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Research article

On the power sums problem of bi-periodic Fibonacci and Lucas polynomials

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Abstract: This paper mainly discussed the power sums of bi-periodic Fibonacci and Lucas polynomials. In addition, we generalized these results to obtain several congruences involving the divisible properties of bi-periodic Fibonacci and Lucas polynomials.

Keywords: bi-periodic Fibonacci polynomial; bi-periodic Lucas polynomal; power sum; divisible property
Mathematics Subject Classification: 11B39, 11B37

1. Introduction

Fibonacci polynomials and Lucas polynomials are important in various fields such as number theory, probability theory, numerical analysis, and physics. In addition, many well-known polynomials, such as Pell polynomials, Pell Lucas polynomials, Tribonacci polynomials, etc., are generalizations of Fibonacci polynomials and Lucas polynomials. In this paper, we extend the linear recursive polynomials to nonlinearity, that is, we discuss some basic properties of the bi-periodic Fibonacci and Lucas polynomials.

The bi-periodic Fibonacci $\{f_n(t)\}$ and Lucas $\{l_n(t)\}$ polynomials are defined recursively by

$$f_0(t) = 0, \quad f_1(t) = 1, \quad f_n(t) = \begin{cases} ayf_{n-1}(t) + f_{n-2}(t) & n \equiv 0 \pmod 2, \\ byf_{n-1}(t) + f_{n-2}(t) & n \equiv 1 \pmod 2, \end{cases} \quad n \geq 2,$$

and

$$l_0(t) = 2, \quad l_1(t) = at, \quad l_n(t) = \begin{cases} byl_{n-1}(t) + l_{n-2}(t) & n \equiv 0 \pmod 2, \\ ayl_{n-1}(t) + l_{n-2}(t) & n \equiv 1 \pmod 2, \end{cases} \quad n \geq 2,$$

where a and b are nonzero real numbers. For $t = 1$, the bi-periodic Fibonacci and Lucas polynomials are, respectively, well-known bi-periodic Fibonacci $\{f_n\}$ and Lucas $\{l_n\}$ sequences. We let

$$\varsigma(n) = \begin{cases} 0 & n \equiv 0 \pmod{2}, \\ 1 & n \equiv 1 \pmod{2}, \end{cases} \quad n \geq 2.$$

In [1], the scholars give the Binet formulas of the bi-periodic Fibonacci and Lucas polynomials as follows:

$$f_n(t) = \frac{a^{\varsigma(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\sigma^n(t) - \tau^n(t)}{\sigma(t) - \tau(t)} \right), \tag{1.1}$$

and

$$l_n(t) = \frac{a^{\varsigma(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\sigma^n(t) + \tau^n(t) \right), \tag{1.2}$$

where $n \geq 0$, $\sigma(t)$, and $\tau(t)$ are zeros of $\lambda^2 - abt\lambda - ab$. This is $\sigma(t) = \frac{abt + \sqrt{a^2b^2t^2+4ab}}{2}$ and $\tau(t) = \frac{abt - \sqrt{a^2b^2t^2+4ab}}{2}$. We note the following algebraic properties of $\sigma(t)$ and $\tau(t)$:

$$\sigma(t) + \tau(t) = abt, \quad \sigma(t) - \tau(t) = \sqrt{a^2b^2t^2 + 4ab}, \quad \sigma(t)\tau(t) = -ab.$$

Many scholars studied the properties of bi-periodic Fibonacci and Lucas polynomials; see [2–6]. In addition, many scholars studied the power sums problem of second-order linear recurrences and its divisible properties; see [7–10].

Taking $a = b = 1$ and $t = 1$, we obtain the Fibonacci $\{F_n\}$ or Lucas $\{L_n\}$ sequence. Melham [11] proposed the following conjectures:

Conjecture 1. *Let $m \geq 1$ be an integer, then the sum*

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1}$$

can be represented as $(F_{2n+1} - 1)^2 R_{2m-1}(F_{2n+1})$, including $R_{2m-1}(t)$ as a polynomial with integer coefficients of degree $2m - 1$.

