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Research article

Numerical solutions of Troesch's problem based on a faster iterative scheme with an application

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Abstract: The purpose of this manuscript was to introduce a new iterative approach based on Green's function for approximating numerical solutions of the Troesch's problem. A Banach space of continuous functions was considered for establishing the main outcome. First, we set an integral operator using a Green's function and embedded this new operator into a three-step iterative scheme. We proved the main convergence result with the help of some mild assumptions on the parameters involved in our scheme and in the problem. Moreover, we proved that the new iterative approach was weak w^2-stable. The high accuracy and stability of the scheme was confirmed by several numerical simulations. As an application of the main result, we solved a class of fractional boundary value problems (BVPs). The main results improved and unified several known results of the literature.

Keywords: Troesch's problem; fractional BVPs; numerical solution; iterative scheme; Green's function
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1. Introduction

In the literature, one deals with different linear and nonlinear problems for which the value of the exact solution is always desirable [1]. On the other hand, finding the exact solutions for such problems is not an easy task to deal with and, in some cases, it is impossible (see, e.g., [2, 3]). Accordingly, the value of approximate numerical solutions in such cases is always requested. Various numerical procedures are available in the literature for finding numerical solutions of various linear and nonlinear problems. However, these procedures require a very complicated set of parameters to start and, often, there is no known criteria to establish their convergence theoretically. Alternatively, fixed point theory suggests the existence and approximation of sought solutions in a new way. To achieve the aim, we set an operator having a property that its fixed point set and the set of solutions for the given problem [4] are the same. In this case, finding existence and approximation of the sought solution for a given problem becomes same as the existence and approximation of the fixed point for this operator. One of the most powerful fixed point theorems for the such operators was given by Banach [5] in 1922, which states that "if Y is a Banach space, then any mapping $P : Y \to Y$ that satisfies the property $\|Py - Pz\| \leq \theta\|y - z\|$ for any $y, z \in Y$ and $\theta \in [0, 1)$ admits a fixed point, namely, $y^* \in Y$ and the sequence of Picard approximations [6] $y_{i+1} = Py_i$ is convergent to y^* for any choice of initial approximation $y_0 \in Y$". Notice that, when P is essentially any nonexpansive mapping, that is, $P : Y \to Y$ that satisfies the property, $\|Py - Pz\| \leq \|y - z\|$ for any $y, z \in Y$, then the Picard approximations do not converge to a fixed point. A simple example of such a case is as follows: Consider $Y = [0, 1]$, then $Py = -y + 1$ is nonexpansive on Y with a unique fixed point $y^* = 0.5$; nevertheless, the Picard approximations in this case read as, $y_0, -y_0 + 1, y_0, -y_0 + 1, ...,$ which fails to reach $y^* = 0.5$ for all $y_0 \in Y - \{0.5\}$. Thus, to overcome this case, Krasnoselskii [7] generalized the Picard approximation scheme [6] as follows:

$$\begin{cases} y_0 \in Y, \\ y_{i+1} = (1 - \kappa)y_i + \kappa P y_i \ (i = 0, 1, 2, ...), \end{cases} \tag{1.1}$$

where $0 \leq \kappa \leq 1$.

Mann [8] introduced a new iterative scheme, which contains the Krasnoselskii iteration (1.1) as a special case. The Mann iteration reads as follows:

$$\begin{cases} y_0 \in Y, \\ y_{i+1} = (1 - \kappa_i)y_i + \kappa_i P y_i \ (i = 0, 1, 2, ...), \end{cases} \tag{1.2}$$

where the sequence $\{\kappa_i\}$ controls the convergence with $0 \leq \kappa_i \leq 1$.

In 2013, Khan [9] suggested the Picard-Mann hybrid (PMH) iterative scheme as follows:

$$\begin{cases} y_0 \in Y, \\ z_i = (1 - \kappa_i)y_i + \kappa_i P y_i, \\ y_{i+1} = P z_i \ (i = 0, 1, 2, ...), \end{cases} \tag{1.3}$$

where the sequence $\{\kappa_i\}$ controls the convergence with $0 \leq \kappa_i \leq 1$.

Abbas et al. [10] proposed the following new type of iterative scheme that is independent of the above schemes:

$$\begin{cases} y_0 \in Y, \\ h_i = P y_i, \\ z_i = P h_i, \\ y_{i+1} = (1 - \kappa_i)z_i + \kappa_i P z_i \ (i = 0, 1, 2, ...), \end{cases} \tag{1.4}$$

where the sequence $\{\kappa_i\}$ controls the convergence with $0 \leq \kappa_i \leq 1$.

Remark 1.1. *The above mentioned iterative schemes are extensively studied by many authors for approximating numerical solutions for fixed point problems. There is a very little ammount of work available in the literature that deals with the numerical solutions of boundary value problems (BVPs) based on these schemes. The purpose of this manuscript is to extend the study of Abbas-iteration to the case of numerical solutions of BVPs in a Banach space setting.*

Now we consider the following Troesch's BVPs:

$$y''(t) = \mu \sinh[\mu y(t)], \tag{1.5}$$

where $\mu \geq 0$ and the associated boundary conditions (BCs) are

$$y(0) = 0, \quad y(1) = 1. \tag{1.6}$$

The Troesch's BVPs (1.5) and (1.6) first appeared in the paper [11]. After this, it has been found that this problem has many applications in many areas of applied sciences (see, e.g., [12, 13] and others). The closed form solution for this problem was first studied by Roberts and Shipman [14], while the numerical solution for this problem was first investigated by Troesch [15] under the shooting techniques. After this, the problem got the name as a Troesch's problem and many authors suggested different methods for solving this problem (see, e.g., [16–22] and others). Recently, Kafri et al. [23] suggested a new approach for numerically solving the problems (1.5) and (1.6) using Picard-Green's and Krasnoselskii-Green's iterative scheme by embedding the Green's function into the Picard [6] and Mann (1.2) iterative schemes. They proved that these new schemes converge faster to the sought solutions compared to the all classical approaches. However, they did not study qualitative aspects of their like stability analysis. In recent years, fixed point iterative schemes have been modified using Green's functions (see, e.g., [24–29] and others). In [30], the authors studied PMH-Green's iterative scheme for numerical solutions of a class of singular BVPs. It has been shown that the Green's function based iterative schemes are faster convergent to the solution of many nonlinear problems. Motivated by these Green's function based iterative approaches, we introduce the Abbas-Green's iterative approach by embedding a Green's function associated with linear terms of the problems (1.5) and (1.6) and prove that this new approach converges repidly to the sought solution than the Picard-Green's and Krasnoselskii-Green's approaches. In this way, we carry new outcomes and improve all the classical results concerning the problems (1.5) and (1.6).

