X is *weakly p-summable* if the scalar sequences $\{x^*(x_n)\}$ are in ℓ_p for every $x^* \in X^*$. We denote by

$$\ell_p^w(X)$$

the set of all such sequences in X. A Banach space operator $u : X \to Y$ is called p-integral $(1 \leqslant p \leqslant \infty)$ if there are a probability measure μ and bounded linear operators $a : L_p(\mu) \to Y^{**}$ and $b : X \to L_\infty(\mu)$ such that $Q_Y u = a i_p b$ where $i_p : L_\infty(\mu) \to L_p(\mu)$ and $Q_Y : Y \to Y^{**}$ is the canonical isometric embedding. A Banach space operator $u : X \to Y$ is called strictly p-integral $(1 \leqslant p \leqslant \infty)$ if there is a probability measure μ and bounded linear operators $a : L_p(\mu) \to Y$ and $b : X \to L_\infty(\mu)$ such that $u = a i_p b$. A Banach space operator $u : X \to Y$ is called p-nuclear $(1 \leqslant p < \infty)$ if there are operators $a \in \mathcal{L}(\ell_p, Y)$, $b \in \mathcal{L}(X, \ell_\infty)$ and a sequence $\lambda \in \ell_p$ such that

$$u = aM_\lambda b,$$

where $M_\lambda : \ell_\infty \to \ell_p : (\xi_n) \to (\lambda_n \xi_n)$. We define the p-nuclear norm as

$$\nu_p(u) := \inf \|a\| \cdot \|M_\lambda\| \cdot \|b\|,$$

the infimum being extended over all factorizations above. We denote by

$$\mathcal{N}_p(X, Y)$$

the space of all p-nuclear operators from X into Y with the norm ν_p. Throughout the rest of the paper, p^* is the Hölder's conjugate of p, i.e., $1/p + 1/p^* = 1$.

Now, we provide a new characterization of the weak RNP property for the dual version.

Theorem 3.1. *X^* has the weak RNP if and only if $R \in \mathcal{I}(C(K), X^*)$, then $R \in \mathcal{N}_p(C(K), X^*)$ for all $p > 1$. That is, there exist $(z_n^*) \in \ell_p(C(K)^*)$ and $(x_n^*) \in \ell_{p^*}^w(X^*)$ such that*

$$R = \sum_{n=1}^{\infty} z_n^* \otimes x_n^*$$

and the series converges in $\mathcal{L}(C(K), X^*)$.

Proof. Suppose that if $R \in \mathcal{I}(C(K), X^*)$, then $R \in \mathcal{N}_p(C(K), X^*)$ for all $p > 1$. Take any $R \in \mathcal{I}(C(K), X^*)$ and $p > 1$. Then, R is p-nuclear. It is clear that R is compact. By ([7], Lemma 3.1), X^* has the weak RNP.

Conversely, suppose that X^* has the weak RNP. Let $R : C(K) \to X^*$ be an integral operator and $p > 1$. By ([10], Proposition 5.28), the representing measure μ is a vector measure of bounded variation with values in X^* and $R = S \circ J$ where $J : C(K) \to L_1(|\mu|_1)$ is the natural injection and $S : L_1(|\mu|_1) \to X^*$ is a bounded linear operator via

$$S(f) = \int_K f d|\mu|, \quad \forall f \in L_1(|\mu|_1).$$

By the assumption, we have $\ell_1 \not\subseteq X$, and μ has a relatively compact range by Lemma 3.53 in [12]. By ([12], Proposition 3.56), S is completely continuous. Also, since J is a Piesch integral operator, J is strictly p-integral. By ([13], p.124), R is p-nuclear.

Now, we are in a position to state our main theorem.

Theorem 3.2. *Suppose that X^{**} has the weak RNP. If $\phi \in (\mathcal{K}(X,Y), \|\cdot\|)^*$, then there exists $(z_n^*) \in \ell_p(C(K)^*)$ and $(x_n^{**}) \in \ell_{p^*}^w(X^{**})$ such that*

$$\phi(U) = \sum_{n=1}^{\infty} x_n^{**}((U^* i^*(z_n^*)) \text{ for every } U \in \mathcal{K}(X,Y),$$

where $K = B_{Y^*}$ and $i : Y \to C(K)$ by

$$i(y)(y^*) = \delta_y(y^*) = y^*(y), \forall y^* \in B_{Y^*}$$

is an isometry and the series

$$\sum_{n=1}^{\infty} z_n^* \otimes x_n^{**}$$

converges in $\mathcal{L}(C(K), X^{**})$.

