Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 8386–8405.
DOI:10.3934/math.2024408
Received: 23 November 2023
Revised: 14 December 2023
Accepted: 25 December 2023
Published: 28 February 2024

Research article

Weighted L^p norms of Marcinkiewicz functions on product domains along surfaces

Badriya Al-Azri[1,*] **and Ahmad Al-Salman**[1,2]

[1] Sultan Qaboos University, College of Science, Department of Mathematics, Muscat, Oman
[2] Department of Mathematics, Yarmouk University, Irbid, Jordan

* **Correspondence:** Email: badriyaazri0@gmail. com.

Abstract: We prove a weighted L^p boundedness of Marcinkiewicz integral operators along surfaces on product domains. For various classes of surfaces, we prove the boundedness of the corresponding operators on the weighted Lebsgue space $L^p(\mathbb{R}^n \times \mathbb{R}^m,\ \omega_1(x)dx,\ \omega_2(y)dy)$, provided that the weights ω_1 and ω_2 are certain radial weights and that the kernels are rough in the optimal space $L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. In particular, we prove the boundedness of Marcinkiewicz integral operators along surfaces determined by mappings that are more general than polynomials and convex functions. Also, in this paper we prove the weighted L^p boundedness of the related square and maximal functions. Our weighted L^p inequalities extend as well as generalize previously known L^p boundedness results.

Keywords: Marcinkiewicz integral operators on product domains; weighted L^p norm; maximal functions; Hardy Littlewood maximal function; convex functions
Mathematics Subject Classification: 42B15, 42B20

1. Introduction and statement of results

Let $\mathbb{R}^n (n \geq 2)$ be an n-dimensional Euclidean space, $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ equipped with normalized Lebesgue measure $d\sigma$, and set $\mathbb{R}_+ = (0, \infty)$. Furthermore, we let $y' = \dfrac{y}{|y|} \in \mathbb{S}^{n-1}$ for $y \neq 0$ and let $\Omega \in L^1(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ that satisfies

$$\int_{\mathbb{S}^{n-1}} \Omega(y')d\sigma(y') = 0. \tag{1.1}$$

The classical Marcinkiewicz integral operator introduced by E. M. Stein in [1] is given by

$$\mu_\Omega(f)(x) = \left(\int_{-\infty}^{\infty} \left| \int_{|y|<2^t} f(x-y)\frac{\Omega(y')}{|y|^{n-1}}\, dy \right|^2 \frac{dt}{2^{2t}} \right)^{\frac{1}{2}}. \tag{1.2}$$

When $\Omega \in Lip_{\alpha}(\mathbb{S}^{n-1})(0 < \alpha \leq 1)$, Stein [1] proved that μ_{Ω} maps $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $1 < p \leq 2$. In [2], A. Benedek, A. Calderón, and R. Panzone proved that μ_{Ω} is bounded on L^p for all $1 < p < \infty$ provided that $\Omega \in C^1\left(\mathbb{S}^{n-1}\right)$. In [3], Walsh proved that μ_{Ω} is bounded on $L^2(\mathbb{R}^n)$ under the weak condition $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbb{S}^{n-1})$. Moreover, he showed that the L^2 boundedness of μ_{Ω} may fail if the condition $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbb{S}^{n-1})$ is replaced by $\Omega \in L(\log L)^{\frac{1}{2}-\varepsilon}(\mathbb{S}^{n-1})$ for some $\varepsilon > 0$. In 2002, Al-Salman et al. [4] improved Walsh's result by showing that the condition $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbb{S}^{n-1})$ is also sufficient for the L^p boundedness of μ_{Ω} for all $p \in (1, \infty)$. For further results and background information about the operator μ_{Ω}, we refer readers to [4–9] and references therein, among others.

In 1990, Torchinsky and Wang studied the L^p boundedness of the operator μ_{Ω} on weighted spaces. In fact, they showed in [10] that μ_{Ω} is bounded on $L^p(\omega)$ $(1 < p < \infty)$ if $\Omega \in Lip_{\alpha}(\mathbb{S}^{n-1})\,(0 < \alpha \leq 1)$ and $\omega \in A_p$ (the Muckenhoupt weight class, see [11]). Subsequently, Ding et al. [12] proved that μ_{Ω} is bounded on $L^p(\omega)$ for $p \in (1, \infty)$ provided that $\Omega \in L^q(\mathbb{S}^{n-1}), q > 1$, and $\omega^{q'} \in A_p(\mathbb{R}^n)$. In [13], Lee et al. proved a weighted norm inequality for μ_{Ω} under the assumption that Ω is in the Hardy space $H^1(\mathbb{S}^{n-1})$ and the weight ω is in the class $\tilde{A}_p^I(\mathbb{R}^n)$ of radial weights introduced by Duoandikoetxea in [14]. In [15], Al-Salman studied weighted inequalities of the generalized operator

$$\mu_{\Omega,\Psi}(f)(x) = \left(\int_{-\infty}^{\infty} \left| \int_{|y|<2^t} f(x - \Psi(|y|)y') \frac{\Omega(y')}{|y|^{n-1}}\, dy \right|^2 \frac{dt}{2^{2t}} \right)^{\frac{1}{2}}, \tag{1.3}$$

where $\Psi : (0, \infty) \to \mathbb{R}$ is a smooth function satisfying the following growth conditions

$$|\Psi(t)| \leqslant C_1 t^d, \qquad |\Psi''(t)| \geqslant C_2 t^{d-2}, \tag{1.4}$$

$$C_3 t^{d-1} \leq |\Psi'(t)| \leq C_4 t^{d-1} \tag{1.5}$$

for some $d \neq 0$ and $t \in (0, \infty)$ where C_1, C_2, C_3, and C_4 are positive constants independent of t. We shall let $\mathcal{G}$ be the class of all smooth mappings $\Psi : (0, \infty) \to \mathbb{R}$ that satisfy the growth conditions (1.4)–(1.5). It is clear that $\mathcal{G}$ contains all power functions $t^{\alpha}(\alpha \neq 0)$. It is shown in [15] that $\mu_{\Omega,\Psi}$ is bounded on $L^p(\omega)$ for $p \in (1, \infty)$ provided that $\omega \in \tilde{A}_p^I$ and that Ω is in the optimal space $L(\log^+ L)^{\frac{1}{2}}(\mathbb{S}^{n-1})$. Here, we remark that for any $q > 1$ and $0 < \alpha \leq 1$, the following inclusions hold and that they are proper

$$Lip_{\alpha}(\mathbb{S}^{n-1}) \subset L^q(\mathbb{S}^{n-1}) \subset L(\log L)(\mathbb{S}^{n-1}) \subset H^1(\mathbb{S}^{n-1}),$$

and

$$L(\log^+ L)^s(\mathbb{S}^{n-1}) \subset L(\log^+ L)^r(\mathbb{S}^{n-1}) \qquad \text{whenever } r < s.$$

In [8], Ding considered the analogy of the operator μ_{Ω} on the product domain setting. For $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ satisfying

$$\int_{\mathbb{S}^{n-1}} \Omega(u', .)d\sigma(u') = \int_{\mathbb{S}^{m-1}} \Omega(., v')d\sigma(v') = 0, \tag{1.6}$$

$$\Omega(tx, sy) = \Omega(x, y), \text{ for any } t, s > 0, \tag{1.7}$$

consider the Marcinkiewicz integral operator on the product domains $\mathcal{U}_{\Omega}$ defined by

$$\mathcal{U}_{\Omega} f(x, y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| F_{t',s'}(f)(x, y) \right|^2 \frac{dt'\, ds'}{2^{2(t'+s')}} \right)^{\frac{1}{2}}; \tag{1.8}$$

where

$$F_{t',s'}(f)(x,y) = \int\int_{\Lambda(t',s')} f(x-u, y-v)\ \frac{\Omega(u',v')}{|u|^{n-1}\,|v|^{m-1}} du\, dv \tag{1.9}$$

and

$$\Lambda(t',s') = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^m : |u| \le 2^{t'} \text{ and } |v| \le 2^{s'}\}.$$

Ding proved that $\mathcal{U}_\Omega$ is bounded on $L^2(\mathbb{R}^n{\times}\mathbb{R}^m)$ when Ω satisfies the additional assumption of $\Omega \in L(\log^+ L)^2(\mathbb{S}^{n-1}{\times}\mathbb{S}^{m-1})$, i.e.,

$$\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{m-1}} |\Omega(u',v')|\,(\log(2 + |\Omega(u',v')|)^2\, d\sigma(u')d\sigma(v') < \infty.$$

In 2002, Chen et al. [7] improved the result of Ding and showed that $\mathcal{U}_\Omega$ is bounded on $L^p(\mathbb{R}^n{\times}\mathbb{R}^m)$ $(1 \le p < \infty)$ under the same condition on Ω. Later, Choi [16] proved that the L^2 boundedness of $\mathcal{U}_\Omega$ still holds under the very weak condition $\Omega \in L(\log L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$. Subsequently, Al-Qassem et al. [17] substantially improved Choi's result by showing that $\mathcal{U}_\Omega$ is bounded on $L^p(\mathbb{R}^n{\times}\mathbb{R}^m)$ for all $1 < p < \infty$ under the same condition $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Moreover, they proved that the condition $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ is nearly optimal in the sense that the L^2 boundedness may fail if the function is assumed to be in $L(\log L)^\alpha(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})\backslash L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for any $\alpha < 1$. For further results for Marcinkiewicz integral operators on product domains, we cite [16–22], among others.

Motivated by the work in [15] and [18], we consider the weighted L^p boundedness of the Marcinkiewicz integral operator on product domains along surfaces. For suitable mappings $\Phi, \Psi : [0,\infty) \to \mathbb{R}$, consider the $\mathcal{U}_{\Omega,\Phi,\Psi}$ given by

$$\mathcal{U}_{\Omega,\Phi,\Psi}f(x,y) = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|F^{\Phi,\Psi}_{t',s'}(f)(x,y)\right|^2 \frac{dt'\,ds'}{2^{2(t'+s')}}\right)^{\frac{1}{2}}; \tag{1.10}$$

where

$$F^{\Phi,\Psi}_{t',s'}(f)(x,y) = \int\int_{\Lambda(t',s')} f(x - \Phi(|u|)u', y - \Psi(|v|)v')\ \frac{\Omega(u',v')}{|u|^{n-1}\,|v|^{m-1}} du\, dv, \tag{1.11}$$

and $\Lambda(t',s') = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^m : |u| \le 2^{t'} \text{ and } |v| \le 2^{s'}\}$. By specializing to the case $\Phi(t) = \Psi(t) = t$, the operator reduces to the classical Marcinkiewicz integral operator $\mathcal{U}_\Omega$ on product domains. Integral operators on product domains along surfaces have been considered by several authors. For background information, we advise the readers to consult [17–23] and references therein.

