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Research article

Cyclic codes over non-chain ring $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ and their applications to quantum and DNA codes

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Abstract: Let $s \geq 1$ be a fixed integer. In this paper, we focus on generating cyclic codes over the ring $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$, where $\alpha_i \in \mathbb{F}_q \backslash \{0\}$, $1 \leq i \leq s$, by using the Gray map that is defined by the idempotents. Moreover, we describe the process to generate an idempotent by using the formula (2.1). As applications, we obtain both optimal and new quantum codes. Additionally, we solve the DNA reversibility problem by introducing $\mathbb{F}_q$ reversibility. The aim to introduce the $\mathbb{F}_q$ reversibility is to describe IUPAC nucleotide codes, and consequently, 5 IUPAC DNA bases are considered instead of 4 DNA bases (A, T, G, C).

Keywords: cyclic code; quantum code; Gray map; non-chain ring; DNA code; DNA reversibility
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1. Introduction

Throughout the manuscript, we denote by $\mathbb{F}_q$ a field of order q, where q is an odd prime power, i.e., $q = p^m$ and $m \geq 1$ is a fixed integer. Moreover, we consider non-zero elements α_i belonging to the field $\mathbb{F}_q$, where $1 \leq i \leq s$ and $s \geq 1$ is a fixed integer. We study over the ring

$$\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s) = \mathbb{F}_q[u_1, u_2, \ldots, u_s]/\langle u_i^2 - (\alpha_i)^2, u_iu_j - u_ju_i\rangle,$$

where $s \geq 1$ and $1 \leq i, j \leq s$. It can be readily verified that $\mathcal{R}$ is a non-chain semi-local ring of order q^{2^s}. Linear codes over finite rings have garnered significant attention since the seminal work of Hammons et al. [1]. The advantages of linearity, such as facilitating encoding and decoding algorithms [2, 3], along with the ease of determining parameters like minimum distance, have made linear codes a focal point of research. A linear code of length n over a finite ring (resp. field) R is an R-submodule (resp. subspace) of R^n. The representation $[n, k, d]$, where n is the code's length, k is its dimension (number of information bits), and d is the minimum distance. Coding theory aims to explore codes with large code rates and minimum distances. That being said, this raises the following queries: What is a code's maximum rate for the specified length and distance? How far can a code go at a given length and rate? Numerous theoretical constraints on $[n, k, d]$ have been established [3] in order to address these issues. These include the Singleton bound, Plotkin bound, Hamming bound, Griesmer bound, and others. An optimal code under a certain bound is defined as a code $[n, k, d]$ that achieves that bound. There are a few online databases that catalog the parameters of optimal and best-known codes. The database [4] is one of the popular platforms that contains the parameters of various types of linear codes over finite fields of size up to 9. In 2010, Zhu et al. [5] conducted a study on cyclic codes over a finite non-chain ring, specifically $\mathbb{F}_2 + u\mathbb{F}_2$, where $u^2 = u$. They demonstrated that these codes are principally generated.

Cyclic codes have demonstrated their significant utility in the field of quantum-error-correcting (QEC) codes, which differ notably from classical-error-correcting (CEC) codes. Shor [6] invented the first quantum code in 1995. These codes are employed to regulate the errors that occur when sending quantum data via a quantum channel. In 1998, Calderbank et al. [7] addressed the challenge of creating QEC codes by utilizing CEC codes over the finite field GF(4). They also introduced a method for constructing QEC codes based on CEC codes. In 2009, Qian et al. [8] investigated binary quantum codes derived from odd-length cyclic codes over a finite ring $\mathbb{F}_2 + u\mathbb{F}_2$ with $u^2 = 0$. In 2011, to obtain quantum codes over $\mathbb{F}_4$, Kai and Zhu [9] employed dual containing cyclic codes over a finite chain ring $\mathbb{F}_4 + u\mathbb{F}_4, u^2 = 0$. In 2015, Gao [10] developed innovative quantum codes over the field $\mathbb{F}_q + v\mathbb{F}_q + v^2\mathbb{F}_q + v^3\mathbb{F}_q$, where $q = p^m$, p is a prime with $3|(p-1)$, $v^4 = v$, and m is a positive integer. Subsequently, Ozen et al. [11] derived ternary quantum codes from cyclic codes over $\mathbb{F}_3 + u\mathbb{F}_3 + v\mathbb{F}_3 + uv\mathbb{F}_3$ with $u^2 = 1,\ v^2 = 1$, and $uv = vu$. In 2021, Ashraf et al. [12] discovered enhanced quantum and LCD codes over the ring $\mathbb{F}_{p^m} + v\mathbb{F}_{p^m}$, where $v^2 = 1$. Recently, researchers have focused on cyclic and constacyclic codes over finite non-chain rings to get good quantum codes over $\mathbb{F}_q$ [13–19]. In 2023, Ali et al. [13] obtained cyclic codes and new quantum codes over finite commutative ring. In [14], the structural properties of cyclic codes were explored over the ring $\frac{\mathbb{F}_{2^m}[v_1, v_2, \ldots, v_k]}{\langle v_i^2 - \alpha_i v_i, v_i v_j - v_j v_i \rangle}$, for $i, j = 1, 2, 3, \ldots, k$. Further, they obtained optimal linear codes and quantum codes over the same ring.

Adleman [20] initially showcased the successful utilization of the DNA structure in tackling a combinatorial issue. In this instance, he employed DNA molecules to effectively address a directed salesman problem with seven nodes. The methodology, he employed hinged on the inherent Watson Crick Complement property of DNA strands. Researchers have long explored the correlation between DNA and error-correcting codes. Liebovitch et al. [21] conducted studies in this realm, focusing on the quest for error-correcting codes within actual DNA sequences. Additionally, Brandao et al. [22] delved into this connection in their recent work. Common constraints employed in DNA codes include the Hamming distance constraint, the reverse-complement constraint, the reverse constraint, and the fixed GC-content constraint. These constraints are frequently referenced in works such as [23–26]. A DNA code is defined when the DNA correspondence of the code C satisfies the criteria of reversibility or a reversible complementarity. In such cases, the code C, or its DNA correspondence, is termed a DNA code. The purpose of obtaining reversible DNA codes is to find the optimal solution in Adleman's experiment so that the DNA sequences are as different as possible from their reversible and reversible complements. Since the aim of algebraic codes is to ensure that the codes are different from each other, studies on the relationship between these two structures have begun. The number of elements of algebraic structures used in studies on DNA codes in the literature is 4 and the power of 4. Because there are 4 bases in DNA: adenine (A), guanine (G), cytosine (C) and thymine (T). The first study in which reversible DNA codes with more than 4 elements were produced without deleting elements is due to Oztas and Siap [27]. In 2013, Oztas and Siap [27] introduced and studied the reversibility problem with more than four elements. Consequently, this issue arises in algebraic structures with more than four elements, where each element corresponds to DNA multiple bases. The reversibility problem shows us, that if we have reversible codes over the algebraic structures with more than four elements, then we cannot obtain reversible DNA codes only with a map. For instance, consider a ring R with $|R| = 16$, where each element corresponds to DNA double bases (or 2 bases), and let's consider $a, b, c \in R$ and let DNA 2-bases.

Let $\tau(a) \rightarrow TG$, $\tau(b) \rightarrow AC$, and $\tau(c) \rightarrow GA$ with a map τ. Let $(a, b, c) \in R^3$ be a vector, it corresponds to (TG, AC, GA) (or $(TGACGA)$). The reverse of (a, b, c) is given as $(a, b, c)^r = (c, b, a)$. Moreover, (c, b, a) corresponds to (GA, AC, TG) (or $(GAACTG)$). $(AG, CA, GT) \neq (GA, AC, TG)$, despite of $(a, b, c)^r = (c, b, a)$. Consequently, when we obtain the reverse of the vector (as $(a, b, c)^r = (c, b, a)$, we can not obtain the reverse of DNA correspondences (i.e., $(TGACGA)^r \neq (GAACTG)$). The reversibility problem is given by $(a, b, c)^r = (c, b, a)$, but $(\tau(a), \tau(b), \tau(c))^r \neq (\tau(c), \tau(b), \tau(a))$. That means, if we have a reverse of a codeword, we cannot obtain the exact DNA reverse of the codeword. Therefore, we must generate special designs and maps to obtain the reversible DNA codes. The reversible codes are not enough to generate reversible DNA codes. In this paper, $\mathbb{F}_q$ reversibility means that the code is reversible over $\mathbb{F}_q$. DNA reversibility means that the code is a reversible DNA code. The general version of this method is presented in [28]. Some studies that solve the DNA reversibility problem for rings with 4 elements and more than 4 elements were discussed in [29–38]. As of the current research, attempts have been made to obtain reversible DNA codes over 4 bases. However, when DNA sequencing is performed, the letter N is sometimes used in DNA strings. This letter N indicates that its location can contain any base. Therefore, by trying to generate DNA codes with the letters A, T, G, C and N, codes that can be directly matched with DNA sequences in the NCBI (The National Center for Biotechnology Information) system or other open sources can be produced. By generating reversible DNA codes, different DNA string structures can be created for the Adleman experiment. Finding the optimal distance for the diversity between strings within the generated DNA code is an open problem.