Conjecture 2. *Let $m \geq 1$ be an integer, then the sum*

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} L_{2k}^{2m+1}$$

can be represented as $(L_{2n+1} - 1) Q_{2m}(L_{2n+1})$, where $Q_{2m}(t)$ is a polynomial with integer coefficients of degree $2m$.

In [12], the authors completely solved the Conjecture 2 and discussed the Conjecture 1. Using the definition and properties of bi-periodic Fibonacci and Lucas polynomials, the power sums problem and their divisible properties are studied in this paper. The results are as follows:

Theorem 1. *We get the identities*

$$\sum_{k=1}^{n} f_{2k}^{2m+1}(t) = \frac{a^{2m+1}}{b\left(a^2b^2t^2+4ab\right)^m} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \left(\frac{f_{(2n+1)(2j+1)}(t) - f_{2j+1}(t)}{l_{2j+1}(t)} \right), \tag{1.3}$$

$$\sum_{k=1}^{n} f_{2k+1}^{2m+1}(t) = \frac{(ab)^m}{\left(a^2b^2t^2+4ab\right)^m} \sum_{j=0}^{m} \binom{2m+1}{m-j} \left(\frac{f_{(2n+2)(2j+1)}(t) - f_{2(2j+1)}(t)}{l_{2j+1}(t)} \right), \tag{1.4}$$

$$\sum_{k=1}^{n} l_{2k}^{2m+1}(t) = \sum_{j=0}^{m} \binom{2m+1}{m-j} \left(\frac{l_{(2n+1)(2j+1)}(t) - l_{2j+1}(t)}{l_{2j+1}(t)} \right), \tag{1.5}$$

$$\sum_{k=1}^{n} l_{2k+1}^{2m+1}(t) = \frac{a^{m+1}}{b^{m+1}} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \left(\frac{l_{(2n+2)(2j+1)}(t) - l_{2(2j+1)}(t)}{l_{2j+1}(t)} \right), \tag{1.6}$$

where n and m are positive integers.

Theorem 2. *We get the identities*

$$\sum_{k=1}^{n} f_{2k}^{2m}(t) = \frac{a^{2m}}{\left(a^2b^2t^2+4ab\right)^m} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m}{m-j} \frac{f_{2j(2n+1)}(t)}{f_{2j}(t)} - \frac{a^{2m}}{\left(a^2b^2t^2+4ab\right)^m} \binom{2m}{m} (-1)^m \left(n+\frac{1}{2}\right), \tag{1.7}$$

$$\sum_{k=1}^{n} f_{2k+1}^{2m}(t) = \frac{(ab)^m}{\left(a^2b^2t^2+4ab\right)^m} \sum_{j=0}^{m} \binom{2m}{m-j} \left(\frac{f_{2j(2n+2)}(t) - f_{4j}(t)}{f_{2j}(t)} \right) - \frac{(ab)^m}{\left(a^2b^2t^2+4ab\right)^m} \binom{2m}{m} n, \tag{1.8}$$

$$\sum_{k=1}^{n} l_{2k}^{2m}(t) = \sum_{j=0}^{m} \binom{2m}{m-j} \frac{f_{2j(2n+1)}(t)}{l_{2j+1}(t)} - 2^{2m-1} - \binom{2m}{m} \left(n+\frac{1}{2}\right), \tag{1.9}$$

$$\sum_{k=1}^{n} l_{2k+1}^{2m}(t) = \frac{a^m}{b^m} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m}{m-j} \left(\frac{f_{2j(2n+2)}(t) - f_{4j}(t)}{f_{2j}(t)} \right) - \frac{a^m}{b^m} \binom{2m}{m} (-1)^m n, \tag{1.10}$$

where n and m are positive integers.

As for application of Theorem 1, we get the following:

Corollary 1. *We get the congruences:*

$$bl_1(t)\, l_3(t) \cdots l_{2m+1}(t) \sum_{k=1}^{n} f_{2k}^{2m+1}(t) \equiv 0 \pmod{f_{2n+1}(t) - 1}, \tag{1.11}$$

and

$$al_1(t)\, l_3(t) \cdots l_{2m+1}(t) \sum_{k=1}^{n} l_{2k}^{2m+1}(t) \equiv 0 \pmod{l_{2n+1}(t) - at}, \tag{1.12}$$

where n and m are positive integers.