In order to state our results in this paper, we recall the definition of radial weights $\tilde{A}^I_p(\mathbb{R}^n)$ introduced in [14]:

Definition 1.1. *Let $\omega(t) \ge 0$; and $\omega \in L^1_{loc}(\mathbb{R}_+)$. For $1 < p < \infty$, we say that $\omega \in A_p(\mathbb{R}_+)$ if there is a positive constant C such that, for any interval $I \subseteq \mathbb{R}_+$,*

$$\left(|I|^{-1}\int_I \omega(t)dt\right)\left(|I|^{-1}\int_I \omega(t)^{-\frac{1}{p-1}}dt\right)^{p-1} \le C < \infty.$$

We say that $\omega \in A_1(\mathbb{R}_+)$ if there is a positive constant C such that

$$\omega^*(t) \le C\,\omega(t) \qquad for\ \ a.e.\ t \in \mathbb{R}_+,$$

where ω^* is the Hardy-Littlewood maximal function of ω on $\mathbb{R}_+$.

Definition 1.2. *Let* $1 \le p \le \infty$. *We say that* $\omega \in \tilde{A}_p(\mathbb{R}_+)$ *if*

$$\omega(x) = \nu_1(|x|)\,\nu_2(|x|)^{1-p},$$

where either $\nu_i \in A_1(\mathbb{R}_+)$ is decreasing or $\nu_i^2 \in A_1(\mathbb{R}_+)$, $\ i = 1, 2$.

Definition 1.3. *For* $1 < p < \infty$, *we let*

$$\bar{A}_p(\mathbb{R}_+) = \{\omega(x) = \omega(|x|) : \omega(t) > 0,\ \omega(t) \in L^1_{loc}(\mathbb{R}_+)\ and\ \omega^2(t) \in A_p(\mathbb{R}_+)\}.$$

Let $A_p^I(\mathbb{R}^n)$ be the weight class defined by exchanging the cubes in the definition of A_p for all n-dimensional intervals with sides parallel to coordinate axes. It is well known that $\bar{A}_p(\mathbb{R}_+) \subseteq \tilde{A}_p(\mathbb{R}_+)$ (see [24]). Moreover, if $\omega(t) \in \bar{A}_p(\mathbb{R}_+)$, then $\omega(|x|)$ is the Mukenhoupt weighted class $A_p(\mathbb{R}^n)$ whose definition can be found in [14]. We let $\tilde{A}_p^I = \tilde{A}_p \cap A_p^I$.

We shall need the following lemma:

Lemma 1.4. *If* $1 < p < \infty$, *then the weight class* $\tilde{A}_p^I(\mathbb{R}_+)$ *has the following properties:*

(i) $\tilde{A}_{p_1}^I \subset \tilde{A}_{p_2}^I$, if $1 \le p_1 < p_2 < \infty$;

(ii) For any $\omega \in \tilde{A}_p^I$, there exists an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \tilde{A}_p^I$;

(iii) For any $\omega \in \tilde{A}_p^I$ and $p > 1$, there exists an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $\omega \in \tilde{A}_{p-\varepsilon}^I$;

(iv) $\omega \in \tilde{A}_p^I$ if and only if $\omega^{1-p'} \in \tilde{A}_{p'}^I$.

For any weights ω_1 and ω_2, we let $L^p(\mathbb{R}^n \times \mathbb{R}^m,\ \omega_1(x)dx,\ \omega_2(y)dy)$ $(1 < p < \infty)$ be the weighted L^p space associated with the weight ω_1 and ω_2, i.e., $L^p(\mathbb{R}^n \times \mathbb{R}^m, \omega_1(x)dx, \omega_2(y)dy) = L^p(\omega_1, \omega_2)$ consists of all measurable functions f with $\|f\|_{L^p(\omega_1,\omega_2)} < \infty$, where

$$\|f\|_{L^p(\omega_1,\omega_2)} = \left(\iint_{\mathbb{R}^n\times\mathbb{R}^m} |f(x,y)|^p\,\omega_1(x)\,\omega_2(y)\,dx\,dy\right)^{\frac{1}{p}}. \tag{1.12}$$

In light of the above discussion, the following natural question arises:

Question: *Let* $\mathcal{U}_{\Omega,\Phi,\Psi}$ *be given by (1.8) and assume that* $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ *satisfying (1.6)–(1.7). Assume that* $\Phi,\Psi \in \mathcal{G}$, $\omega_1 \in \tilde{A}_p^I(\mathbb{R}^n)$ *and* $\omega_2 \in \tilde{A}_p^I(\mathbb{R}^m)$ *for some* $1 < p < \infty$. *Is* $\mathcal{U}_{\Omega,\Phi,\Psi}$ *bounded on* $L^p(\omega_1, \omega_2)$?

In the following we shall answer the above question in the affirmative. In fact, we shall prove that the weighted L^p boundedness holds for various classes of mappings Φ and Ψ.

Theorem 1.5. *Suppose that* $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ *satisfying (1.6)–(1.7),* $\omega_1 \in \tilde{A}_p^I(\mathbb{R}^n)$, *and* $\omega_2 \in \tilde{A}_p^I(\mathbb{R}^m)$. *If* $\Phi,\Psi \in \mathcal{G}$, *then* $\mathcal{U}_{\Omega,\Phi,\Psi}$ *is bounded on* $L^p(\omega_1, \omega_2)$ *for* $1 < p < \infty$.

We remark here that, by specializing to the case $\Phi(t) = \Psi(t) = t$, we obtain that the classical operator $\mathcal{U}_\Omega$ is bounded on $L^p(\omega_1, \omega_2)$ for $1 < p < \infty$. This result, as far as we know, is not known previously. We shall prove in this paper that the weighted boundedness in Theorem 1.5 holds for a more mappings Φ and Ψ. In order to state our second result, we recall the following class of mappings introduced in [5]:

Definition 1. 6. *A function* $\psi : [0,\infty) \to \mathbb{R}$ *is said to belong to the class* $\mathcal{PC}_\lambda(d)$ $(d > 0)$ *if there exist* $\lambda \in \mathbb{R}$, *a polynomial* P, *and* $\varphi \in C^{d+1}[0,\infty)$ *such that*

$$\begin{array}{l} (i)\, \psi(t) = P(t) + \lambda\varphi(t) \\ (ii)\, P(0) = 0 \;\text{ and }\; \varphi^{(j)}(0) = 0 \;\text{ for }\; 0 \leqslant j \leqslant d \\ (iii)\, \varphi^{(j)} \;\textit{ is positive nondecreasing on }\; (0,\infty) \;\text{ for }\; 0 \leqslant j \leqslant d+1. \end{array} \tag{1.13}$$

say that In fact, we prove the following:

The class $\mathcal{PC}_\lambda(d)$ was introduced in [5]. It is shown in [5] that the class $\cup_{d\geq 0}\mathcal{PC}_\lambda(d)$ properly contains the class of polynomials $\mathcal{P}_d$ of degree less than or equal d as well as the class of convex increasing functions. Examples of functions in $\cup_{d\geq 0}\mathcal{PC}_\lambda(d)$ that are neither convex nor polynomial are widely available. A particular example is the function $\theta(t) = -t^2 + t^2\ln(1+t)$. Our second result in this paper is the following:

Theorem 1.7. *Suppose that* $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ *satisfying (1.6)–(1.7),* $\omega_1 \in \tilde{A}_p^I(\mathbb{R}^n)$, *and* $\omega_2 \in \tilde{A}_p^I(\mathbb{R}^m)$. *If* $\Phi \in \mathcal{PC}_\lambda(d)$, $\Psi \in \mathcal{PC}_\alpha(b)$ *for* d, $b > 0$ *and* λ, $\alpha \in \mathbb{R}$, *then* $\mathcal{U}_{\Omega,\Phi,\Psi}$ *is bounded on* $L^p(\omega_1,\omega_2)$ *for* $1 < p < \infty$ *with* L^p *bounds independent of* $\lambda,\alpha \in \mathbb{R}$*and the coefficients of the particular polynomials involved in the standard representations of* Φ *and* Ψ.

We remark here that; Theorem 1.7 is the analogy of Theorem 1.3 [15] in the product domain setting. On the other hand, Theorem 1.7 is a generalization of the corresponding result in [18]. More specifically, if $\omega_1(x) = \omega_2(x) = 1$, then Theorem 1.7 reduces to Theorem 1.3 in [18].

We point out here that the method employed in this paper is based on interpolation between good L^2 estimates and crude L^p estimates. The L^2 estimates depend heavily on the nature of the involved surface. This is clearly expressed interns of the obtained oscillatory estimates. On the other hand, the L^p estimates depend on proving the boundedness of the corresponding maximal functions. The the method employed can be used to study the weighted L^p boundedness of more general classes of Marcinkiewicz integral operators along surfaces.

Throughout this paper, the letter C will stand for a constant that may vary at each occurrence, but it is independent of the essential variables.

2. Weighted estimates for certain square and maximal functions

This section is devoted to obtaining weighted estimates of certain square functions and maximal functions. For positive real numbers a and b and a Schwartz function $\Phi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, we let

$$S_{\Phi,a,b}(f)(x,y) = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Phi_{a^t,b^s} * f(x,y)|^2 dtds \right)^{\frac{1}{2}} \tag{2.1}$$

where

$$\Phi_{a^t,b^s}(x,y) = a^{-nt}\, b^{-ms}\Phi(a^{-t}x, b^{-s}y).$$

It can be observed here that if $\Phi(x,y) = \Phi^{(1)}(x)\Phi^{(2)}(y)$ and $f(x,y) = f_1(x)f_2(y)$, then

$$S_{\Phi,a,b}(f)(x,y) = S_{\Phi^{(1)},a}(f_1)(x)S_{\Phi^{(2)},b}(f_2)(y)$$

where $S_{\Phi^{(1)},a}$ and $S_{\Phi^{(2)},b}$ are the square functions in the one parameter setting defined in [15]. Thus, by Lemma 2.1 in [15], it follows that for two Muckenhoupt weights $\omega_1, \omega_2 \in A_p$, we have

$$\|S_{\Phi,a,b}(f_1 f_2)\|_{L^p(\omega_1,\omega_2)} \leq C_p \, \|f_1\|_{L^p(\omega_1)} \|f_2\|_{L^p(\omega_2)} = C_p \|f_1 f_2\|_{L^p(\omega_1,\omega_2)}. \tag{2.2}$$

Therefore, it is natural to question if (2.2) holds for general $\Phi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and $f \in L^p(\omega_1, \omega_2)$. In the following lemma, which is analogues to Lemma 2.1 in [15], we answer this question in the affirmative:
Lemma 2.1. *Given a, $b > 2$ and let ψ, θ be C^∞ functions on $\mathbb{R}$ that satisfy the following conditions:*
(i) $supp(\psi) \subseteq \left[\dfrac{4}{5a}, \dfrac{5a}{4}\right]$ and $supp(\theta) \subseteq \left[\dfrac{4}{5b}, \dfrac{5b}{4}\right]$.
(ii) $\left|\dfrac{d^l\psi}{du^l}(u)\right|, \left|\dfrac{d^l\theta}{du^l}(u)\right| \leq \dfrac{C_l}{u^l}$ for all u and $l \geq 0$ where C_l is independent of a and b.