The main objectives of this article is to develop QEC codes over the ring $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ and investigate the structural characteristics of cyclic codes over this finite ring. When $q = 5$, we introduce a reversibility problem over the ring for correspondence over $\mathbb{F}_q$. The main contributions of this paper are the following:

(i) To investigate some optimal codes over $\mathcal{R}(1, 1)$, see Table 1;

(ii) To provide new quantum codes over $\mathcal{R}(1, 1, 1)$, see Table 2;

(iii) to introduce a method for solving the $\mathbb{F}_q$ reversibility problem to DNA codes;

(iv) To solve the DNA reversibility problem over the ring $\mathcal{R}_q(\alpha_1, \alpha_2, \ldots, \alpha_s)$, where $\alpha_i = \alpha_{s-i}$ with $0 \leq i \leq \lfloor s/2 \rfloor$ and s is an even integer which is a special case of the ring $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$;

(v) To solve the DNA reversibility problem, by using 5 IUPAC DNA bases (A, T, G, C, N) in [39].

Table 1. Gray images of cyclic codes over $\mathcal{R}(1,1)$.

n	$g_0(x)$	$g_1(x)$	$g_2(x)$	$g_3(x)$	$\varrho(\mathscr{C})$	*Remarks*
3	$(x+2)^2$	$x+2$	$x+2$	1	$[12,8,3]_3$	optimal
4	(x^2+1)	$x+1$	$x+1$	$x+1$	$[16,11,4]_3$	optimal
5	$(x^4+x^3+x^2+x+1)$	$x+2$	$x+2$	1	$[20,14,4]_3$	optimal
6	$(x+1)^2(x+2)$	$x+2$	$x+2$	1	$[24,19,3]_3$	optimal
7	$(x+2)(x^6+x^5+x^4$ $+x^3+x^2+x+1)$	$(x+2)(x^6+x^5+x^4$ $+x^3+x^2+x+1)$	$(x+2)(x^6+x^5+x^4$ $+x^3+x^2+x+1)$	$(x^6+x^5+x^4$	$[28,1,28]_3$	optimal
8	$(x^2+1)(x^2+2x+2)$	$x+2$	$x+2$	1	$[32,26,4]_3$	optimal
11	$(x^5+x^4+2x^3+x^2+2)$	$x+2$	$x+2$	$x+2$	$[44,36,4]_3$	optimal

Table 2. New quantum codes (NQC) from cyclic codes over $\mathcal{R}(1,1,1)$.

n	*Ring*	M	*Factors*	$\varrho(\mathscr{C})$	$[[n,k,d]]_q$	*Remark*
9	$\mathcal{R}(1,1,1)$	M_8	$g_0(x)=(x+2)^4$, $g_i=x+2,\ for$ $1\le i\le 6,\ g_7(x)=1$	$[72,62,3]$	$[[72,52,3]]_3$	*NQC*
13	$\mathcal{R}(1,1,1)$	M_8	$g_0(x)=g_3(x)=$ x^3+2x^2+2x+2, $g_1(x)=g_4(x)=x^3+2x+2$, $g_2(x)=g_5(x)=$ $g_6(x)=g_7(x)=1$	$[104,92,4]$	$[[104,80,4]]_3$	...
15	$\mathcal{R}(1,1,1)$	M_8	$g_0(x)=(x+2)^2$ $(x^4+x^3+x^2+x+1)$ $g_i=x+2,\ for\ 1\le i\le 6$ $g_7(x)=1$	$[120,108,3]$	$[[120,96,3]]_3$	...
18	$\mathcal{R}(1,1,1)$	M_8	$g_0(x)=(x+1)^4(x+2)$, $g_i=x+1,\ for\ 1\le i\le 6$, $g_7(x)=1$	$[144,133,3]$	$[[144,122,3]]_3$	*NQC*
22	$\mathcal{R}(1,1,1)$	M_8	$g_0(x)=(x^5+2x^4+2x^3+2x^2+1)$, $g_2(x)=(x^5+x^4+2x^3+x^2+2)$, $(x^5+x^4+2x^3+x^2+2)$, $g_4(x)=g_7(x)=$ $(x^5+2x^4+2x^3+2x^2+1)$, $g_1(x)=g_3(x)=$ $g_5(x)=g_6(x)=1$	$[176,151,4]$	$[[176,126,4]]_3$	*NQC*
22	$\mathcal{R}(1,1,1)$	M_8	$g_0(x)=g_1(x)=g_7(x)$ $=(x^5+2x^4+2x^3+2x^2+1)$ $(x^5+x^4+2x^3+x^2+2),\ ,$ $g_2(x)=g_3(x)=g_4(x)=$ $(x^5+x^4+2x^3+2x^2+2)$, $g_6(x)=(x^5+x^4+2x^3+x^2+2)$ $g_6(x)=1$	$[176,126,5]$	$[[176,76,5]]_3$	*NQC*
26	$\mathcal{R}(1,1,1)$	M_8	$g_0(x)=(x^3+2x^2+2x+2)$ (x^3+2x^2+x+1), $g_1(x)=(x^3+2x^2+2x+2)$, $g_2(x)=g_4(x)=(x^3+2x^2+x+1)$ $g_3(x)=g_4(x)=g_5(x)=g_6(x)=1$	$[208,193,4]$	$[[208,178,4]]_3$	*NQC*
45	$\mathcal{R}(1,1,1)$	M_8	$g_0(x)=(x+2)^4(x^4+x^3+x^2+x+1)$, $g_1(x)=(x^4+x^3+x^2+x+1)$, $g_4(x)=g_6(x)=(x+2)$, $g_2(x)=g_3(x)=g_5(x)=g_7(x)=1$	$[360,346,3]$	$[[360,332,2]]_3$	*NQC*
55	$\mathcal{R}(1,1,1)$	M_8	$g_0(x)=(x^{20}+x^{18}+2x^{17}$ $+2x^{16}+2x^{15}+x^{14}$ $+2x^{10}+x^9+2x^8+$ $2x^7+x^5+x^4+$ $2x^3+2x^2+2x+1)$, $g_2(x)=g_4(x)=g_8(x)$ $=(x^5+x^4+2x^3+x^2+2)$ $g_1(x)=g_3(x)=g_5(x)=g_7(x)=1$	$[440,405,4]$	$[[440,370,4]]_3$	*NQC*

The rest of the paper is organized as follows. In Section 2, we introduce a formula to obtain orthogonal idempotents over the defined ring. In Section 3, we give the main theorems that provide the relations between cyclic codes, linear codes, and their Gray images. Moreover, we discuss some examples that give new quantum codes as applications. In Section 4, we introduce the $\mathbb{F}_q$ reversibility problem and give the method to solve it. Finally as an application, we solve the DNA reversibility problem over the ring $\mathcal{R}_q(\alpha_1, \alpha_2, \ldots, \alpha_s)$.

2. Preliminaries

Let $\mathbb{F}_q$ be a finite field of order q, where q be a prime power such that $q = p^m$ for a positive integer m. The ring $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s) = \mathbb{F}_q[u_1, u_2, \ldots, u_s]/\langle u_i^2 - (\alpha_i)^2, u_iu_j - u_ju_i\rangle$ where $s \geq 1$, $1 \leq i, j \leq s$ and α_i is a non-zero element of the finite field $\mathbb{F}_q$. We begin our discussion with some basic definitions:

(i) The number of positions where two vectors differ, i.e., $\mathbf{x} = x_1 \ldots x_n$ and $\mathbf{y} = y_1 \ldots y_n$, is their Hamming distance and is represented by $d(\mathbf{x}, \mathbf{y})$.

(ii) $wt(\mathbf{x})$ represents the Hamming weight of a vector $\mathbf{x} = x_1x_2 \ldots x_n$, which is the number of non-zero x_i.

(iii) The Euclidean inner product of $\mathbf{x}$ and $\mathbf{y}$ is defined as $\mathbf{x} \cdot \mathbf{y} = x_0y_0 + x_1y_1 + \cdots + x_{n-1}y_{n-1}$, given that $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$.