Taking $t = 1$ in Corollary 1, we have the following conclusions for bi-periodic Fibonacci $\{f_n\}$ and Lucas $\{l_n\}$ sequences.

Corollary 2. *We get the congruences:*

$$bl_1l_3\cdots l_{2m+1}\sum_{k=1}^{n} f_{2k}^{2m+1} \equiv 0 \pmod{f_{2n+1}-1}, \tag{1.13}$$

and

$$al_1l_3\cdots l_{2m+1}\sum_{k=1}^{n} l_{2k}^{2m+1} \equiv 0 \pmod{l_{2n+1}-a}, \tag{1.14}$$

where n and m are nonzero real numbers.

Taking $a = b = 1$ and $t = 1$ in Corollary 1, we have the following conclusions for bi-periodic Fibonacci $\{F_n\}$ and Lucas $\{L_n\}$ sequences.

Corollary 3. *We get the congruences:*

$$L_1L_3\cdots L_{2m+1}\sum_{k=1}^{n} F_{2k}^{2m+1} \equiv 0 \pmod{F_{2n+1}-1}, \tag{1.15}$$

and

$$L_1L_3\cdots L_{2m+1}\sum_{k=1}^{n} L_{2k}^{2m+1} \equiv 0 \pmod{L_{2n+1}-1}, \tag{1.16}$$

where n and m are nonzero real numbers.

2. Proofs of theorems

To begin, we will give several lemmas that are necessary in proving theorems.

Lemma 1. *We get the congruence*

$$f_{(2n+1)(2j+1)}(t) - f_{2j+1}(t) \equiv 0 \pmod{f_{2n+1}(t)-1},$$

where n and m are nonzero real numbers.

Proof. We prove it by complete induction for $j \geq 0$. This clearly holds when $j = 0$. If $j = 1$, we note that $abf_{3(2n+1)}(t) = \left(a^2b^2t^2 + 4ab\right) f_{2n+1}^3(t) - 3abf_{2n+1}(t)$ and we obtain

$$\begin{aligned}
&f_{3(2n+1)}(t) - f_3(t) = \left(abt^2+4\right) f_{2n+1}^3(t) - 3f_{2n+1}(t) - \left(abt^2+4\right) f_1^3(t) + 3f_1(t)\\
&= \left(abt^2+4\right)(f_{2n+1}(t) - f_1(t))\left(f_{2n+1}^2(t) + f_{2n+1}(t) f_1(t) + f_1^2(t)\right) - 3\left(f_{2n+1}(t) - f_1(t)\right)\\
&= \left(abt^2+4\right)(f_{2n+1}(t) - 1)\left(f_{2n+1}^2(t) + f_{2n+1}(t) f_1(t) + f_1^2(t)\right) - 3\left(f_{2n+1}(t) - 1\right)\\
&\equiv 0 \pmod{f_{2n+1}(t)-1}.
\end{aligned}$$

This is obviously true when $j = 1$. Assuming that Lemma 1 holds if $j = 1, 2, \ldots, k$, that is,

$$f_{(2n+1)(2j+1)}(t) - f_{2j+1}(t) \equiv 0 \pmod{f_{2n+1}(t)-1}.$$

If $j = k + 1 \geq 2$, we have

$$l_{2(2n+1)}(t) f_{(2n+1)(2j+1)}(t) = f_{(2n+1)(2j+3)}(t) + abf_{(2n+1)(2j-1)}(t),$$

and

$$abl_{2(2n+1)}(t) = \left(a^2b^2t^2 + 4ab\right) f_{2n+1}^2(t) - 2ab \equiv \left(a^2b^2t^2 + 4ab\right) f_1^2(t) - 2ab \pmod{f_{2n+1}(t) - 1}.$$

