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Research article

Dynamic analysis of a stochastic vector-borne model with direct transmission and media coverage

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Abstract: This paper presents a stochastic vector-borne epidemic model with direct transmission and media coverage. It proves the existence and uniqueness of positive solutions through the construction of a suitable Lyapunov function. Immediately after that, we study the transmission mechanism of vector-borne diseases and give threshold conditions for disease extinction and persistence; in addition we show that the model has a stationary distribution that is determined by a threshold value, i.e., the existence of a stationary distribution is unique under specific conditions. Finally, a stochastic model that describes the dynamics of vector-borne diseases has been numerically simulated to illustrate our mathematical findings.

Keywords: vector-borne disease; direct transmission; media coverage; stationary distribution
Mathematics Subject Classification: 60G51, 60G57, 92B05

1. Introduction

Vector-borne disease seriously threatens global health, and it is usually caused by vector-borne parasites, viruses, and bacteria that transmits pathogens between humans or from animals to humans. According to the World Health Organization, the disease accounts for 17% of all infectious and has caused 700,000 deaths annually [1]. Despite scientific and technological advances and the growing influence in all regions, vector-borne diseases remain one of the leading causes of global disease. Mathematical modeling has become an essential method for studying epidemics. Since the first attempt to model malaria transmission by Ross [2] and subsequent modifications by MacDonald [3], a series of vector-borne disease models have been proposed [4–7]. Various disease models based on influencing factors (e.g., time delay, vaccination, age structure, etc.) have been extensively studied [8–11].

It is commonly known that direct and indirect transmissions are two significant ways that various diseases are spread. Although vector-borne diseases are mainly transmitted by vectors, i.e., indirect transmission, vector-borne diseases are often transmitted directly through blood transfusions, organ transplantation, laboratory exposure, or mother-to-baby during pregnancy, childbirth, or breastfeeding. Furthermore Zika can be transmitted through sexual contact [12]. Thus, direct transmission plays a vital role in the dynamics of vector-borne diseases and has attracted widespread attention [13–16]. In the deterministic model proposed by Wei et al. [16], the host population is assumed to be divided into three subpopulations, i.e., susceptible, infected, and recovered individuals. The infected individuals will not relapse once recovered, i.e., the recovered individuals will not become susceptible or infected. Let $S(t), I(t)$, and $R(t)$ be the numbers of susceptible, infected, and recovered individuals at time t. The vector population is divided into two parts, i.e., susceptible and infected vectors, denoted by $M(t)$ and $V(t)$ as the corresponding numbers at time t. The newly recruited vectors are susceptible when vertical transmission is ignored. On the other hand, media coverage is a crucial factor in the control of the spread of epidemics [17]. The media helps people to understand the progress of an epidemic and provide beneficial guidance [18]. Many scholars have studied the impact of media coverage on disease transmission from the perspective of mathematical modeling [19, 20].

Based on the above discussion, we introduce media coverage into the epidemic model and investigate the dynamics of vector-borne diseases with direct transmission. Let β_1 be the transmission rate without media intervention, and $\beta_2 I/(m + I)$ be the effect of media coverage on transmission, where $\beta_1 > \beta_2$ and m measures how quickly people react to media reports [21]. During the spread of the vector-borne epidemic, two transmission rates can lead to the susceptible becoming infected : the rate denoted by β_3 from an infected vector to a susceptible person, and the one denoted by β_4 from an infected person to a susceptible vector. We propose a vector-borne model with direct transmission and media coverage as follows

$$\begin{cases} \mathrm{d}S = \left(\Lambda_1 - \left(\beta_1 - \dfrac{\beta_2 I}{m+I}\right)\dfrac{SI}{1+\alpha_1 I} - \dfrac{\beta_3 SV}{1+\alpha_2 V} - d_1 S\right)dt, \\ \mathrm{d}I = \left(\left(\beta_1 - \dfrac{\beta_2 I}{m+I}\right)\dfrac{SI}{1+\alpha_1 I} + \dfrac{\beta_3 SV}{1+\alpha_2 V} - (\mu + d_1 + \gamma)I\right)dt, \\ \mathrm{d}R = (\gamma I - d_1 R)\,dt, \\ \mathrm{d}M = \left(\Lambda_2 - \dfrac{\beta_4 MI}{1+\alpha_3 I} - d_2 M\right)dt, \\ \mathrm{d}V = \left(\dfrac{\beta_4 MI}{1+\alpha_3 I} - d_2 V\right)dt, \end{cases} \tag{1.1}$$

where Λ_i is a recruitment rate, d_i $(i = 1, 2)$, and μ are the natural, and disease-related death rates of people and vector population, α_i $(i = 1, 2, 3)$ denotes the saturated constants during different transmission processes, and γ is the recovery rate of infected people. Here, $\beta_1 SI$, $\beta_3 SV$, and $\beta_4 MI$ measure the contagiousness of the vector-borne disease, and $1/(1 + \alpha_1 I)$, $1/(1 + \alpha_2 V)$, and $1/(1 + \alpha_3 I)$ reflect the behavioral change of susceptible individuals. The basic reproduction number is $R_0 = \frac{\beta_1 \Lambda_1}{d_1(d_1+\gamma+\mu)} + \frac{\beta_3\beta_4\Lambda_1\Lambda_2}{d_1 d_2^2(d_1+\gamma+\mu)}$, which determines whether the epidemic occurs. If $R_0 < 1$, system (1.1) has a unique disease-free equilibrium $E_0 = \left(\frac{\Lambda_1}{d_1}, 0, 0, \frac{\Lambda_2}{d_2}, 0\right)$. This represents no infected individuals in either population. If $R_0 > 1$, then model (1.1) has two equilibria: a disease-free equilibrium E_0 and an endemic equilibrium $E^* = (S^*, I^*, R^*, M^*, V^*)$. This means that some individuals of both populations have been infected.

In the real world, random fluctuations are essential to ecosystems [22–24]. Random factors, such as temperature and humidity, inevitably affect the epidemic's spread. Many stochastic models have been studied in recent years [25–27]. Considering the complex environmental changes, Liu and Jiang claimed that the random perturbation may depend on the square of the state variables S and I in the system [28, 29]. Recently, nonlinear perturbations have received much attention [30–32]. In addition to this, sometimes ecosystems are also affected by violent random perturbations such as typhoons and tsunamis. To reflect reality better, Levy jumps were introduced into the model [33, 34]. However, this noise differs in detail and often leads to different results. It is worth noting that in the model of vector-borne diseases, Jovanović and Krstić [35] proposed that the random perturbation is proportional to the distance. Ran et al. [36] studied the dynamics of a stochastic vector-borne model with age structure and saturation incidence, considering the environmental noise on the mosquito bite rate and transmission rate between vector and host. Son and Denu [37] provided another stochastic vector-borne model with direct transmission, in which environmental noise affects the mortality of hosts and vectors. We did not want to add complex perturbations to make the model unmanageable; simple perturbations are more likely to reveal the inherent nature of the model. In our work, suppose that the environmental white noise is proportional to the number of subpopulations [38, 39]. Next, we extend the deterministic model (1.1) to a stochastic model. The recovered class is decoupled from the others in the model and then neglected. Then, we propose the following stochastic model

$$\begin{cases} \mathrm{d}S = \left(\Lambda_1 - \left(\beta_1 - \dfrac{\beta_2 I}{m+I}\right)\dfrac{SI}{1+\alpha_1 I} - \dfrac{\beta_3 SV}{1+\alpha_2 V} - d_1 S\right)dt + \sigma_1 S\,\mathrm{d}B_1(t), \\ \mathrm{d}I = \left(\left(\beta_1 - \dfrac{\beta_2 I}{m+I}\right)\dfrac{SI}{1+\alpha_1 I} + \dfrac{\beta_3 SV}{1+\alpha_2 V} - (\mu + d_1 + \gamma)I\right)dt + \sigma_2 I\mathrm{d}B_2(t), \\ \mathrm{d}M = \left(\Lambda_2 - \dfrac{\beta_4 MI}{1+\alpha_3 I} - d_2 M\right)dt + \sigma_3 M\mathrm{d}B_3(t), \\ \mathrm{d}V = \left(\dfrac{\beta_4 MI}{1+\alpha_3 I} - d_2 V\right)dt + \sigma_4 V\mathrm{d}B_4(t), \end{cases} \tag{1.2}$$

where $B_i(t)$ $(i = 1, 2, 3, 4)$ denotes independent standard Brownian motions, σ_i $(i = 1, 2, 3, 4)$ represents the white noise intensity, and the remaining parameters are the same as in model (1.1).

The rest of this paper is organized as follows. Section 2 reviews some basic concepts and valuable lemmas used later. The uniqueness and positivity of the solution are proved in Section 3. Section 4 provides sufficient conditions for determining whether a disease is extinct. In Section 5, we explore the persistence in the mean. In Section 6, we prove the existence of a unique ergodic stationary distribution under certain conditions. In Section 7, we validate the results of analysis through numerical simulations. A brief conclusion is given in the last section.

2. Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ that satisfies the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Denote $\mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y_i > 0, 1 \leq i \leq n\}$. Consider an n-dimensional stochastic differential equation of the following form [40]

$$dy(t) = f(y(t), t)dt + g(y(t), t)dB(t) \tag{2.1}$$

with the initial value $y(0) = y_0 \in \mathbb{R}^n$, where $B(t)$ denotes an n-dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Define a differential operator $\mathscr{L}$ of Eq (2.1) as follows

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(y,t)\frac{\partial}{\partial y_i} + \frac{1}{2}\sum_{i,j=1}^{n}\left[g^T(y,t)g(y,t)\right]_{ij}\frac{\partial^2}{\partial y_i \partial y_j}.$$

If $\mathscr{L}$ acts on a nonnegative function $\mathscr{V} \in C^{2,1}(\mathbb{R}^n \times [t_0,\infty];\mathbb{R}_+)$, then

$$\mathscr{L}\mathscr{V}(y,t) = \mathscr{V}_t(y,t) + \mathscr{V}_y(y,t)f(y,t) + \frac{1}{2}\operatorname{trace}\left[g^T(y,t)\mathscr{V}_{yy}(y,t)g(y,t)\right],$$

where $\mathscr{V}_t = \frac{\partial \mathscr{V}}{\partial t}$, $\mathscr{V}_y = (\frac{\partial \mathscr{V}}{\partial y_1}, \ldots, \frac{\partial \mathscr{V}}{\partial y_n})$, $\mathscr{V}_{yy} = (\frac{\partial^2 \mathscr{V}}{\partial y_i \partial y_j})_{n\times n}$. By Itô's formula, it follows that

$$d\mathscr{V}(y(t),t) = \mathscr{L}\mathscr{V}(y(t),t)dt + \mathscr{V}_y(y(t),t)g(y(t),t)dB(t), \quad y(t) \in \mathbb{R}^n.$$

Lemma 1. *(Strong law of large numbers, [41])* Let $M = \{M_t\}_{t\geqslant 0}$ be a real- valued continuous local martingale vanishing at $t = 0$. Then

$$\lim_{t\to\infty}\langle M, M\rangle_t = \infty \quad \text{a.s.} \Rightarrow \lim_{t\to\infty}\frac{M_t}{\langle M, M\rangle_t} = 0. \quad \text{a.s.}$$
$$\limsup_{t\to\infty}\frac{\langle M, M\rangle_t}{t} < \infty \quad \text{a.s.} \Rightarrow \lim_{t\to\infty}\frac{M_t}{t} = 0. \text{ a.s.}$$

Let $Y(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^n$ in the following form

$$dY(t) = a(Y)dt + \sum_{i=1}^{k}\sigma_i dB_i(t),$$

where the diffusion matrix $\bar{A}(Y) = (b_{ij}(y))$ and $b_{ij}(y) = \sum_{r=1}^{k}\sigma_r^i(y)\sigma_r^j(y)$.

Lemma 2. *[42]* The Markov process $Y(t)$ has a unique stationary distribution $\pi(\cdot)$ if there is a bounded domain $D \in \mathbb{R}^n$ with a regular boundary and the following holds

(i) There is a positive number M such that $\sum_{i,j=1}^{n} b_{ij}(y)\xi_i\xi_j \geq M|\xi|^2, y \in D, \xi \in \mathbb{R}^n$.

(ii) There exists a nonnegative C^2 -function $\mathscr{V}$ such that $\mathscr{L}\mathscr{V}$ is negative for any $\mathbb{R}^n \setminus D$; then

$$P\left\{\limsup_{T\to\infty}\frac{1}{T}\int_0^T f(Y(t))dt = \int_{\mathbb{R}^n} f(y)\pi(dy)\right\} = 1.$$

3. Uniqueness and positivity of solution

Theorem 1. For a given initial value $Y(0) = (S(0), I(0), M(0), V(0)) \in \mathbb{R}_+^4$, the solution $Y(t) = (S(t), I(t), M(t), V(t))$ of model (1.2) is unique on $t \geq 0$ and will maintain in $\mathbb{R}_+^4$ with probability one.

Proof. For a given initial value $(S(0), I(0), M(0), V(0)) \in \mathbb{R}_+^4$, the coefficient in the model (1.2) satisfies the local Lipschitz continuity condition. Hence, there is a unique local solution when $t \in [0, \tau_e)$, where τ_e is the explosion time [43,44]. To obtain the global property of the solution, we need to prove that $\tau_e = \infty$ almost surely (a.s.). Suppose that $k_0 \geq 1$ is sufficiently large such that $S(0), I(0), M(0)$ and $V(0)$ all lie within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, define a stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : \min\{S(t), I(t), M(t), V(t)\} \leq 1/k \ \ or \ \ \max\{S(t), I(t), M(t), V(t)\} \geq k\}, \tag{3.1}$$

where $\emptyset$ is an empty set and $\inf \emptyset = \infty$. It can be seen that τ_k increases as $k \to \infty$ and $\tau_\infty = \lim_{k\to\infty} \tau_k$ with $0 \leq \tau_\infty \leq \tau_e$ a.s. In other words, if $\tau_e = \infty$ a.s. does not hold, there must exists constants $T, k_1 > 0$ and $\epsilon \in (0, 1)$ such that $P\{\tau_k \leq T\} > \epsilon$ for all $k \geq k_1$. Define a C^2-function $W : \mathbb{R}^4 \to \mathbb{R}_+$ and

$$\begin{aligned} W(S(t), I(t), M(t), V(t)) =&(S(t) - 1 - \log S(t)) + (I(t) - 1 - \log I(t)) + (M(t) - 1 - \log M(t)) \\ &+ (V(t) - 1 - \log V(t)). \end{aligned} \tag{3.2}$$

Obviously, W is a non-negative function. Applying Itô's formula to Eq (3.2) yields

$$\begin{aligned} dW(S(t), I(t), M(t), V(t)) =&\Bigg[\left(1 - \frac{1}{S}\right)\left(\Lambda_1 - (\beta_1 - \frac{\beta_2 I}{m + I})\frac{SI}{1 + \alpha_1 I} - \frac{\beta_3 SV}{1 + \alpha_2 V} - d_1 S\right) + \frac{1}{2}\sigma_1^2 \\ &+ \left(1 - \frac{1}{I}\right)\left((\beta_1 - \frac{\beta_2 I}{m + I})\frac{SI}{1 + \alpha_1 I} + \frac{\beta_3 SV}{1 + \alpha_2 V} - (\mu + d_1 + \gamma)I\right) + \frac{1}{2}\sigma_2^2 \\ &+ \left(1 - \frac{1}{M}\right)\left(\Lambda_2 - \frac{\beta_4 MI}{1 + \alpha_3 I} - d_2 M\right) + \frac{1}{2}\sigma_3^2 \\ &+ \left(1 - \frac{1}{V}\right)\left(\frac{\beta_4 MI}{1 + \alpha_3 I} - d_2 V\right) + \frac{1}{2}\sigma_4^2\Bigg] dt + \sigma_1 (S - 1)\, dB_1(t) \\ &+ \sigma_2 (I - 1)\, dB_2(t) + \sigma_3 (M - 1)\, dB_3(t) + \sigma_3 (V - 1)\, dB_4(t) \\ =&\mathscr{L} W dt + \sigma_1 (S - 1)\, dB_1(t) + \sigma_2 (I - 1)\, dB_2(t) + \sigma_3 (M - 1)\, dB_3(t) \\ &+ \sigma_4 (V - 1)\, dB_4(t), \end{aligned}$$

where $\mathscr{L} W : \mathbb{R}_+^4 \to \mathbb{R}_+$ can be written in the following form

$$\begin{aligned} &\mathscr{L} W(S(t), I(t), M(t), V(t)) \\ =&\Bigg[\left(1 - \frac{1}{S}\right)\left(\Lambda_1 - (\beta_1 - \frac{\beta_2 I}{m + I})\frac{SI}{1 + \alpha_1 I} - \frac{\beta_3 SV}{1 + \alpha_2 V} - d_1 S\right) + \frac{1}{2}\sigma_1^2 \\ &+ \left(1 - \frac{1}{I}\right)\left((\beta_1 - \frac{\beta_2 I}{m + I})\frac{SI}{1 + \alpha_1 I} + \frac{\beta_3 SV}{1 + \alpha_2 V} + (\mu + d_1 + \gamma)I\right) + \frac{1}{2}\sigma_2^2 \\ &+ \left(1 - \frac{1}{M}\right)\left(\Lambda_2 - \frac{\beta_4 MI}{1 + \alpha_3 I} - d_2 M\right) + \frac{1}{2}\sigma_3^2 + \left(1 - \frac{1}{V}\right)\left(\frac{\beta_4 MI}{1 + \alpha_3 I} - d_2 V\right) + \frac{1}{2}\sigma_4^2\Bigg] \\ \leq&\Lambda_1 + \Lambda_2 + 2d_1 + 2d_2 + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} + \mu + \gamma + \frac{\beta_1 I}{1 + \alpha_1 I} + \frac{\beta_3 V}{1 + \alpha_2 V} \\ &- \frac{\beta_1 S}{1 + \alpha_1 I} - \frac{\beta_3 SV}{(1 + \alpha_2 V)I} + \frac{\beta_4 I}{1 + \alpha_3 I} - \frac{\beta_4 MI}{(1 + \alpha_3 I)V} + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 + \frac{1}{2}\sigma_4^2 \\ \leq&\Lambda_1 + \Lambda_2 + 2d_1 + 2d_2 + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} + \mu + \gamma + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 + \frac{1}{2}\sigma_4^2 =: \kappa. \end{aligned}$$

Hence, we have

$$dW(S(t), I(t), M(t), V(t)) \leq \kappa dt + \sigma_1(S-1)dB_1(t) + \sigma_2(I-1)dB_2(t) + \sigma_3(M-1)dB_3(t) + \sigma_4(V-1)dB_4(t). \tag{3.3}$$

Integrate both sides of Eq (3.3) from 0 to $\tau_k \wedge T$. It is easy to get that

$$\begin{aligned}&\int_0^{\tau_k\wedge T} dW\left(S(u), I(u), M(u), V(u)\right)\\ \leq &\int_0^{\tau_k\wedge T} \kappa du + \int_0^{\tau_k\wedge T} \left\{\sigma_1(S-1)dB_1 + \sigma_2(I-1)dB_2 + \sigma_3(M-1)dB_3 + \sigma_4(V-1)dB_4\right\}.\end{aligned} \tag{3.4}$$

Setting $\Omega = \{\tau_k \leq T\}$ for $k \geq k_1$ and by using Eq (3.1), we get that $P(\Omega_k) \geq \epsilon$. Further, every ω from Ω is associated with at least one among $S(\tau_k, \omega), I(\tau_k, \omega), M(\tau_k, \omega)$, and $V(\tau_k, \omega)$ that is equal to k or $1/k$. Hence, $W(S(\tau_k), I(\tau_k), M(\tau_k), V(\tau_k))$ is not less than $k - 1 - \log k$ or $\frac{1}{k} - 1 + \log k$. That is to say,

$$W(S(\tau_k), I(\tau_k), M(\tau_k), V(\tau_k)) \geq (k - 1 - \log k) \wedge (\frac{1}{k} - 1 - \log\frac{1}{k}). \tag{3.5}$$

Combining Eqs (3.4) and (3.5), we have

$$\begin{aligned}W(S(0), I(0), M(0), V(0)) + \kappa(\tau_k \wedge T) &\geq E[1_{\Omega(\omega)} W(S(\tau_k), I(\tau_k), M(\tau_k), V(\tau_k))]\\ &\geq \epsilon(k - 1 - \log k) \wedge (\frac{1}{k} - 1 - \log\frac{1}{k}),\end{aligned}$$

where $1_{\Omega(\omega)}$ denotes an indicator function of set Ω. Letting $k \to \infty$ leads to the following contradiction

$$\infty \geq W(S(0), I(0), M(0), V(0)) + \kappa(\tau_k \wedge T) = \infty.$$

It implies that $\tau_e = \infty$ a.s. The proof is complete.