(iv) When $\mathscr{C} = \mathscr{C}^\perp$, a code $\mathscr{C}$ is considered self-dual; when $\mathscr{C} \subseteq \mathscr{C}^\perp$, it is considered self-orthogonal; and when $\mathscr{C}^\perp \subseteq \mathscr{C}$, it is considered dual.

(v) A linear code $\mathscr{C}$ is said to be reversible if

$$\mathbf{c}^r = (c_{n-1}, c_{n-2}, \ldots, c_0) \in \mathscr{C}$$

whenever $\mathbf{c} = (c_0, c_1, c_2, \ldots, c_{n-1}) \in \mathscr{C}$.

(vi) [14] Let $\mathscr{C}$ be a linear code of length n over R. Then $\mathscr{C}$ is called a complement if for any

$$\mathbf{z} = (z_0, z_1, \ldots, z_{n-1}) \in \mathscr{C},\ \mathbf{z}^c = (\overline{z_0}, \overline{z_1}, \ldots, \overline{z_{n-1}}) \in \mathscr{C},$$

reversible-complement if for any $\mathbf{z} \in \mathscr{C}, \mathbf{z}^{rc} \in \mathscr{C}$.

(vii) [14] Let $\mathscr{C}$ be a linear code of length n over R. Then $\mathscr{C}$ (or the DNA correspondence of $\mathscr{C}$) is called a reversible (reversible complement) DNA code if the DNA correspondence of $\mathscr{C}$ satisfies the properties of being reversible (reversible compliment).

κ represents the collection of positions occupied by non-zero digits within a binary number, forming a mapping to an integer ([14]). $\kappa(e) = \kappa(e = (a_n...a_2a_1)_2) = \{j | a_j \neq 0\}$, where $e \in Z^+ \cup \{0\}$. $\kappa(23) = \kappa(23 = (10111)_2) = \{1, 2, 3, 5\}$ is an example for it.

Definition 2.1. The self orthogonal idempotent form over the ring $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ can be obtained by using the following formula:

$$I_j = \frac{\prod_{i=1}^{s} \begin{cases} \alpha_i - u_i, & \text{if } i \in \kappa(j) \\ \alpha_i + u_i, & \text{if } i \notin \kappa(j) \end{cases}}{2^s \prod_{i=1}^{s} \alpha_i} \tag{2.1}$$

where $0 \leq j \leq 2^s - 1$. These idempotents satisfy that $(I_j)^2 = I_j$, $\sum_{j=0}^{2^s-1} I_j = 1$ and $I_{j_1}I_{j_2} = 0$, where $0 \leq j_1, j_2 \leq 2^s - 1$.

According to the Chinese remainder theorem,

$$\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s) = \bigoplus_{j=0}^{2^s-1} I_j \mathbb{F}_q.$$

Thus, element of $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ can be expressed as $r = \sum_{j=0}^{2^s-1} I_j r_j$, where $r_j \in \mathbb{F}_q$ and $0 \leq j \leq 2^s - 1$.

Example 2.2. Let us consider the construction of the idempotent set over the ring $\mathcal{R}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathbb{F}_q[u_1, u_2, u_3, u_4]/\langle u_1^2 - \alpha_1^2, u_2^2 - \alpha_2^2, u_3^2 - \alpha_3^2, u_4^2 - \alpha_4^2, u_1u_2 - u_2u_1, u_1u_3 - u_3u_1, u_1u_4 - u_4u_1, u_2u_3 - u_3u_2, u_2u_4 - u_4u_2, u_3u_4 - u_4u_3\rangle$ over $\mathbb{F}_q$. As stipulated by the definition outlined above, the idempotents can be characterized as follows:

$$I_0 = \frac{(\alpha_1 + u_1)(\alpha_2 + u_2)(\alpha_3 + u_3)(\alpha_4 + u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_1 = \frac{(\alpha_1 - u_1)(\alpha_2 + u_2)(\alpha_3 + u_3)(\alpha_4 + u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_2 = \frac{(\alpha_1 + u_1)(\alpha_2 - u_2)(\alpha_3 + u_3)(\alpha_4 + u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_3 = \frac{(\alpha_1 - u_1)(\alpha_2 - u_2)(\alpha_3 + u_3)(\alpha_4 + u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_4 = \frac{(\alpha_1 + u_1)(\alpha_2 + u_2)(\alpha_3 - u_3)(\alpha_4 + u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_5 = \frac{(\alpha_1 - u_1)(\alpha_2 + u_2)(\alpha_3 - u_3)(\alpha_4 + u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_6 = \frac{(\alpha_1 + u_1)(\alpha_2 - u_2)(\alpha_3 - u_3)(\alpha_4 + u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_7 = \frac{(\alpha_1 - u_1)(\alpha_2 - u_2)(\alpha_3 - u_3)(\alpha_4 + u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_8 = \frac{(\alpha_1 + u_1)(\alpha_2 + u_2)(\alpha_3 + u_3)(\alpha_4 - u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_9 = \frac{(\alpha_1 - u_1)(\alpha_2 + u_2)(\alpha_3 + u_3)(\alpha_4 - u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_{10} = \frac{(\alpha_1 + u_1)(\alpha_2 - u_2)(\alpha_3 + u_3)(\alpha_4 - u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_{11} = \frac{(\alpha_1 - u_1)(\alpha_2 - u_2)(\alpha_3 + u_3)(\alpha_4 - u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_{12} = \frac{(\alpha_1 + u_1)(\alpha_2 + u_2)(\alpha_3 - u_3)(\alpha_4 - u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_{13} = \frac{(\alpha_1 - u_1)(\alpha_2 + u_2)(\alpha_3 - u_3)(\alpha_4 - u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_{14} = \frac{(\alpha_1 + u_1)(\alpha_2 - u_2)(\alpha_3 - u_3)(\alpha_4 - u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$I_{15} = \frac{(\alpha_1 - u_1)(\alpha_2 - u_2)(\alpha_3 - u_3)(\alpha_4 - u_4)}{2^4\alpha_1\alpha_2\alpha_3\alpha_4}.$$

The Gray map over the ring $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ is defined as follows:

$$\varrho : \mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s) \longrightarrow \mathbb{F}_q^{2^s},$$
$$\sum_{j=0}^{2^s-1} I_j r_j \longmapsto (r_0, r_1, \ldots, r_{2^s-1})M,$$

where $r_j \in \mathbb{F}_q$, $0 \leq j \leq 2^s - 1$, $M \in GL_{2^s}(\mathbb{F}_q)$ and $MM^T = dI_{(2^s)\times(2^s)}$ $(d \in \mathbb{F}_q - \{0\})$. The above Gray map can be extended from components of $\mathcal{R}^n(\alpha_1, \alpha_2, \ldots, \alpha_s)$ to $\mathbb{F}_q^{2^s}$. The Lee weight of $r \in \mathcal{R}^n(\alpha_1, \alpha_2, \ldots, \alpha_s)$ is defined as $w_L(r) = w_H(\varrho(r))$, where w_H gives the Hamming weight over $\mathbb{F}_q$. Our next result gives a characterization of the above mentioned Gray map.

Proposition 2.3. *The map ϱ is an $\mathbb{F}_q$-linear and distance preserving map from $(\mathcal{R}^n(\alpha_1, \alpha_2, \ldots, \alpha_s), d_L)$ to $(\mathbb{F}_q^{2^s n}, d_H)$, where $d_H = d_L$.*

Proof. Let $r, r' \in (\mathcal{R}^n(\alpha_1, \alpha_2, \ldots, \alpha_s), d_L)$ such that $r = \sum_{j=0}^{2^s-1} I_j r_j$ and $r' = \sum_{j=0}^{2^s-1} I_j r'_j$, where $r_j r'_j \in \mathbb{F}_q$ and $0 \leq j \leq 2^s - 1$. Then, we have

$$\begin{aligned}
\varrho(r + r') &= \varrho(\sum_{j=0}^{2^s-1} I_j r_j + \sum_{j=0}^{2^s-1} I_j r'_j) \\
&= \varrho(\sum_{j=0}^{2^s-1} I_j (r_j + r'_j)) \\
&= (r_0 + r'_0, r_1 + r'_1, \ldots, r_{2^s-1} + r'_{2^s-1})M \\
&= (r_0, r_1, \ldots, r_{2^s-1})M + (r'_0, r'_1, \ldots, r'_{2^s-1})M \\
&= \varrho(r) + \varrho(r') \text{ for all } r, r' \in \mathcal{R}^n(\alpha_1, \alpha_2, \ldots, \alpha_s)
\end{aligned}$$

and for all $e \in \mathbb{F}_q$

$$\begin{aligned}
\varrho(er) &= \varrho(\sum_{j=0}^{2^s-1} eI_j r_j) \\
&= (er_0, er_1, \ldots, er_{2^s-1})M \\
&= e(r_0, r_1, \ldots, r_{2^s-1})M \\
&= e\varrho(r) \text{ for all } r \in \mathcal{R}^n(\alpha_1, \alpha_2, \ldots, \alpha_s).
\end{aligned}$$