The motivation of our work is as follows: First, we shall construct a Green's function with the linear term of the problems (1.5) and (1.6), then construct an integral operator that has the same fixed points set as the solution set of the problems (1.5) and (1.6). Using this operator, we suggest a new iterative scheme based on the Abbas-iterative scheme (1.4), and the final scheme we name as an Abbas-Green's iterative scheme. Using Banach fixed point theorem [5], the constructed operator is a contraction and, therefores, it will admit a unique fixed point, which is the unique solution for the problems (1.5) and (1.6). We establish a strong convergence of our new scheme without imposing strong conditions. We also establish the stability result for our scheme. Eventually, we prove numerical effectiveness of our scheme over other schemes of the literature under various values of sets of parameters involved in our scheme and problem.

2. Preliminaries

The following notions and definitions will be used in the main results.

Definition 2.1. *[31] Suppose we have a selfmap P on a given Banach space Y such that the sequence $\{y_i\}$ of the map P is produced by*

$$\begin{cases} y_0 \in Y, \\ y_{i+1} = U(P, y_i), \end{cases} \tag{2.1}$$

where y_0 is the initial approximation and U is an appropriate function. In such a case, we assume that the sequence $\{y_i\}$ is strongly convergent to some element $y^* \in F_P$, where the set F_P contains all fixed points of P, then $\{y_i\}$ is known as stable of P-stable, provided that for any $\{a_i\}$ in Y, it follows that

$$\lim_{i\to\infty} \|a_{i+1} - U(P, a_i)\| = 0 \Rightarrow \lim_{i\to\infty} a_i = y^*.$$

Definition 2.2. *[32] A given sequence $\{a_i\}$ is known as an equivalent sequence of $\{y_i\}$ in a setting of a Banach space, provided that*

$$\lim_{i\to\infty} \|a_i - y_i\| = 0.$$

The following concept of stability is due to Timis [33]. This notion of stability uses the concept of equivalent sequences opposed to the concept of classical stability that is based on the concept of arbitrary sequences.

Definition 2.3. *[33] Suppose we have a selfmap P defined on a Banach space, namely, Y such that $\{y_i\}$ forms a sequence produced by (2.1) in P. Assume that $\{y_i\}$ is strongly convergent to some element $y^* \in F_P$, then $\{y_i\}$ is known as weak w^2-stable with respect to P if one has an equivalent sequence, namely, $\{a_i\} \subseteq Y$ for $\{y_i\}$, such that*

$$\lim_{i\to\infty} \|a_{i+1} - U(P, a_i)\| = 0 \Rightarrow \lim_{i\to\infty} a_i = y^*.$$

3. Green's function construction for the Troesch's problem

In this paper, we first construct a Green's function for the problems (1.5) and (1.6) as follows: Assume that L denotes the linear term such that

$$L[y] = y''. \tag{3.1}$$

In this case, it is known that if y_1 and y_2 are any two linearly independent solutions for $L = 0$, then the Green's function is denoted often by G and reads as follows:

$$G(t, s) = \begin{cases} c_1 y_1 + c_2 y_2 & \text{when } a < t < s, \\ d_1 y_1 + d_2 y_2 & \text{when } s < t < b. \end{cases}$$

The real constants c_1, c_2, d_1 and d_2 in above function will be found by using the following facts about Green's functions. Notice that G satisfies the following BVPs:

$$L[G(t, s)] = \delta(t - s), \tag{3.2}$$

subjected to the BCs as

$$B_1[G(t, s)] = 0 = B_2[G(t, s)]. \tag{3.3}$$

In this case, the following characterizations of G are well known.

(c1) At $t = s$, the function G is always continuous, that is,

$$G(t, s)_{t\to s^+} = G(t, s)_{t\to s^-}. \tag{3.4}$$

From the definition of $G(t, s)$ and (3.4), we have

$$c_1 y_1(s) + c_2 y_2(s) = d_1 y_1(s) + d_2 y_2(s). \tag{3.5}$$

(c2) The function G' posseses a unit jump discontinuity.

In the differential equation given in (3.2), the function G'' admits essentially a singularity just like the function δ. Thus, one has essentially a lower order derivative in (3.2) and at least one of that possesses such type of singularity; it follows that the function G'' is more singular compared to the function δ and, thus, one will have nothing essentially on the right side of the given equation in order to compare this type of singularity. Now, comparable to passing successively from the function δ to the Heaviside function H to a ramp function, it can be seen that the function G' possesses only a unit jump discontinuity and the function $G(t, s)$ is essentially continuous.

Now integrate (3.2) as follows:

$$\int_{s_-}^{s^+} G''(t, s)dt = \int_{s_-}^{s^+} \delta(t - s)dt. \tag{3.6}$$

Equation (3.2) is equivalent to the following:

$$G'(t, s)|_{s^-}^{s^+} = H(t - s)|_{s^-}^{s^+}, \ \ G'(s^+, s) - G'(s^-, s) = 1. \tag{3.7}$$

It follows that

$$d_1 y_1'(s) + d_2 y_2'(s) - c_1 y_1'(s) - c_2 y_2'(s) = 1. \tag{3.8}$$

We now consider (3.1) and, more generally, we consider the following equation:

$$L[y] = y'' = f(t, y, y', y''), \tag{3.9}$$

with the general BCs:

$$B_1[y] = a_0 y(a) + a_1 y'(a) = 0, \ B_2[y] = b_0 y(b) + b_1 y'(b) = 1, \tag{3.10}$$

where $t \in [a, b]$.

