Research article

Coefficient bounds for certain families of bi-Bazilevič and bi-Ozaki-close-to-convex functions

Muajebah Hidan¹, Abbas Kareem Wanas²,*, Faiz Chaseb Khudher³, Gangadharan Murugusundaramoorthy⁴ and Mohamed Abdalla⁵

¹ Mathematics Department, College of Science, King Khalid University, Abha, Saudi Arabia
² Department of Mathematics, College of Science, University of Al-Qadisiyah, Al Diwaniyah 58001, Al-Qadisiyah, Iraq
³ Department of Mathematics, College of Science, University of Al-Qadisiyah, Al-Qadisiyah, Iraq
⁴ Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology(VIT), Vellore 632014, Tamil Nadu, India
⁵ Department of Mathematics, Faculty of Science, South Valley University, Qena 83523, Egypt

* Correspondence: Email: abbas.kareem.w@qu.edu.iq.

Abstract: The aim of this work is to introduce two families, \( \mathcal{B}_\Sigma(\varphi; \vartheta) \) and \( \mathcal{O}_\Sigma(\kappa; \vartheta) \), of holomorphic and bi-univalent functions involving the Bazilevič functions and the Ozaki-close-to-convex functions, by using generalized telephone numbers. We determine upper bounds on the Fekete-Szegő type inequalities and the initial Taylor-Maclaurin coefficients for functions in these families. We also highlight certain edge cases and implications for our findings.

Keywords: bi-univalent function; holomorphic function; Bazilevič function; Ozaki-close-to-convex functions; upper bounds; telephone numbers; Fekete-Szegő problem

Mathematics Subject Classification: 30C45, 30C50

1. Introduction and preliminaries

Indicate by \( \mathcal{A} \) the family of holomorphic functions in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) of the form

\[
f(z) = z + \sum_{k \geq 2} a_k z^k.
\]  \hspace{1cm} (1.1)

The \( \mathcal{A} \) subfamily of functions that are univalent in \( U \) is denoted by \( S \).

We denote by \( S^* \), \( C \), and \( K \) the families of functions that starlike, convex, and close-to-convex in \( U \), respectively, and given as follows:
A function \( f \in S \) is said to be starlike if
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in U,
\]
a function \( f \in S \) is said to be convex if
\[
\Re \left( \frac{(zf'(z))^2}{f'(z)} \right) > 0, \quad z \in U,
\]
and a function \( f \in S \) is said to be a close-to-convex functions if
\[
\Re \left( \frac{zf'(z)}{g(z)} \right) > 0; \quad g \in S^*, \quad z \in U.
\]

Note that, \( C \subset S^* \subset K \).
The families of functions that are starlike of order \( \aleph (0 \leq \aleph < 1) \) and convex of order \( \aleph (0 \leq \aleph < 1) \) are denoted by \( S^*(\aleph) \) and \( C(\aleph) \), respectively, and are given as follows:
\[
S^*(\aleph) = \left\{ f \in S : \Re \left( \frac{zf'(z)}{f(z)} \right) > \aleph, \quad z \in U \right\}
\]
and
\[
C(\aleph) = \left\{ f \in S : \Re \left( \frac{(zf'(z))^2}{f'(z)} \right) > \aleph, \quad z \in U \right\}.
\]
A well known subclass of \( S \), named as class of Bazilevič function [36], is defined as:
\[
\mathcal{B}(\varphi) = \left\{ f \in \mathcal{A} : \Re \left( \frac{z^{1-\varphi}f'(z)}{(f(z))^{1-\varphi}} \right) > 0; \quad z \in U; \quad \varphi \geq 0 \right\}.
\]
For \( 0 \leq \alpha \leq \pi \), Kaplan [25] defined a subfamily of \( A \) called close-to-convex functions, given as:
\[
\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \Re \left( e^{i\alpha}zf'(z) \right) > 0; \quad g \in S^*; \quad z \in U \right\}.
\]
In 1935, Ozaki [34] considered functions in \( \mathcal{A} \) satisfying the condition
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2}, \quad z \in U,
\]
whose members are known to be close-to-convex and therefore univalent.