We have

$$\begin{aligned}
& f_{(2n+1)(2k+3)}(t) - f_{2k+3}(t) \\
&= l_{2(2n+1)}(t) f_{(2n+1)(2k+1)}(t) - abf_{(2n+1)(2k-1)}(t) - l_2(t) f_{2k+1}(t) + abf_{2k-1}(t) \\
&\equiv \left(\left(abt^2 + 4\right) f_1^2(t) - 2\right) f_{(2n+1)(2k+1)}(t) - abf_{(2n+1)(2k-1)}(t) \\
&\quad - \left(\left(abt^2 + 4\right) f_1^2(t) - 2\right) f_{2k+1}(t) + abf_{2k-1}(t) \\
&\equiv \left(\left(abt^2 + 4\right) f_1^2(t) - 2\right) \left(f_{(2n+1)(2k+1)}(t) - f_{2k+1}(t)\right) - ab\left(f_{(2n+1)(2k-1)}(t) - f_{2k-1}(t)\right) \\
&\equiv 0 \pmod{f_{2n+1}(t) - 1}.
\end{aligned}$$

This completely proves Lemma 1. □

Lemma 2. *We get the congruence*

$$al_{(2n+1)(2j+1)}(t) - al_{2j+1}(t) \equiv 0 \pmod{l_{2n+1}(t) - at},$$

where n and m are nonzero real numbers.

Proof. We prove it by complete induction for $j \geq 0$. This clearly holds when $j = 0$. If $j = 1$, we note that $al_{3(2n+1)}(t) = bl_{2n+1}^3(t) + 3al_{2n+1}(t)$ and we obtain

$$\begin{aligned}
& al_{3(2n+1)}(t) - al_3(t) = bl_{2n+1}^3(t) + 3al_{2n+1}(t) - bl_1^3(t) - 3al_1(t) \\
&= \left(l_{2n+1}(t) - l_1(t)\right)\left(bl_{2n+1}^2(t) + bl_{2n+1}(t) l_1(t) + bl_1^2(t)\right) - 3a\left(l_{2n+1}(t) - l_1(t)\right) \\
&= \left(l_{2n+1}(t) - at\right)\left(bl_{2n+1}^2(t) + bayl_{2n+1}(t) + ba^2t^2\right) - 3a\left(l_{2n+1}(t) - at\right) \\
&\equiv 0 \pmod{l_{2n+1}(t) - at}.
\end{aligned}$$

This is obviously true when $j = 1$. Assuming that Lemma 2 holds if $j = 1, 2, \ldots, k$, that is,

$$al_{(2n+1)(2j+1)}(t) - al_{2j+1}(t) \equiv 0 \pmod{l_{2n+1}(t) - at}.$$

If $j = k + 1 \geq 2$, we have

$$l_{2(2n+1)}(t)\, l_{(2n+1)(2j+1)}(t) = l_{(2n+1)(2j+3)}(t) + l_{(2n+1)(2j-1)}(t),$$

and

$$al_{2(2n+1)}(t) = bl_{2n+1}^2(t) + 2a \equiv bl_1^2(t) + 2a \pmod{l_{2n+1}(t) - at}.$$

We have

$$\begin{aligned}
& al_{(2n+1)(2k+3)}(t) - al_{(2k+3)}(t) \\
&= a\left(l_{2(2n+1)}(t)\, l_{(2n+1)(2k+1)}(t) - l_{(2n+1)(2k-1)}(t)\right) - a\left(l_2(t)\, l_{2k+1}(t) - l_{2k-1}(t)\right) \\
&\equiv \left(bl_1^2(t) + 2a\right) l_{(2n+1)(2k+1)}(t) - al_{(2n+1)(2k-1)}(t) - \left(bl_1^2(t) + 2a\right) l_{2k+1}(t) + al_{2k-1}(t) \\
&\equiv \left(abt^2 + 2\right)\left(al_{(2n+1)(2k+1)}(t) - al_{2k+1}(t)\right) - \left(al_{(2n+1)(2k-1)}(t) - al_{2k-1}(t)\right) \\
&\equiv 0 \pmod{l_{2n+1}(t) - at}.
\end{aligned}$$

This completely proves Lemma 2. □

Proof of Theorem 1. We only prove (1.3), and the proofs for other identities are similar.