Let $\Upsilon \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ be given by $\widehat{\Upsilon}(\xi, \eta) = \psi(|\xi|^2).\, \theta(|\eta|^2)$ and let $S_{\Upsilon,a,b}$ be the square function $S_{\Upsilon,a,b}$ given by (2.1) with Φ is replaced by Υ. Then, for $1 < p < \infty$, $\omega_1 \in A_p(\mathbb{R}^n)$, and $\omega_2 \in A_p(\mathbb{R}^m)$, there exists a constant C_p independent of a, b such that

$$\|S_{\Upsilon,a,b}(f)\|_{L^p(\omega_1,\omega_2)} \leq C_p \, \|f\|_{L^p(\omega_1,\omega_2)}. \tag{2.3}$$

Proof: For $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$, let

$$m_{a,b}(\xi, \eta, t', s') = \widehat{\Upsilon}(\xi, \eta) = \psi(|a^{t'} \xi|^2)\, \theta(|b^{s'} \eta|^2).$$

By the assumption (*ii*), we have

$$\left(\int_{-\infty}^{\infty} \left| \frac{\partial^\alpha m_{a,b}(\xi, \eta, t', s')}{\partial \xi^\alpha} \right|^2 dt' \right)^{\frac{1}{2}} \leq C_\alpha \, |\xi|^{-\alpha} \tag{2.4}$$

and

$$\left(\int_{-\infty}^{\infty} \left| \frac{\partial^\beta m_{a,b}(\xi, \eta, t', s')}{\partial \eta^\beta} \right|^2 ds' \right)^{\frac{1}{2}} \leq C_\beta \, |\eta|^{-\beta} \tag{2.5}$$

for every multi-index α, β with $|\alpha|, |\beta| \geq 0$, where C_α, C_β are constants independent of a and b. We set

$$K(x, y, t', s') = \Upsilon_{a^{t'}, b^{s'}}(x, y) = a^{-nt'}\, b^{-ms'}\, \Upsilon(a^{-t'} x, b^{-s'} y).$$

Then, by (2.4)–(2.5), and a vector-valued analogy of the argument in [25, p. 245–246], we obtain

$$\left(\int_{-\infty}^{\infty} \left| \frac{\partial^\alpha K(x, y, t', s')}{\partial\, x^\alpha} \right|^2 dt' \right)^{\frac{1}{2}} \leq C \, |x|^{-n-|\alpha|}, \tag{2.6}$$

$$\left(\int_{-\infty}^{\infty} \left| \frac{\partial^\beta K(x, y, t', s')}{\partial\, y^\beta} \right|^2 ds' \right)^{\frac{1}{2}} \leq C \, |y|^{-m-|\beta|}, \tag{2.7}$$

for $|\alpha| \leq 1$ and $|\beta| \leq 1$ where C is a constant independent of a and b.

Now, let

$$g_{\Upsilon,a,b}(f)(x,y) = |\Upsilon_{a^{t'},b^{s'}} * f(x,y)|,$$
$$g_{\Upsilon,a}(f)(x,.) = |\Upsilon_{a^{t'}} * f(x,.)|, \tag{2.8}$$

and

$$g_{\Upsilon,b}(f)(.,y) = |\Upsilon_{b^{s'}} * f(.,y)|. \tag{2.9}$$

Then,

$$g_{\Upsilon,a,b}(f)(x,y) \leqslant g_{\Upsilon,a}(g_{\Upsilon,b}(f))(x,y). \tag{2.10}$$

By Plancherel's theorem, we obtain

$$\|g_{\Upsilon,a}(f)(x,.)\|_{L^2(\mathbb{R}^m)} \leq C\,\|f\|_{L^2(\mathbb{R}^m)} \tag{2.11}$$

and

$$\|g_{\Upsilon,b}(f)(.,y)\|_{L^2(\mathbb{R}^n)} \leq C\,\|f\|_{L^2(\mathbb{R}^n)}. \tag{2.12}$$

Hence, by the Corollary on page 205 in [25], and (2.4), (2.6), and (2.11), we have

$$\int_{\mathbb{R}^n} |g_{\Upsilon,a}(f)(x,.)|^p w_1(x)\,dx \leqslant C \int_{\mathbb{R}^n} |(f)(x,.)|^p\, w_1(x)\,dx \tag{2.13}$$

for $w_1(x) \in A_p(\mathbb{R}^n)$.

Thus, by (2.13) and following similar arguments as in [26], we get

$$\int_{\mathbb{R}^n} |g_{\Upsilon,a}(f)(x,y)|^p\, w_1(x)\, w_2(y)\,dx \leqslant C \int_{\mathbb{R}^n} |(f)(x,y)|^p w_1(x)\, w_2(y)dx. \tag{2.14}$$

for each $y \in \mathbb{R}^m$ with C independent of y. Then, by integration over $\mathbb{R}^m$, we get

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^n} |g_{\Upsilon,a}(f)(x,y)|^p\, w_1(x)\, w_2(y)\,dx\,dy \leqslant C \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} |(f)(x,y)|^p w_1(x)\, w_2(y)dx\,dy. \tag{2.15}$$

By repeating the argument between (2.13) and (2.15) for $g_{\Upsilon,b}(f)(.,y)$, and replacing x by y, we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^m} |g_{\Upsilon,b}(f)(x,y)|^p\, w_1(x)\, w_2(y)\,dx\,dy \leqslant C \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} |(f)(x,y)|^p w_1(x)\, w_2(y)dx\,dy. \tag{2.16}$$

Finally, by (2.10), inequality (2.3) follows vector-valued analogues of the argument in the proof of the Theorem 3 in [26, p. 128], and (2.15)–(2.16). This ends the proof of Lemma 2.1.

Now, for $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ and suitable mappings $\Phi, \Psi : (0,\infty) \longrightarrow \mathbb{R}$, we define the family of measures $\{\sigma_{\Omega,\Phi,\Psi,a^{t'},b^{s'}} : t',\ s' \in \mathbb{R}\}$ by

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} f\,d\sigma_{\Omega,\Phi,\Psi,a^{t'},b^{s'}} = a^{-t'}\, b^{-s'} \iint\limits_{\substack{|u|<a^{t'}\\ |v|<b^{s'}}} f(x-\Phi(|u|)\,u', y-\Psi(|v|)\,v')\frac{\Omega(u',v')}{|u|^{n-1}\,|v|^{m-1}}\,dudv. \tag{2.17}$$

We let $\mathcal{M}_{\Omega,\Phi,\Psi,a,b}$ be the maximal function corresponding to the family $\{\sigma_{\Omega,\Phi,\Psi,a^{t'},b^{s'}} : t',\ s' \in \mathbb{R}\}$, i.e.,

$$\mathcal{M}_{\Omega,\Phi,\Psi,a,b}(f)(x,y) = \sup_{t',\,s' \in \mathbb{R}} \left|\sigma_{\Omega,\Phi,\Psi,a^{t'},b^{s'}} * f(x,y)\right|. \tag{2.18}$$

Then, we have the following lemma:

Lemma 2.2. *Suppose that $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ satisfying (1.7). For $a, b > 2$ and suitable $\Phi, \Psi : (0,\infty) \longrightarrow \mathbb{R}$, let $\mathcal{M}_{\Omega,\Phi,\Psi,a,b}$ be the maximal function defined by (2.18). Suppose that (i) $\Phi,\Psi \in \mathcal{G}$; or (ii) $\Phi \in \mathcal{PC}_\lambda(d_1)$, $\Psi \in \mathcal{PC}_\alpha(d_2)$ for d_1, $d_2 > 0$ and λ, $\alpha \in \mathbb{R}$. Then, for $1 < p < \infty$ and $\omega_1 \in A_p)(\mathbb{R}^n)$, $\omega_2 \in A_p)(\mathbb{R}^m)$, there exists a constant C_p independent of Ω, a, and b such that*

$$\|\mathcal{M}_{\Omega,\Phi,\Psi,a,b}(f)\|_{L^p(\omega_1,\omega_2)} \leq C_p \,\|\Omega\|_{L^1}\, \|f\|_{L^p(\omega_1,\omega_2)}. \tag{2.19}$$

Proof: We shall start by verifying (2.19) under assumption (ii) on the functions Φ and Ψ. Notice that

$$\mathcal{M}_{\Omega,\Phi,\Psi,a,b}(f) \leq \mathcal{M}_{\Omega,\Phi,\Psi}(f) = \sup_{t',\,s' \in \mathbb{R}} \left|\sigma_{\Omega,\Phi,\Psi,2^{t'},2^{s'}} * f(x,y)\right|. \tag{2.20}$$

Thus, it is enough to show that

$$\|\mathcal{M}_{\Omega,\Phi,\Psi}(f)\|_{L^p(\omega_1,\omega_2)} \leq C_p \,\|\Omega\|_{L^1}\, \|f\|_{L^p(\omega_1,\omega_2)}. \tag{2.21}$$

We define the one parameter maximal functions

$$\mathcal{M}_{\Omega,\Psi}(f)(\cdot,y) = \sup_{s' \in \mathbb{R}} \left| 2^{-s'} \int_{|v|<2^{s'}} f(.,y - \Psi(|v|)\,v') \frac{\Omega(\cdot,v')}{|v|^{m-1}}\,dv \right|,$$

and

$$\mathcal{M}_{\Omega,\Phi}(f)(x,\cdot) = \sup_{t' \in \mathbb{R}} \left| 2^{-t'} \int_{|u|<2^{t'}} f(x - \Phi(|u|)\,u',.) \frac{\Omega(u',\cdot)}{|u|^{n-1}}\,du \right|.$$