Lately, Kargar and Ebadian [26] considered the generalization of Ozaki’s condition as the following:

**Definition 1.1.** Let \( f \in \mathcal{A} \) be locally univalent for \( z \in U \) and let \( -\frac{1}{2} \leq \zeta \leq 1 \). A function \( f \) is called an Ozaki-close-to-convex function in \( U \) if
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \frac{1}{2} - \zeta, \quad z \in U.
\]
We denote by $F(\kappa)$ the family of all Ozaki-close-to-convex functions. It is clear that, for $-\frac{1}{2} \leq \kappa \leq \frac{1}{2}$, we have $F(\kappa) \subset K \subset S^*$. In $U$, a function $f \in A$ is considered bi-univalent if $f$ and $f^{-1}$ are both univalent functions. Denote by $\Sigma$ all bi-univalent functions in $U$. We revisit a few functions in the family $\Sigma$ from Srivastava et al.'s work [41]. We notice that the family $\Sigma$ consists of $z - \frac{z^2}{1 - z}$, $-\log(1 - z)$ and $\frac{1}{2} \log\left(\frac{1 + z}{1 - z}\right)$ and so is not empty, however the Koebe function is not a member of $\Sigma$. Also, the functions $z - \frac{z^2}{1 - z}$ and $\frac{z}{1 - z}$ are not bi-univalent. A very large number of works related to the class $\Sigma$ have been presented in the recent papers (one may see [1, 3, 6, 8, 10–12, 14, 16, 18, 19, 22–24, 27–29, 31–33, 38–40, 42–46, 49–51]).

The Littlewood-Paley conjecture (see [21]) that the coefficients of odd univalent functions are bounded by unity has been refuted by Fekete and Szegö, a fact widely recognized in the field of Geometric Function Theory (GFT). Consequently, given $f \in S$, we get the Fekete-Szegö problem $|a_3 - \beta a_2^2|$. Many authors studied and deduced the Fekete-Szegö inequality for different families of functions (see [2–5, 9, 13, 17, 30, 35, 38, 40, 41, 47, 48, 53]) in GFT.

2. Generalized telephone numbers (GTN)

The recurrence relation quantifies traditional telephone numbers:

$$X(\ell) = X(\ell - 1) + (\ell - 1)X(\ell - 2), \quad \ell \geq 2.$$ 

Initial conditions:

$$X(0) = X(1) = 1.$$ 

For integers $\ell \geq 0$ and $\varsigma \geq 1$, Wloch and Wolowiec-Musial [52] defined generalized telephone numbers $X(\varsigma, \ell)$ by the recurrence relation:

$$X(\varsigma, \ell) = \varsigma X(\varsigma, \ell - 1) + (\ell - 1)X(\varsigma, \ell - 2),$$

with initial conditions

$$X(\varsigma, 0) = 1 \quad \text{and} \quad X(\varsigma, 1) = \varsigma.$$ 

Bednarz and Wolowiec-Musial [7] recently examined the accessible generalization of phone numbers using the formula

$$X_\varsigma(\ell) = X_\varsigma(\ell - 1) + \varsigma(\ell - 1)X_\varsigma(\ell - 2),$$

where $\ell \geq 2$ and $\varsigma \geq 1$ with initial conditions

$$X_\varsigma(0) = X_\varsigma(1) = 1.$$ 

According to Deniz [17], who conducted this investigation very recently, $X_\varsigma(\ell)$ has an exponential generating function

$$e^{(r + \varsigma^2 z^2)} = \sum_{\ell=0}^{\infty} X_\varsigma(\ell) \frac{r^\ell}{\ell!}.$$
As $ς = 1$, classical phone numbers $X_κ(ℓ) \equiv X(ℓ)$ are evident.

Now, we study the function

$$\vartheta(z) = e^{\left(z + \frac{ςz^2}{2}\right)} = 1 + z + \frac{ςz^2}{2} + \frac{1 + 3ςz^3}{6} + \cdots$$

(2.1)

with its domain of definition as the open unit disk $U$. Note that the holomorphic function $\vartheta(z)$ in $U$, with a positive real part such that $\vartheta(0) = 1$, $\vartheta'(0) > 0$, and $\vartheta$ maps $U$ onto an area that is symmetric with respect to the real axis and starlike with respect to 1. Using the generalized telephone numbers, we now give the following subfamilies of holomorphic bi-Ozaki-close-to-convex and bi-Bazilević functions.