Also, we have

$$\begin{aligned}
d_L(r, r') &= w_L(r - r') \\
&= w_H(\varrho(r - r')) \\
&= w_H(\varrho(r) - \varrho(r')) \\
&= d_H(\varrho(r), \varrho(r')).
\end{aligned}$$

Hence, ϱ is an $\mathbb{F}_q$-linear map and it satisfies the condition for being distance-preserving. □

The operations $\oplus$ and $\otimes$ are used as $\Gamma_1 \otimes \Gamma_2 \otimes \cdots \otimes \Gamma_s = \{(\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_s) \mid \gamma_i \in \Gamma_i :\ 1 \le i \le s\}$ and $\Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_s = \{(\gamma_1 + \gamma_2 + \cdots + \gamma_s) \mid \gamma_i \in \Gamma_i :\ 1 \le i \le s\}$. The following linear codes of length n are defined by using code $\mathscr{C}$ of length n over $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$:

$$\mathscr{C}_i = \{r_i | \sum_{j=0}^{2^s-1} I_j r_j,\ \text{where } r_i \in \mathbb{F}_q, 0 \le i \le 2^s - 1\},$$

where $0 \le i \le 2^s - 1$. Therefore, we can represent any linear code of length n as $\mathscr{C} = \bigoplus_{j=0}^{2^s-1} I_j \mathscr{C}_j$ and $|\mathscr{C}| = \prod_{j=0}^{2^s-1} |\mathscr{C}_j|$ over $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$. If G_j is a generator matrix of $\mathscr{C}_j$ where $0 \le j \le 2^s - 1$, then the generator matrix G of $\mathscr{C}$ is given by

$$G = \begin{pmatrix} I_0 G_0 \\ I_1 G_1 \\ \vdots \\ I_{2^s-1} G_{2^s-1} \end{pmatrix}$$

and the generator matrix of $\varrho(\mathscr{C})$ is given by

$$\varrho(G) = \begin{pmatrix} \varrho(I_0 G_0) \\ \varrho(I_1 G_1) \\ \vdots \\ \varrho(I_{2^s-1} G_{2^s-1}) \end{pmatrix}.$$

Proposition 2.4. *Let* $\mathscr{C} = \bigoplus_{j=0}^{2^s-1} I_j \mathscr{C}_j$ *be a linear code of length n over* $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$. *Then,* $\varrho(\mathscr{C})$ *is a* $[2^s n,\ \sum_{j=0}^{2^s-1} k_j,\ d]$*-linear code over* $\mathbb{F}_q$ *for* $0 \le j \le 2^s - 1$*, where each* $\mathscr{C}_j$ *is an* $[n,\ k_j,\ d]$*-code.*

Proof. The proof follows easily with the property of the Gray map. □

Proposition 2.5. *If* $\mathscr{C}$ *is a linear code of length n over* $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$*, then*

$$\varrho(\mathscr{C}) = \bigotimes_{j=0}^{2^s-1} \mathscr{C}_j.$$

Proof. The result follows by [40, Theorem 4.1]. □

Theorem 2.6. *Let* $\mathscr{C}$ *be a self-orthogonal linear code of length n over* $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ *and M be a* $2^s \times 2^s$ *non-singular matrix over* $\mathbb{F}_q$ *which has the property* $MM^T = \epsilon I_{2^s}$*, where* I_{2^s} *is the identity matrix,* $0 \neq \epsilon \in \mathbb{F}_q$*, and* M^T *is the transpose of matrix M. Then, the Gray image* $\varrho(\mathscr{C})$ *is a self-orthogonal linear code of length* $2^s n$ *over* $\mathbb{F}_q$*.*

Proof. Suppose that $\mathscr{C}$ is a self-orthogonal linear code of length n over $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$, i.e., $\mathscr{C} \subseteq \mathscr{C}^\perp$. Now, let $Q, S \in \varrho(\mathscr{C})$ such that

$$Q = \varrho(q) = (q_0 M, q_1 M, \ldots, q_{n-1} M)$$

and

$$S = \varrho(s) = (s_0M, s_1M, \ldots, s_{n-1}M).$$

We have to show that $\varrho(\mathscr{C})$ is self-orthogonal, that is, $Q \cdot S = 0$. Since $\mathscr{C}$ is self-orthogonal, $q \cdot s = \sum_{j=0}^{n-1} q_j.s_j = 0$. Therefore,

$$Q \cdot S = QS^{\perp} = \sum_{j=0}^{n-1} q_j MM^T s_j^{\perp} = \epsilon \sum_{j=0}^{n-1} q_j.s_j = 0.$$

Suppose that Q and S are arbitrary, then $\varrho(\mathscr{C}) \subseteq \varrho(\mathscr{C}^{\perp})$. Thus, $\varrho(\mathscr{C})$ is a self-orthogonal linear code of length $2^s n$ over $\mathbb{F}_q$. □

3. Cyclic codes and quantum codes over $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$

On the ring $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$, as previously outlined, we investigate several structural characteristics of cyclic codes and substantiate certain findings. We use a cyclic operator defined as $\eta((c_0, c_1, c_2, \ldots, c_{n-1})) = (c_{n-1}, c_0, c_1, c_2, \ldots, c_{n-2})$ for codewords.

Theorem 3.1. *Let $\mathscr{C} = \bigoplus_{j=0}^{2^s-1} I_j\mathscr{C}_j$ be a linear code of length n over $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$. Then each $\mathscr{C}_j$ is a cyclic code over $\mathbb{F}_q$, where $0 \le i \le 2^s - 1$ if and only if $\mathscr{C}$ is a cyclic code over $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$.*

Proof. Let $c \in \mathscr{C}$ be a codeword of length n. Also, let $c = \sum_{j=0}^{2^s-1} I_j t_j$ where $t_j \in \mathbb{F}_q$. If $\varrho(t_j) \in \mathscr{C}_j$, then $\varrho(c) = \sum_{j=0}^{2^s-1} I_j \varrho(t_j)$ and $\varrho(c) \in \varrho(\mathscr{C})$. The reverse direction of the proof is also clear. □

Theorem 3.2. *Let $\mathscr{C} = \bigoplus_{j=0}^{2^s-1} I_j\mathscr{C}_j$ be a cyclic code of length n over $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ and $g_j(x)$ be the generator polynomial of $\mathscr{C}_j$. Then, $\mathscr{C} = \langle g(x) \rangle$ and $|\mathscr{C}| = q^{(2^s)n - \sum_{i=0}^{2^s-1} \deg(g_i(x))}$, where $g(x) = \sum_{j=0}^{2^s-1} I_j g_j(x)$.*

Proof. Given $\mathscr{C}_j = \langle g_j(x) \rangle$, where $0 \le i \le 2^s - 1$ and $\mathscr{C} = \bigoplus_{j=0}^{2^s-1} I_j\mathscr{C}_j$. Let $c \in \mathscr{C}$ be such that $c = \{c(x) \mid \sum_{j=0}^{2^s-1} I_j g_j(x) \text{ for } g_j(x) \in \mathscr{C}_j\}$. Therefore,

$$\mathscr{C} \subseteq \langle I_0 g_0(x), I_1 g_1(x),\ I_2 g_2(x),\ \ldots, I_{2^s-1} g_{2^s-1}(x) \rangle \subseteq \mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)[x]/\langle x^n - 1 \rangle.$$

For any

$$\sum_{j=0}^{2^s-1} I_j f_j(x) g_j(x) \in \langle \sum_{j=0}^{2^s-1} I_j g_j(x) \rangle \subseteq \mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)[x]/\langle x^n - 1 \rangle,$$

where

$$f_j(x) \in \mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)[x]/\langle x^n - 1 \rangle.$$

Hence,

$$\langle I_0 g_0(x), I_1 g_1(x), I_2 g_2(x), \ldots,\ I_{2^s-1} g_{2^s-1}(x) \rangle \subseteq \mathscr{C}.$$

This implies

$$\langle I_0 g_0(x), I_1 g_1(x), I_2 g_2(x), \ldots,\ I_{2^s-1} g_{2^s-1}(x) \rangle = \mathscr{C}.$$