It is clear that model (1.1) has a disease-free equilibrium $E_0 = (\Lambda_1/d_1 0, \Lambda_2/d_2, 0)$ whereby the disease tends to become extinct within the time limit. However, there is no disease-free equilibrium in the stochastic version of the model, which requires other ways to consider its extinction. Define a threshold value

$$\mathscr{R}_0^S = \frac{1}{\mu_1 + \sigma_*^2/2}\left(\frac{\beta\Lambda_1}{d_1} + \frac{\beta_4\Lambda_2}{d_2}\right), \qquad \sigma_* = \min\{\sigma_2, \sigma_4\}.$$

4. Disease extinction

Theorem 2. Assume that $d_1 > \frac{\sigma_1^2 \vee \sigma_2^2}{2}$ and $d_2 > \frac{\sigma_1^3 \vee \sigma_2^4}{2}$. Let $(S(t), I(t), M(t), V(t))$ be the solution of system (1.2) with any initial value $(S(0), I(0), M(0), V(0))$. If $\mathscr{R}_0^S < 1$, then

$$\limsup_{t\to\infty} \frac{\log(I+V)}{t} \leq \left(\mu_1 + \frac{\sigma_*^2}{2}\right)(\mathscr{R}_0^S - 1) < 0, \quad a.s..$$

Moreover,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(u)du = \frac{\Lambda_1}{d_1}, \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t I(u)du = 0 \qquad a.s.,$$
$$\lim_{t\to\infty} \frac{1}{t}\int_0^t M(u)du = \frac{\Lambda_2}{d_2}, \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t V(u)du = 0 \qquad a.s..$$

Proof. According to Reference [45], we have

$$\lim_{t\to\infty}\frac{S(t)}{t}=\lim_{t\to\infty}\frac{I(t)}{t}=\lim_{t\to\infty}\frac{M(t)}{t}=\lim_{t\to\infty}\frac{V(t)}{t}=0,\ \text{ a.s.,} \tag{4.1}$$

$$\begin{aligned}&\lim_{t\to\infty}\frac{\int_0^t S(u)\mathrm{d}B_1(u)}{t}=\lim_{t\to\infty}\frac{\int_0^t I(u)\mathrm{d}B_2(u)}{t}=\lim_{t\to\infty}\frac{\int_0^t M(u)\mathrm{d}B_3(u)}{t}\\&=\lim_{t\to\infty}\frac{\int_0^t V(u)\mathrm{d}B_4(u)}{t}=0\ \ a.s..\end{aligned} \tag{4.2}$$

We integrate both sides of the proposed model (1.2) and obtain

$$\begin{aligned}\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t}=&\Lambda_1-\frac{d_1}{t}\int_0^t S(u)du-\frac{\mu+d_1+\gamma}{t}\int_0^t I(u)du+\frac{\sigma_1}{t}\int_0^t S(u)dB_1(u)\\&+\frac{\sigma_2}{t}\int_0^t I(u)dB_2(u).\end{aligned}$$

It is obvious that

$$\begin{aligned}\frac{1}{t}\int_0^t S(u)du=&\frac{\Lambda_1}{d_1}-\frac{(d_1+\mu+\gamma)}{d_1t}\int_0^t I(u)du+\frac{\sigma_1}{d_1t}\int_0^t S(u)dB_1(t)+\frac{\sigma_2}{d_1t}\int_0^t I(u)dB_2(t)\\&-\frac{S(t)-S(0)}{d_1t}-\frac{I(t)-I(0)}{d_1t}.\end{aligned} \tag{4.3}$$

From Eqs (4.1) and (4.2), the limit of Eq (4.3) given by

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(u)du=\frac{\Lambda_1}{d_1}-\lim_{t\to\infty}\left(\frac{d_1+\mu+\gamma}{d_1t}\int_0^t I(u)du\right). \tag{4.4}$$

Similarly, we integrate on both sides of the last two equations of the model (1.2). Hence,

$$\begin{aligned}\frac{M(t)-M(0)}{t}+\frac{V(t)-V(0)}{t}=&\Lambda_2-\frac{d_2}{t}\left(\int_0^t M(u)du+\int_0^t V(u)du\right)+\frac{\sigma_3}{t}\int_0^t M(u)dB_3(u)\\&+\frac{\sigma_4}{t}\int_0^t V(u)dB_4(u).\end{aligned}$$

Combining (4.1) and (4.2), we can get the following equation

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t V(u)du=\frac{\Lambda_2}{d_2}-\lim_{t\to\infty}\frac{1}{t}\int_0^t M(u)du. \tag{4.5}$$

On the other hand, by Itô's formula, it follows that

$$\begin{aligned}d\log(I+V)=&\frac{\beta_4MI}{(1+\alpha_1I)(I+V)}dt+\frac{\beta_3SV}{(1+\alpha_2V)(I+V)}dt+(\beta_1-\frac{\beta_2I}{m+I})\frac{SI}{(1+\alpha_1I)(I+V)}dt\\&-\sigma_2^2\frac{I^2}{(I+V)^2}dt-\sigma_4^2\frac{V^2}{(I+V)^2}dt+\sigma_2\frac{I}{I+V}dB_2(t)+\sigma_4\frac{V}{I+V}dB_4(t)\\&-(\mu+d_1+\gamma)\frac{I}{I+V}dt-d_2\frac{V}{I+V}dt.\\\le&\frac{\beta_4MI}{(I+V)}dt+\beta\frac{S(I+V)}{(I+V)}dt+\frac{\sigma_2}{2}\frac{I}{I+V}dB_2(t)+\frac{\sigma_4}{2}\frac{V}{I+V}dB_4(t)\\&-\mu_1\frac{I+V}{I+V}dt-\sigma_*^2\frac{(I+V)^2}{2(I+V)^2}dt.\end{aligned}$$

The last term here uses the inequality $2IV \leq (I+V)^2$. Integrate on both sides of the equation and divide it by t. Thus,

$$\begin{aligned}\frac{1}{t}\log(I+V) \leq & \frac{\beta}{t}\int_0^t S(u)du + \frac{\beta_4}{t}\int_0^t M(u)du + \frac{1}{t}\int_0^t \sigma_2\frac{I}{I+V}dB_2(u) + \frac{1}{t}\int_0^t \sigma_4\frac{I}{I+V}dB_4(u)\\ & - \frac{1}{t}\int_0^t \frac{\sigma_*^2}{2}dt - \frac{1}{t}\int_0^t \mu_1 du,\end{aligned}$$

where $\mu_1 = \min\{\mu + d_1 + \gamma, d_2\}, \sigma_* = \min\{\sigma_2, \sigma_4\}$, and $\beta = \max\{\beta_1, \beta_3\}$. From Eqs (4.4) and (4.5), we can get

$$\begin{aligned}\frac{1}{t}\log(I+V) \leq & \beta_1\left(\frac{\Lambda_1}{d_1} - \frac{(d_1+\mu)}{d_1 t}\int_0^t I(u)du\right) + \beta_4\left(\frac{\Lambda_2}{d_2} - \frac{1}{t}\int_0^t V(u)du\right)\\ & + \frac{1}{t}\int_0^t \sigma_2\frac{I}{I+V}dB_2(u) + \frac{1}{t}\int_0^t \sigma_4\frac{I}{I+V}dB_4(u) - \frac{1}{t}\int_0^t \frac{\sigma_*^2}{2}dt - \frac{1}{t}\int_0^t \mu_1 du.\end{aligned} \tag{4.6}$$

According to Lemma 1, it is obtained that

$$\lim_{t\to\infty}\left(\frac{1}{t}\int_0^t \sigma_4\frac{V}{I+V}dB_4(u) + \frac{1}{t}\int_0^t \sigma_2\frac{I}{I+V}dB_2(u)\right) = 0, \qquad a.s.. \tag{4.7}$$

By using Eqs (4.6) and (4.7), we have

$$\limsup_{t\to\infty}\frac{\log(I+V)}{t} \leq \left(\mu_1 + \frac{\sigma_*^2}{2}\right)(\mathscr{R}_0^S - 1) < 0, \quad a.s..$$

It means that $\lim_{t\to\infty}\frac{1}{t}\int_0^t I(u)du = 0$ and $\lim_{t\to\infty}\frac{1}{t}\int_0^t V(u)du = 0, a.s.$ Combining Eqs (4.4) and (4.5), it is obvious that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(u)du = \frac{\Lambda_1}{d_1}, \quad \lim_{t\to\infty}\frac{1}{t}\int_0^t M(u)du = \frac{\Lambda_2}{d_2}, \ \ a.s..$$

The proof is complete.

5. Persistence in the mean of the disease

The most interesting aspect in the study of epidemic modeling is the extinction and persistence of the epidemic; in the previous section we studied disease extinction and in this section we will show that diseases are persistent in the mean.