**Definition 2.1.** If a function $f \in \Sigma$ satisfies the following subordinations, it belongs to the family $B_\Sigma(\varphi; \vartheta)$:

$$\frac{z^{1-\varphi}f'(z)}{(f(z))^{1-\varphi}} < e^{\left(z + \frac{ςz^2}{2}\right)} =: \vartheta(z)$$

and

$$\frac{w^{1-\varphi}g'(w)}{(g(w))^{1-\varphi}} < e^{\left(w + \frac{ςw^2}{2}\right)} =: \vartheta(w),$$

where $\varphi$ is non-negative integer, $ς \geq 1$, and $g(w) = f^{-1}(w)$.

**Remark 2.1.** If we take $\varphi = 0$ in Definition 2.1, the family $B_\Sigma(\varphi; \vartheta)$ reduces to the family $S_\Sigma(\vartheta)$, which was studied recently by Cotîrîlă and Wanas (see [15]).

**Definition 2.2.** The family $O_\Sigma(\kappa; \vartheta)$ contains all the functions $f \in \Sigma$ if the next subordinations satisfy:

$$\frac{2\kappa - 1}{2\kappa + 1} + \frac{2}{2\kappa + 1} \left(1 + \frac{zf''(z)}{f'(z)}\right) < e^{\left(z + \frac{ςz^2}{2}\right)} =: \vartheta(z)$$

and

$$\frac{2\kappa - 1}{2\kappa + 1} + \frac{2}{2\kappa + 1} \left(1 + \frac{wg''(w)}{g'(w)}\right) < e^{\left(w + \frac{ςw^2}{2}\right)} =: \vartheta(w),$$

where $\frac{1}{2} \leq \kappa \leq 1$, $ς \geq 1$, and $g(w) = f^{-1}(w)$.

**Remark 2.2.** If we take $\kappa = \frac{1}{2}$ in Definition 2.2, the family $O_\Sigma(\kappa; \vartheta)$ reduces to the family $C_\Sigma(\vartheta)$, which was introduced recently by Cotîrîlă and Wanas (see [15]).

In the following sections we determine the upper bounds on the Fekete-Szegő type inequalities and the initial Taylor-Maclaurin coefficients for functions in these families in Definitions 2.1 and 2.2.

### 3. Initial Taylor coefficient estimates

We recall the following lemma where we obtain the upper bounds on the Fekete-Szegő type inequalities and the initial Taylor-Maclaurin coefficients for functions in $f \in B_\Sigma(\varphi; \vartheta)$, where $\varphi$ is non-negative integer.

**Lemma 3.1.** ([20], p.41) Let $h \in P$ be given by the following series:

$$h(z) = 1 + c_1z + c_2z^2 + \cdots , \quad \text{where} \quad z \in U$$

then

$$|c_n| \leq 2 , \quad \text{for all} \quad n \in \mathbb{N}.$$
Theorem 3.1. If \( f \) given by (1.1) is in the family \( B_{\Sigma}(\varphi; \vartheta) \), where \( \varphi \) is non-negative integer, then

\[
|a_2| \leq \min \left\{ \frac{1}{\varphi + 1}, \sqrt{\frac{2}{(\varphi + 2) (\varphi + 1) + (1 - \varphi) (\varphi + 1)^2}} \right\}
\]

and

\[
|a_3| \leq \min \left\{ \frac{\varphi + \zeta + 2}{(\varphi + 2) (\varphi + 1)^2}, \frac{1}{\varphi + 2} \right\}.
\]

Proof. Assume that \( f \in B_{\Sigma}(\varphi; \vartheta) \) and that \( f^{-1} = g \). We consider the holomorphic functions \( \Theta, \Upsilon : U \rightarrow U \), where \( \Theta(0) = \Upsilon(0) = 0 \), satisfying the following criteria:

\[
\frac{z^{1-\varphi} f'(z)}{(f(z))^{1-\varphi}} = \vartheta(\Theta(z)), \quad z \in U
\]

and

\[
\frac{w^{1-\varphi} g'(w)}{(g(w))^{1-\varphi}} = \vartheta(\Upsilon(w)), \quad w \in U.
\]

Define the functions \( m \) and \( n \) by

\[
m(z) = \frac{1 - \Theta(z)}{1 - \Theta(z)} = 1 + m_1 z + m_2 z^2 + \cdots
\]

and

\[
n(z) = \frac{1 + \Upsilon(z)}{1 - \Upsilon(z)} = 1 + n_1 z + n_2 z^2 + \cdots.
\]

It follows that \( m, n \) are analytic functions in \( U \), where \( m(0) = 1 = n(0) \). Then, we get \( \Theta, \Upsilon : U \rightarrow U \), where \( m \) and \( n \) are functions with a positive real part in \( U \).