Then,

$$\mathcal{M}_{\Omega,\Phi,\Psi}(f)(x,y) \leq \mathcal{M}_{\Omega,\Phi}(\mathcal{M}_{\Omega,\Psi}(f)(\cdot,y)(x,\cdot). \tag{2.22}$$

By polar coordinates, we have

$$\mathcal{M}_{\Omega,\Psi} f(\cdot,y) \leq \int_{\mathbb{S}^{m-1}} |\Omega(\cdot,v')|\ \mathcal{M}_{\Psi,v'} f(\cdot,y) d\sigma(v'), \tag{2.23}$$

where

$$\mathcal{M}_{\Psi,v'}(f)(.,y) = \sup_{s' \in \mathbb{R}} 2^{-s'} \int_0^{2^{s'}} |f(\cdot,y - \Psi(r')v')|dr'.$$

Now, we have

$$\mathcal{M}_{\Psi,v'}(f)(.,y) \leq \sum_{j=0}^{\infty} 2^{-j} \left(\sup_{s' \in \mathbb{R}} 2^{-s'+j} \int_{2^{s'-j-1}}^{2^{s'-j}} |f(.,y - \Psi(r')v')|dr' \right)$$

$$\leq C \sum_{j=0}^{\infty} 2^{-j} \left(\sup_{z>0} \frac{1}{z} \int_{0}^{cz} |f(.,y - r'\, v')| dr' \right) \tag{2.24}$$

$$= C \sup_{z>0} \frac{1}{z} \int_{0}^{cz} |f(.,y - r'\, v')| dr'; \tag{2.25}$$

where (2.24) follows by change of variables and (1.4). By (8) in [14] and since $\omega_2 \in \tilde{A}_p(\mathbb{R}^m)$, we get

$$\|\mathcal{M}_{\Psi,v'}(f)\|_{L^p(\omega_2)} \leq C_p \, \|f\|_{L^p(\omega_2)}; \tag{2.26}$$

where C_p is a constant independent of v'. By a similar argument, for $\omega_1 \in \tilde{A}_p(\mathbb{R}^n)$, we get

$$\|\mathcal{M}_{\Phi,u'}(f)\|_{L^p(\omega_1)} \leq C_p \, \|f\|_{L^p(\omega_1)}, \tag{2.27}$$

where

$$\mathcal{M}_{\varphi,u'}(f)(x,.) \leq C \sup_{s>0} \frac{1}{s} \int_{0}^{cs} |f(x - t\, u')| dt.$$

Thus, by (2.23), (2.26), and Minkowski's inequality, we get

$$\|\mathcal{M}_{\Omega,\,\Psi}(f)\|_{L^p(\omega_2)} \leq C_p \, \|\Omega\|_{L^1} \, \|f\|_{L^p(\omega_2)}. \tag{2.28}$$

Similarly, for $\omega_1 \in \tilde{A}_p(\mathbb{R}^n)$, we get

$$\|\mathcal{M}_{\Omega,\,\Phi}(f)\|_{L^p(\omega_1)} \leq C_p \, \|\Omega\|_{L^1} \|f\|_{L^p(\omega_1)}. \tag{2.29}$$

Now, by (2.28) and following a similar argument as in [26], we have

$$\int_{\mathbb{R}^m} \mathcal{M}_{\Omega,\phi}(f)(x,y)|^p \, w_1(x)\, w_2(y)\, dy \leqslant C \, \|\Omega\|_{L^1} \int_{\mathbb{R}^m} |f(x,y)|^p w_1(x)\, w_2(y)\, dy, \tag{2.30}$$

for $x \in \mathbb{R}^n$ where C is a constant independent of x. Then, by integration with respect to x, we get

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \left|\mathcal{M}_{\Omega,\Psi}(f)(x,y)\right|^p w_1(x)\, w_2(y)\, dx\, dy \leqslant C \, \|\Omega\|_{L^1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |f(x,y)|^p w_1(x)\, w_2(y) dx\, dy. \tag{2.31}$$

Thus, by following a similar argument as in (2.30)–(2.31) on $\mathcal{M}_{\Omega,\Phi}$, replacing x by y, we get

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |\mathcal{M}_{\Omega,\Phi}(f)(x,y)|^p \, w_1(x)\, w_2(y)\, dx\, dy \leqslant C \, \|\Omega\|_{L^1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |f(x,y)|^p w_1(x)\, w_2(y) dx\, dy. \tag{2.32}$$

Thus, by (2.22), we have

$$\|\mathcal{M}_{\Omega,\,\Phi,\,\Psi}(f)\|_{L^p(\omega_1,\omega_2)} \leq C_p \, \|\Omega\|_{L^1} \, \|f\|_{L^p(\omega_1,\omega_2)}. \tag{2.33}$$

Hence, (2.19) follows by (2.20) and (2.33). This ends the proof of (2.19) under assumption (ii) on the functions Φ and Ψ. To prove (2.19) under assumption (i) on the functions Φ and Ψ, we follow a similar argument as above and make use of estimates developed in the proof of Lemma 2.2 in [15]. We omit the details. This ends the proof of the lemma.

Next, we prove the following weighted inequalities for square functions:

Lemma 2.3. *Suppose that* $\|\Omega\|_{L^1} \leq 1$. *Suppose also that* a, b, Ψ, Φ, *and* Υ *are as in Lemma 2.2. Let For* $t, s \in \mathbb{R}$, *let* $\sigma_{\Omega,\Phi,\Psi,a,b}$ *be given by (2.17) where* t' *and* s' *are replaced by* t *and* s, *respectively. Assume that (i)* $\Phi, \Psi \in \mathcal{G}$; *or (ii)* $\Phi \in \mathcal{PC}_\lambda(d_1)$, $\Psi \in \mathcal{PC}_\alpha(d_2)$ *for* $d_1, d_2 > 0$ *and* $\lambda, \alpha \in \mathbb{R}$. *Then, for* $1 < p < \infty$, $j, k \in \mathbb{Z}$, $\omega_1 \in \tilde{A}_p^I(\mathbb{R}^n)$, *and* $\omega_2 \in \tilde{A}_p^I(\mathbb{R}^m)$, *and there exists a constant* C_p *independent of* a, b, j, k, *and* Ω *such that*

$$\left\|\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\sigma_{\Omega,\Phi,\Psi,a^t,b^s} * \Upsilon_{a^{t+j},b^{s+k}} * f(x,y)|^2 dt\,ds\right)^{\frac{1}{2}}\right\|_{L^p(\omega_1,\omega_2)} \leq C\,\|f\|_{L^p(\omega_1,\omega_2)}. \tag{2.34}$$

Proof: Notice that

$$\begin{aligned}\sup_{t,s\in\mathbb{R}}\left|\sigma_{\Omega,\Phi,\Psi,a^t,b^s} * \Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right| &= \mathcal{M}_{\Omega,\Phi,\Psi,a,b}(\Upsilon_{a^{t+j},b^{s+k}} * f)(x,y)\\ &\leq \mathcal{M}_{\Omega,\Phi,\Psi,a,b}\left(\sup_{t,s\in\mathbb{R}}\left|\Upsilon_{a^{t+j},b^{s+k}} * f\right|\right)(x,y).\end{aligned}$$

Next, by Lemma 2.2, we have

$$\begin{aligned}&\left\|\sup_{t,s\in\mathbb{R}}\left|\sigma_{\Omega,\Phi,\Psi,a^t,b^s} * \Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right|\right\|_{L^p(\omega_1,\omega_2)}\\ &\leq \left\|\mathcal{M}_{\Omega,\Phi,\Psi,a,b}\left(\sup_{t,s\in\mathbb{R}}|\Upsilon_{a^{t+j},b^{s+k}} * f(x,y)|\right)\right\|_{L^p(\omega_1,\omega_2)}\\ &\leq C\left\|\sup_{t,s\in\mathbb{R}}|\Upsilon_{a^{t+j},b^{s+k}} * f(x,y)|\right\|_{L^p(\omega_1,\omega_2)}.\end{aligned} \tag{2.35}$$

Now, by duality, choose a non-negative function $g(x,y)$ with $\|g\|_{L^{p'}(\omega_1^{1-p'},\omega_2^{1-p'})} \leq 1$ such that

$$\begin{aligned}&\left\|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sigma_{\Omega,\Phi,\Psi,a^t,b^s} * \Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right|dt\,ds\right\|_{L^p(\omega_1,\omega_2)}\\ &\leq \iint_{\mathbb{R}^n\times\mathbb{R}^m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sigma_{\Omega,\Phi,\Psi,a^t,b^s} * \Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right| g(x,y)dt\,ds\,dx\,dy\\ &\leq C\iint_{\mathbb{R}^n\times\mathbb{R}^m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right|\left(\sup_{t,s\in\mathbb{R}}|\sigma_{\Omega,\Phi,\Psi,a^t,b^s} * g(x,y)|\right)dt\,dsdx\,dy\\ &\leq C\iint_{\mathbb{R}^n\times\mathbb{R}^m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right|\mathcal{M}_{\Omega,\Phi,\Psi,a,b}(\tilde{g})(-x,-y)\,dt\,ds\,dx\,dy,\end{aligned} \tag{2.36}$$

where $\tilde{g}(x,y) = g(-x,-y)$. Thus, by Lemma 2.1, (2.36), and Hölder's inequality, we get

$$\begin{aligned}&\left\|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sigma_{\Omega,\Phi,\Psi,a^t,b^s} * \Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right|\right\|_{L^p(\omega_1,\omega_2)} \\ &\leq \left\|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right| dt\,dr\right\|_{L^p(\omega_1,\omega_2)} \left\|\mathcal{M}_{\Omega,\varphi,\phi,a,b}(\tilde{g})\right\|_{L^{p'}(\omega_1^{1-p'},\omega_2^{1-p'})}.\end{aligned} \tag{2.37}$$

By an application of Lemma 2.2, we get

$$\begin{aligned}&\left\|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sigma_{\Omega,\Phi,\Psi,a^t,b^s} * \Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right| dt\,dr\right\|_{L^p(\omega_1,\omega_2)} \\ &\leq C\left\|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right| dtds\right\|_{L^p(\omega_1,\omega_2)}.\end{aligned} \tag{2.38}$$

Hence, by interpolation between (2.35) and (2.38) in a vector-valued setting, we get

$$\begin{aligned}&\left\|\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sigma_{\Omega,\Phi,\Psi,a^t,b^s} * \Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right|^2 dt\,ds\right)^{\frac{1}{2}}\right\|_{L^p(\omega_1,\omega_2)} \\ &\leq C\left\|\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\Upsilon_{a^{t+j},b^{s+k}} * f(x,y)\right|^2 dtds\right)^{\frac{1}{2}}\right\|_{L^p(\omega_1,\omega_2)} \\ &\leq C\,\|f\|_{L^p(\omega_1,\omega_2)},\end{aligned} \tag{2.39}$$

where the last inequality is obtained by Lemma 2.1. This completes the proof of Lemma 2.3.