Theorem 3. Assume that $d_1 > \frac{\sigma_1^2 \wedge \sigma_2^2}{2}$ and $d_2 > \frac{\sigma_1^3 \wedge \sigma_2^4}{2}$. If

$$\mathscr{R}_1^S = \frac{9\sqrt[9]{\Lambda_1^2\Lambda_2 d_1^2 d_2(\beta_1-\beta_2)\beta_3\beta_4} + \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2\right)}{4d_1+3d_2+\mu+\gamma} > 1,$$

then for any given initial value $(S(0), I(0), M(0), V(0)) \in R_+^4$, the solution of system (1.2) has the following properties

$$(i) \lim_{t\to\infty} \frac{1}{t}\int_0^t S(u)du \geq \frac{\Lambda_1}{d_1 + \beta_1/\alpha_1 + \beta_2/\alpha_2}, \quad a.s..$$
$$(ii) \lim_{t\to\infty} \frac{1}{t}\int_0^t M(u)du \geq \frac{\Lambda_2}{d_2 + \beta_4/\alpha_3}, \quad a.s..$$
$$(iii) \lim_{t\to\infty} \frac{1}{t}\int_0^t I(u)\mathrm{d}u + \lim_{t\to\infty} \frac{1}{t}\int_0^t V(u)\mathrm{d}u \geq \frac{4d_1 + 3d_2 + \mu + \gamma}{(\beta_1 + \beta_4 + d_1\alpha_3 + d_1\alpha_1) \wedge (\beta_3 + d_2\alpha_2)}(\mathscr{R}_1^S - 1), \quad a.s..$$

Proof. (i) From the first equation of system (1.2) integrating the above inequality and dividing both sides by t, we get

$$\begin{aligned}\frac{S(t) - S(0)}{t} =& \Lambda_1 - \frac{1}{t}\int_0^t \left(\beta_1 - \frac{\beta_2 I(t)}{I(t) + m}\right)\frac{S(u)I(u)}{1 + \alpha_1 I(u)}du - \frac{1}{t}\int_0^t \frac{\beta_3 S(u)V(u)}{1 + \alpha_2 V(u)}du \\ &- \frac{1}{t}\int_0^t d_1 S(u)du - \frac{\sigma_1}{t}\int_0^t S(u)dB_1(u).\end{aligned}$$

In view of Theorem 1, for any initial value $(S(0), I(0), M(0), V(0)) \in \mathbb{R}_+^4$, there is a unique global solution $(S(t), I(t), M(t), V(t)) \in \mathbb{R}_+^4$. Thus,

$$\frac{S(t) - S(0)}{t} + \frac{\sigma_1}{t}\int_0^t S(u)dB_1(u) \geq \Lambda_1 - \frac{1}{t}\int_0^t \frac{\beta_1 S(u)}{\alpha_1}du - \frac{1}{t}\int_0^t \frac{\beta_3 S(u)}{\alpha_2}du - \frac{1}{t}\int_0^t d_1 S(u)du. \quad (5.1)$$

Through the strong law of large numbers for local martingales, we have

$$\lim_{t\to\infty}\left(\frac{S(t) - S(0)}{t} + \frac{\sigma_1}{t}\int_0^t S(u)dB_1(u)\right) = 0 \quad \text{a.s.,}$$

which together with Eq (5.1) yields

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(u)du \geq \frac{\Lambda_1}{d_1 + \beta_1/\alpha_1 + \beta_2/\alpha_2} \quad \text{a.s..}$$

This is the required assertion (i).

(ii) From the third equation of system (1.2), integrating the above inequality and dividing both sides by t, we get

$$\frac{M(t) - M(0)}{t} = \Lambda_2 - \frac{1}{t}\int_0^t \frac{\beta_4 M(u)I(u)}{1 + \alpha_3 I(u)}du - \frac{1}{t}\int_0^t d_2 M(u)du - \frac{\sigma_3}{t}\int_0^t M(u)dB_3(u).$$

Then

$$\frac{M(t) - M(0)}{t} + \frac{\sigma_3}{t}\int_0^t M(u)dB_3(u) \geq \Lambda_2 - \frac{1}{t}\int_0^t \frac{\beta_4 M(u)}{\alpha_3}du - \frac{1}{t}\int_0^t d_2 M(u)du. \quad (5.2)$$

According to the strong law of large numbers of local martingales, we have

$$\lim_{t\to\infty}\left(\frac{M(t) - M(0)}{t} + \frac{\sigma_3}{t}\int_0^t M(u)dB_3(u)\right) = 0 \quad \text{a.s.,}$$

which together with Eq (5.2) yields

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t M(u)du \geq \frac{\Lambda_2}{d_2+\beta_4/\alpha_3} \quad \text{a.s..}$$

This is the required assertion (ii).

(iii) First, define a function $W_2(S,I,V) = -\ln S - \ln I - \ln M - \ln V$. According to the Itô's formula:

$$\mathrm{d}W_2(t) = \mathscr{L}W_2(t)\mathrm{d}t - \sigma_1\,\mathrm{d}B_1(t) - \sigma_2\,\mathrm{d}B_2(t) - \sigma_3\,\mathrm{d}B_3(t) - \sigma_4\,\mathrm{d}B_4(t),$$

where

$$\begin{aligned}
\mathscr{L}W_2(t) =& -\frac{1}{S}\left(\Lambda_1 - (\beta_1 - \frac{\beta_2 I}{m+I})\frac{SI}{1+\alpha_1 I} - \frac{\beta_3 SV}{1+\alpha_2 V} - d_1 S\right)\\
&- \frac{1}{I}\left(\left(\beta_1 - \frac{\beta_2 I}{m+I}\right)\frac{SI}{1+\alpha_1 I} + \frac{\beta_3 SV}{1+\alpha_2 V} - (\mu + d_1 + \gamma)I\right)\\
&- \frac{1}{M}(\Lambda_2 - \frac{\beta_4 MI}{1+\alpha_3 I} - d_2 M) - \frac{1}{V}(\frac{\beta_4 MI}{1+\alpha_3 I} - d_2 V) - \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2\right)\\
=& -\frac{\Lambda_1}{S} + \frac{\beta_1 I}{1+\alpha_1 I} - (\frac{\beta_2 I}{m+I})\frac{I}{1+\alpha_1 I} + \frac{\beta_3 V}{1+\alpha_2 V} + d_1\\
&- \frac{\beta_3 SV}{I(1+\alpha_2 V)} - \left(\beta_1 - \frac{\beta_2 I}{m+I}\right)\frac{S}{1+\alpha_1 I} + (d_1+\mu+\gamma)\\
&- \frac{\Lambda_2}{M} + \frac{\beta_4 I}{(1+\alpha_3 I)} - \frac{\beta_4 MI}{V(1+\alpha_3 I)} + 2d_2 - d_2(1+\alpha_2 V) - d_1(1+\alpha_3 I)\\
&- d_1(1+\alpha_1 I) + d_2(1+\alpha_2 V) + d_1(1+\alpha_3 I) + d_1(1+\alpha_1 I) - \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2\right)\\
\leq& (\beta_1+\beta_4+d_1\alpha_3+d_1\alpha_1)I + (\beta_3+d_2\alpha_2)V - \frac{\Lambda_1}{2S} - \frac{\Lambda_2}{M} - \frac{\beta_4 MI}{V(1+\alpha_3 I)} - \frac{\beta_3 SV}{I(1+\alpha_2 V)}\\
&- (\beta_1-\beta_2)\frac{S}{1+\alpha_1 I} - \frac{\Lambda_1}{2S} - d_2(1+\alpha_2 V) - d_1(1+\alpha_3 I) - d_1(1+\alpha_1 I)\\
&+ (4d_1+3d_2+\mu+\gamma) - \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2\right)\\
\leq& (\beta_1+\beta_4+d_1\alpha_3+d_1\alpha_1)I + (\beta_3+d_2\alpha_2)V - 9\sqrt[9]{\Lambda_1^2\Lambda_2 d_1^2 d_2(\beta_1-\beta_2)\beta_3\beta_4}\\
&+ (4d_1+3d_2+\mu+\gamma) - \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2\right).
\end{aligned}$$

We integrate the above inequality in the interval $(0,t)$, divide it by t, and take the limit to t. Thus,

$$\begin{aligned}
0 \leq& -9\sqrt[9]{\Lambda_1^2\Lambda_2 d_1^2 d_2(\beta_1-\beta_2)\beta_3\beta_4} + (4d_1+3d_2+\mu+\gamma) - \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2\right)\\
&+ \lim_{t\to\infty}\frac{(\beta_1+\beta_4+d_1\alpha_3+d_1\alpha_1)}{t}\int_0^t I(u)\mathrm{d}u + \lim_{t\to\infty}\frac{(\beta_3+d_2\alpha_2)}{t}\int_0^t V(u)\mathrm{d}u\\
&- \lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_1\,\mathrm{d}B_1(u) - \lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_2\,\mathrm{d}B_2(u) - \lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_3\,\mathrm{d}B_3(u) - \lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_4\,\mathrm{d}B_4(u).
\end{aligned} \tag{5.3}$$

According to the law of large numbers for a martingale,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_1\, \mathrm{d}B_1(u) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_2\, \mathrm{d}B_2(u) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_3\, \mathrm{d}B_3(u) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_4\, \mathrm{d}B_4(u) = 0.$$

It follows that Eq (5.3) becomes

$$\begin{aligned}&\lim_{t\to\infty}\frac{(\beta_1+\beta_4+d_1\alpha_3+d_1\alpha_1)}{t}\int_0^t I(u)\mathrm{d}u + \lim_{t\to\infty}\frac{(\beta_3+d_2\alpha_2)}{t}\int_0^t V(u)\mathrm{d}u\\ &\geq 9\sqrt[9]{\Lambda_1^2\Lambda_2 d_1^2 d_2(\beta_1-\beta_2)\beta_3\beta_4} - (4d_1+3d_2+\mu+\gamma) + \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2\right) \quad a.s.\end{aligned}$$

The proof is complete.