Since $|\mathscr{C}| = \prod_{j=0}^{2^s-1} |\mathscr{C}_j|$, we have

$$|\mathscr{C}| = q^{(2^s)n - \sum_{i=0}^{2^s-1} \deg(g_i(x))}.$$

□

Theorem 3.3. *Let* $\mathscr{C} = \bigoplus_{j=0}^{2^s-1} I_j\mathscr{C}_j$ *be a cyclic code of length n over* $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$*. Then, there exists a unique monic polynomial* $g(x) \in \mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)[x]$ *such that* $\mathscr{C} = \langle g(x)\rangle$ *and* $g(x)$ *divides* $(x^n - 1)$*. Moreover, if* $g_j(x)$ *is the generator polynomial of* $\mathscr{C}_j$*, then* $g(x) = \sum_{j=0}^{2^s-1} I_j g_j(x)$*.*

Proof. The proof follows by the reverse direction of the proof of Theorem 3.2. □

Theorem 3.4. *Let* $\mathscr{C} = \bigoplus_{j=0}^{2^s-1} I_j\mathscr{C}_j$ *be a cyclic code of length n over* $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$*. Then,* $\mathscr{C}^\perp = \bigoplus_{j=0}^{2^s-1} I_j\mathscr{C}_j^\perp$ *is also a cyclic code of length n over* $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$*.*

Proof. Let $\mathscr{C} = \bigoplus_{j=0}^{2^s-1} I_j\mathscr{C}_j$ be a cyclic code. By Theorem 3.1, $\mathscr{C}_j$ is a cyclic code. Thus, $\mathscr{C}_j^\perp$ is a cyclic code. Hence, by Theorem 3.1, $\mathscr{C}^\perp = \bigoplus_{j=0}^{2^s-1} I_j\mathscr{C}_j^\perp$ is a cyclic code. □

Lemma 3.5 ([7])**.** *Let* $\mathscr{C} = \langle g(x)\rangle$ *be a cyclic code of length n over* $\mathbb{F}_q$*. Then* $\mathscr{C}^\perp \subseteq \mathscr{C}$ *if and only if* $x^n - 1 \equiv 0 \ (mod\ g(x)g^*(x))$*, where the reciprocal polynomial of* $g(x)$ *is denoted by* $g^*(x)$*.*

Theorem 3.6. *Let* $\mathscr{C} = \bigoplus_{j=0}^{2^s-1} I_j\mathscr{C}_j$ *be a cyclic code of length n over* $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ *and* $\mathscr{C} = \langle g(x)\rangle = \langle \sum_{j=0}^{2^s-1} I_j g_j(x)\rangle$*, where* $\mathscr{C}_j = \langle g_j(x)\rangle$*. Then,* $\mathscr{C}^\perp \subseteq \mathscr{C}$ *if and only if*

$$x^n - 1 \equiv 0 \ (mod\ g_j(x)g_j^*(x)).$$

Proof. In view of Lemma 3.5 and Theorem 3.1, we conclude the result. □

Now, we give a definition and a lemma about QEC codes. We begin with the following definition:

Definition 3.7 ([19])**.** A quantum code represented by $[[n, k, d]]_q$ is defined as a subspace of $H(\mathbb{C})^n = \underbrace{H \otimes H \otimes \ldots \otimes H}_{n-times}$ (is also a q^n-dimensional Hilbert space) with dimension q^k and minimum distance d. Moreover, we consider $[[n, k, d]]_q$ to be better than $[[n', k', d']]_q$ if either or both of the following conditions hold:

(i) $d > d'$ whenever the code rate $\frac{k}{n} = \frac{k'}{n'}$ (larger distance);

(ii) $\frac{k}{n} > \frac{k'}{n'}$ whenever the distance $d = d'$ (larger code rate).

Lemma 3.8 ([41])**.** *(Theorem 3) (CSS construction) Let* $\mathscr{C}_1 = [n, k_1, d_1]_q$ *and* $\mathscr{C}_2 = [n, k_2, d_2]_q$ *be two linear codes over GF(q) with* $\mathscr{C}_2^\perp \subseteq \mathscr{C}_1$*. Furthermore, let*

$$d = min\{wgt(v) : v \in (\mathscr{C}_1 \backslash \mathscr{C}_2^\perp) \cup (\mathscr{C}_2 \backslash \mathscr{C}_1^\perp)\} \geq min(d_1, d_2).$$

Then, there exists a QEC code with the parameters $[[n, k_1 + k_2 - n, d]]_q$*. In particular, if* $\mathscr{C}_1^\perp \subseteq \mathscr{C}_1$*, then there exists a QEC code with the parameters* $[[n, 2k_1 - n, d_1]]_q$*, where* $d_1 = min\{wgt(v) : v \in (\mathscr{C}_1 \backslash \mathscr{C}_1^\perp)\}$*.*

Theorem 3.9. *Let* $\mathscr{C} = \bigoplus_{j=0}^{2^s-1} I_j\mathscr{C}_j$ *be a cyclic code of length n over* $\mathcal{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$*, and next let the parameters of its Gray image be* $[2^s n, \sum_{j=0}^{2^s-1} k_j, d_H]$*. If* $\mathscr{C}^\perp \subseteq \mathscr{C}$*, then there exists a QEC code* $[[2^s n, 2\sum_{j=0}^{2^s-1} k_j - 2^s n, d_H]]_q$ *over* $\mathbb{F}_q$*.*

In the following examples, we obtain optimal as well as new quantum codes.

Example 3.10. Let $\mathscr{R}(1,1) = \mathbb{F}_3[u_1,u_2]/\langle u_1^2-(1)^2, u_2^2-(1)^2, u_1u_2-u_2u_1\rangle$ be a finite commutative ring. For $n=4$, we have that $x^4-1=(x+1)(x+2)(x^2+1)\in\mathbb{F}_3[x]$. Let $g_0(x)=(x^2+1)$, $g_1(x)=(x+1)$, $g_2(x)=(x+1)$, $g_3(x)=(x+1)$ and

$$M_4 = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix},$$

where $M_4M_4^T = 2I_4$. Then, cyclic code $\mathscr{C}$ is of length 4 over $\mathscr{R}(1,1)$ and its Gray image is $[16,11,4]_3$ which is an optimal code as per the database made available by [4].

Example 3.11. Let $\mathscr{R}(1,1,1) = \mathbb{F}_3[u_1,u_2,u_3]/\langle u_1^2-(1)^2, u_2^2-(1)^2, u_3^2-(1)^2, u_1u_2-u_2u_1, u_1u_3-u_3u_1, u_3u_2-u_2u_3\rangle$ be a finite commutative ring. Then, for $n=26$, we have $x^{26}-1=(x+1)(x+2)(x^3+2x+1)(x^3+2x+2)(x^3+x^2+2)(x^3+x^2+x+2)(x^3+x^2+2x+1)(x^3+2x^2+1)(x^3+2x^2+x+1)(x^3+2x^2+2x+2)\in\mathbb{F}_3[x]$. Let $g_0(x)=(x^3+2x^2+2x+2)(x^3+2x^2+x+1)$, $g_1(x)=(x^3+2x^2+2x+2)$, $g_2(x)=(x^3+2x^2+x+1)$, $g_3(x)=1$, $g_4(x)=(x^3+2x^2+x+1)$, $g_5(x)=1$, $g_6(x)=1$, $g_7(x)=1$ and

$$M_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix},$$

where $M_5M_5^T = 2I_8$. Then, cyclic code $\mathscr{C}$ is of length 26 over $\mathscr{R}(1,1,1)$ and its Gray image is $[208,193,4]_3$. Moreover, $x^{26}-1\equiv 0(mod g_i(x)g^*(x))$ for $0\le i\le 7$. Hence, we obtain $[[208,178,4]]_3$ quantum code, that is, a new quantum code according to the database [42].

In Table 1, we obtain optimal codes over $\mathscr{R}(1,1)$ and in Table 2, we obtain new quantum codes (NQC) over $\mathscr{R}(1,1,1)$. The Magma computation system [43] is used to complete all of the computations in the above examples and tables.

4. DNA codes

In the present section, we give a general reversibility problem for $\mathbb{F}_q$ reversibility by using the ring $\mathscr{R}_q(\alpha_1,\alpha_2,\ldots,\alpha_s)$, where $\alpha_i=\alpha_{s-i}$ and $0\le i\le\lfloor s/2\rfloor$ and s is even positive integer. Moreover, a special case of $\mathbb{F}_q$ reversibility corresponds to DNA reversibility over $\mathbb{F}_5$.