3. Preliminary estimates

This section is establish some preliminary estimates that are needed to prove our results.

Lemma 3.1. *Let $\Omega \in L^2(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ satisfying (1.6)–(1.7) with $\|\Omega\|_1 \leq 1$ and $\|\Omega\|_2 \leq A$ for some $A > 2$. Suppose that $\Phi,\Psi \in \mathcal{G}$ with powers d_1, d_1 in (1.4)–(1.5). For $t,\ s \in \mathbb{R}$, let $\sigma_{A,t,s}^{(\Phi,\Psi)}$ be the measure defined via the Fourier transform by*

$$\hat{\sigma}_{A,t,s}^{(\Phi,\Psi)}(\xi,\eta) = \frac{1}{A^{t+s}}\iint_{\Gamma(A^t,A^s)} e^{-i(\Phi(|u|)\,\xi.u'+\Psi(|v|)\,\eta.v')}\frac{\Omega_\kappa(u',v')}{|u|^{n-1}\,|v|^{m-1}}\,du\,dv, \tag{3.1}$$

where

$$\Gamma(A^t,A^s) = \{(u,v)\in\mathbb{R}^n\times\mathbb{R}^m : A^{t-1} < |u| \leq A^t \text{ and } A^{s-1} < |v| \leq A^s\}. \tag{3.2}$$

Then, there exists $\varepsilon \in (0,\frac{1}{2})$ such that

$$\left|\hat{\sigma}_{A,t,s}^{(\Phi,\Psi)}(\xi,\eta)\right| \leq 1 \tag{3.3}$$

$$\left|\hat{\sigma}_{A,t,s}^{(\Phi,\Psi)}(\xi,\eta)\right| \leq C\ |A^{d_1t}\,\xi\,|^{-\frac{\varepsilon}{2\log_2 A}}\ \ |A^{d_2s}\,\eta\,|^{-\frac{\varepsilon}{2\log_2 A}}; \tag{3.4}$$

$$\left|\hat{\sigma}_{A,t,s}^{(\Phi,\Psi)}(\xi,\eta)\right| \leq C\ |A^{d_1t}\,\xi\,|^{-\frac{\varepsilon}{2\log_2 A}}\ \ |A^{d_2s}\,\eta\,|^{\frac{\varepsilon}{2\log_2 A}} \tag{3.5}$$

$$\left|\hat{\sigma}_{A,t,s}^{(\Phi,\Psi)}(\xi,\eta)\right| \leq C\ |A^{d_1t}\,\xi\,|^{\frac{\varepsilon}{2\log_2 A}}\ \ |A^{d_2s}\,\eta\,|^{-\frac{\varepsilon}{2\log_2 A}} \tag{3.6}$$

$$\left|\hat{\sigma}_{A,t,s}^{(\Phi,\Psi)}(\xi,\eta)\right| \leq C\ |A^{d_1t}\,\xi\,|^{\frac{\varepsilon}{2\log_2 A}}\ \ |A^{d_2s}\,\eta\,|^{\frac{\varepsilon}{2\log_2 A}} \tag{3.7}$$

where the constant C is independent of A, s, and t.

Proof: We shall assume that $d_1, d_2 > 0$. The other cases follows by similar argument. The estimate (3.3) is clear. To see the estimate (3.4), notice that

$$\left|\hat{\sigma}_{A,t,s}^{(\Phi,\Psi)}(\xi,\eta)\right| \leq \int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{m-1}} |\Omega(u',v')|\, g(A,\Phi,\xi) g(A,\Psi,\eta) d\sigma(u') d\sigma(v') \tag{3.8}$$

where

$$g(A,\Phi,\xi) = \left|\int_{\frac{1}{A}}^{1} e^{-i\Phi(A^t r)\,\xi.u'} dr\right|,$$

and $g(A,\Psi,\eta)$ has similar definition as $g(A,\Phi,\xi)$. By integration by parts along with the assumptions (1.4)–(1.5), and the observations $g(A,\Phi,\xi) \leq 1$ and $g(A,\Psi,\eta) \leq 1$, there exists $\varepsilon \in (0,\frac{1}{2})$ such that

$$g(A,\Phi,\xi) \leq C\,|A^{d_1t}\,\xi\cdot u'\,|^{-\varepsilon} \tag{3.9}$$

$$g(A,\Psi,\eta) \leq C\,|A^{d_2s}\,\eta\cdot v'\,|^{-\varepsilon}. \tag{3.10}$$

By (3.8), (3.9)–(3.10), Hölder's inequality, and assumption on Ω, we have

$$\left|\hat{\sigma}_{A,t,s}^{(\Phi,\Psi)}(\xi,\eta)\right| \leq A|A^{d_1t}\,\xi|^{-\varepsilon}|A^{d_2s}\,\eta\,|^{-\varepsilon}C\tilde{C}_\varepsilon, \tag{3.11}$$

where

$$\tilde{C}_\varepsilon = \sup_{\xi'\in\mathbb{S}^{n-1}}\left(\int_{\mathbb{S}^{n-1}} |\xi'\cdot u'\,|^{-2\varepsilon} d\sigma(u')\right)^{\frac{1}{2}} \sup_{\eta'\in\mathbb{S}^{m-1}}\left(\int_{\mathbb{S}^{m-1}} |\eta'\cdot v'\,|^{-2\varepsilon} d\sigma(v')\right)^{\frac{1}{2}}.$$

Since $\varepsilon < 1/2$, we have $\tilde{C}_\varepsilon < \infty$. Thus, by (3.9)–(3.10), (3.8), and an interpolation, we get

$$\left|\hat{\sigma}_{A,t,s}^{(\Phi,\Psi)}(\xi,\eta)\right| \leq A^{\frac{1}{2\log_2 A}}|A^{d_1t}\,\xi|^{-\frac{\varepsilon}{2\log_2 A}}|A^{d_2s}\,\eta\,|^{-\frac{\varepsilon}{2\log_2 A}}C,$$

which implies (3.4) since $A^{\frac{1}{2\log_2 A}} < 2$. To verify (3.5), we first notice that

$$\frac{1}{A^s}\left|\int_{A^{s-1}}^{A^s}\left(e^{-i\Psi(A^s r)\,\eta.v'}-1\right)dr\right| \leq \min\{1, C_1|A^{d_2s}\,\eta\,|\}, \tag{3.12}$$

which by interpolation implies

$$\frac{1}{A^s}\left|\int_{A^{s-1}}^{A^s}\left(e^{-i\Psi(A^s r)\,\eta.v'}-1\right)dr\right| \leq C|A^{d_2s}\,\eta\,|^{\varepsilon}. \tag{3.13}$$

By the cancellation property (1.6), Hölder's inequality, the assumption on Ω, (3.9), and (3.13), we have

$$\begin{aligned}
\left|\hat{\sigma}^{(\Phi,\Psi)}_{A,t,s}(\xi,\eta)\right| &\leq \frac{1}{A^{t+s}}\left|\iint_{\Gamma(A^t,A^s)}\left(e^{-i(\Phi(|u|)\,\xi.u'+\Psi(|v|)\,\eta.v')}-e^{-i\Phi(|u|)\,\xi.u'}\right)\frac{\Omega_\kappa(u',v')}{|u|^{n-1}\,|v|^{m-1}}\,du\,dv\right| \\
&\leq C|A^{d_2 s}\,\eta\,|^{\varepsilon}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{m-1}}|\Omega(u',v')|\,g(A,\Phi,\xi)d\sigma(u')d\sigma(v') \\
&\leq C|A^{d_2 s}\,\eta\,|^{\varepsilon}\,\|\Omega\|_2\left|\mathbb{S}^{m-1}\right|\left(\int_{\mathbb{S}^{n-1}}|g(A,\Phi,\xi)|^2\,d\sigma(u')\right)^{\frac{1}{2}} \\
&\leq C|A^{d_2 s}\,\eta\,|^{\varepsilon}\,\|\Omega\|_2\left|\mathbb{S}^{m-1}\right|\;|A^{d_1 t}\,\xi|^{-\varepsilon}\sup_{\xi'\in\mathbb{S}^{n-1}}\left(\int_{\mathbb{S}^{n-1}}|\xi'\cdot u'\,|^{-2\varepsilon}d\sigma(u')\right)^{\frac{1}{2}} \\
&\leq CA|A^{d_2 s}\,\eta\,|^{\varepsilon}|A^{d_1 t}\,\xi|^{-\varepsilon}, \qquad (3.14)
\end{aligned}$$

where the last inequality follows by the same reasoning for $\tilde{C}_\varepsilon$ above. Thus, (3.5) follows by (3.14), (3.3), and an interpolation. The verifications of other estimates follows by a similar argument with minor modifications. We omit the details. This completes the proof of the lemma.