Suppose y_h is the solution of the homogenous part satisfying the given BCs and y_p denotes the particular solution satisfying (3.9) such that it satisfies the BCs $B_k = 0$, $k = 1, 2$. We show that the problems (3.9) and (3.10) have a general solution y with $y = y_h + y_p$. Since G denotes the Green's function and y_p denotes the particular solution, it follows that

$$y_p = \int_0^1 G(t, s) f(s, y(s), y'(s), y''(s))ds. \tag{3.11}$$

Notice that the function G solves (3.2) and (3.3). Now, we consider L on $y_h + y_p$ as follows:

$$\begin{aligned}
L[y_h + y_p] &= L[y_h] + L[\int_0^1 G(t, s) f(s, y(s), y'(s), y''(s))ds] \\
&= \int_0^1 L[G(t, s)] f(s, y(s), y'(s), y''(s))ds \\
&= \int_0^1 \delta(t - s) f(s, y(s), y'(s), y''(s))ds \\
&= f(t, y(t), y'(t), y''(t)).
\end{aligned}$$

$$\begin{aligned} B_k[y_h + y_p] &= B_k[y_h] + B_k[y_p] \\ &= B_k[\int_0^1 G(t,s) f(s, y(s), y'(s), y''(s)) ds] \\ &= \int_0^1 B_k[G(t,s)] f(s, y(s), y'(s), y''(s)) ds. \end{aligned}$$

Since $B_1[G(s,t)] = 0$ and $B_2[G(s,t)] = 1$, it follows that $B_1[y_h + y_p] = 0$ and $B_2[y_h + y_p] = 1$. Hence, we conclude that $y = y_h + y_p$ is a solution of (3.9) and (3.10).

4. A brief description of the iteration

In this section, our aim is to obtain a new modified iterative scheme based on the Green's function. In fact, we combine the Abbas-iterative scheme (1.4) with an integral operator containing a Green's function associated with the problems (1.5) and (1.6). This integral operator has a property that its fixed point set is the same as the solution set of the problems (1.5) and (1.6).

To obtain our objective, we start by considering the following integral operator:

$$M[y] = y_h + \int_0^1 G(t,s) y''(s) ds. \tag{4.1}$$

Now, adding as well as subtracting the term $f(s, y, y', y'')$ within the integral in (4.1), one has

$$M[y] = y_h + \int_0^1 G(t,s)(y''(s) - f(s, y, y', y'')) ds + \int_0^1 G(t,s) f(s, y, y', y'') ds. \tag{4.2}$$

Keep (3.11) in mind, and one has from (4.2):

$$M[y] = y_h + \int_0^1 G(t,s)(y''(s) - f(s, y, y', y'')) ds + y_p. \tag{4.3}$$

However, $y_h + y_p = y$ and, therefore, (4.3) gives us

$$M[y] = y + \int_0^1 G(t,s)(y''(s) - f(s, y, y', y'')) ds. \tag{4.4}$$

Now we apply the Abbas-iterative scheme (1.4) on the operator M in (4.4), and we have

$$\begin{cases} h_i = M[y_i], \\ z_i = M[h_i], \\ y_{i+1} = (1-\kappa_i) z_i + \kappa_i M[z_i] \ (i = 0, 1, 2, 3, \ldots). \end{cases} \tag{4.5}$$

It follows that

$$\begin{cases} h_i = y_i + \int_0^1 G(t,s)(y_i'' - f(s, y_i, y_i', y_i'')) ds, \\ z_i = h_i + \int_0^1 G(t,s)(h_i'' - f(s, h_i, h_i', h_i'')) ds, \\ y_{i+1} = z_i + \kappa_i \int_0^1 G(t,s)(z_i'' - f(s, z_i, z_i', z_i'')) ds \ (i = 0, 1, 2, 3, \ldots). \end{cases} \tag{4.6}$$

5. Convergence analysis and stability results

This section includes the convergence of our scheme. We divide the problem into two cases as follows:

Case I: In this case, we consider the value of $\mu \leq 1$.

To find the value of $G(t, s)$, we consider the two linearly independent solutions of $y'' = 0$ as $1, t$. Therefore,

$$G(t,s) = \begin{cases} c_1 + c_2 t & \text{when } 0 < t < s, \\ d_1 + d_2 t & \text{when } s < t < 1. \end{cases}$$

Using the properties of Green's function, we get the values of the constant $c_1,\ c_2,\ d_1$, and d_2, and we take its adjoint, which makes the function $G(t, s)$ as follows:

$$G(t,s) = \begin{cases} s(1-t) & \text{when } 0 < t < s, \\ t(1-s) & \text{when } s < t < 1. \end{cases}$$

Suppose $Y = C[0, 1]$, which is a Banach space with the supremum norm. Now, set $P_G : Y \to Y$ by

$$P_G y = y + \int_0^1 G(t,s)(y'' - f(y, y', y''))ds. \tag{5.1}$$

Hence, (4.6) takes the following form:

$$\begin{cases} h_i = P_G y_i, \\ z_i = P_G h_i, \\ y_{i+1} = (1-\kappa_i) z_i + \kappa_i P_G z_i \ (i = 0, 1, 2, ...). \end{cases} \tag{5.2}$$

The iterative scheme (5.12) is our desired Abbas-Green's iterative scheme. We now prove its convergence theoretically as follows.

Theorem 5.1. *Consider the operator $P_G : Y \to Y$ in (5.1) and suppose $g(y) = \sinh(\mu y)$ is any function that has essentially a derivative bound, namely, y and y being a Lipschitz function having a constant L_y. Furthermore, if ($L_y + \frac{\mu^2 L_c}{8} < 1$, where $L_c = \max_{t\in[0,1]} |\cosh(\mu y(t))|$ and $\{y_i\}$ is the sequence of iterates due to the Abbas-Green's iteration (5.12) with $\sum \kappa_i = \infty$, then $\{y_i\}$ is strongly convergent to the unique fixed point of P_G and, hence, to the unique sought solution of the problems (1.5) and (1.6).*

Proof. The operator P_G is a contraction. As integrating each term in the definition of P_G by using integration by parts formula, one has

$$P_G = (1-t)y(0) + ty(1) - \int_0^1 G(s,t)\mu \sinh[\mu y(s)]ds. \tag{5.3}$$

Notice that $\int_0^1 G(t, s) = \frac{t-t^2}{2}$ and the function $g(t) = \frac{t-t^2}{2}$ on $[0, 1]$ has a maximum $\frac{1}{8}$. Accordingly, $\int_0^1 |G(t, s)| \leq \frac{1}{8}$. From (5.3) and by keeping the triangular inequality in mind, we have

$$\begin{aligned} |P_G(y) - P_G(z)| &= |[(1-t)y(0) + ty(1) - \int_0^1 G(s,t)\mu \sinh[\mu y(s)]ds] \\ &\quad -[(1-t)z(0) + tz(1) - \int_0^1 G(s,t)\mu \sinh[\mu z(s)]ds]| \end{aligned}$$

$$\begin{aligned}
&= |(1-t)[y(0)-z(0)] + [ty(1)-tz(1)] - \int_0^1 G(s,t)[\mu\sinh(\mu y(s)) \\
&\quad -\mu\sinh(\mu z(s))ds]| \\
&\leq (1-t)|y(0)-z(0)| + t|y(1)-z(1)| + |\int_0^1 G(s,t)[\mu\sinh(\mu y(s)) \\
&\quad -\mu\sinh(\mu z(s))ds]|. \qquad (5.4)
\end{aligned}$$