But, we have

\[
\Theta(z) = \frac{1 - m(z)}{m(z) + 1} = \frac{1}{2} \left[ m_1 z + \left( m_2 - \frac{m_1^2}{2} \right) z^2 + \cdots, z \in U \right]
\]

and

\[
\Upsilon(z) = \frac{1 - n(z)}{n(z) + 1} = \frac{1}{2} \left[ n_1 z + \left( n_2 - \frac{n_1^2}{2} \right) z^2 + \cdots, z \in U. \right.
\]

By substituting (3.3) and (3.4) into (3.1) and (3.2) and applying (2.1), we get

\[
\frac{z^{1-\varphi} f'(z)}{(f(z))^{1-\varphi}} = \vartheta(\Theta(z)) = e^{\left( \frac{m(0)-1}{m(0)+1} + \frac{(m(0)-1)^2}{m(0)+1} \right)} z^2 + \cdots
\]

and

\[
\frac{w^{1-\varphi} g'(w)}{(g(w))^{1-\varphi}} = \vartheta(\Upsilon(w)) = e^{\left( \frac{n(0)-1}{n(0)+1} + \frac{(n(0)-1)^2}{n(0)+1} \right)} w^2 + \cdots
\]

Equating the coefficients in (3.5) and (3.6) yields

\[
(\varphi + 1) a_2 = \frac{1}{2} m_1,
\]
\[(\varphi + 2)a_3 + \frac{1}{2}(\varphi + 2)(\varphi - 1)a_2^2 = \frac{m_2}{2} + \frac{(\zeta - 1)m_1^2}{8}, \quad (3.8)\]

\[- (\varphi + 1)a_2 = \frac{1}{2}n_1 \quad (3.9)\]

and

\[(\varphi + 2)(2a_2^2 - a_3) + \frac{1}{2}(\varphi + 2)(\varphi - 1)a_2^2 = \frac{n_2}{2} + \frac{(\zeta - 1)n_1^2}{8}. \quad (3.10)\]

From (3.7) and (3.9), we have

\[m_1 = -n_1 \quad (3.11)\]

and

\[2(\varphi + 1)^2a_2^2 = \frac{1}{4}(m_1^2 + n_1^2). \quad (3.12)\]

If we add (3.8) to (3.10), we obtain

\[(\varphi + 2)(\varphi + 1)a_2^2 = \frac{1}{2}(m_2 + n_2) + \frac{1}{8}(\zeta - 1)(m_1^2 + n_1^2). \quad (3.13)\]

Substituting from (3.12) the value of \(m_1^2 + n_1^2\) in the relation (3.13), we get

\[a_2^2 = \frac{m_2 + n_2}{2[\varphi + 2]^2 + (1 - \zeta)(\varphi + 1)^2]} \quad (3.14)\]

Applying Lemma 3.1 for the coefficients \(m_1, m_2, n_1, n_2\) in (3.12) and (3.14), we get

\[|a_2| \leq \frac{1}{\varphi + 1}, \quad |a_2| \leq \sqrt{\frac{2}{[\varphi + 2](\varphi + 1) + (1 - \zeta)(\varphi + 1)^2}}. \]

Applying (3.11) and subtracting (3.10) from (3.8) yields \(m_1^2 = n_1^2\), which is the bound on \(|a_3|\),

\[2(\varphi + 2)(a_3 - a_2^2) = \frac{1}{2}(m_2 - n_2). \quad (3.15)\]

Substituting \(a_2^2\) from (3.12) into (3.15) yields the following result:

\[a_3 = \frac{m_1^2 + n_1^2}{8(\varphi + 1)^2} + \frac{m_2 - n_2}{4(\varphi + 2)}. \]

So, we have

\[|a_3| \leq \frac{1}{(\varphi + 1)^2} + \frac{1}{\varphi + 2}. \]