$$\begin{aligned}
\sum_{k=1}^{n} f_{2k}^{2m+1}(t) &= \sum_{k=1}^{n}\left(\frac{a^{\varsigma(2k+1)}}{(ab)^{\lfloor\frac{2k}{2}\rfloor}}\cdot\left(\frac{\sigma^{2k}(t)-\tau^{2k}(t)}{\sigma(t)-\tau(t)}\right)\right)^{2m+1} \\
&= \frac{a^{2m+1}}{(\sigma(t)-\tau(t))^{2m+1}}\sum_{k=1}^{n}\frac{\left(\sigma^{2k}(t)-\tau^{2k}(t)\right)^{2m+1}}{(ab)^{(2m+1)k}} \\
&= \frac{a^{2m+1}}{(\sigma(t)-\tau(t))^{2m+1}}\sum_{k=1}^{n}\sum_{j=0}^{2m+1}(-1)^j\binom{2m+1}{j}\frac{\sigma^{2k(2m+1-j)}(t)\,\tau^{2kj}(t)}{(ab)^{(2m+1)k}} \\
&= \frac{a^{2m+1}}{(\sigma(t)-\tau(t))^{2m+1}}\sum_{j=0}^{2m+1}(-1)^j\binom{2m+1}{j}\left(\frac{1-\frac{\sigma^{2n(2m+1-2j)}(t)}{(ab)^{(2m+1-2j)n}}}{\frac{(ab)^{2m+1-2j}}{\sigma^{2(2m+1-2j)}(t)}-1}\right) \\
&= \frac{a^{2m+1}}{(\sigma(t)-\tau(t))^{2m+1}}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{j}\left(\frac{1-\frac{\sigma^{2n(2m+1-2j)}(t)}{(ab)^{(2m+1-2j)n}}}{\frac{(ab)^{2m+1-2j}}{\sigma^{2(2m+1-2j)}(t)}-1}-\frac{1-\frac{\sigma^{2n(2j-1-2m)}(t)}{(ab)^{(2j-1-2m)n}}}{\frac{(ab)^{2j-1-2m}}{\sigma^{2(2j-1-2m)}(t)}-1}\right) \\
&= \frac{a^{2m+1}}{(\sigma(t)-\tau(t))^{2m+1}}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{j}\left(\frac{\frac{\sigma^{2(2m+1-2j)}(t)}{(ab)^{2m+1-2j}}-\frac{\sigma^{(2n+2)(2m+1-2j)}(t)}{(ab)^{(n+1)(2m+1-2j)}}+1-\frac{\sigma^{2n(2j-1-2m)}(t)}{(ab)^{(2j-1-2m)n}}}{1-\frac{\sigma^{2(2m+1-2j)}(t)}{(ab)^{(2m+1-2j)}}}\right) \\
&= \frac{a^{2m+1}}{(\sigma(t)-\tau(t))^{2m+1}}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{j} \\
&\quad\times\left(\frac{\sigma^{2m+1-2j}(t)-\tau^{2m+1-2j}(t)-\frac{\sigma^{(2n+1)(2m+1-2j)}(t)}{(ab)^{(2m+1-2j)n}}+\frac{\tau^{(2n+1)(2m+1-2j)}(t)}{(ab)^{(2m+1-2j)n}}}{-\sigma^{2m+1-2j}(t)-\tau^{2m+1-2j}(t)}\right) \\
&= \frac{a^{2m+1}}{b\left(a^2b^2t^2+4ab\right)^m}\sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\left(\frac{f_{(2n+1)(2j+1)}(t)-f_{2j+1}(t)}{l_{2j+1}(t)}\right).
\end{aligned}$$

□

Proof of Theorem 2. We only prove (1.7), and the proofs for other identities are similar.