Since $|y(0)-z(0)| \leq L_y\|u-y\|$ and $|y(1)-z(1)| \leq L_y\|u-y\|$, and from (5.4) and keeping $\int_0^1 |G(t,s)| \leq \frac{1}{8}$ in mind, we have

$$\begin{aligned}
|P_G(y)-P_G(z)| &\leq (1-t)L_y\|y-z\| + tL_y\|y-z\| + \frac{1}{8}|\int_0^1 |\mu\sinh(\mu y(s)) - \mu\sinh(\mu z(s))ds]| \\
&\leq L_y\|y-z\| + \frac{1}{8}|\int_0^1 |\mu\sinh(\mu y(s)) - \mu\sinh(\mu z(s))ds]| \\
&= L_y\|y-z\| + \frac{|\mu|}{8}\int_0^1 |\sinh(\mu y(s)) - \sinh(\mu z(s))|ds. \qquad (5.5)
\end{aligned}$$

By using the mean value theorem on g, then from (5.5), we have

$$\begin{aligned}
|P_G(y)-P_G(z)| &\leq L_y\|y-z\| + \frac{|\mu|}{8}\max_{[0,1]}|\sinh(\mu y(t)) - \sinh(\mu z(t))| \\
&\leq L_y\|y-z\| + \frac{\mu^2}{8}L_c\|y-z\| \\
&= (L_y + \frac{\mu^2}{8}L_c)\|y-z\| \\
&\leq \theta\|y-z\|,
\end{aligned}$$

where $\theta = L_y + \frac{\mu^2 L_c}{8} < 1$, and it follows that P_G is a θ-contraction. Hence, by the Banach result [5], P_G has a unique fixed point in $Y = C[0,1]$, namely, y^*, which is the unique solution for the problems (1.5) and (1.6). We now prove the convergence of our scheme (5.12). For this, we have

$$\begin{aligned}
\|h_i - y^*\| &= \|P_G y_i - y^*\| \\
&= \|P_G y_i - P_G y^*\| \\
&\leq \theta\|y_i - y^*\|.
\end{aligned}$$

Consequently, we obtained

$$\|h_i - y^*\| \leq \theta\|y_i - y^*\|. \qquad (5.6)$$

Also,

$$\begin{aligned}
\|z_i - y^*\| &= \|P_G h_i - y^*\| \\
&= \|P_G h_i - P_G y^*\| \\
&\leq \theta\|h_i - y^*\|.
\end{aligned}$$

Hence,

$$\|z_i - y^*\| \leq \theta\|h_i - y^*\|. \qquad (5.7)$$

Using (5.6) and (5.7), we have

$$
\begin{aligned}
\|y_{i+1} - y^*\| &= \|(1-\kappa_i)z_i + \kappa_i P_G z_i - y^*\| \\
&= \|(1-\kappa_i)(z_i - y^*) + \kappa_i (P_G z_i - z^*)\| \\
&\leq (1-\kappa_i)\|z_i - y^*\| + \kappa_i \|P_G z_i - y^*\| \\
&= (1-\kappa_i)\|z_i - y^*\| + \kappa_i \|P_G z_i - P_G y^*\| \\
&\leq (1-\kappa_i)\|z_i - y^*\| + \kappa_i \theta \|z_i - y^*\| \\
&= [1 - \kappa_i(1-\theta)]\|z_i - y^*\| \\
&\leq \theta[1 - \kappa_i(1-\theta)]\|h_i - y^*\| \\
&\leq \theta^2[1 - \kappa_i(1-\theta)]\|y_i - y^*\|.
\end{aligned}
$$

Accordingly,

$$
\begin{aligned}
\|y_{i+1} - y^*\| &= \theta^2[1 - \kappa_i(1-\theta)]\|y_i - y^*\| \\
&\leq \theta^4[1 - \kappa_i(1-\theta)][1 - \kappa_{i-1}(1-\theta)]\|y_{i-1} - y^*\| \\
&\leq \theta^6[1 - \kappa_i(1-\theta)][1 - \kappa_{i-1}(1-\theta)][1 - \kappa_{i-2}(1-\theta)]\|y_{i-2} - y^*\| \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
&\leq (\theta)^{2i+2}\prod_{k=0}^{i}[1 - \kappa_k(1-\theta)]\|y_0 - y^*\|.
\end{aligned}
$$

Since $0 \leq \kappa_i \leq 1$ with $\sum \kappa_i = \infty$, then $\|y_i - y^*\| \to 0$ as $i \to \infty$, and $\{y_i\}$ converges to the unique fixed point y^* of P_G which is the unique solution of the problems (1.5) and (1.6). □

Case II: In this case, we consider the value of $\mu > 1$.

Since $\mu > 1$, we may face a difficulty due to the existence of boundary layer when solving the problems (1.5) and (1.6). Hence, we must use the following transformation:

$$e(t) = \tanh\left(\frac{\mu e(t)}{4}\right). \tag{5.8}$$

In this case, the problems (1.5) and (1.6) take the following form:

$$(1 - e^2(t))e''(t) + 2e(t)(e'(t))^2 = \mu^2 e(t)(1 + e^2(t)), \tag{5.9}$$

and the new BCs become:

$$e(0) = 0, \quad e(1) = \tanh(\frac{\mu}{4}). \tag{5.10}$$

Now, consider the linear term $L(e) = e'' - \mu^2 e = 0$ and the nonlinear term $N(e) = g(e, e', e'') = -e^2 e'' + 2e(e')^2 - \mu^2 e^3$. Since the linear term here is different from Case I, we cannot apply Theorem 5.1. That is we need to construct a new Green's function with this linear term. To do this, we consider the two linearly independent solutions $\{\sinh(\mu t), \sinh(\mu(1-t))\}$. In this, the Green's function attains the following form:

$$G(t,s)=\begin{cases}\frac{\sinh(\mu s)\sinh(\mu(1-t))}{\mu\sinh(\mu)} & \text{when } \ 0<t<s,\\ \frac{\sinh(\mu t)\sinh(\mu(1-s))}{\mu\sinh(\mu)} & \text{when } \ s<t<1.\end{cases}$$