Now, by the same argument as in [18], we have the following lemma:

Lemma 3.2. *Let* $\Omega \in L^2(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ *satisfying (1.6)–(1.7) with* $\|\Omega\|_1 \leq 1$ *and* $\|\Omega\|_2 \leq A$ *for some* $A > 2$. *Let* $\Phi \in \mathcal{PC}_\lambda(d_1)$, $\Psi \in \mathcal{PC}_\alpha(d_2)$ *for* $d_1,\, d_2 > 0$ *and* $\lambda,\, \alpha \in \mathbb{R}$. *Suppose that*

$$\Phi(w) = P(w) + \lambda\varphi_1(w) \qquad \textit{and} \qquad \Psi(z) = Q(z) + \alpha\varphi_2(z),$$

where P, Q, φ_1, and φ_2 are as in the definition of the spaces $\mathcal{PC}_\lambda(d_1)$ and $\Psi \in \mathcal{PC}_\alpha(d_2)$. For $t,\ s \in \mathbb{R}$, let $\sigma^{(\Phi,\Psi)}_{A,t,s}, \sigma^{(\Phi,Q)}_{A,t,s}, \sigma^{(P,\Psi)}_{A,t,s}$, and $\sigma^{(P,Q)}_{A,t,s}$ be the measures defined by (3.1) with proper modifications. Then,

(i) $\|\sigma^{(\Phi,\Psi)}_{A,t,s}\| \leq C$;

(ii) $\left|\widehat{\sigma}^{(\Phi,\Psi)}_{A,t,s}(\xi,\eta)\right| \leq C\;|\lambda\,\varphi_1\,(A^{t-1})\,\xi\,|^{-\frac{1}{2\,(d_1+1)\log_2 A}}\;\;|\alpha\,\varphi_2\,(A^{s-1})\,\eta\,|^{-\frac{1}{2\,(d_2+1)\log_2 A}}$;

(iii) $\left|\widehat{\sigma}^{(\Phi,\Psi)}_{A,t,s}(\xi,\eta)-\widehat{\sigma}^{(P,\Psi)}_{A,t,s}(\xi,\eta)\right| \leq C\;\left|\lambda\varphi_1(A^t)\,\xi\right|^{\frac{1}{2\log_2 A}}\;\left|\alpha\varphi_2(A^{s-1})\eta\right|^{-\frac{1}{2(d_2+1)\,\log_2 A}}$;

(iv) $\left|\widehat{\sigma}^{(\Phi,\Psi)}_{A,t,s}(\xi,\eta)-\widehat{\sigma}^{(\Phi,Q)}_{A,t,s}(\xi,\eta)\right| \leq C\;\left|\,\lambda\,\varphi_1(A^{t-1})\,\xi\right|^{-\frac{1}{2\,(d_1+1)\,\log_2 A}}\;\;|\,\alpha\,\varphi_2(A^s)\,\eta|^{\frac{1}{2\log_2 A}}$;

(v) $\left|\widehat{\sigma}^{(\Phi,\Psi)}_{A,t,s}(\xi,\eta)-\widehat{\sigma}^{(P,\Psi)}_{A,t,s}(\xi,\eta)-\widehat{\sigma}^{(\Phi,Q)}_{A,t,s}(\xi,\eta)+\widehat{\sigma}^{(P,Q)}_{A,t,s}(\xi,\eta)\right|$
$\leq C|\lambda\varphi_1(A^t)\,\xi|^{\frac{1}{2\log_2 A}}\,|\alpha\varphi_2(A^s)\,\eta|^{\frac{1}{2\log_2 A}}$;

(vi) $\left|\widehat{\sigma}^{(\Phi,Q)}_{A,t,s}(\xi,\eta)-\widehat{\sigma}^{(P,Q)}_{A,t,s}(\xi,\eta)\right| \leq C\,|\lambda\varphi_1(A^t)\,\xi|^{\frac{1}{2\log_2 A}}$;

(vii) $\left|\widehat{\sigma}^{(P,\Psi)}_{A,t,s}(\xi,\eta)-\widehat{\sigma}^{(P,Q)}_{A,t,s}(\xi,\eta)\right| \leq C\;|\,\alpha\,\varphi_2(A^s)\,\eta|^{\frac{1}{2\log_2 A}}$,

where C is independent of κ and $(\xi,\eta) \in (\mathbb{R}^n,\mathbb{R}^m)$.

We end this section by the following estimates contained in the argument in [18].

Lemma 3.3. *Let* $\Omega \in L^2(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ *satisfying (1.6)–(1.7) with* $\|\Omega\|_1 \leq 1$ *and* $\|\Omega\|_2 \leq A$ *for some* $A > 2$. *Suppose that* $P(w) = \sum_{k=0}^{d_1} c_{k,1}\,w^k$ *and* $Q(z) = \sum_{k=0}^{d_2} c_{k,2}\,z^k$ *are polynomials of degrees* d_1 *and* d_2, *respectively. For* $0 \leq l \leq d_1$ *and* $0 \leq s \leq d_2$, *let*

$$P_l(w) = \sum_{k=0}^{l} c_{k,1}\,w^k \quad \text{and} \quad Q_o(z) = \sum_{k=0}^{o} c_{k,2}\,z^k$$

with the convention that $\sum_{j\in\emptyset} = 0$. *For* $t,\ s \in \mathbb{R}$, $0 \le l \le d_1$, and $0 \le o \le d_2$, let $\sigma_{A,t,s}^{(l,o)}$ be defined by (3.1) where Φ and Ψ are replaced by P_l and Q_o, respectively. For $0 \le l \le d_1$, $0 \le o \le d_2$, let $\sigma_{A,t,s}^{(l,o)} = \sigma_{A,t,s}^{(P_l,Q_o)}$. Then, for $1 \le l \le d_1$ and $1 \le o \le d_2$, we have

(i)$\|\sigma_{A,t,s}^{(l,o)}\| \le C$;

(ii) $\left|\widehat{\sigma}_{A,t,s}^{(l,o)}(\xi,\eta)\right| \le C\ |c_{l,1}A^{l(t-1)}\ l!\,\xi|^{-\frac{1}{2l\log_2 A}}\,|\,c_{o,2}\,(A^{o(s-1)}\ o!\eta|^{-\frac{1}{2o\log_2 A}}$;

(iii) $\left|\widehat{\sigma}_{A,t,s}^{(l,o)}(\xi,\eta) - \widehat{\sigma}_{A,t,s}^{(l-1,o)}(\xi,\eta)\right| \le C\ |c_{l,1}\,A^{lt}\xi|^{\frac{1}{2\log_2 A}}\ |c_{o,2}\,A^{o(s-1)}\,o!\ \eta|^{-\frac{1}{2o\log_2 A}}$;

(iv) $\left|\widehat{\sigma}_{A,t,s}^{(l,o)}(\xi,\eta) - \widehat{\sigma}_{A,t,s}^{(l,o-1)}(\xi,\eta)\right| \le C\ \left|c_{l,1}\,A^{l(t-1)}\,l!\,\xi\right|^{-\frac{1}{2l\log_2 A}}\ \left|c_{o,2}A^{os}\,\eta\right|^{\frac{1}{2\log_2 A}}$;

(v) $\left|\widehat{\sigma}_{A,t,s}^{(l,o)}(\xi,\eta) - \widehat{\sigma}_{A,t,s}^{(l-1,o)}(\xi,\eta) - \widehat{\sigma}_{A,t,s}^{(l,o-1)}(\xi,\eta) + \widehat{\sigma}_{A,t,s}^{(l-1,o-1)}(\xi,\eta)\right|$
$\le C\,|c_{l,1}A^{lt}\,\xi|^{\frac{1}{\kappa+1}}\ |c_{o,2}\,A^{os}\,\eta|^{\frac{1}{2\log_2 A}}$;

(vi)$\left|\widehat{\sigma}_{A,t,s}^{(l,o-1)}(\xi,\eta) - \widehat{\sigma}_{A,t,s}^{(l-1,o-1)}(\xi,\eta)\right| \le C\ |c_{l,1}\,A^{lt}\,\xi|^{\frac{1}{2\log_2 A}}$;

(vii)$\left|\widehat{\sigma}_{A,t,s}^{(l-1,o)}(\xi,\eta) - \widehat{\sigma}_{A,t,s}^{(l-1,o-1)}(\xi,\eta)\right| \le C\ \left|c_{o,2}A^{os}\,\eta\right|^{\frac{1}{2\log_2 A}}$,

where C is independent of A and $(\xi,\eta) \in (\mathbb{R}^n, \mathbb{R}^m)$.

4. Proof of results

This section is devoted for the proofs of Theorems 1.5 and 1.7. To this end, we prove the following proposition:

Proposition 4.1. *Suppose that* $\Omega \in L^2(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ *satisfying (1.6)–(1.7) with* $\|\Omega\|_1 \le 1$ *and that* $\|\Omega\|_2 \le A$ *for some* $A > 2$. *Suppose also that* $\omega_1 \in \tilde{A}_p^I(\mathbb{R}^n)$ *and* $\omega_2 \in \tilde{A}_p^I(\mathbb{R}^m)$, $1 < p < \infty$. *Assume that the mappings* Φ, Ψ *satisfies (i)* $\Phi,\Psi \in \mathcal{G}$*; or (ii)* $\Phi \in \mathcal{PC}_\lambda(d_1)$, $\Psi \in \mathcal{PC}_\alpha(d_2)$ *for* $d_1,\ d_2 > 0$ *and* $\lambda,\ \alpha \in \mathbb{R}$. *Then, for* $1 < p < \infty$, *we have*

$$\left\|\mathcal{U}_{\Omega,\Phi,\Psi}(f)\right\|_{L^p(\omega_1,\omega_2)} \le (\log_2 A)\,C_p\,\|f\|_{L^p(\omega_1,\omega_2)} \tag{4.1}$$

with constants C_p independent of A.

Proof: We shall prove (4.1) under the assumption (ii) on the mappings Φ and Ψ. The proof under the assumption (i) follows by similar argument with minor modifications. We write Φ and Ψ as

$$\Phi(w) = P(w) + \lambda\varphi_1(w) \quad \text{and} \quad Psi(z) = Q(z) + \alpha\varphi_2(z), \tag{4.2}$$

where P and Q are polynomials of degrees d_1 and d_2 as in the statement of Lemma 3.3. We let $\{c_{k,1}\}, \{c_{k,2}\}, P_l, Q_o$, and $\sigma_{A,t,s}^{(l,o)}$ be as in Lemma 3.3. Let $\sigma_{A,t,s}^{(d_1+1,d_2+1)}$ be the measure $\sigma_{A,t,s}^{(\Phi,\Psi)}$ in Lemma 3.1. By simple change of variables, we have

$$\mathcal{U}_{\Omega,\Phi,\Psi}(f)(x,y) = (\log_2 A)\,\mathcal{U}_{A,\Phi,\Psi}f(x,y), \tag{4.3}$$

where

$$\mathcal{U}_{A,\Phi,\Psi}f(x,y) = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|F_{A,t,s}^{(\Phi,\Psi)}(f)(x,y)\right|^2 2^{-2(\log_2 A)(t+s)}dt\,ds\right)^{\frac{1}{2}}, \tag{4.4}$$

$$F_{A,t,s}^{(\Phi,\Psi)}(f)(x,y) = \int\int_{\Lambda(A^t,A^s)} f(x-\Phi(|u|)u', y-\Psi(|v|)v')\ \frac{\Omega(u',v')}{|u|^{n-1}\,|v|^{m-1}}du\,dv,$$

and

$$\Lambda(A^t, A^s) = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m : |u| \leq A^t \text{ and } |v| \leq A^s\}.$$