If the reverse writing of each codeword of a code (or DNA code) is included in the code, then this code is called a reversible code (or reversible DNA code). The reverse of a codeword c is denoted by c^r. Moreover, the $\mathbb{F}_q$ reversibility problem is similar to the DNA reversibility problem for the rings that its decomposition has more than one field as $\mathscr{R}_q(\alpha_1,\alpha_2,\ldots,\alpha_s) = \bigoplus_{j=0}^{2^s-1} I_j\mathbb{F}_q$.

Let $(a_0,a_1,a_2) \in \mathcal{R}_q(\alpha_1,\alpha_2)$ and (a_0,a_1,a_2) correspond to $(\underbrace{a_{0,0},a_{0,1},a_{0,2},a_{0,3}}_{a_0},\underbrace{a_{1,0},a_{1,1},a_{1,2},a_{1,3}}_{a_1},$ $\underbrace{a_{2,0},a_{2,1},a_{2,2},a_{2,3}}_{\alpha_2}) = t_1$ over $\mathbb{F}_5$. Also, $(a_0,a_1,a_2)^r = (a_2,a_1,a_0)$ corresponds to $(\underbrace{a_{2,0},a_{2,1},a_{2,2},a_{2,3}}_{a_2},$ $\underbrace{a_{1,0},a_{1,1},a_{1,2},a_{1,3}}_{a_1},\underbrace{a_{0,0},a_{0,1},a_{0,2},a_{0,3}}_{a_0}) = t_2$. But, the reverse of t_1 is not equal to t_2, then $t_1^r \neq t_2$. It is called as $\mathbb{F}_q$ reversibility problem.

We give a map as follow to define a correspondence between DNA and $\mathbb{F}_5$, that is,

$$\begin{aligned} \theta : \mathbb{F}_5 &\longrightarrow \{A,T,G,C,N\}, \\ A &\longrightarrow 1, \\ G &\longrightarrow 2, \\ T &\longrightarrow 3, \\ C &\longrightarrow 4, \\ N &\longrightarrow 0. \end{aligned}$$

Each element $\alpha \in \mathcal{R}_q(\alpha_1,\alpha_2,\dots,\alpha_s)$ is expressed by $\alpha = a_0I_0 + a_1I_1 + \dots + a_{2^s-1}I_{2^s-1}$, where $a_0,a_1,\dots,a_{2^s-1} \in \mathbb{F}_q$. By using the structure $\mathcal{R}_q(\alpha_1,\alpha_2,\dots,\alpha_s) = I_0\mathbb{F}_q \oplus I_1\mathbb{F}_q \oplus \dots \oplus I_{2^s-1}\mathbb{F}_q$ for any element of $\mathcal{R}_q(\alpha_1,\alpha_2,\dots,\alpha_s)$, we use the Gray map as follows:

$$\begin{aligned} \phi : \mathcal{R}_q(\alpha_1,\alpha_2,\dots,\alpha_s) &\longrightarrow \mathbb{F}_q^{2^s}, \\ \alpha &\longrightarrow (a_0,a_1,\dots,a_{2^s-1}). \end{aligned}$$

The map can be extended to n-tuples coordinate-wise. We define an automorphism as follows to satisfy the conditions of reversibility for DNA codes:

$$\begin{aligned} \varphi : \mathcal{R}_q(\alpha_1,\alpha_2,\dots,\alpha_s) &\longrightarrow \mathcal{R}_q(\alpha_1,\alpha_2,\dots,\alpha_s), \\ u_i &\longrightarrow -u_i \ \forall\ i \ \in \{0,\dots,s-1\}. \end{aligned}$$

By using the automorphism φ, we can write the following lemma.

Lemma 4.1. *$\varphi(I_i) = I_{2^s-1-i}$ $\forall$, where I_i are idempotents of $\mathcal{R}_q(\alpha_1,\alpha_2,\dots,\alpha_s)$ and ($i \in \{0,1,\dots,2^s-1\}$).*

The following theorem gives the rule for obtaining the reverse for the maps of the elements of $\mathcal{R}_q(\alpha_1,\alpha_2,\dots,\alpha_s)$.

Theorem 4.2. *Let $\alpha \in \mathcal{R}_q(\alpha_1,\alpha_2,\dots,\alpha_s)$, where $\alpha_i = \alpha_{s-i}$ $(0 \le i \le \lfloor s/2 \rfloor)$ and $\alpha = a_0I_0 + a_1I_1 + \dots + a_{2^s-1}I_{2^s-1}$. $\phi(\alpha) = (a_0,a_1,\dots,a_{2^s-1})$. $(\phi(\alpha))^r = \varphi(\phi(\alpha))$. Then, $\varphi(\phi(\alpha))$ is a reverse order of $\phi(\alpha)$ over $\mathbb{F}_q$.*

Proof. Proof of the Theorem 4.2 follows directly from the Definition 2.1, and Lemma 4.1. □

Corollary 4.3. *Let $\alpha \in \mathcal{R}_5(\alpha_1,\alpha_2,\dots,\alpha_s)$, where $\alpha_i = \alpha_{s-i}$ $(0 \le i \le \lfloor s/2 \rfloor)$. Then, $\theta(\phi(\varphi(\alpha)))$ is the DNA reverse of $\theta(\phi(\alpha))$. Moreover, each element of $\mathcal{R}_5(\alpha_1,\alpha_2,\dots,\alpha_s)$ corresponds to 2^s-DNA bases (or called DNA 2^s bases).*

The maps φ, ϕ and θ can be extended for vectors over $\mathcal{R}_q(\alpha_1, \alpha_2, \ldots, \alpha_s)$. Let us consider $\mathcal{R}_5(\alpha_1, \alpha_2, \ldots, \alpha_s) = \mathbb{F}_5[u_1, u_2, \ldots, u_s]/\langle u_i^2 - (\alpha_i)^2, u_i u_j - u_j u_i\rangle$, where $\alpha_i = \alpha_{s-i}$ $(0 \leq i \leq \lfloor s/2 \rfloor)$. Each element correspond to a 2^s-DNA base (that has a length of 2^s) in the ring $\mathcal{R}_5(\alpha_1, \alpha_2, \ldots, \alpha_s)$. We apply Theorem 4.2 for DNA in the following example.

Example 4.4. Let us consider the ring $\mathcal{R}_5(1, 1, 1, 1)$. Then, we have

$$\alpha = 2I_0 + 3I_1 + 4I_2 + I_3 + 2I_4 + 2I_5 + 4I_6 + 3I_7 + 2I_8 + I_9 + 0I_{10} + 2I_{11} + I_{12} + 3I_{13} + 0I_{14} + 3I_{15} \in \mathcal{R}_5(1, 1, 1, 1);$$

$$\phi(\alpha) = (2, 3, 4, 1, 2, 2, 4, 3, 2, 1, 0, 2, 1, 3, 0, 3);$$

$$\theta(\phi(\alpha)) = (GTCAGGCTGANGATNT);$$

$$\varphi(\alpha) = 2I_{15} + 3I_{14} + 4I_{13} + I_{12} + 2I_{11} + 2I_{10} + 4I_9 + 3I_8 + 2I_7 + I_6 + 0I_5 + 2I_4 + I_3 + 3I_2 + 0I_1 + 3I_0;$$

$$\phi(\varphi(\alpha)) = (3, 0, 3, 1, 2, 0, 1, 2, 3, 4, 2, 2, 1, 4, 3, 2).$$

Thus, $\phi(\varphi(\alpha))$ is a reverse of $\phi(\alpha)$ over $\mathbb{F}_5$. This gives

$$\theta(\phi(\varphi(\alpha))) = (TNTAGNAGTCGGACTG).$$

Hence, $\theta(\phi(\varphi(\alpha)))$ is the DNA reverse of $\theta(\phi(\alpha))$ and these are DNA 16 bases.

We introduce a method to generate $\mathbb{F}_q$ reversible codes and discuss one of the applications to generating the reversible DNA code by using $\mathcal{R}_5(\alpha_1, \alpha_2, \ldots, \alpha_s)$. Following definition a version of coterm polynomial [35]. The following definition is used for generating $\mathbb{F}_q$ reversible and DNA reversible codes over $\mathcal{R}_q(\alpha_1, \alpha_2, \ldots, \alpha_s)$ and $\mathcal{R}_5(\alpha_1, \alpha_2, \ldots, \alpha_s)$, respectively.