Again, we suppose $Y = C[0,1]$, which is a Banach space with the supremum norm and G is a Green's function given above. Set $S_G : Y :\to Y$ by

$$S_G e = e + \int_0^1 G(t,s)[(1-e^2)e'' + 2e(e')^2 - \mu^2 e(1+e^2)]ds. \tag{5.11}$$

In this case, (4.6) takes the following form:

$$\begin{cases} h_i = S_G e_i,\\ z_i = S_G h_i,\\ e_{i+1} = (1-\kappa_i)z_i + \kappa_i S_G z_i \ (i = 0,1,2,...),\end{cases} \tag{5.12}$$

where S_G is the operator given in (5.11). The convergence theorem in this case is the following.

Theorem 5.2. *Consider the operator $S_G : Y \to Y$ in (5.11) and suppose $g(e,e',e'') = -e^2e'' + 2e(e')^2 - \mu^2 e^3$ is a function with derivative bound "e" and "e" being a Lipschitz function with the constant L_y. Assume that $(L_y + \frac{\max_{[0,1]\times\mathbb{R}\times\mathbb{R}} |\frac{\partial g}{\partial e}| \cosh(\frac{\mu}{2}) - 1}{\mu^2 \cosh(\frac{\mu}{2})} < 1$ and $\{y_i\}$ is the sequence of iterates due to the Abbas-Green's iteration (5.12). Subsequently, $\{e_i\}$ is strongly convergent to the unique fixed point of S_G and hence to the unique sought solution of the transformed problems (5.9) and (5.10).*

Proof. Put $L_y + \frac{\mu^2 \max_{t\in[0,1]} |\cosh(\mu y(t))|}{8} = \theta$, and it follows that S_G is a θ-contraction. Hence, by Banach result [5], S_G has a unique fixed point in $Y = C[0,1]$, namely, y^*, which is the unique solution for the tranformed problems (5.9) and (5.10). The remaining proof is now same as the proof of Theorem 5.1 and, hence, is omitted. □

As one knows in the theory of fixed points, a fixed point iteration scheme sometimes is not numerically stable as we apply it on finding approximate fixed points (see the work published in [34,35] and the references therein). Notice that a given fixed point iteration scheme is called a stable fixed point scheme when the difference between two iterative steps does not affect the strong convergence of the scheme to a fixed point. The paper by Urabe [36] suggested for the first time the notion of stability. After this, Harder and Hicks [31] constructed formal definition for the concept of stability. After this, authors studied the concept of stability more deeply and introduced a more general definition of stability called weak w^2-stability. In this paper, we use the new and nature notion of stability, the so-called weak w^2-stability associated with our scheme (5.12).

Theorem 5.3. *Let us take Y, P_G, and $\{y_i\}$ as in Theorem 5.1. Subsequently, the convergence $\{y_i\}$ is always w^2-stable for the mapping P_G.*

Proof. Assume that $\{\bar{y}_i\}$ denotes an equivalent sequence for the sequence of iterates $\{y_i\}$, that is, $\{\bar{y}_i\}$ admits the property $\lim_{i\to\infty} \|\bar{y}_i - y_i\| = 0$. To obtain the purpose, we set

$$\epsilon_i = \|\bar{y}_{i+1} - [(1-\kappa_i)\bar{z}_i + \kappa_i P_G \bar{z}_i]\|, \tag{5.13}$$

where $\bar{z}_i = P_G \bar{h}_i$ and $\bar{h}_i = P_G \bar{y}_i$.

Now, we suppose that $\lim_{i\to\infty} \epsilon_i = 0$ and show that $\lim_{i\to\infty} \|\bar{y}_{i+1} - y^*\| = 0$. For this, we have

$$\|\bar{h}_i - h_i\| = \|P_G\bar{y}_i - P_G y_i]\| \le \theta\|\bar{y}_i - y_i\|.$$

Consequently, we get

$$\|\bar{h}_i - h_i\| \le \theta\|\bar{y}_i - y_i\|. \tag{5.14}$$

Also,

$$\|\bar{z}_i - z_i\| = \|P_G\bar{h}_i - P_G h_i]\| \le \theta\|\bar{h}_i - h_i\|.$$

Hence,

$$\|\bar{z}_i - z_i\| \le \theta\|\bar{h}_i - h_i\|. \tag{5.15}$$

Using (5.14) and (5.15), we have

$$\begin{array}{rcl}
\|\bar{y}_{i+1} - y_i\| & \le & \|\bar{y}_{i+1} - y_{i+1}\| + \|y_{i+1} - y^*\| \\
& \le & \|\bar{y}_{i+1} - [(1-\kappa_i)\bar{z}_i + \kappa_i P_G\bar{z}_i]\| + \|[(1-\kappa_i)\bar{z}_i + \kappa_i P_G\bar{z}_i] - y_{i+1}\| + \|y_{i+1} - y^*\| \\
& = & \epsilon_i + \|[(1-\kappa_i)\bar{z}_i + \kappa_i P_G\bar{z}_i] - y_{i+1}\| + \|y_{i+1} - y^*\| \\
& = & \epsilon_i + \|[(1-\kappa_i)\bar{z}_i + \kappa_i P_G\bar{z}_i] - [(1-\kappa_i)z_i + \kappa_i P_G z_i]\| + \|y_{i+1} - y^*\| \\
& = & \epsilon_i + \|[(1-\kappa_i)(\bar{z}_i - z_i) + \kappa_i(P_G\bar{z}_i - P_G z_i)]\| + \|y_{i+1} - y^*\| \\
& \le & \epsilon_i + (1-\kappa_i)\|\bar{z}_i - z_i\| + \kappa_i\|P_G\bar{z}_i - P_G z_i\| + \|y_{i+1} - y^*\| \\
& \le & \epsilon_i + (1-\kappa_i)\|\bar{z}_i - z_i\| + \kappa_i\theta\|\bar{z}_i - z_i\| + \|y_{i+1} - y^*\| \\
& = & \epsilon_i + [1 - \kappa_i(1-\theta)]\|\bar{z}_i - z_i\| + \|y_{i+1} - y^*\| \\
& \le & \epsilon_i + \theta[1 - \kappa_i(1-\theta)]\|\bar{h}_i - h_i\| + \|y_{i+1} - y^*\| \\
& \le & \epsilon_i + \theta^2[1 - \kappa_i(1-\theta)]\|\bar{y}_i - y_i\| + \|y_{i+1} - y^*\|.
\end{array}$$