$$\begin{aligned}
\sum_{k=1}^{n} f_{2k}^{2m}(t) &= \sum_{k=1}^{n}\left(\frac{a^{\varsigma(2k+1)}}{(ab)^{\lfloor\frac{2k}{2}\rfloor}}\cdot\left(\frac{\sigma^{2k}(t)-\tau^{2k}(t)}{\sigma(t)-\tau(t)}\right)\right)^{2m} \\
&= \frac{a^{2m}}{(\sigma(t)-\tau(t))^{2m}}\sum_{k=1}^{n}\frac{\left(\sigma^{2k}(t)-\tau^{2k}(t)\right)^{2m}}{(ab)^{2mk}} \\
&= \frac{a^{2m}}{(\sigma(t)-\tau(t))^{2m}}\sum_{k=1}^{n}\sum_{j=0}^{2m}(-1)^j\binom{2m}{j}\frac{\sigma^{2k(2m-j)}(t)\,\tau^{2kj}(t)}{(ab)^{2mk}} \\
&= \frac{a^{2m}}{(\sigma(t)-\tau(t))^{2m}}\sum_{j=0}^{2m}(-1)^j\binom{2m}{j}\left(\frac{1-\frac{\sigma^{2n(2m-2j)}(t)}{(ab)^{(2m-2j)n}}}{\frac{(ab)^{2m-2j}}{\sigma^{2(2m-2j)}(t)}-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a^{2m}}{(\sigma(t)-\tau(t))^{2m}} \sum_{j=0}^{m} (-1)^j \binom{2m}{j} \left(\frac{1-\frac{\sigma^{2n(2m-2j)}(t)}{(ab)^{(2m-2j)n}}}{\frac{(ab)^{2m-2j}}{\sigma^{2(2m-2j)}(t)}-1} + \frac{1-\frac{\sigma^{2n(2j-2m)}(t)}{(ab)^{(2j-2m)n}}}{\frac{(ab)^{2j-2m}}{\sigma^{2(2j-2m)}(t)}-1} \right) \\
&\quad + \frac{a^{2m}}{(\sigma(t)-\tau(t))^{2m}} (-1)^{m+1} \binom{2m}{m} n \\
&= \frac{a^{2m}}{(\sigma(t)-\tau(t))^{2m}} \sum_{j=0}^{m} (-1)^j \binom{2m}{j} \left(\frac{\frac{\sigma^{2(2m-2j)}(t)}{(ab)^{2m-2j}} - \frac{\sigma^{(2n+2)(2m-2j)}(t)}{(ab)^{(n+1)(2m-2j)}} - 1 + \frac{\sigma^{2n(2j-2m)}(t)}{(ab)^{(2j-2m)n}}}{1-\frac{\sigma^{2(2m-2j)}(t)}{(ab)^{2m-2j}}} \right) \\
&\quad + \frac{a^{2m}}{(\sigma(t)-\tau(t))^{2m}} (-1)^{m+1} \binom{2m}{m} n \\
&= \frac{a^{2m}}{(\sigma(t)-\tau(t))^{2m}} \sum_{j=0}^{m} (-1)^j \binom{2m}{j} \left(\frac{\sigma^{2m-2j}(t) - \tau^{2m-2j}(t) - \frac{\sigma^{(2n+1)(2m-2j)}(t)}{(ab)^{n(2m-2j)}} + \frac{\tau^{(2n+1)(2m-2j)}(t)}{(ab)^{n(2m-2j)}}}{\tau^{2m-2j}(t) - \sigma^{2m-2j}(t)} \right) \\
&\quad + \frac{a^{2m}}{(\sigma(t)-\tau(t))^{2m}} (-1)^{m+1} \binom{2m}{m} n \\
&= \frac{a^{2m}}{(a^2b^2t^2+4ab)^m} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m}{m-j} \left(\frac{f_{2j(2n+1)}(t) - f_{2j}(t)}{f_{2j}(t)} \right) + \frac{a^{2m}}{(a^2b^2t^2+4ab)^m} (-1)^{m+1} \binom{2m}{m} n.
\end{aligned}
$$