Definition 4.5. Let $g(x) = \beta_0 + \beta_1 x + \cdots + \beta_{n-1}x^{n-1} \in \mathcal{R}_5(\alpha_1, \alpha_2, \ldots, \alpha_s)[x]/(x^n - 1)$ be a polynomial, where $\alpha_i = \alpha_{s-i}$ $(0 \leq i \leq \lfloor s/2 \rfloor)$. If for all $1 \leq i \leq \lfloor n/2 \rfloor$ such that $\beta_i = \beta_{n-i}$, then $g(x)$ is called a coterm polynomial over $\mathcal{R}_q(\alpha_1, \alpha_2, \ldots, \alpha_s)$. Moreover,

$$\begin{aligned} \Theta : \mathcal{R}_q(\alpha_1, \alpha_2, \ldots, \alpha_s)[x] &\longrightarrow \mathcal{R}_q^n(\alpha_1, \alpha_2, \ldots, \alpha_s), \\ g(x) &\longrightarrow (\beta_0, \beta_1, \ldots, \beta_{n-1}). \end{aligned}$$

$\Theta(g(x)) = (\beta_0, \beta_1, \ldots, \beta_{n-1})$ is called coterm tuple.

Theorem 4.6. *Let c be a coterm tuple of coterm polynomial $g(x) = \beta_0 + \beta_1 x + \cdots + \beta_{n-1}x^{n-1}$ over $\mathcal{R}_q(\alpha_1, \alpha_2, \ldots, \alpha_s)[x]/(x^n - 1)$, where $\alpha_i = \alpha_{s-i}$ $(0 \leq i \leq \lfloor s/2 \rfloor)$. Then, the coterm φ-generation set*

$$T_g^t = \{\varphi(c^{-t-1}), \cdots, \varphi(c^{-2}), \varphi(c^{-1}), c^0, c^1, \ldots, c^t\},$$

where $t < \lfloor (n-1)/2 \rfloor$. And, the coterm φ-generation matrix:

$$GT_g^t = \begin{bmatrix} \varphi(c^{-t-1}) \\ \vdots \\ \varphi(c^{-1}) \\ c^0 \\ c^1 \\ \vdots \\ c^t \end{bmatrix} = \begin{bmatrix} \varphi(\beta_{t+1}) & \varphi(\beta_{t+2}) & \cdots & \cdots & \varphi(\beta_t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi(\beta_1) & \cdots & \cdots & \varphi(\beta_{n-1}) & \varphi(\beta_0) \\ \beta_0 & \beta_1 & \cdots & \cdots & \beta_{n-1} \\ \beta_{n-1} & \beta_0 & \cdots & \cdots & \beta_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{n-t} & \beta_{n-t-1} & \cdots & \cdots & \beta_{n-(t+1)} \end{bmatrix}.$$

Thus, $\mathscr{C} = \langle T_g^t \rangle$ ($\mathscr{C} = \langle GT_g^t \rangle$) generate codes over $\mathscr{R}_q(\alpha_1, \alpha_2, \ldots, \alpha_s)$ and $\phi(\mathscr{C})$ correspond the $\mathbb{F}_q$ reversible codes. Next, let code $\mathscr{C}$ be over $\mathscr{R}_5(\alpha_1, \alpha_2, \ldots, \alpha_s)$. Then, $\theta(\phi(\mathscr{C}))$ correspond reversible DNA codes over $\mathbb{F}_5$.

Proof. Let $g(x)$ be a coterm polynomial and c be a coterm tuple of $g(x)$. In the coterm φ-generation set (or matrix), each row has its reverse and each codewords has its reverse as follows $(\phi(\alpha c)^i)^r = (\varphi(\phi(\alpha c^{-i-1}))$, where $0 \le i \le t$. Thus, $\phi(\mathscr{C})$ corresponds to the $\mathbb{F}_q$ reversible codes because of the spanning set conditions of ([35], Lemma 3.2.) Moreover, $\theta(\phi(\mathscr{C}))$ correspond reversible DNA codes over $\mathbb{F}_5$ when $\mathscr{C}$ over $\mathscr{R}_5(\alpha_1, \alpha_2, \ldots, \alpha_s)$. □

Theorem 4.7. *If all letter N is deleted and $\theta(\phi(\mathscr{C}))$ is a reversible DNA set with different length DNA strings, then DNA reversibility is protected.*

Proof. Because of the structure, the coterm φ generation matrix (set), zeros (letter N) don't effect the DNA reversibility. Hence, the proof follows. □

Example 4.8. Let $g(x) = g(x) = \beta_0+\beta_1 x+\beta_2 x^2+\beta_3 x^3+\beta_4 x^4+\beta_5 x^5+\beta_6 x^6$ over $\mathscr{R}_5(1,1) = \mathbb{F}_5[u_1, u_2]/\langle u_1^2-1, u_2^2-1, u_1u_2-u_2u_1\rangle$ where $\beta_0 = I_0+I_1+I_2+I_3, \beta_1 = \beta_6 = I_0+2I_1+3I_2+I_3, \beta_2 = \beta_5 = 2I_0+4I_1+I_2+3I_3$, $\beta_3 = \beta_4 = I_0 + 2I_1 + 4I_2 + 3I_3$, and

$$
\begin{aligned}
I_0 &= \frac{(1+u_1)(1+u_2)}{2^2},\\
I_1 &= \frac{(1-u_1)(1+u_2)}{2^2},\\
I_2 &= \frac{(1+u_1)(1-u_2)}{2^2},\\
I_3 &= \frac{(1-u_1)(1-u_2)}{2^2}.
\end{aligned}
$$

Let t=0 be chosen. Then,

$$
GT_g^0 = \begin{bmatrix} \varphi(c^{-1}) \\ c^0 \end{bmatrix} = \begin{bmatrix} \varphi(\beta_1) & \varphi(\beta_2) & \varphi(\beta_3) & \varphi(\beta_4) & \varphi(\beta_5) & \varphi(\beta_6) & \varphi(\beta_0) \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \end{bmatrix}.
$$

$[7, 2, 3]_{\mathscr{R}_5(1,1)}$ code is obtained and $\theta(\phi(\mathscr{C}))$ correspond reversible DNA codes over $\mathbb{F}_5$, where $\mathscr{C} =< GT_g^0 >$.

5. Conclusions

In the present paper, we studied the structural properties of cyclic codes over the ring $\mathscr{R}(\alpha_1, \alpha_2, \ldots, \alpha_s)$, where $\alpha_i \in \mathbb{F}_q\backslash\{0\}$, $1 \le i \le s$. Also, we introduced a method to generate idempotent for the above mentioned ring. Furthermore, we generated new quantum codes according to the database made available by [42] as well as optimal codes according to the database [4]. We also introduced a method to solving the reversibility problem both for $\mathbb{F}_q$ and DNA, respectively. The aim to introduce the $\mathbb{F}_q$ reversibility is to describe the IUPAC nucleotide codes (see [39] for details). Previous studies examined the DNA reversibility problem for 4 DNA bases (A, T, G, C). In this study, IUPAC nucleotide codes were considered 5 IUPAC DNA bases instead of 4 DNA bases. Here, DNA reversibility is expressed as $\mathbb{F}_q$ reversibility by pairing the prime number of bases with $\mathbb{F}_q$ and is solved. DNA reversibility solutions, which are special cases of $\mathbb{F}_q$ reversibility, are also presented in this study on 5 IUPAC DNA bases (A, T, G, C, N). Moreover, even if all N strands are deleted from a DNA Code, the remaining cluster is a cluster of different lengths that still provides DNA reversibility. This studies of quantum codes construction can be generalized from Hermitian-dual containing cyclic and constacyclic codes over a non-chain ring. Also, finding the proper distance constraint is a open problem for future studies.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

References

1. A. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, P. Solé, The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes, *IEEE Trans. Inform. Theory*, **40** (1994), 301–319. https://doi.org/10.1109/18.312154
2. S. Ling, C. Xing, *Coding theory: a first course*, Cambridge University Press, 2004.
3. F. J. MacWilliams, N. J. A. Sloane, *The theory of error-correcting codes*, North-Holland Mathematical Library, Elsevier, 1977.
4. M. Grassl, Code tables: bounds on the parameters of various types of codes. Available from: `http://www.codetables.de/`.
5. S. Zhu, Y. Wang, M. Shi, Some results on cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$, *IEEE Trans. Inform. Theory*, **56** (2010), 1680–1684. https://doi.org/10.1109/TIT.2010.2040896
6. P. W. Shor, Scheme for reducing decoherence in quantum computer memory, *Phys. Rev. A.*, **52** (1995), 2493–2496. https://doi.org/10.1103/PhysRevA.52.R2493

7. A. R. Calderbank, E. M. Rains, P. M. Shor, N. J. A. Sloane, Quantum error-correction via codes over GF(4), *IEEE Trans. Inf. Theory*, **44** (1998), 1369–1387. https://doi.org/10.1109/18.681315

8. J. Qian, W. Ma, W. Gou, Quantum codes from cyclic codes over finite ring, *Int. J. Quantum Inf.*, **50** (2009), 1277–1283. https://doi.org/10.1142/S0219749909005560