Subsequently, we obtained

$$\|\bar{y}_{i+1} - y^*\| \le \epsilon_i + \theta^2[1 - \kappa_i(1-\theta)]\|\bar{y}_i - y_i\| + \|y_{i+1} - y^*\|. \tag{5.16}$$

As assumed, $\lim_{i\to\infty} \epsilon_i = 0$ and also $\lim_{i\to\infty} \|\bar{y}_i - y_i\| = 0$ as $\{\bar{y}_i\}$ is an equivalent sequence for $\{y_i\}$ and also, $\lim_{k\to\infty} \|y_i - y^*\| = 0$ because $\{y_i\}$ is strongly convergent to y^*. Thus, (5.16) gives $\lim_{i\to\infty} \|\bar{y}_i - y^*\| = 0$. Hence, $\{y_i\}$ produced by the Abbas-Green's iterations (5.12) is eventually weak w^2-stable for the mapping P_G. □

Theorem 5.4. *Let us take Y, S_G, and $\{e_i\}$ as in Theorem 5.2. Subsequently, the convergence $\{e_i\}$ is always w^2-stable for the mapping S_G.*

Proof. The proof of this theorem can be complete using the same arguments in Theorem 5.3. Therefore, we omit the proof. □

6. Numerical computations

The purpose of this section is to varify numerically the convergence of our scheme. These numerical results show that our new iterative scheme converges faster to the numerical solution. To achieve our objective, we assume $\mu = 0.5$ and $\kappa_i = 0.9 \in (0, 1)$ and obtain the following results in Tables 1–3 for the variable $t = 0.2, 0.5, 0.8$, respectively. In each case, our scheme converges to the sought solution.

Table 1. Numerical convergence of the schemes when $t = 0.2$.

i	**Picard-Green**	**Mann-Green**	**PMH-Green**	**Abbas-Green**
0	0.200000	0.200000	0.200000	0.200000
1	0.191895	0.192706	0.192111	0.192129
2	0.192135	0.192171	0.192129	0.192129
3	0.192129	0.192132	0.192129	0.192129
4	0.192129	0.192129	0.192129	0.192129
5	0.192129	0.192129	0.192129	0.192129
6	0.192129	0.192129	0.192129	0.192129
7	0.192129	0.192129	0.192129	0.192129
8	0.192129	0.192129	0.192129	0.192129
9	0.192129	0.192129	0.192129	0.192129
10	0.192129	0.192129	0.192129	0.192129

Table 2. Numerical convergence of the schemes when $t = 0.5$.

i	**Picard-Green**	**Mann-Green**	**PMH-Green**	**Abbas-Green**
0	0.500000	0.500000	0.500000	0.500000
1	0.484129	0.485716	0.484515	0.484548
2	0.484558	0.484635	0.484547	0.484547
3	0.484547	0.484555	0.484547	0.484547
4	0.484547	0.484548	0.484547	0.484547
5	0.484547	0.484547	0.484547	0.484547
6	0.484547	0.484547	0.484547	0.484547
7	0.484547	0.484547	0.484547	0.484547
8	0.484547	0.484547	0.484547	0.484547
9	0.484547	0.484547	0.484547	0.484547
10	0.484547	0.484547	0.484547	0.484547

Table 3. Numerical convergence of the schemes when $t = 0.8$.

i	Picard-Green	Mann-Green	PMH-Green	Abbas-Green
0	0.800000	0.800000	0.800000	0.800000
1	0.787752	0.788977	0.787996	0.788017
2	0.788023	0.788094	0.788017	0.788017
3	0.788016	0.788023	0.788017	0.788017
4	0.788017	0.788017	0.788017	0.788017
5	0.788017	0.788017	0.788017	0.788017
6	0.788017	0.788017	0.788017	0.788017
7	0.788017	0.788017	0.788017	0.788017
8	0.788017	0.788017	0.788017	0.788017
9	0.788017	0.788017	0.788017	0.788017
10	0.788017	0.788017	0.788017	0.788017

Now we plot the convergence of our proposed scheme in Figure 1 for different choices of t, where $\mu = 0.5$ and $\kappa_i = 0.9$. This graphs show that the convergence of our scheme is stable.

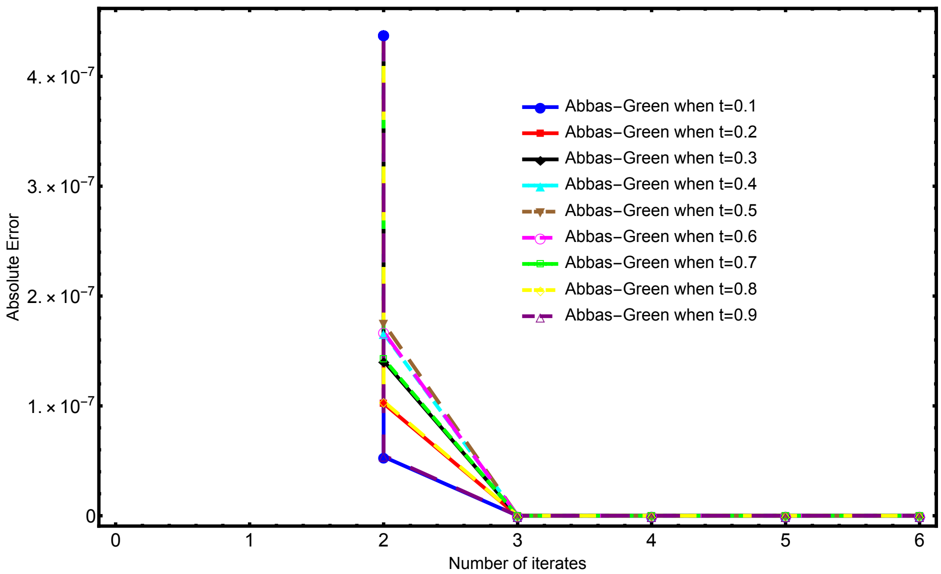

Figure 1. Behaviors of Abbas-Green's iterations for different choices of t.