□

Proof of Corollary 1. First, from the definition of $f_n(t)$ and binomial expansion, we easily prove $\left(f_{2n+1}(t)-1, a^2b^2t^2+4ab\right) = 1$. Therefore, $\left(f_{2n+1}(t)-1, \left(a^2b^2t^2+4ab\right)^m\right) = 1$. Now, we prove (1.11) by Lemma 1 and (1.3):

$$
\begin{aligned}
& bl_1(t)\, l_3(t) \cdots l_{2m+1}(t) \sum_{k=1}^{n} f_{2k}^{2m+1}(t) \\
&= l_1(t)\, l_3(t) \cdots l_{2m+1}(t) \left(\frac{a^{2m+1}}{(\sigma(t)-\tau(t))^{2m}} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \left(\frac{f_{(2n+1)(2j+1)}(t) - f_{2j+1}(t)}{l_{2j+1}(t)} \right) \right) \\
&\equiv 0 \pmod{f_{2n+1}(t)-1}.
\end{aligned}
$$

Now, we use Lemma 2 and (1.5) to prove (1.12):

$$
\begin{aligned}
& al_1(t)\, l_3(t) \cdots l_{2m+1}(t) \sum_{k=1}^{n} l_{2k}^{2m+1}(t) \\
&= l_1(t)\, l_3(t) \cdots l_{2m+1}(t) \left(\sum_{j=0}^{m} \binom{2m+1}{m-j} \left(\frac{al_{(2n+1)(2j+1)}(t) - al_{2j+1}(t)}{l_{2j+1}(t)} \right) \right) \\
&\equiv 0 \pmod{l_{2n+1}(t)-at}.
\end{aligned}
$$

□

3. Conclusions

In this paper, we discuss the power sums of bi-periodic Fibonacci and Lucas polynomials by Binet formulas. As corollaries of the theorems, we extend the divisible properties of the sum of power of linear Fibonacci and Lucas sequences to nonlinear Fibonacci and Lucas polynomials. An open problem is whether we extend the Melham conjecture to nonlinear Fibonacci and Lucas polynomials.

Use of AI tools declaration

The authors declare that they did not use Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. N. Yilmaz, A. Coskun, N. Taskara, On properties of bi-periodic Fibonacci and Lucas polynomials, *AIP. Conf. P.*, **1863** (2017), 310002. https://doi.org/10.1063/1.4992478
2. Y. Choo, On the reciprocal sums of products of two generalized bi-periodic Fibonacci numbers, *Mathematics*, **9** (2021), 178. https://doi.org/10.3390/math9020178
3. T. Du, Z. Wu, Some identities involving the bi-periodic Fibonacci and Lucas polynomials, *AIMS Math.*, **8** (2023), 5838–5846. https://doi.org/ 10.3934/math2023294
4. H. H. Leung, Some binomial-sum identities for the generalized bi-periodic Fibonacci sequences, *Notes Number Theory Discrete Math.*, **26** (2020), 199–208. https://doi.org/10.7546/nntdm.2020.26.1.199-208
5. T. Du, Z. Wu, On the reciprocal products of generalized Fibonacci sequences, *J. Inequal. Appl.*, **2022** (2022), 154. https://doi.org/10.1186/s13660-022-02889-8
6. Y. Choo, Relations between generalized bi-periodic Fibonacci and Lucas sequences, *Mathematics*, **8** (2020), 1527. https://doi.org/10.3390/math8091527
7. X. Li, Some identities involving chebyshev polynomials, *Math. Probl. Eng.*, **5** (2015), 950695. https://doi.org/10.1155/2015/950695
8. L. Chen, W. Zhang, Chebyshev polynomials and their some interesting applications, *Adv. Differ. Equ.*, **2017** (2017), 303. https://doi.org/10.1186/s13662-017-1365-1

9. T. Wang, H. Zhang, Some identities involving the derivative of the first kind cebyshev polynomials, *Math. Probl. Eng.*, **7** (2015), 146313. https://doi.org/10.1155/2015/146313
10. X. Wang, On the power sum problem of Lucas polynomials and its divisible property, *Open Math.*, **16** (2018), 698–703. https://doi.org/10.1515/math-2018-0063
11. R. S. Melham, Some conjectures concerning sums of odd powers of Fibonacci and Lucas numbers, *Fibonacci Quart.*, **46/47** (2008/2009), 312–315.
12. T. Wang, W. Zhang, Some identities involving Fibonacci, Lucas polynomials and their applications, *Bull. Math. Soc. Sci. Math. Roumanie*, **55** (2012), 95–103.