Thus, to prove (4.1), it suffices to show that

$$\left\|\mathcal{U}_{A,\Phi,\Psi}(f)\right\|_{L^p(\omega_1,\omega_2)} \leq C_p \|f\|_{L^p(\omega_1,\omega_2)} \tag{4.5}$$

with constant C_p independent of A. Let $\{\sigma_{A,t,s}^{(l,o)} : 0 \leq l \leq d_1,\ 0 \leq o \leq d_2\}$ be as in Lemma 3.3. Notice that

$$\widehat{\sigma}_{A,t,s}^{(0,0)} = \widehat{\sigma}_{A,t,s}^{(0,d_2+1)} = \widehat{\sigma}_{A,t,s}^{(d_1+1,0)} = 0. \tag{4.6}$$

Following the same arguments in [18], for $1 \leq l \leq d_1$, $1 \leq o \leq d_2$, $1 < p < \infty$, $j, k \in \mathbb{Z}$, $\omega_1 \in \tilde{A}_p^I(\mathbb{R}^n)$, and $\omega_2 \in \tilde{A}_p^I(\mathbb{R}^m)$, we can find linear transformations $L_l : \mathbb{R}^n \to \mathbb{R}^n$ and $Q_s : \mathbb{R}^m \to \mathbb{R}^m$ and measures $\{\tau_{A,t,s}^{(l,o)} : t,\ s \in \mathbb{R},\}$ such that

$$\left|\widehat{\tau}_{A,t,s}^{(l,o)}(\xi,\eta)\right| \leq C\ |A^{lt} L_l(\xi)|^{-\frac{1}{2\beta_l \log_2 A}}\ |A^{os} Q_o(\eta)|^{-\frac{1}{2\delta_o \log_2 A}}; \tag{4.7}$$

$$\left|\widehat{\tau}_{A,t,s}^{(l,s)}(\xi,\eta) - \widehat{\tau}_{A,t,s}^{(l-1,o)}(\xi,\eta)\right| \leq C\ |A^{lt}\, L_l(\xi)|^{\frac{1}{2\log_2 A}}\ |A^{os} Q_o(\eta)|^{-\frac{1}{2\delta_o \log_2 A}}; \tag{4.8}$$

$$\left|\widehat{\tau}_{A,t,s}^{(l,s)}(\xi,\eta) - \widehat{\tau}_{A,t,s}^{(l,o-1)}(\xi,\eta)\right| \leq C\ \left|A^{lt}\, L_l(\xi)\right|^{-\frac{1}{2\beta_l \log_2 A}}\ |A^{os} Q_o(\eta)|^{\frac{1}{2\log_2 A}}; \tag{4.9}$$

$$\begin{aligned} &\left|\widehat{\tau}_{A,t,s}^{(l,s)}(\xi,\eta) - \widehat{\tau}_{A,t,s}^{(l-1,o)}(\xi,\eta) - \widehat{\tau}_{A,t,s}^{(l,s-1)}(\xi,\eta) + \widehat{\tau}_{A,t,s}^{(l-1,s-1)}(\xi,\eta)\right| \\ &\leq C\, |A^{lt}\, L_l(\xi)|^{\frac{1}{2\log_2 A}}\ |A^{os} Q_o(\eta)|^{\frac{1}{2\log_2 A}}; \end{aligned} \tag{4.10}$$

$$\left|\widehat{\tau}_{A,t,s}^{(l,o-1)}(\xi,\eta) - \widehat{\tau}_{A,t,s}^{(l-1,o-1)}(\xi,\eta)\right| \leq C\ |A^{lt}\, L_l(\xi)|^{\frac{1}{2\log_2 A}}; \tag{4.11}$$

$$\left|\widehat{\tau}_{A,t,s}^{(l-1,o)}(\xi,\eta) - \widehat{\tau}_{A,t,s}^{(l-1,o-1)}(\xi,\eta)\right| \leq C\ |A^{os} Q_o(\eta)|^{\frac{1}{2\log_2 A}}; \tag{4.12}$$

$$\left\|\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\tau_{A,t,s}^{(l,o)} * \Upsilon_{a^{t+j},b^{s+k}} * f(x,y)|^2 dt\, ds\right)^{\frac{1}{2}}\right\|_{L^p(\omega_1,\omega_2)} \leq C_{\,p}\|f\|_{L^p(\omega_1,\omega_2)}; \tag{4.13}$$

and

$$\sum_{l=1}^{d_1+1}\sum_{o=1}^{d_2+1} \tau_{A,t,s}^{(l,o)} = \sigma_{A,t,s}^{(d_1+1,d_2+1)}; \tag{4.14}$$

where

$$\beta_l = \begin{cases} d_1 + 1, & l = d_1 + 1; \\ l, & l \neq d_1 + 1, \end{cases}$$

and

$$\delta_o = \begin{cases} d_2 + 1, & o = d_2 + 1; \\ o, & o \neq d_2 + 1. \end{cases}$$

Thus, by (4.14) and Minkowski's inequality, we obtain that

$$\left\|\mathcal{U}_{A,\Phi,\Psi}(f)\right\|_{L^p(\omega_1,\omega_2)} \leq C_p \sum_{l=1}^{d_1+1}\sum_{o=1}^{d_2+1} \left\|S_{A,l,o}(f)\right\|_{L^p(\omega_1,\omega_2)}; \tag{4.15}$$

where

$$S_{A,l,o}(f)(x,y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \left(\tau_{A,t,s}^{(l,o)} * (f)(x,y) \right) \right|^2 dtds \right)^{\frac{1}{2}}.$$

Now, by a similar argument as in [27], choose two collections of C^∞ functions $\{\varpi_i^{(l)}\}_{i\in\mathbb{Z}}$ and $\{\varpi_i^{(o)}\}_{i\in\mathbb{Z}}$ on $(0,\infty)$ satisfying the following properties:

$$supp(\varpi_i^{(l)}) \subseteq \left[A^{-l(i+1)}, A^{-l(i-1)} \right] \text{ and } supp(\varpi_i^{(o)}) \subseteq \left[A^{-o(i+1)}, A^{-o(i-1)} \right]; \tag{4.16}$$

$$0 \le \varpi_i^{(l)}, \varpi_i^{(o)} \le 1; \tag{4.17}$$

$$\sum_{i\in\mathbb{Z}} \varpi_i^{(l)})(u) = \sum_{i\in\mathbb{Z}} \varpi_i^{(o)})(u) = 1; \tag{4.18}$$

$$\left| \frac{d^r \varpi_i^{(l)}}{du^r}(u) \right|, \left| \frac{d^r \varpi_i^{(o)}}{du^r}(u) \right| \le \frac{C_r}{u^r}, \tag{4.19}$$

where C_r is independent of A. Define the measures $\{\upsilon_i^{(l)} : i \in \mathbb{Z}\}$ on $\mathbb{R}^n$ and $\{\upsilon_i^{(o)} : i \in \mathbb{Z}\}$ on $\mathbb{R}^m$ by

$$\widehat{(\upsilon_i^{(l)})}(x) = \varpi_i^{(l)}(|x|^2) \text{ and } \widehat{(\upsilon_i^{(o)})}(y) = \varpi_i^{(o)}(|y|^2).$$

By (4.18), we immediately obtain

$$\begin{aligned} (\widehat{\tau_{A,t,s}^{(l,o)} * f})(\xi,\eta) &= \widehat{\tau}_{A,t,s}^{(l,o)}(\xi,\eta) \,.\, \widehat{f}(\xi,\eta) \sum_{j\in\mathbb{Z}} \widehat{\upsilon}_j^{(l)}(\xi) \,.\, \sum_{i\in\mathbb{Z}} \widehat{\upsilon}_i^{(o)}(\eta) \\ &= \widehat{\tau}_{A,t,s}^{(l,o)}(\xi,\eta) \,.\, \widehat{f}(\xi,\eta) \sum_{j\in\mathbb{Z}} \widehat{\upsilon}_{\lfloor t\rfloor+j}^{(l)}(\xi) \,.\, \sum_{i\in\mathbb{Z}} \widehat{\upsilon}_{\lfloor s\rfloor+i}^{(o)}(\eta), \end{aligned} \tag{4.20}$$

where $\lfloor t\rfloor$ is the greatest integer function such that $t-1 < \lfloor t\rfloor < t$, and similarly for $\lfloor s\rfloor$ (see [6, 20]). Hence, by taking the inverse Fourier transform for (4.20), we get

$$(\tau_{A,t,s}^{(l,o)} * f)(x,y) = \sum_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} \left(\upsilon_{\lfloor t\rfloor+j}^{(l)} \otimes \upsilon_{\lfloor s\rfloor+i}^{(o)} \right) * \tau_{A,t,s}^{(l,o)} * f(x,y). \tag{4.21}$$

Thus, by (4.21), we obtain

$$S_{A,l,o}(f)(x,y) \le C \sum_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} I_{A,i,j}^{(l,o)}(f)(x,y) \tag{4.22}$$

where

$$I_{A,i,j}^{(l,o)}(f)(x,y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \left(\upsilon_{\lfloor t\rfloor+j}^{(l)} \otimes \upsilon_{\lfloor s\rfloor+i}^{(s)} \right) * \tau_{A,t,s}^{(l,o)} * f(x,y) \right|^2 dt\, ds \right)^{\frac{1}{2}}. \tag{4.23}$$

By (4.7)–(4.12) and the Plancherel theorem, we get

$$\|I_{A,i,j}^{(l,o)}(f)\|_2 \le \Theta_{i,j} \|f\|_2, \tag{4.24}$$

where

$$\Theta_{i,j} = \begin{cases} 2^{\frac{is+jl}{l+j}}, & \text{if } i,j \le -2; \\ 2^{-i-j}, & \text{if } i,j \ge 3; \\ 2^{\frac{i-jl}{l}}, & \text{if } i \le -2 \text{ and } j \ge 3; \\ 2^{\frac{-is+j}{s}}, & \text{if } i \ge 3 \text{ and } j \le -2; \\ 1, & \text{if } i \ge -2 \text{ and } j \le 3. \end{cases}$$

Next, by (4.13), for $1 < p < \infty$ and $\omega_1 \in \tilde{A}^I_p(\mathbb{R}^n)$, $\omega_2 \in \tilde{A}^I_p(\mathbb{R}^m)$, there exists a positive constant C_p independent of i, j, and A such that

$$\|I^{(l,o)}_{A,i,j}(f)\|_{L^p(\omega_1,\omega_2)} \le C_p \,\|f\|_{L^p(\omega_1,\omega_2)}. \tag{4.25}$$

Now, we have three cases:

Case 1. $p > 2$. Choose a $q > p$ and $\varepsilon > 0$, such that $\omega_1^{1+\varepsilon} \in \tilde{A}^I_p(\mathbb{R}^n) \subset \tilde{A}^I_q(\mathbb{R}^n)$ and $\omega_2^{1+\varepsilon} \in \tilde{A}^I_p(\mathbb{R}^m) \subset \tilde{A}^I_q(\mathbb{R}^m)$. Thus, by (4.25) we get

$$\|I^{(l,o)}_{A,i,j}(f)\|_{L^q(\omega_1^{1+\varepsilon},\omega_2^{1+\varepsilon})} \le C_p \,\|f\|_{L^q(\omega_1^{1+\varepsilon},\omega_2^{1+\varepsilon})}, \tag{4.26}$$

which when combined with (4.24) and the interpolation theorem with change of measures, we have

$$\|I^{(l,o)}_{A,i,j}(f)\|_{L^p(\omega_1,\omega_2)} \le C^{1-\gamma}\,\Theta^{\gamma}_{i,j}\|f\|_{L^p(\omega_1,\omega_2)} \tag{4.27}$$

for $0 < \gamma < 1$ and $p > 2$.