When the exact solution is not known, then it is not possible to check directly the rate of convergence of a scheme. Therefore, we compare the rate of convergence of a scheme using absolute errors. A small value of absolute error means that the iterative values are very close to the exact solution. We now provide some result comparisons of absolute errors obtained from our proposed scheme and other schemes of the literature in Tables 4 and 5, which shows that our proposed scheme converges faster to the numerical solution as compared to the other schemes of the literature.

Table 4. Comparison of absolute error, i.e., $|y_i - y_{i+1}|$ $\mu = 0.5$ and $\kappa_i = 0.9$ for third iteration.

t	**Picard-Green**	**Mann-Green**	**PMH-Green**	**Abbas-Green**
0.1	8.99915×10^{-8}	1.40678×10^{-6}	3.90601×10^{-11}	7.21645×10^{-13}
0.2	1.71531×10^{-7}	2.77891×10^{-6}	7.44839×10^{-11}	1.36002×10^{-15}
0.3	2.36885×10^{-7}	4.07244×10^{-6}	1.02933×10^{-10}	1.83187×10^{-15}
0.4	2.79714×10^{-7}	5.22423×10^{-6}	1.21652×10^{-10}	1.83187×10^{-15}
0.5	2.9566×10^{-7}	6.14108×10^{-6}	1.2872×10^{-10}	2.22045×10^{-15}
0.6	2.82801×10^{7}	6.68705×10^{-6}	1.23255×10^{-10}	2.27596×10^{-15}
0.7	2.41943×10^{-7}	6.66837×10^{-6}	1.05556×10^{-10}	1.9984×10^{-15}
0.8	1.76683×10^{-7}	5.81505×10^{-6}	7.7151×10^{-11}	1.33227×10^{-15}
0.9	9.32398×10^{-8}	3.75795×10^{-6}	4.07379×10^{-11}	6.66134×10^{-16}

Table 5. Comparison of absolute error, i.e., $|y_i - y_{i+1}|$ for $\mu = 0.8$ and $\kappa_i = 0.9$ for third iteration.

t	**Picard-Green**	**Mann-Green**	**PMH-Green**	**Abbas-Green**
0.1	4.50678×10^{-6}	4.18547×10^{-7}	1.9991×10^{-10}	7.29833×10^{-14}
0.2	8.60806×10^{-6}	6.7069×10^{-7}	3.83862×10^{-10}	1.39333×10^{-13}
0.3	0.0000119279	6.072×10^{-7}	5.36424×10^{-10}	1.92957×10^{-13}
0.4	0.0000141488	1.17772×10^{-7}	6.43367×10^{-10}	2.28706×10^{-13}
0.5	0.0000150385	8.37319×10^{-7}	6.92655×10^{-10}	2.42917×10^{-13}
0.6	0.0000144745	2.17856×10^{-6}	6.75806×10^{-10}	2.33591×10^{-13}
0.7	0.0000124641	3.6314×10^{-6}	5.8962×10^{-10}	2.01061×10^{-13}
0.8	9.158×10^{-6}	4.60791×10^{-6}	4.38101×10^{-10}	1.47438×10^{-13}
0.9	4.85605×10^{-6}	4.02199×10^{-6}	2.34107×10^{-10}	7.84928×10^{-14}

Remark 6.1. *In Tables 1–5, we see that our new Abbas-Green's iterative approach suggests small absolute error. This means that our iterative scheme moves faster to the numerical solutions corresponding to the other Green's function based iterative schemes.*

Finally, we compare our proposed scheme with other schemes of the literature in Figures 2 and 3.

Remark 6.2. *Figures 2 and 3 show that, for small values of t, Picard-Green's iterative scheme is better than the Mann-Green's iterative scheme and for large values of t, Mann-Green's iterative scheme is better than the Picard-Green's iterative scheme. On the other hand, our novel Abbas-Green's fixed point scheme showed a significant rates of convergence as compared to the one-step scheme of Picard and Mann fixed point schemes and two step PMH scheme for different choices of t.*

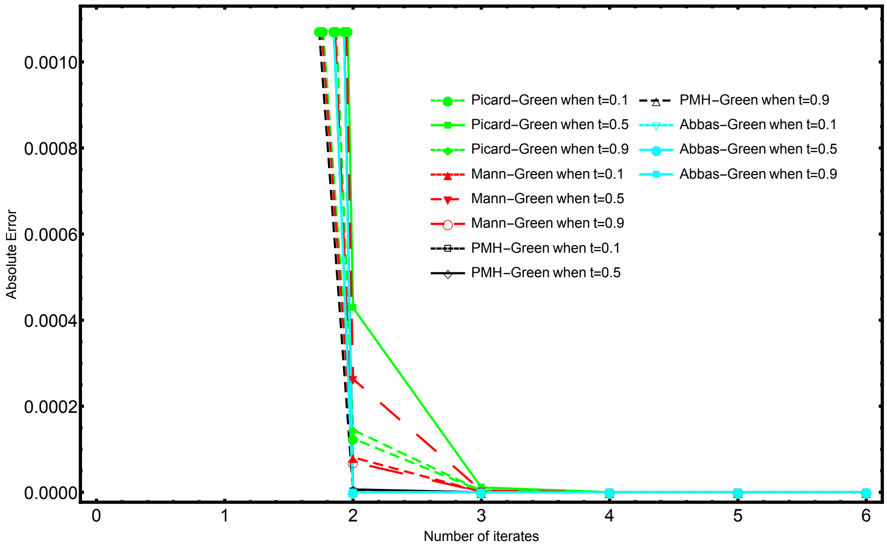

Figure 2. Behaviors of different iterations for different choices of t, where $\mu = 0.5$ and $\kappa_i = 0.99$.