9. X. Kai, S. Zhu, Quaternary construction of quantum codes from cyclic codes over $F_4 + uF_4$, *Int. J. Quantum Inf.*, **9** (2011), 689–700. https://doi.org/10.1142/S0219749911007757

10. J. Gao, Quantum codes from cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q + v^2\mathbb{F}_q + v^3\mathbb{F}_q$, *Int. J. Quantum Inf.*, **13** (2015), 1550063. https://doi.org/10.1142/S021974991550063X

11. M. Özen, N. T. Özzaim, H. Ince, Quantum codes from cyclic codes over $F_3 + uF_3 + vF_3 + uvF_3$, *J. Phys.*, **766** (2016), 012020. https://doi.org/10.1088/1742-6596/766/1/012020

12. M. Ashraf, N. Khan, G. Mohammad, New quantum and LCD codes over the finite field of odd characteristic, *Int. J. Theor. Phys.*, **60** (2021), 2322–2332. https://doi.org/10.1007/s10773-021-04849-2

13. S. Ali, A. S. Alali, M. Jeelani, M. Kurulay, E. S. Oztas, P. Sharma, On the construction of quantum and LCD codes from cyclic codes over the finite commutative rings, *Axioms*, **12** (2023), 367. https://doi.org/10.3390/axioms12040367

14. S. Ali, S. A. Alali, E. S. Oztas, P. Sharma, Construction of quantum codes over the class of commutative rings and their applications to DNA codes, *Mathematics*, **11** (2023), 1430. https://doi.org/10.3390/math11061430

15. M. Ashraf, G. Mohammad, Quantum codes from cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, *Quantum Inf. Process.*, **15** (2016), 4089. https://doi.org/10.1007/s11128-016-1379-8

16. M. Ashraf, G. Mohammad, Quantum codes over F_p from cyclic codes over $F_p[u, v]/\langle u^2 - 1,\ v^3 - v,\ uv - vu\rangle$, *Cryptogr. Commun.*, **11** (2019), 325–335. https://doi.org/10.1007/s12095-018-0299-0

17. S. Bhardwaj, M. Goyal, M. Raka, New quantum codes from constacyclic codes over a general non-chain ring, *arXiv*, 2022. https://doi.org/10.48550/arXiv.2212.02821

18. H. Islam, O. Prakash, Construction of LCD and new quantum codes from cyclic codes over a finite non-chain ring, *Cryptogr. Commun.*, **14** (2022), 59–73. https://doi.org/10.1007/s12095-021-00516-9

19. H. Islam, O. Prakash, New quantum and LCD codes over the finite field of even characteristic, *Def. Sci. J.*, **71** (2020), 656–661. https://doi.org/10.14429/dsj.71.16641

20. L. Adleman, Molecular computation of solutions to combinatorial problems, *Science*, **266** (1994), 1021–1024. https://doi.org/10.1126/science.7973651

21. L. S. Liebovitch, Y. Tao, A. T. Todorov, L. Levine, Is there an error correcting code in the base sequence in DNA, *Biophys. J.*, **71** (1996), 1539–1544. https://doi.org/10.1016/S0006-3495(96)79356-6

22. M. M. Brandao, L. Spoladore, L. C. B. Faria, A. S. L. Rocha, M. C. Silva-Filho, R. Palazzo, Ancient DNA sequence revealed by error-correcting codes, *Sci. Rep.*, **5** (2015), 12051. https://doi.org/10.1038/srep12051

23. A. G. Frutos, Q. Liu, A. J. Thiel, A. M. W. Sanner, A. E. Condon, L. M. Smith, et al., Demonstration of a word design strategy for DNA computing on surfaces, *Nucleic Acids Res.*, **25** (1997), 4748–4757. https://doi.org/10.1093/nar/25.23.4748

24. O. D. King, Bounds for DNA codes with constant GC-content, *Electron. J. Comb.*, **10** (2003), R33. https://doi.org/10.37236/1726

25. M. Li, H. J. Lee, A. E. Condon, R. M. Corn, DNA word design strategy for creating sets of non-interacting oligonucleotides for DNA microarrays, *Langmuir*, **18** (2002), 805–812. https://doi.org/10.1021/la0112209

26. A. Marathe, A. E. Condon, R. M. Corn, On combinatorial DNA word design, *J. Comput. Biol.*, **8** (2001), 201–220. https://doi.org/10.1089/106652701752530818

27. E. S. Oztas, I. Siap, Lifted polynomials over F_{16} and their applications to DNA codes, *Filomat*, **27** (2013), 459–466. https://doi.org/10.2298/FIL1303459O

28. E. S. Oztas, I. Siap, On a generalization of lifted polynomials over finite fields and their applications to DNA codes, *Int. J. Comput. Math.*, **92** (2015), 1976–1988. https://doi.org/10.1080/00207160.2014.930449

29. T. Abualrub, A. Ghrayeb, X. Zeng, Construction of cyclic codes over $GF(4)$ for DNA computing, *J. Franklin Inst.*, *343* (2006), 448–457. https://doi.org/10.1016/j.jfranklin.2006.02.009

30. H. Q. Dinh, S. Pathak, A. K. Upadhyay, W. Yamaka, New DNA codes from cyclic codes over mixed alphabets, *Mathematics*, **8** (2020), 1977. https://doi.org/10.3390/math8111977

31. H. Q. Dinh, A. K. Singh, S. Pattanayak, S. Sriboonchitta, Construction of cyclic DNA codes over the ring $Z_4[u]/\langle u^2-1\rangle$ based on the deletion distance, *Theor. Comput. Sci.*, **773** (2019), 27–42. https://doi.org/10.1016/j.tcs.2018.06.002

32. H. Q. Dinh, A. K. Singh, S. Pattanayak, S. Sriboonchitta, Cyclic DNA codes over the ring $F_2+uF_2+vF_2+uvF_2+v^2F_2+uv^2F_2$, *Des. Codes Cryptogr.*, **86** (2018), 1451–1467. https://doi.org/10.1007/s10623-017-0405-x

33. K. Gowdhaman, S. Gupta, C. Mohan, K. Guenda, D. Chinnapillai, Cyclic DNA codes over the ring $Z_4+uZ_4+u^2Z_4$, *J. Algebra Comb. Discrete Appl.*, **8** (2021), 219–231.

34. F. Gursoy, E. S. Oztas, B. Yildiz, Reversible DNA codes over a family of non-chain rings $R_{k,s}$, *arXiv*, 2017. https://doi.org/10.48550/arXiv.1711.02385

35. E. S. Oztas, B. Yildiz, I. Siap, A novel approach for constructing reversible codes and applications to DNA codes over the ring $F_2[u]/(u^{2k}-1)$, *Finite Fields Appl.*, **46** (2017), 217–234. https://doi.org/10.1016/j.ffa.2017.04.001

36. I. Siap, T. Abulraub, A. Ghrayeb, Similarity cyclic DNA codes over rings, *2008 2nd International Conference on Bioinformatics and Biomedical Engineering*, IEEE, 2008. https://doi.org/10.1109/ICBBE.2008.149

37. I. Siap, T. Abulraub, A. Ghrayeb, Cyclic DNA codes over the ring $\mathbb{F}_2[u]/(u^2-1)$ based on the deletion distance, *J. Franklin Inst.*, **346** (2009), 731–740. https://doi.org/10.1016/j.jfranklin.2009.07.002

38. B. Yildiz, I. Siap, Cyclic codes over $\mathbb{F}_2[u]/(u^4-1)$ and applications to DNA codes, *Comput. Math. Appl.*, **63** (2012), 1169–1176. https://doi.org/10.1016/j.camwa.2011.12.029

39. A. D. Johnson, An extended IUPAC nomenclature code for polymorphic nucleic acids, *Bioinformatics*, **26** (2010), 1386–1389. https://doi.org/10.1093/bioinformatics/btq098

40. T. Bag, A. K. Upadhyay, Study on negacyclic codes over the ring $Z_p[u]/\langle u^{k+1} - u\rangle$, *J. Appl. Math. Comput.*, **59** (2019), 693–700. https://doi.org/10.1007/s12190-018-1197-5

41. M. Grassl, T. Beth, M. Röetteler, On optimal quantum codes, *Int. J. Quantum Inf.*, **2** (2004), 55–64. https://doi.org/10.1142/S0219749904000079

42. N. Aydin, P. Liu, B. Yoshino, A database of quantum codes, *arXiv*, 2021. https://doi.org/10.48550/arXiv.2108.03567

43. W. Bosma, J. Cannon, *Handbook of magma functions*, University of Sydney, 1995.