Proof. Suppose that X^{**} has the weak RNP. Let K be B_{Y^*}. Then K is weak* compact subset of Y^*. Define $i : Y \to C(K)$ by

$$i(y) = \delta_y(y^*) = y^*(y), \forall y^* \in B_{Y^*}.$$

Then, i is the canonical isometry of Y into $C(K)$. Thus the map $J_1 : \mathcal{K}(X,Y) \to \mathcal{K}(X, C(K))$ defined by $J_1(T) = iT$ is an isometry. Let $J_2 : \mathcal{K}(X, C(K)) \to \mathcal{K}_{adj}(C(K)^*, X^*) := \{T^* : T \in \mathcal{K}(X, C(K))\}$ be the isometry via $J_2(T) = T^*$. Since $C(K)$ has the metric approximation property, $\mathcal{K}(X, C(K))) = \overline{\mathcal{F}(X, C(K)))}^{\|\cdot\|}$ and so $\mathcal{K}_{adj}(X, C(K))) = \overline{\mathcal{F}_{adj}(X, C(K)))}^{\|\cdot\|}$ which is isometrically isomorphic to the injective tensor product $C(K)\hat{\otimes}_\varepsilon X^*$ via the canonical isometry J_3.

$$\mathcal{K}(X,Y) \xrightarrow{J_1} \mathcal{K}(X, C(K)) \xrightarrow{J_2} \mathcal{K}_{adj}(C(K)^*, X^*) \xrightarrow{J_3} C(K)\hat{\otimes}_\varepsilon X^*.$$

Let $J := J_3 J_2 J_1$. Now, suppose that $\phi \in (\mathcal{K}(X), \|\cdot\|)^*$. Then $\phi J^{-1} \in (J(\mathcal{K}(X,Y)), \|\cdot\|_\varepsilon)^*$. Choose a Hahn-Banach extension $\widehat{\phi J^{-1}} \in (C(K)\hat{\otimes}_\varepsilon X^*)^*$ of ϕJ^{-1}. Let

$$\psi : \mathcal{I}(C(K), X^{**}) \to (C(K)\hat{\otimes}_\varepsilon X^*)^*$$

be the canonical isometry ([10], Section 3). Let $R := \psi^{-1}(\widehat{\phi J^{-1}}) \in \mathcal{I}(C(K), X^{**})$. By the well-known results of Grothendieck ([2], p. 99), we have $\mathcal{P}(C(K), X^{**}) = \mathcal{I}(C(K), X^{**})$ as Banach spaces. Then, by Theorem 3.1, there exist $(z_n^*) \in \ell_p(C(K)^*)$ and $(x_n^{**}) \in \ell_{p^*}^w(X^{**})$ such that

$$R = \sum_{n=1}^{\infty} z_n^* \otimes x_n^{**}$$

converges in $\mathcal{L}(C(K), X^{**})$. Then, we have

$$R(f) = \sum_{n=1}^{\infty} z_n^*(f) x_n^{**}, \quad \forall f \in C(K).$$

For each $m \in \mathbb{N}$, let $S_m = \sum_{n=1}^{m} z_n^* \otimes x_n^{**}$. From Theorem 2.3, it is clear that S_m converges to R pointwisely in $\mathcal{I}(C(K), X^*)$ and $\|S_m\| \le \|R\|_I$. Then, we obtain that $S_m \xrightarrow{\tau_c} R$ as $m \to \infty$.

For every $V \in \mathcal{I}(C(K), X^{**})$, denote by $[\psi(V), \ \cdot\ ]$ the dual action on $C(K)\hat{\otimes}_\varepsilon X^*$. Now, let $U \in \mathcal{K}(X, Y)$. Since ψ is an isometric isomorphism and S_m converges to R pointwisely, we have

$$\lim_m [\psi(S_m), J(U)] = [\psi(R), J(U)].$$

For every $m \in \mathbb{N}$, by the definition of J and the dual action $[\psi(\cdot), \ \cdot\ ]$ on $C(K)\hat{\otimes}_\varepsilon X^*$,

$$[\psi(S_m), J(U)] = \sum_{n=1}^{m} x_n^{**}((iU)^*(z_n^*)).$$

Hence, we have

$$\begin{aligned} \phi(U) &= \phi J^{-1}(J(U)) \\ &= \psi\psi^{-1}(\widehat{\phi J^{-1}})(J(U)) \\ &= [\psi(R), J(U)] \\ &= \lim_m [\psi(S_m), J(U)] \\ &= \sum_{n=1}^{\infty} x_n^{**}((iU)^*(z_n^*)). \end{aligned}$$

4. Conclusions

In this work, we provide new representations of the dual of a space of compact operators $K(X, Y)$ under the separable condition of Y or weak RNP condition of X^{**}. The dual of $K(X, Y)$ can be represented by the integral operator and the elements of $C[0, 1]$ if Y is separable. On the other hand, the dual of $K(X, Y)$ can be represented by the trace of some tensor products if X^{**} has the weak RNP.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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