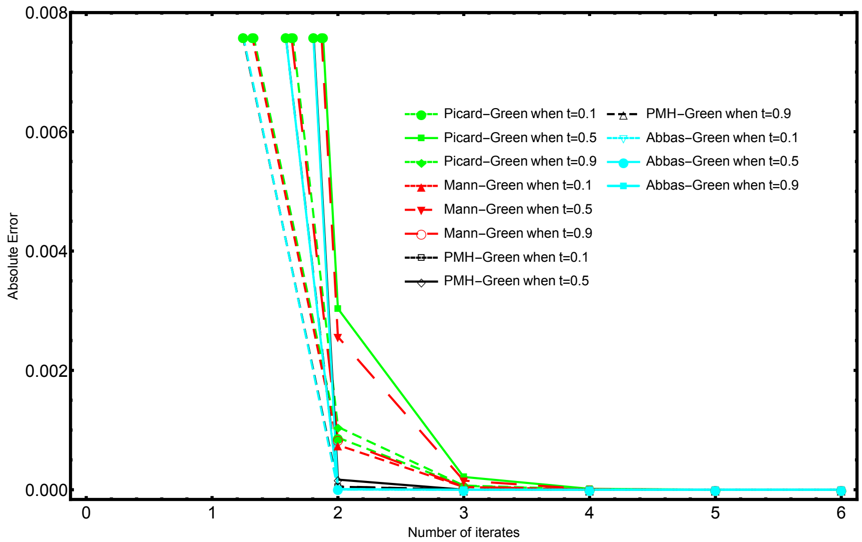

Figure 3. Behaviors of different iterations for different choices of t, where $\mu = 0.8$ and $\kappa_i = 0.99$.

7. Application to a class of fractional BVPs

There are various classes of fractional BVPs in the literature, but the exact solutions for such BVPs are not easy to compute by using the techniques of ordinary BVPs. On the other hand, fractional calculus finds its application in all areas of applied sciences, especially in mathematical modeling and nonlinear real-world problems. The purpose of this section is to obtain a new application in a larger class of fractional BVPs of our novel Green's function iteration procedure. To obtain the main goal, set the following fractional order BVPs:

$$D^{\alpha}y(t) + D^{\beta}y(t) = f(t, y(t)), \tag{7.1}$$

with the BCs:

$$y(0) = y(1) = 0, \tag{7.2}$$

where $t \in [0, 1]$, f is a continuous mapping, $\alpha, \beta \in (0, 1)$ with D^{α} and D^{β} denotes the fractional derivatives in the sense of Caputo of orders α and β, respectively.

The Green's function with the problems (7.1) and (7.2) is given by the following rule:

$$G(t) = t^{\alpha-1}M_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}),$$

where $M_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})$ is known as the Mittag-Leffler function in the literature and given in [37].

We now express the solution of the (7.1) and (7.2) as a fixed point of the function P_G on the Banach space $C[0, 1]$ as follows:

$$P_G y = \int_0^1 G(t-s)[f(s, y(s)]ds. \tag{7.3}$$

Based on (7.3), our scheme becomes

$$\begin{cases} y_0 \in C[0, 1], \\ h_i = \int_0^1 G(t-s)[f(s, y_i(s)]ds, \\ z_i = \int_0^1 G(t-s)[f(s, h_i(s)]ds, \\ y_{i+1} = (1-\kappa_i)z_i + \kappa_i \int_0^1 G(t-s)[f(s, z_i(s)]ds. \end{cases} \tag{7.4}$$

Now using (7.4), we provide the main result of this section.

Theorem 7.1. *Consider the selfmap P_G defined in (7.3). Assume that $|f(t, a) - f(t, b)| \leq k|a - b|$ for some $k < \alpha$ with $\frac{k}{\alpha} < 1$. If the sequence $\{\kappa_i\}$ satisfies the condition $\sum \kappa_i = \infty$, then $\{y_i\}$ produced by Abbas-Green's iterative scheme (7.4) is strongly convergent to unique fixed point of P_G and, hence, to the unique solution of the problems (7.1) and (7.2).*

Proof. For any y and z in $C[0, 1]$, we have

$$\begin{aligned} |P_G y(t) - P_G z(t)| &= |\int_0^1 G(t-s)[f(s, y(s)]ds - \int_0^1 G(t-s)[f(s, z(s)]ds| \\ &= |\int_0^1 G(t-s)[f(s, y(s) - f(s, z(s)]ds| \\ &\leq \int_0^1 G(t-s)|f[(s, y(s) - f(s, z(s)]|ds \\ &\leq \int_0^1 G(t-s)k|y(s) - z(s)|ds \\ &\leq \sup_{t\in[0,1]} \int_0^1 G(t-s)dsk|y(t) - z(t)| \\ &\leq (\frac{1}{\alpha})k|y(t) - z(t)| \leq \frac{k}{\alpha}|y(t) - z(t)|. \end{aligned}$$

Hence, $\|P_G y - P_G z\| \leq \frac{k}{\alpha}\|y - z\|$ for all $y, h \in C[0,1]$. Thus, P_G forms a contraction and admits a unique fixed point, namely, y^*, which forms a unique solution for the problems (7.1) and (7.2). In the view of of our main outcome, the scheme (7.4) is strongly convergent to the unique solution of the problems (7.1) and (7.2) and the convergence is weak w^2-stable. □

8. Conclusions and future plans

In this section, we list some conclusions of our results and leave two open questions for readers.

(i) We proposed a novel iterative scheme by embedding a Green's function into the Abbas-iterative scheme for approximating numerical solutions of Troesch's BVPs.
(ii) We obtained a convergence result under possible mild conditions.
(iii) We proved the weak w^2-stability result for our scheme.
(iv) We proved numerically and graphically that our scheme converges faster to the numerical solution as compared to the classical approaches.
(v) The main outcome of the paper is used for solving a class of fractional BVPs of a broad class.
(vi) Thus, our results are new/extend the classical results of the literature.

In our future work, we will extend the research of this paper to the following cases.

(a) We will investigate the approximate solutions for BVPs arising in the fluid dynamics based on Green's function and fixed point schemes approach.
(b) We will advance the study of Green's function schemes to the setting of partial differential equations (PDEs) and fractional order BVPs that arise in various fields of mathematical modeling.
(c) We will extend the study of our new scheme to the higher order BVPs.

We provide interesting open questions to readers as follows:
Open question 1. The underlying space used in this paper is a Banach space of continuous function, which satisfies the symmetric condition $\|f - g\| = \|g - f\|$ for any function f and g. Can we replace this space by asymmetric space in the results of this paper?

Recently, Berinde [38] introduced the concept of enriched contractions and proved that the concept of enriched contractions is more general than the concept of contraction in the sense of Banach. Hence, we have the following open question:
Open question 2. Can we prove/extend the results of this paper to the setting of enriched contractions?

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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