6. Stationary distribution

The ergodic property for an epidemic model means that the stochastic model has a unique stationary distribution that forecasts the permanence of the epidemic in the future. That means the disease persists for all time regardless of the initial condition [46].

In this section, we provide a sufficient condition for the existence of a stationary distribution in the model (1.2). Denote

$$\mathscr{R}_2^S = \min\{\mathscr{R}_3^S, \mathscr{R}_4^S\}, \quad \mathscr{R}_3^S = \frac{7}{r_1}\sqrt[7]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4}}, \quad \mathscr{R}_4^S = \frac{8}{r_2}\sqrt[8]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4}},$$

where $r_1 = \frac{1}{2}\sum_{i=1}^4 \sigma_i^2 + 2d_1 + 3d_2 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3}$, and $r_2 = \frac{1}{2}\sum_{i=1}^4 \sigma_i^2 + 4d_1 + 2d_2 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3}$.

Theorem 4. If $\mathscr{R}_2^S > 1$, *then model (1.2) has a unique stationary distribution* $\pi(\cdot)$ *with ergodicity.*

Proof. The diffusion matrix for model (1.2) is given by

$$\bar{A} = \begin{pmatrix} \sigma_1^2 S^2 & 0 & 0 & 0\\ 0 & \sigma_2^2 I^2 & 0 & 0\\ 0 & 0 & \sigma_3^2 M^2 & 0\\ 0 & 0 & 0 & \sigma_4^2 V^2\end{pmatrix}.$$

Denote $M = \min_{(S,I,M,V)\in D\subset\mathbb{R}_+^4}\left\{\sigma_1^2 S^2, \sigma_2^2 I^2, \sigma_3^2 M^2, \sigma_4^2 V^2\right\}$. It follows that

$$\sum_{i,j=1}^4 b_{ij}(S,I,M,V)\xi_i\xi_j = \sigma_1^2 S^2\xi_1^2 + \sigma_2^2 E^2\xi_2^2 + \sigma_3^2 I^2\xi_3^2 + \sigma_4^2 R^2\xi_4^2 \geq M|\xi|^2$$

for $(S,I,M,V)\in D, \xi = (\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{R}^4$, where $D = \left(\frac{1}{k},k\right)\times\left(\frac{1}{k},k\right)\times\left(\frac{1}{k},k\right)\times\left(\frac{1}{k},k\right)$ and k is a sufficiently large integer. Therefore, the condition (i) in Lemma 2 is satisfied. Next, we prove the condition (ii) in Lemma 2. Let

$$\begin{aligned}\mathscr{V}_1 &= -\log S - \log M - \log V - \log I + \alpha_2 d_2 V,\\ \mathscr{V}_2 &= -\log S - \log M - \log V + (\alpha_1+\alpha_3)(S+I),\\ \mathscr{V}_3 &= -\log S - \log M, \quad \mathscr{V}_4 = \frac{1}{\theta+2}(S+I+M+V)^{\theta+2}.\end{aligned}$$

Denote $\lambda_i = r_i(\mathscr{R}_i - 1)$ $(i = 1, 2)$, $\sigma^2 = \sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2$, $b = 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2+\sigma_3^2}{2}$ and $d = d_1 \wedge d_2$. We construct a C^2-function $\widetilde{\mathscr{V}} : \mathbb{R}_+^4 \to \mathbb{R}$ as follows

$$\widetilde{\mathscr{V}}(S, I, M, V) = \Theta_1 \mathscr{V}_1 + \Theta_2 \mathscr{V}_2 + \mathscr{V}_3 + \mathscr{V}_4,$$

where Θ_i $(i = 1, 2)$ denotes sufficiently large positive constants satisfying $-\Theta_1\lambda_1 + F_2 \le -3$, $-\Theta_2\lambda_2 + F_3 \le -2$, and

$$\begin{aligned}
F_2 &= \sup_{(S,I,M,V)\in\mathbb{R}_+^4} \left\{ \frac{1}{2}(\alpha_2 d_2 \beta_4)^2 M^2 - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) + b + B \right\}, \\
F_3 &= \sup_{(S,I,M,V)\in\mathbb{R}_+^4} \left\{ \Theta_1 \alpha_2 d_2 \beta_4 MI - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) + b + B \right\}, \\
B &= \sup_{(S,I,M,V)\in\mathbb{R}_+^4} \left\{ (\Lambda_1 + \Lambda_2)(S + I + M + V)^{\theta+1} - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)(S + I + M + V)^{\theta+2} \right\} < \infty,
\end{aligned}$$

Here, θ is a positive constant satisfying that $d > (\theta+1)\sigma^2/2$. It means that

$$\liminf_{k\to\infty,(S,I,M,V)\in\mathbb{R}_+^4\setminus D} \widetilde{\mathscr{V}}(S, I, M, V) = \infty,$$

and $\widetilde{\mathscr{V}}(S, I, M, V)$ is a continuous function. Then the function $\widetilde{\mathscr{V}}(S, I, M, V)$ must have a minimum point $\left(\bar{S}, \bar{I}, \bar{M}, \bar{V}\right) \in \mathbb{R}_+^4$. Further, we construct a nonnegative C^2-function $\mathscr{V} : \mathbb{R}_+^4 \to \mathbb{R}_+$ in the following form

$$\mathscr{V}(S, I, M, V) = \Theta_1 \mathscr{V}_1 + \Theta_2 \mathscr{V}_2 + \mathscr{V}_3 + \mathscr{V}_4 - \widetilde{\mathscr{V}}\left(\bar{S}, \bar{I}, \bar{M}, \bar{V}\right).$$

Applying Itô's formula to $\mathscr{V}_1$, we get

$$\begin{aligned}
\mathscr{L}\mathscr{V}_1 =& -\frac{1}{S}\left[\Lambda_1 - d_1 S - \left(\beta_1 - \beta_2 \frac{I}{I+m}\right)\frac{SI}{1+\alpha_1 I} - \beta_3 \frac{SV}{1+\alpha_2 V}\right] + \frac{\sigma_1^2}{2} \\
& - \frac{1}{I}\left[-(d_1 + \mu + \gamma)I + \left(\beta_1 - \beta_2 \frac{I}{I+m}\right)\frac{SI}{1+\alpha_1 I} + \beta_3 \frac{SV}{1+\alpha_2 V}\right] + \frac{\sigma_2^2}{2} \\
& - \frac{1}{M}\left[\Lambda_1 - \beta_4 \frac{MI}{1+\alpha_3 I} - d_2 M\right] + \frac{\sigma_3^2}{2} - \frac{1}{V}\left[\beta_4 \frac{MI}{1+\alpha_3 I} - d_2 V\right] + \frac{\sigma_4^2}{2} - \frac{\alpha_2 d_2 \beta_4 MI}{(1+\alpha_3 I)} - d_2 \alpha_2 V \\
\le& -\frac{\Lambda_1}{2S} - \frac{\Lambda_2}{M} - (\beta_1 - \beta_2)\frac{S}{1+\alpha_1 I} - \frac{\Lambda_1}{2S} - \frac{\beta_3 SV}{(1+\alpha_2 V)I} + \frac{\beta_4 MI}{(1+\alpha_3 I)V} + d_2(1 + \alpha_2 V)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}+\frac{\sigma_4^2}{2}+2d_1+3d_2+(\mu+\gamma)+\frac{\beta_1}{\alpha_1}+\frac{\beta_3}{\alpha_2}+\frac{\beta_4}{\alpha_3}+\alpha_2 d_2\beta_4 MI\\
\leq&-7\sqrt[7]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4(1+\alpha_1 I)(1+\alpha_3 I)}}+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}+\frac{\sigma_4^2}{2}+2d_1+3d_2+(\mu+\gamma)\\
&+\frac{\beta_1}{\alpha_1}+\frac{\beta_3}{\alpha_2}+\frac{\beta_4}{\alpha_3}+\alpha_2 d_2\beta_4 MI,\\
\mathscr{L}\mathscr{V}_2=&-\frac{1}{S}\left[\Lambda_1-d_1S-\left(\beta_1-\beta_2\frac{I}{I+m}\right)\frac{SI}{1+\alpha_1 I}-\beta_3\frac{SV}{1+\alpha_2 V}\right]+\frac{\sigma_1^2}{2}\\
&-\frac{1}{I}\left[-(d_1+\mu+\gamma)I+\left(\beta_1-\beta_2\frac{I}{I+m}\right)\frac{SI}{1+\alpha_1 I}+\beta_3\frac{SV}{1+\alpha_2 V}\right]+\frac{\sigma_2^2}{2}\\
&-\frac{1}{M}\left[\Lambda_1-\beta_4\frac{MI}{1+\alpha_3 I}-d_2M\right]+\frac{\sigma_4^2}{2}-\frac{1}{V}\left[\beta_4\frac{MI}{1+\alpha_3 I}-d_2V\right]+\frac{\sigma_4^2}{2}\\
&-d_1(\alpha_1+\alpha_3)I-d_1(\alpha_1+\alpha_3)S-(\mu+\gamma)(\alpha_1+\alpha_3)I\\
\leq&-\frac{\Lambda_1}{2S}-\frac{\Lambda_2}{M}-(\beta_1-\beta_2)\frac{S}{1+\alpha_1 I}-\frac{\Lambda_1}{2S}-\frac{\beta_3 SV}{(1+\alpha_2 V)I}-\frac{\beta_4 MI}{(1+\alpha_3 I)V}-d_1(1+\alpha_1 I)\\
&-d_1(1+\alpha_3 I)+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}+\frac{\sigma_4^2}{2}+4d_1+2d_2+(\mu+\gamma)+\frac{\beta_1}{\alpha_1}+\frac{\beta_3}{\alpha_2}+\frac{\beta_4}{\alpha_3}\\
\leq&-8\sqrt[8]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4(1+\alpha_3 V)}}+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}+\frac{\sigma_4^2}{2}+4d_1+2d_2+(\mu+\gamma)\\
&+\frac{\beta_1}{\alpha_1}+\frac{\beta_3}{\alpha_2}+\frac{\beta_4}{\alpha_3},
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{L}\mathscr{V}_3=&-\frac{1}{S}\left[\Lambda_1-d_1S-\left(\beta_1-\beta_2\frac{I}{I+m}\right)\frac{SI}{1+\alpha_1 I}-\beta_3\frac{SV}{1+\alpha_2 V}\right]+\frac{\sigma_1^2}{2}\\
&-\frac{1}{M}\left[\Lambda_2-\beta_4\frac{MI}{1+\alpha_3 I}-d_2M\right]+\frac{\sigma_3^2}{2}\\
\leq&-\frac{\Lambda_1}{S}-\frac{\Lambda_2}{M}+2d_1+\mu+\gamma+\frac{\beta_1}{\alpha_1}+\frac{\beta_4}{\alpha_3}+\frac{\beta_3}{\alpha_2}+\frac{\sigma_1^2+\sigma_3^2}{2},\\
\mathscr{L}\mathscr{V}_4=&(S+I+M+V)^{\theta+1}\left[\Lambda_1-d_1(S+I+R)-(\mu+\gamma)I+\Lambda_2-d_2(M+V)\right]\\
&+(\theta+1)(S+I+M+V)^{\theta}\left[\frac{\sigma_1^2}{2}S^2+\frac{\sigma_3^2}{2}I^2+\frac{\sigma_3^2}{2}M^2+\frac{\sigma_4^2}{2}V^2\right]\\
\leq&(S+I+M+V)^{\theta+1}(\Lambda_1+\Lambda_2)-(S+I+R+M+V)^{\theta+2}(d_1\wedge d_2)\\
&+\frac{1}{2}(\theta+1)(S+I+M+V)^{\theta}\sigma^2(S+I+M+V)^2\\
\leq&(S+I+M+V)^{\theta+1}(\Lambda_1+\Lambda_2)-\left(d-\frac{(\theta+1)\sigma^2}{2}\right)(S+I+M+V)^{\theta+2}\\
\leq&B-\frac{1}{2}\left(d-\frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2}+I^{\theta+2}+M^{\theta+2}+V^{\theta+2}\right),
\end{aligned}
$$