Case 2. $1 < p < 2$. Choose a $1 < q < p$ and $\varepsilon > 0$ such that $\omega_1 \in \tilde{A}^I_p(\mathbb{R}^n)$, $\omega_2 \in \tilde{A}^I_p(\mathbb{R}^m)$ and $\omega_1^{1+\varepsilon} \in \tilde{A}^I_q(\mathbb{R}^n)$, $\omega_2^{1+\varepsilon} \in \tilde{A}^I_q(\mathbb{R}^m)$. Thus, by (4.25) we get

$$\|I^{(l,o)}_{A,i,j}(f)\|_{L^q(\omega_1^{1+\varepsilon},\omega_2^{1+\varepsilon})} \le C_p \,\|f\|_{L^q(\omega_1^{1+\varepsilon},\omega_2^{1+\varepsilon})} \tag{4.28}$$

for some positive constant C_p independent of A. Then, by the same argument as in Case 1, we obtain (4.27) for $0 < \gamma < 1$ and $1 < p < 2$.

Case 3. $p = 2$. We choose $\varepsilon > 0$ such that $\omega_1^{1+\varepsilon} \in \tilde{A}^I_2(\mathbb{R}^n), \omega_2^{1+\varepsilon} \in \tilde{A}^I_2(\mathbb{R}^n)$. Then, we follow a similar argument as in the previous two cases and get

$$\|I^{(l,o)}_{A,i,j}(f)\|_{L^p(\omega_1,\omega_2)} \le C^{1-\gamma}\,\Theta^{\gamma}_{i,j}\|f\|_{L^p(\omega_1,\omega_2)} \tag{4.29}$$

for $0 < \gamma < 1$ and $p = 2$.

Finally, by (4.15), (4.22), and (4.27)–(4.29), we get (4.5). This completes the proof of Proposition 4.1.

Proof (of Theorem 1.5): Assume that $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. We write Ω as

$$\Omega(x, y) = \sum_{k=0}^{\infty} \theta_k \Omega_k(x, y), \tag{4.30}$$

where Ω_k satisfies (1.6)–(1.7), $\|\Omega_\kappa\|_1 \le 4, \|\Omega_k\|_2 \le 2^{2(k+1)}$, and the estimate

$$\sum_{k=0}^{\infty}(k+1)\,\theta_\kappa \le \|\Omega\|_{L(\log L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}. \tag{4.31}$$

By (4.30) and Minkowski's inequality, we have

$$\left\|\mathcal{U}_{A,\Phi,\Psi}(f)\right\|_{L^p(\omega_1,\omega_2)} \le \sum_{k=0}^{\infty} \theta_k \left\|\mathcal{U}_{2^{2(k+1)},\Phi,\Psi}(f)\right\|_{L^p(\omega_1,\omega_2)}.$$

Thus, by Proposition 4.1 with $A = 2^{2(k+1)}$, we have

$$\begin{aligned}\left\|\mathcal{U}_{A,\Phi,\Psi}(f)\right\|_{L^p(\omega_1,\omega_2)} &\leq \sum_{k=0}^{\infty} \log_2(2^{2(k+1)})\theta_k \|f\|_{L^p(\omega_1,\omega_2)}\\ &= \left(\sum_{k=0}^{\infty} 2(k+1)\theta_k\right)\|f\|_{L^p(\omega_1,\omega_2)}\\ &\leq 2\|\Omega\|_{L(\log L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \|f\|_{L^p(\omega_1,\omega_2)}.\end{aligned}$$

This completes the proof.

Proof (of Theorem 1.7): The proof follows a similar argument as in the proof of Theorem 1.5. We omit the details.

5. Conclusions

In this paper, we proved the weighted L^p boundedness of Marcinkiewicz integral operators along surfaces. We considered surfaces that are determined by functions satisfying some growth conditions or mappings that are more general than polynomials and convex functions. We proved the weighted L^p boundedness of related square functions and maximal functions. The argument in this paper can be used to treat more general integral operators. This shall be the topic of future research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Authors cordially thank the reviewers for their useful comments on the manuscript. The authors would like to thank Sultan Qaboos University for paying the APC.

Conflict of interest

The authors declare no conflicts of interest.

References

1. E. M. Stein, On the function of Littlewood-Paley, Lusin and Marcinkiewicz, *T. Am. Math. Soc.*, **88** (1958), 430–466. https://doi.org/10.2307/1993226
2. A. Benedek, A. Calderon, R. Panzone, Convolution operators on Banach space valued functions, *P. Natl. Acad. Sci. U. S. A.*, **48** (1962), 356–365. https://doi.org/10.1073/pnas.48.3.356
3. T. Walsh, On the function of Marcinkiewicz, *Stud. Math.*, **44** (1972), 203–217. https://doi.org/10.1090/S0002-9947-1958-0112932-2

4. H. Al-Qassem, A. Al-Salman, L. Cheng, Y. Pan, L^p bounds for the functions of Marcinkiewicz, *Math. Res. Lett.*, **9** (2002), 697–700. https://doi.org/10.4310/MRL.2002.v9.n5.a11
5. A. Al-Salman, Marcinkiewicz functions with Hardy space kernels, *Math. Inequal. Appl.*, **21** (2018), 553–567. https://doi.org/10.7153/mia-2018-21-40
6. A. Al-Salman, On Marcinkiewicz integrals along flat surfaces, *Turk. J. Math.*, **29** (2005), 111–120.
7. J. Chen, D. Fan, Y. Ying, Rough Marcinkiewicz integrals with $L(\log L)^2$ kernels, *Adv. Math. (China)*, **30** (2001), 179–181.
8. Y. Ding, L^2 boundedness of Marcinkiewicz integral with rough kernel, *Hokkaido Math. J.*, **27** (1998), 105–115. https://doi.org/10.14492/hokmj/1351001253
9. S. Sato, Estimates for Littlewood-Paley functions and extrapolation, *Integr. Equ. Oper. Theory*, **62** (2008), 429–440. https://doi.org/10.1007/s00020-008-1631-4
10. A. Torchinsky, S. Wang, A note on the Marcinkiewicz integral, *Colloq. Math.*, **60** (1990), 235–243. https://doi.org/10.4064/cm-60-61-1-235-243
11. J. Garcia-Cuerva, J. L. R. de Francia, *Weighted norm inequalities and related topics*, Amsterdam: Elsevier, 2011.
12. Y. Ding, D. Fan, Y. Pan, Weighted boundedness for a class of rough Marcinkiewicz integrals, *Indiana Univ. Math. J.*, **48** (1999), 1037–1056. https://doi.org/10.1512/iumj.1999.48.1696
13. M. Y. Lee, C. C. Lin, Weighted L^p boundedness of Marcinkiewicz integral, *Integr. Equ. Oper. Theory*, **49** (2004), 211–220. https://doi.org/10.1007/s00020-002-1204-x
14. J. Duoandikoetxea, Weighted norm inequalities for homogeneous singular integrals, *T. Am. Math. Soc.*, **336** (1993), 869–880. https://doi.org/10.2307/2154381
15. A. Al-Salman, Certain weighted L^p bounds for the functions of marcinkiewicz, *SE Asian B. Math.*, **30** (2006), 609–620.
16. Y. Choi, Marcinkiewicz integrals with rough homogeneous kernels of degree zero in product domains, *J. Math. Anal. Appl.*, **261** (2001), 53–60. https://doi.org/10.1006/jmaa.2001.7465
17. H. Al-Qassem, A. Al-Salman, L. C. Cheng, Y. Pan, Marcinkiewicz integrals on product spaces, *Stud. Math.*, **167** (2005), 227–234. https://doi.org/10.4064/sm167-3-4
18. B. Al-Azriyah, A. Al-Salman, Singular and marcinkiewicz integral operators on product domains, *Commun. Korean Math. S.*, **38** (2023), 401–430. https://doi.org/10.4134/CKMS.c210421
19. B. Al-Azriyah, A. Al-Salman, A note on marcinkiewicz integral operators on product domains, *Kyungpook Math. J.*, **63** (2023), 577–591. https://doi.org/10.5666/KMJ.2023.63.4.577
20. A. Al-Salman, Rough Marcinkiewicz integrals on product spaces, *Int. Math. Forum*, **2** (2007), 1119–1128. https://doi.org/10.12988/imf.2007.07097
21. A. Al-Salman, Marcinkiewicz integrals along subvarieties on product domains, *International Journal of Mathematics and Mathematical Sciences*, **2004** (2004), 730308. https://doi.org/10.1155/S0161171204401264
22. J. Chen, D. Fan, Y. Ying, The method of rotation and Marcinkiewicz integrals on product domains, *Stud. Math.*, **153** (2002), 41–58. https://doi.org/10.4064/sm153-1-4

23. H. Al-Qassem, Y. Pan, L^p boundedness for singular integrals with rough kernels on product domains, *Hokkaido Math. J.*, **31** (2002), 555–613. https://doi.org/10.14492/hokmj/1350911903
24. D. Fan, Y. Pan, D. Yang, A weighted norm inequality for rough singular integrals, *Tohoku Math. J.*, **51** (1999), 141–161. https://doi.org/10.2748/tmj/1178224808
25. E. M. Stein, *Harmonic analysis: Real-variable mathods, orthogonality and oscillatory integrals*, Princeton: Princeton University Press, 1993.
26. R. Fefferman, E. M. Stein, Singular integrals on product spaces, *Adv. Math.*, **45** (1982), 117–143. https://doi.org/10.1016/S0001-8708(82)80001-7
27. A. Al-Salman, Y. Pan, Singular integrals with rough kernels in $Llog^+L(\mathbb{S}^{n-1})$, *J. Lond. Math. Soc.*, **66** (2002), 153–174. https://doi.org/10.1112/S0024610702003241