Also, substituting the value of \(a_2^2\) from (3.13) into (3.15), we get

\[a_3 = \frac{m_2 - n_2}{4(\varphi + 2)} + \frac{m_2 + n_2}{2(\varphi + 2)(\varphi + 1)} + \frac{(\zeta - 1)(m_1^2 + n_1^2)}{8(\varphi + 2)(\varphi + 1)} \]

and we have

\[|a_3| \leq \frac{\varphi + \zeta + 2}{(\varphi + 2)(\varphi + 1)}. \]
Theorem 3.2. Let \( f \in O_{\Sigma} (\kappa; \vartheta) \left( \frac{1}{2} \leq \kappa \leq 1 \right) \) and \( f \) be given by (1.1). Then,

\[
|a_2| \leq \min \left\{ \frac{(2\kappa + 1)^2}{16}, \frac{2\kappa + 1}{2\sqrt{2(k - \varsigma) + 3}} \right\}
\]

and

\[
|a_3| \leq \min \left\{ \frac{(2\kappa + 1)(3\varsigma + 5)}{24}, \frac{2\kappa + 1}{12} + \frac{(2\kappa + 1)^2}{16} \right\}.
\]

Proof. Assume that \( f \in O_{\Sigma} (\kappa; \vartheta) \) and \( g = f^{-1} \). Specifically, there exist holomorphic functions \( \Theta, \Upsilon : U \rightarrow U \), hence

\[
\frac{2\kappa - 1}{2\kappa + 1} + \frac{2}{2\kappa + 1} \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \vartheta(\Theta(z)), \quad \text{where} \quad z \in U
\]

(3.16)

and

\[
\frac{2\kappa - 1}{2\kappa + 1} + \frac{2}{2\kappa + 1} \left( 1 + \frac{wg''(w)}{g'(w)} \right) = \vartheta(\Upsilon(w)), \quad \text{where} \quad w \in U,
\]

(3.17)

where \( \Theta(z) \) and \( \Upsilon(z) \) have the forms (3.3) and (3.4). From (3.16), (3.17), and (2.1), we deduce that

\[
\frac{2\kappa - 1}{2\kappa + 1} + \frac{2}{2\kappa + 1} \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \vartheta(\Theta(z)) = e^{\left( \frac{\frac{m_1 - 1}{m_1 + 1} \left( \frac{\frac{m_1 - 1}{m_1 + 1}}{2} \right)^2 }{m_1} \right)}
\]

(3.18)

\[
= 1 + \frac{1}{2} m_1 z + \left[ \frac{m_2}{2} + \frac{(\varsigma - 1)m_1^2}{8} \right] z^2 + \cdots
\]

and

\[
\frac{2\kappa - 1}{2\kappa + 1} + \frac{2}{2\kappa + 1} \left( 1 + \frac{wg''(w)}{g'(w)} \right) = \vartheta(\Upsilon(w)) = e^{\left( \frac{\frac{n_1 - 1}{n_1 + 1} \left( \frac{\frac{n_1 - 1}{n_1 + 1}}{2} \right)^2 }{n_1} \right)}
\]

(3.19)

\[
= 1 + \frac{1}{2} n_1 w + \left[ \frac{n_2}{2} + \frac{(\varsigma - 1)n_1^2}{8} \right] w^2 + \cdots
\]

Equating the coefficients in (3.18) and (3.19), yields

\[
\frac{4}{2\kappa + 1} a_2 = \frac{1}{2} m_1,
\]

(3.20)

\[
\frac{12}{2\kappa + 1} a_3 - \frac{8}{2\kappa + 1} a_2 = \frac{m_2}{2} + \frac{(\varsigma - 1)m_1^2}{8},
\]

(3.21)

\[
- \frac{4}{2\kappa + 1} a_2 = \frac{1}{2} n_1
\]

(3.22)

and

\[
\frac{12}{2\kappa + 1} (2a_2^2 - a_3) - \frac{8}{2\kappa + 1} a_2^2 = \frac{n_2}{2} + \frac{(\varsigma - 1)n_1^2}{8}.
\]

(3.23)

From (3.20) and (3.22), we have

\[
m_1 = -n_1
\]

(3.24)
and
\[
\frac{32}{(2\kappa + 1)^2}a_2^2 = \frac{1}{4}(m_1^2 + n_1^2).
\] (3.25)

If we add (3.21) to (3.23), we obtain
\[
\frac{8}{2\kappa + 1}a_2^2 = \frac{1}{2}(m_2 + n_2) + \frac{1}{8}(\varsigma - 1)(m_1^2 + n_1^2).
\] (3.26)

Substituting from (3.25) the value of \(m_1^2 + n_1^2\) in the relation (3.26), we deduce that
\[
a_2^2 = \frac{(2\kappa + 1)^2}{16(2(\kappa - \varsigma) + 3)}(m_2 + n_2).
\] (3.27)