where

$$B = \sup_{(S,I,M,V)\in\mathbb{R}_+^4} \left\{(\Lambda_1 + \Lambda_2)(S + I + M + V)^{\theta+1} - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)(S + I + M + V)^{\theta+2}\right\} < \infty.$$

Thus, it follows that

$$\begin{aligned}\mathscr{L}\mathscr{V} \leq & -\varTheta_1 r_1(\mathscr{R}_3^S - 1) - \varTheta_2 r_2(\mathscr{R}_4^S - 1) + 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} + \frac{\sigma_1^2 + \sigma_3^2}{2} \\ & + \varTheta_1 (\alpha_2 d_2 \beta_4) MI + B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) - \frac{\Lambda_1}{S} - \frac{\Lambda_2}{M}.\end{aligned}$$

A closed subset is defined as follows:

$$D = \left\{(S, I, M, V) \in \mathbb{R}_+^4 : \epsilon \leqslant S \leqslant \frac{1}{\epsilon}, \epsilon \leqslant I \leqslant \frac{1}{\epsilon}, \epsilon \leqslant M \leqslant \frac{1}{\epsilon}, \epsilon \leqslant V \leqslant \frac{1}{\epsilon}\right\},$$

where $\epsilon > 0$ represents sufficiently small constants satisfying the following conditions

$$-\frac{\Lambda_1}{\epsilon} + F_3 < 1, \tag{6.1}$$

$$-\varTheta_1 r_3(7\sqrt[7]{\frac{(\beta_1 - \beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4(1+\alpha\epsilon)(1+\alpha\epsilon)}} - 1) + F_2 + \frac{\varTheta_1^2\epsilon^2}{2} < -1, \tag{6.2}$$

$$H + \varTheta_1 (\alpha_2 d_2 \beta_4) \frac{1}{\epsilon^2} - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(\frac{1}{\epsilon}\right)^{\theta+2} < -1, \tag{6.3}$$

$$-\frac{\Lambda_2}{\epsilon} + F_2 < -1, \tag{6.4}$$

$$-\varTheta_2 r_2(8\sqrt[8]{\frac{(\beta_1 - \beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4(1+\alpha_3\epsilon)}} - 1) + F_3 < -1, \tag{6.5}$$

$$H + \varTheta_1 (\alpha_2 d_2 \beta_4) \frac{1}{\epsilon^2} - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(\frac{1}{\epsilon}\right)^{\theta+2} < -1, \tag{6.6}$$

where H is a constant and is determined later. Denote $Y = (S, I, M, V)$. We divide $\mathbb{R}_+^4 \setminus D$ into the following eight cases

$D_1 = \{Y \in \mathbb{R}_+^4, 0 < S < \epsilon\}$, $D_2 = \{Y \in \mathbb{R}_+^4, 0 < I < \epsilon\}$,
$D_3 = \{Y \in \mathbb{R}_+^4, S > \frac{1}{\epsilon}, \epsilon < M < \frac{1}{\epsilon}, \epsilon < I < \frac{1}{\epsilon}\}$, $D_4 = \{Y \in \mathbb{R}_+^4, I > \frac{1}{\epsilon}, \epsilon < M < \frac{1}{\epsilon}\}$,
$D_5 = \{Y \in \mathbb{R}_+^4, 0 < M < \epsilon\}$, $D_6 = \{Y \in \mathbb{R}_+^4, 0 < V < \epsilon_4, \}$,
$D_7 = \{Y \in \mathbb{R}_+^4, \frac{1}{\epsilon} < M, \epsilon < I < \frac{1}{\epsilon}\}$, $D_8 = \{Y \in \mathbb{R}_+^4, \frac{1}{\epsilon} < V, \epsilon < M < \frac{1}{\epsilon}, \epsilon < I < \frac{1}{\epsilon}\}$.

Now, we will prove that $\mathscr{L}V(S, I, M, V) < -1$ on $\mathbb{R}_+^4 \setminus D$; this is equivalent to proving that it is valid on the above eight subsets.

Case 1. When $(S, I, M, V) \in D_1$, we can get

$$\begin{aligned}\mathscr{L}\mathscr{V} \leq& 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 (\alpha_2 d_2 \beta_4) MI \\ &+ B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) - \frac{\Lambda_1}{S} \\ \leq& -\frac{\Lambda_1}{S} + F_3 \leq -\frac{\Lambda_1}{\epsilon} + F_3,\end{aligned}$$

where

$$F_3 = \sup_{(S,I,M,V)\in\mathbb{R}_+^4}\left\{\Theta_1\alpha_2 d_2\beta_4 MI - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) + a + B\right\}.$$

According to Eq (6.1), we have that $\mathscr{L}\mathscr{V}(S, I, M, V) < -1$ for any $(S, I, M, V) \in D_1$.

Case 2. When $(S, I, M, V) \in D_2$, we have

$$\begin{aligned}\mathscr{L}\mathscr{V} \leq& -\Theta_1 r_3 (7\sqrt[7]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4(1+\alpha_1 I)(1+\alpha_3 I)}} - 1) + 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} + \frac{\sigma_1^2 + \sigma_3^2}{2} \\ &+ \frac{(\alpha_2 d_2\beta_4)^2 M^2}{2} + B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) + \frac{\Theta_1^2 I^2}{2} \\ \leq& -\Theta_1 r_3 (7\sqrt[7]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4(1+\alpha_1 \epsilon)(1+\alpha_3 \epsilon)}} - 1) + 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} + \frac{\sigma_1^2 + \sigma_3^2}{2} \\ &+ \frac{(\alpha_2 d_2\beta_4)^2 M^2}{2} + B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) + \frac{\Theta_1^2 \epsilon^2}{2} \\ \leq& -\Theta_1 r_3 (7\sqrt[7]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4(1+\alpha_1 \epsilon)(1+\alpha_3 \epsilon)}} - 1) + F_2 + \frac{\Theta_1^2\epsilon^2}{2},\end{aligned}$$

where

$$F_2 = \sup_{(S,I,M,V)\in\mathbb{R}_+^4}\left\{\frac{1}{2}(\alpha_2 d_2\beta_4)^2 M^2 - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) + a + B\right\}.$$

Given Eq (6.2), we get that $\mathscr{L}\mathscr{V}(S, I, M, V) < -1$ for any $(S, I, M, V) \in D_2$.

Case 3. When $(S, I, M, V) \in D_3$, we have that

$$\begin{aligned}\mathscr{L}\mathscr{V} \leq& d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 (\alpha_2 d_2 \beta_4) MI \\ &+ B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) \\ \leq& H + \Theta_1 (\alpha_2 d_2\beta_4)\frac{1}{\epsilon^2} - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(\frac{1}{\epsilon}\right)^{\theta+2} \\ \leq& H + \Theta_1 (\alpha_2 d_2\beta_4)\frac{1}{\epsilon^2} - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(\frac{1}{\epsilon}\right)^{\theta+2},\end{aligned}$$

where $H = d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2+\sigma_3^2}{2} + B$, and, from Eq (6.3), we have that $\mathscr{L}\mathscr{V}(S, I, M, V) < -1$ for any $(S, I, M, V) \in D_3$.

Case 4. When $(S, I, M, V) \in D_4$, then

$$\begin{aligned}\mathscr{L}\mathscr{V} \leq & 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 (\alpha_2 d_2 \beta_4) MI \\ & + B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) \\ \leq & H + \Theta_1 (\alpha_2 d_2 \beta_4) \frac{1}{\epsilon^2} - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(\frac{1}{\epsilon}\right)^{\theta+2} \\ = & H + \Theta_1 (\alpha_2 d_2 \beta_4) \left(\frac{1}{\epsilon}\right)^2 - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(\frac{1}{\epsilon}\right)^{\theta+2}.\end{aligned}$$

Again, from Eq (6.3), we find that $\mathscr{L}\mathscr{V}(S, I, M, V) < -1$ for any $(S, I, M, V) \in D_4$.

Case 5. When $(S, I, M, V) \in D_5$, we have

$$\begin{aligned}\mathscr{L}\mathscr{V} \leq & 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 (\alpha_2 d_2 \beta_4) MI \\ & + B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) - \frac{\Lambda_2}{M} \\ \leq & -\frac{\Lambda_2}{M} + F_3 \leq -\frac{\Lambda_1}{\epsilon} + F_3.\end{aligned}$$

By means of Eq (6.4) we obtain that $\mathscr{L}\mathscr{V}(S, I, M, V) < -1$ for any $(S, I, M, V) \in D_5$.

Case 6. When $(S, I, M, V) \in D_6$, we can get

$$\begin{aligned}\mathscr{L}\mathscr{V} \leq & -\Theta_2 r_2 (8\sqrt[8]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4(1+\alpha_3 V)}} - 1) + 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} + \frac{\sigma_1^2 + \sigma_3^2}{2} \\ & + \Theta (\alpha_2 d_2 \beta_4) MI + B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) \\ \leq & -\Theta_2 r_2 (8\sqrt[8]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4(1+\alpha\epsilon)}} - 1) + 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} + \frac{\sigma_1^2 + \sigma_3^2}{2} \\ & + \Theta (\alpha_2 d_2 \beta_4) MI + B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) \\ \leq & -\Theta_2 r_2 (8\sqrt[8]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4(1+\alpha\epsilon)}} - 1) + F_3.\end{aligned}$$

By Eq (6.5), we have that $\mathscr{L}\mathscr{V}(S, I, M, V) < -1$ for any $(S, I, M, V) \in D_6$.

Case 7. When $(S, I, M, V) \in D_7$, it follows that

$$\begin{aligned}\mathscr{L}\mathscr{V} \leq & 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 \left(\alpha_2 d_2 \beta_4\right) MI \\ & + B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) \\ \leq & d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 \left(\alpha_2 d_2 \beta_4\right) \frac{1}{\epsilon^2} \\ & + B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(\frac{1}{\epsilon}\right)^{\theta+2} \\ \leq & H + \Theta_1 \left(\alpha_2 d_2 \beta_4\right) \frac{1}{\epsilon^2} - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(\frac{1}{\epsilon}\right)^{\theta+2},\end{aligned}$$

Using Eq (6.6), we have that $\mathscr{L}\mathscr{V}(S, I, M, V) < -1$ for any $(S, I, M, V) \in D_7$.

Case 8. When $(S, I, M, V) \in D_8$, then

$$\begin{aligned}\mathscr{L}\mathscr{V} \leq & 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 \left(\alpha_2 d_2 \beta_4\right) MI \\ & + B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}\right) \\ \leq & 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 \left(\alpha_2 d_2 \beta_4\right) \frac{1}{\epsilon^2} \\ & + B - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(\frac{1}{\epsilon}\right)^{\theta+2} \\ \leq & H + \Theta_1 \left(\alpha_2 d_2 \beta_4\right) \frac{1}{\epsilon^2} - \frac{1}{2}\left(d - \frac{(\theta+1)\sigma^2}{2}\right)\left(\frac{1}{\epsilon}\right)^{\theta+2}.\end{aligned}$$

Again using Eq (6.6), we have that $\mathscr{L}\mathscr{V}(S, I, M, V) < -1$ for any $(S, I, M, V) \in D_8$.

In Cases 1–8, we have chosen sufficiently small values of ϵ such that $\mathscr{L}\mathscr{V}(S, I, M, V) < -1$ for any $(S, I, M, V) \in D_i$ $(i = 1, 2, \ldots, 8)$. Thus, $\mathscr{L}\mathscr{V}(S, I, M, V) < -1$ for all (S, I,M,V) $\in \mathbb{R}_+^4 \backslash D$. Then, the condition (ii) in Lemma 2 is satisfied. According to Lemma 2, we can obtain that system (1.2) is ergodic and has a unique stationary distribution. This completes the proof.

7. Numerical simulations

Numerical simulations are presented to support our theoretical findings of the model (1.2) and reveal the impact of media coverage on the spread of disease. Using the Milstein method mentioned by Higham [47], we consider the discretized equations as follows:

$$
\begin{aligned}
S_{i+1} =& S_i + \left(\Lambda_1 - d_1 S_i - \left(\beta_1 - \frac{\beta_2 I_i}{m+I_i}\right)\frac{S_i I_i}{1+\alpha_1 I_i} - \frac{\beta_3 S_i V_i}{1+\alpha_2 V_i}\right)\Delta t \\
&+ S_i\left(\sigma_1\sqrt{\Delta t}\zeta_{1i} + \frac{\sigma_1}{2}S_i\left(\zeta_{1i}^2 - 1\right)\Delta t\right), \\
I_{i+1} =& I_i + \left(\left(\beta_1 - \frac{\beta_2 I_i}{m+I_i}\right)\frac{S_i I_i}{1+\alpha_1 I_i} + \frac{\beta_3 S_i V_i}{1+\alpha_2 V_i} - (d_1+\mu+\gamma)I_i\right)\Delta t \\
&+ I_i\left(\sigma_2\sqrt{\Delta t}\zeta_{2i} + \frac{\sigma_2^2}{2}I_i\left(\zeta_{2i}^2 - 1\right)\Delta t\right), \\
M_{i+1} =& M_i + \left(\Lambda_2 - \frac{\beta_4 M_i I_i}{1+\alpha_3 I_i} - d_2 M_i\right)\Delta t + M_i\left(\sigma_3\sqrt{\Delta t}\zeta_{3i} + \frac{\sigma_3^2}{2}M_i\left(\zeta_{3i}^2 - 1\right)\Delta t\right), \\
V_{i+1} =& V_i + \left(\frac{\beta_4 M_i I_i}{1+\alpha_3 I_i} - d_2 V_i\right)\Delta t + V_i\left(\sigma_4\sqrt{\Delta t}\zeta_{4i} + \frac{\sigma_4^2}{2}V_i\left(\zeta_{4i}^2 - 1\right)\Delta t\right),
\end{aligned}
$$

where the time increment $\Delta t > 0$ and $\zeta_{1i}, \zeta_{2i}, \zeta_{3i}, \zeta_{4i}$, are mutually independent Gaussian random variables which follow the distribution $N(0,1)$ for $i = 0, 1, 2, ..., n$.

Vector-borne diseases with two transmission routes may be more likely to become endemic than diseases with one transmission route. Therefore, we tend to choose lower transmission rates and recruitment when numerically modeling disease extinction.

Example 1. Let $\Lambda_1 = 100, \Lambda_2 = 100, \beta_1 = 0.000012, \beta_2 = 0.0000018, \beta_3 = 0.000039, \beta_4 = 0.000039$, $\alpha_1 = 0.13$, $\alpha_2 = 0.15$, $\alpha_3 = 0.15$, $\mu = 0.13$, $\gamma = 0.13$, $d_1 = 0.1$, $d_2 = 0.1$, $m = 20$, $\sigma_1 = 0.025$, $\sigma_2 = 0.25$, $\sigma_3 = 0.03$, $\sigma_4 = 0.26$, $\mu_1 = \min\{\mu + d_1 + \gamma, d_2\}$, $\sigma_* = \min\{\sigma_2, \sigma_4\}$, $\beta = \max\{\beta_1, \beta_3\}$, and the initial values $(S(0), I(0), M(0), V(0)) = (1000, 15, 1000, 50)$. So

$$
\mathscr{R}_0^S = \frac{1}{\mu_1 + \sigma_*^2/2}\left(\frac{\beta\Lambda_1}{d_1} + \frac{\beta_4\Lambda_2}{d_2}\right) \approx 0.594 < 1.
$$

According to Theorem 2, the solution of the stochastic model (1.2) will eventually approach zero; this means that the disease will die out almost surely. And, from Figure 1, it is observed that the number of infected individuals tends to zero.

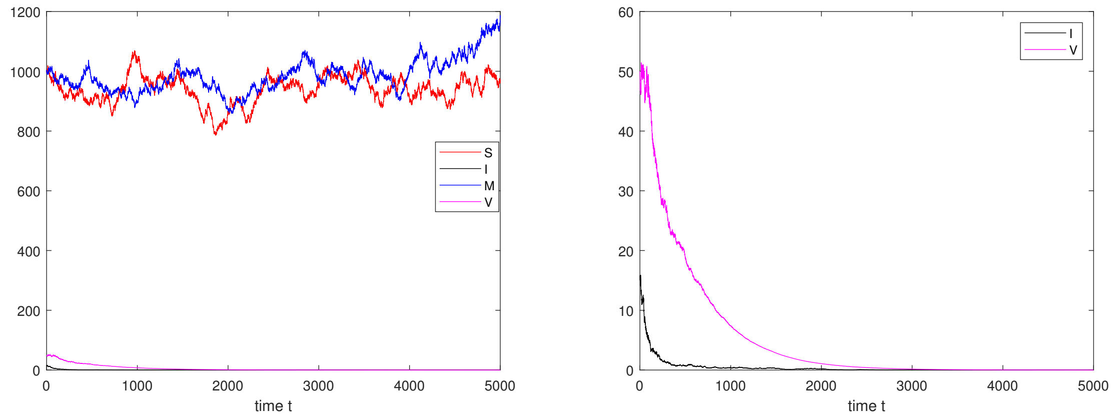

Figure 1. The disease became extinct in both groups at initial values $(S(0), I(0), M(0), V(0))$ =(1000, 15, 10000, 50).

Example 2. We keep the parameters the same as in Example 1, except that $\Lambda_1 = \Lambda_2 = 500$, $\beta_1 = 0.01$ and $\beta_3 = \beta_4 = 0.001$. Then

$$\mathscr{R}_1^S = \frac{9\sqrt[9]{\Lambda_1^2\Lambda_2 d_1^2 d_2(\beta_1-\beta_2)\beta_3\beta_4} + \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2\right)}{4d_1+3d_2+\mu+\gamma} \approx 3.555 > 1.$$

Theorem 3 implies that the disease is persistent in the mean. Interestingly, in Figure 2, it is clear that the number of infected individuals is higher than that of susceptible individuals.