Applying Lemma 3.1 for the coefficients \(m_1, m_2, n_1,\) and \(n_2\) in (3.25) and (3.27), we get
\[
|a_2| \leq \frac{(2\kappa + 1)^2}{16}, \quad |a_2| \leq \frac{2\kappa + 1}{2\sqrt{2(\kappa - \varsigma) + 3}}.
\]

Subtracting (3.23) from relation (3.21) and applying (3.24), we get
\[
|a_3| \leq \frac{2\kappa - 1}{12} + \frac{(2\kappa + 1)^2}{16}.
\]

Then by substituting from (3.25) the value of \(a_2^2\) into (3.28), we get
\[
a_3 = \frac{(2\kappa + 1)(m_2 - n_2)}{48} + \frac{(2\kappa + 1)^2(m_1^2 + n_1^2)}{128}.
\]

So, we have
\[
|a_3| \leq \frac{2\kappa + 1}{12} + \frac{(2\kappa + 1)^2}{16}.
\]

Also, substituting the value of \(a_2^2\) from (3.26) into (3.28), we get
\[
a_3 = \frac{(2\kappa + 1)(m_2 - n_2)}{48} + \frac{(2\kappa + 1)(m_2 + n_2)}{16} + \frac{(2\kappa + 1)(\varsigma - 1)(m_1^2 + n_1^2)}{64}
\]
and we have
\[
|a_3| \leq \frac{(2\kappa + 1)(3\varsigma + 5)}{24}.
\]

Utilizing \(a_2^2\) and \(a_3\) values, and spurred by Zaprawa’s recent work [53], we prove the Fekete-Szegö problem for \(f \in B_\Sigma(\varphi; \theta)\) and \(f \in O_\Sigma(\nu; \theta)\) in the following theorems.

**Theorem 3.3.** For a non-negative integer \(\varphi\) and \(h \in \mathbb{R}\), let \(f \in B_\Sigma(\varphi; \theta)\) be of the form (1.1). Then,
\[
|a_3 - h a_2^2| \leq \begin{cases} \frac{1}{\varphi+2}; & |h - 1| \leq \frac{|(\varphi+2)(\varphi+1)+(1-\varsigma)(\varphi+1)^2|}{2(\varphi+2)}, \\ \frac{2|h-1|}{(\varphi+2)(\varphi+1)(1-\varsigma)(\varphi+1)^2}; & |h - 1| \geq \frac{|(\varphi+2)(\varphi+1)+(1-\varsigma)(\varphi+1)^2|}{2(\varphi+2)}. \end{cases}
\]
Proof. It follows from (3.14) and (3.15) that

\[ a_3 - ha_2^2 = \frac{m_2 - n_2}{4(\varphi + 2)} + (1 - h) a_2^2 \]

\[ = \frac{m_2 - n_2}{4(\varphi + 2)} + \frac{(m_2 + n_2)(1 - h)}{2(\varphi + 1) + (1 - \varsigma)(\varphi + 1)^2} \]

\[ = \frac{1}{2} \left[ \psi(h, \varsigma) + \frac{1}{2(\varphi + 2)} \right] m_2 + \left( \psi(h, \varsigma) - \frac{1}{2(\varphi + 2)} \right) n_2, \]

where

\[ \psi(h, \varsigma) = \frac{1 - h}{(\varphi + 2)(\varphi + 1) + (1 - \varsigma)(\varphi + 1)^2}. \]

According to Lemma 3.1, we find that

\[ |a_3 - ha_2^2| \leq \begin{cases} \frac{1}{\varphi + 2}, & 0 \leq |\psi(h, \varsigma)| \leq \frac{1}{2(\varphi + 2)}, \\ 2|\psi(h, \varsigma)|, & |\psi(h, \varsigma)| \geq \frac{1}{2(\varphi + 2)}. \end{cases} \]

After some computations, we obtain

\[ |a_3 - ha_2^2| \leq \begin{cases} \frac{1}{\varphi + 2}, & |h - 1| \leq \frac{|(\varphi + 2)(\varphi + 1)(1 - \varsigma)(\varphi + 1)^2|}{2(\varphi + 2)}, \\ 0, & |h - 1| \geq \frac{|(\varphi + 2)(\varphi + 1)(1 - \varsigma)(\varphi + 1)^2|}{2(\varphi + 2)}. \end{cases} \]

Putting \( h = 1 \) in Theorem 3.3, we get the next result:

**Corollary 3.1.** If \( f \in \mathcal{B}_E(\varphi; \vartheta) \) is of the form (1.1), then we have that

\[ |a_3 - a_2^2| \leq \frac{1}{\varphi + 2}. \]

**Theorem 3.4.** For \( \frac{1}{2} \leq \kappa \leq 1 \) and \( h \in \mathbb{R} \), let \( f \in \mathcal{O}_E(\kappa; \vartheta) \) be of the form (1.1). Then,

\[ |a_3 - ha_2^2| \leq \begin{cases} \frac{1}{\varphi + 2}, & |h - 1| \leq \frac{2\varsigma - 1}{3}, \\ 0, & |h - 1| \geq \frac{2\varsigma - 1}{3}. \end{cases} \]

**Proof.** It follows from (3.27) and (3.28) that

\[ a_3 - ha_2^2 = \frac{(2\kappa + 1)(m_2 - n_2)}{48} + (1 - h) a_2^2 \]

\[ = \frac{(2\kappa + 1)(m_2 - n_2)}{48} + \frac{(2\kappa + 1)^2 (m_2 + n_2)(1 - h)}{16(2(\kappa - \varsigma) + 3)}. \]
\[
\frac{1}{16} \left( \left( \phi(h, \varsigma) + \frac{2\kappa + 1}{3} \right) m_2 + \left( \phi(h, \varsigma) - \frac{2\kappa + 1}{3} \right) n_2 \right),
\]

where
\[
\phi(h, \varsigma) = \frac{(2\kappa + 1)^2 (1 - h)}{2(\kappa - \varsigma) + 3}.
\]

According to Lemma 3.1, we find that
\[
\left| a_3 - h a_2^2 \right| \leq \begin{cases} 
\frac{2\kappa + 1}{12}, & 0 \leq |\phi(h, \varsigma)| \leq \frac{2\kappa + 1}{3}, \\
\frac{1}{2} |\phi(h, \varsigma)|, & |\phi(h, \varsigma)| \geq \frac{2\kappa + 1}{3}.
\end{cases}
\]

After some computations, we obtain
\[
\left| a_3 - h a_2^2 \right| \leq \begin{cases} 
\frac{2\kappa + 1}{12}, & |h - 1| \leq \frac{(2\kappa + 1)(2(\kappa - \varsigma) + 3)}{3(2\kappa + 1)^2}, \\
\frac{(2\kappa + 1)^2 |h - 1|}{4(2(\kappa - \varsigma) + 3)}, & |h - 1| \geq \frac{(2\kappa + 1)(2(\kappa - \varsigma) + 3)}{3(2\kappa + 1)^2}.
\end{cases}
\]

Fixing \( h = 1 \) in Theorem 3.4, we get the following result:

**Corollary 3.2.** If \( f \in O_2(\kappa; \vartheta) \) is of the form (1.1), then
\[
\left| a_3 - a_2^2 \right| \leq \frac{2\kappa + 1}{12}.
\]

**Remark 3.1.** \( \varphi = 0 \) and \( \kappa = \frac{1}{2} \) give the results of Cotîrlă and Wanas (see [15]).

### 4. Conclusions

Motivated by many recent advances on the Fekete-Szegö functional and Taylor-Maclaurin coefficient estimations, we defined new families of holomorphic bi-univalent functions \( B_{2\kappa}(\varphi; \vartheta) \) and \( O_{2\kappa}(\kappa; \vartheta) \) associated with generalized telephone numbers are presented and thoroughly examined in this article. For functions in these families, we determined Taylor-Maclaurin coefficient inequalities and examined the well-known Fekete-Szegö issue. Furthermore, the generic coefficients \( |a_n|, n \geq 4 \), for the functions of these new classes remain unbounded. We also opted to utilize a significant finding from a recently released evaluate-cum-explanatory paper by Srivastava ( [37], p. 340) to extend our study based on the \( q \)-difference operator. This observation pointed out that using some seemingly parametric and argumentative versions of the extra parameter \( p \) is redundant; the effects for the new or previously mentioned \( q \)--analogs could be easily (and possibly trivially) translated into corresponding effects for the so-called \( (p; q) \)--analogs (with \( 0 < |q| \leq 1 \)). Further, one can obtain the second Hankel determinant inequalities for function classes studied in this article (see [42–47] and references cited therein).
Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through a large group research project under grant number RGP2/432/44.

Conflict of interest

The authors declare that they have no conflicts of interest.

References


©2024 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)