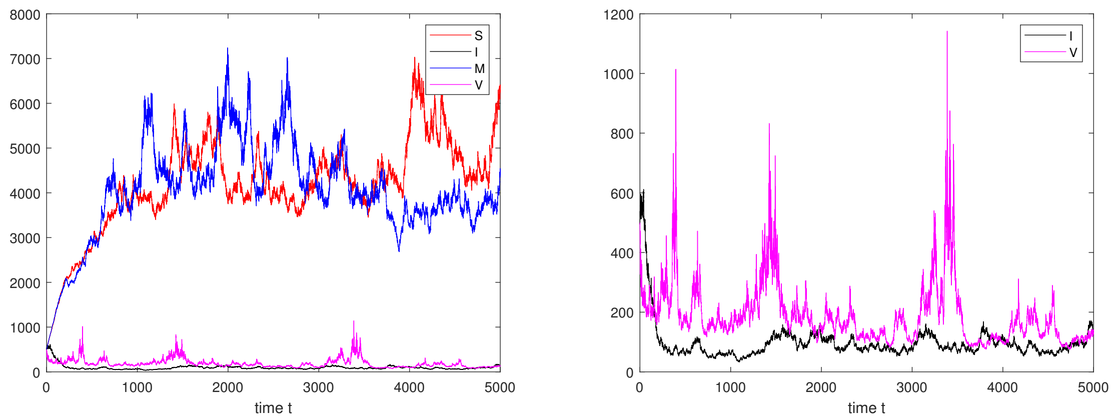

Figure 2. The disease persists in both groups at initial values $(S(0), I(0), M(0), V(0)) = (500, 500, 500, 500)$.

Example 3. Choose the parameters $\Lambda_1 = 35000$, $\Lambda_2 = 30000$, $\beta_1 = 0.05$, $\beta_2 = 0.000002$, $\beta_3 = 0.069$, $\beta_4 = 0.069$, $\alpha_1 = 0.1$, $\alpha_2 = 0.12$, $\alpha_3 = 0.12$, $\mu = 0.23$,$\gamma = 0.2$, $d_1 = 0.5$, $d_2 = 0.058$, $m = 100$, $\sigma_1 = 0.015$, $\sigma_2 = 0.018$, $\sigma_3 = 0.018$, $\sigma_4 = 0.02$, the initial values $\mu_1 = \min\{\mu+d_1+\gamma, d_2\}$, $\sigma = \min\{\sigma_2, \sigma_4\}$, $\beta = \max\{\beta_1, \beta_3\}$, and the initial values $(S(0), I(0), M(0), V(0)) = (20000, 2000, 20000, 2000)$. Using the parameters

$$\mathscr{R}_3^S = 7\sqrt[7]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4}} \Big/ \left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}+\frac{\sigma_4^2}{2}+2d_1+3d_2+(\mu+\gamma)+\frac{\beta_1}{\alpha_1}+\frac{\beta_3}{\alpha_2}+\frac{\beta_4}{\alpha_3}\right),$$

$$\mathscr{R}_4^S = 8\sqrt[8]{\frac{(\beta_1-\beta_2)\beta_3\beta_4\Lambda_1^2\Lambda_2}{4}} \Big/ \left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}+\frac{\sigma_4^2}{2}+4d_1+2d_2+(\mu+\gamma)+\frac{\beta_1}{\alpha_1}+\frac{\beta_3}{\alpha_2}+\frac{\beta_4}{\alpha_3}\right),$$

we can obtain that $\mathscr{R}_2^S = \{\mathscr{R}_3^S, \mathscr{R}_4^S\} \approx 49.720 > 1$ and the conditions of Theorem 4 are satisfied. Figure 3 shows the histograms of solutions of model (1.2) with white noise. Theoretical conclusions and numerical simulations indicate that the disease will eventually prevail and persist for a long time.

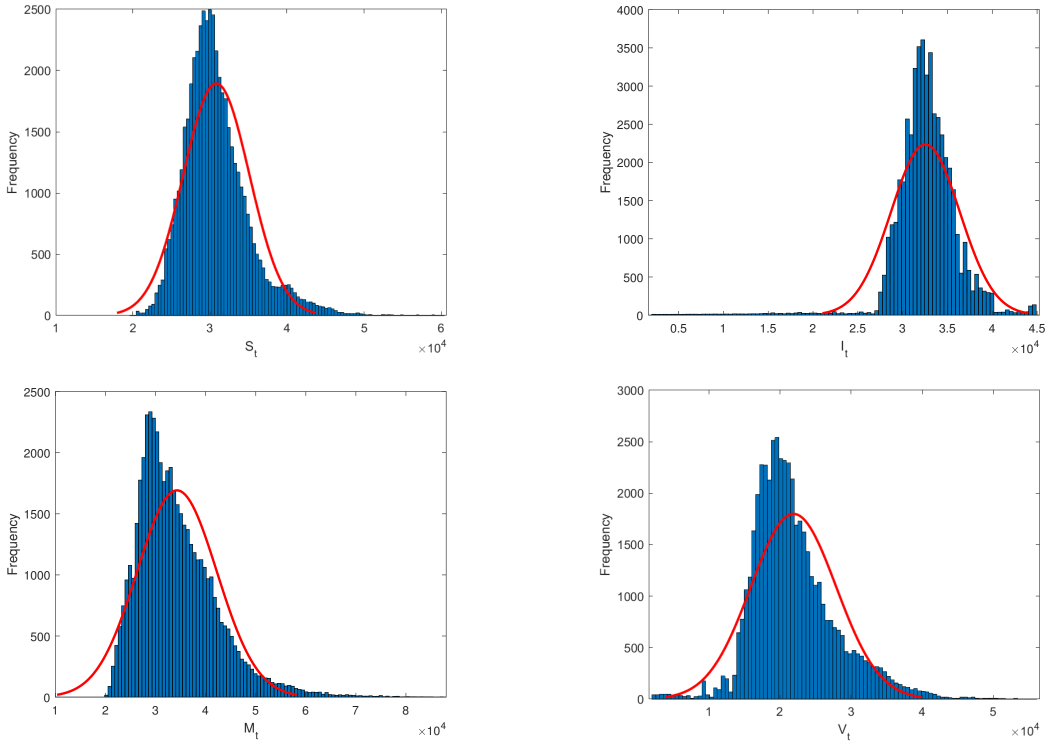

Figure 3. Histograms with 100 bins generated from 50,000 simulations of the model (1.2), where the red curves are the probability density functions.

Example 4. Given $\beta_2 = 0$, different transmission rates β_1, =0, 0.02, 0.05, 0.08, 0.12. When the transmission rate $\beta_1 = 0.05$, we select that β_2=0, 0.006, 0.01, 0.012, 0.016, 0.02. The rest parameters are the same as in the Example 1. Figure 4 shows that as β_1 changes from 0, it significantly impacts the system. As β_2 increases, the numbers of infections decreases. This shows that the existence of direct transmission via this transmission route has an significant influence on disease transmission, and that reducing the rate of human-to-human contact through media coverage can reduce the scale of vector-borne infectious diseases.

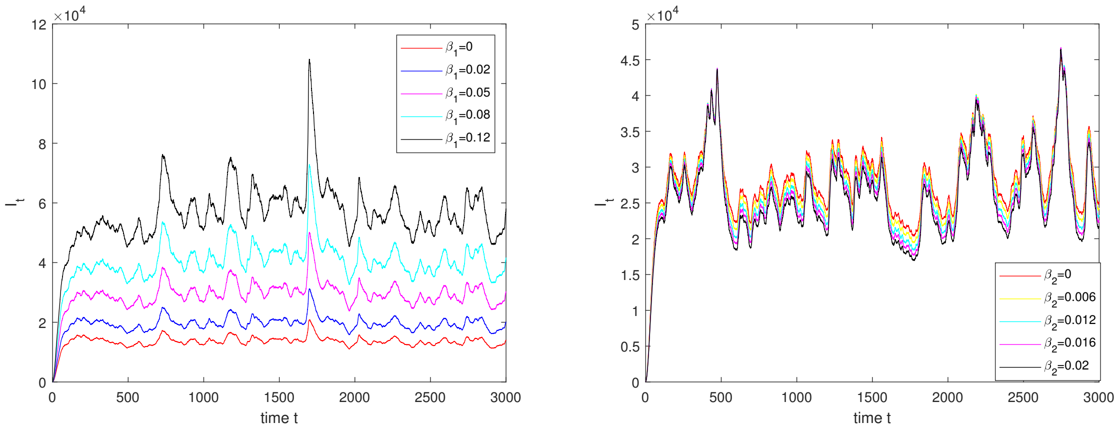

Figure 4. The number of individuals infected given different parameters β_1 and β_2 .

8. Conclusions

This paper presented a direct transmission model that is saturated with stochastic vector-borne disease incidence and the associated dynamical behavior. We obtained the positive definiteness and uniqueness of the solution to the stochastic model. Then, we established sufficient conditions for the extinction of the disease in two populations. Furthermore, we have proven the uniqueness and existence of an ergodic stationary distribution of the model when $\mathscr{R}_2^S > 1$ by choosing a suitable stochastic Lyapunov function.

On the other hand, from the simulation, we found that the disease under the condition of an increasing transmission rate β_1 showed an increasing transmission scale. It reflected that direct transmission i.e., the transmission route, has a critical influence on the spread of vector-borne diseases. In addition, we observed in the numerical experiments that there is indeed an effect on the number of the infected by increasing the value of the β_2. This also validates the inhibitory impact of media coverage on the spread of the disease.

Finally, reviewing the model we built, we have found that model (1.1) becomes the classical SIR model with media coverage if we set $\beta_3 = \beta_4 = 0$. This means that the disease can be endemic in the host population if there is no transmission pathway from the vector to the host. The threshold parameters obtained in this study do not explain this phenomenon. Some algorithms that guarantee the positivity of the solution are more useful when numerical simulations are performed [48,49]. We leave these issues for future research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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