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Research article

The strong consistency and asymptotic normality of the kernel estimator type in functional single index model in presence of censored data

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Abstract: In the present study, we address the nonparametric estimation challenge related to the regression function within the Single Functional Index Model in the random censoring framework. The principal achievement of this investigation lies in the establishment of the asymptotic characteristics of the estimator, including rates of almost complete convergence. Moreover, we establish the asymptotic normality of the constructed estimator under mild conditions. Subsequently, we provide the application of our findings towards the construction of confidence intervals. Lastly, we illuminate the finite-sample performance of both the model and the estimation methodology through the analysis of simulated data and a real-world data example.

Keywords: almost complete convergence; convergence rates; exponential inequality; kernel regression

Mathematics Subject Classification: 62G05, 62G20, 62N02

1. Introduction

Regression analysis has proven to be versatile and has offered a strong statistical modeling framework in a wide range of practical and theoretical situations where the goal is to model the predictive relationship between related responses and predictors. Parametric regression models are valuable tools for analysing real-world data, but they can be subject to significant modeling biases if the model structures are poorly described. This is a common issue in many practical problems. Nonparametric smoothing techniques can be used as an option to address concerns about modeling bias. This article will specifically examine the analysis of estimators that belong to the kernel type.

For $\mathbf{x} \in \mathbb{R}^d$, the regression function, whenever it exists, is defined to be

$$r(\mathbf{x}) := \mathbb{E}(Y \mid \mathbf{X} = \mathbf{x}). \tag{1.1}$$

A well-known estimator for the regression function $r(\cdot)$, often used in nonparametric statistics, is the kernel regression function estimator. This estimator is, under suitable conditions, strongly consistent, i.e., it converges almost surely to the unknown regression function at $\mathbf{x}$. Because of numerous applications and their important role in mathematical statistics, the problem of estimating $m(\cdot, \varphi)$ and $f_{\mathbf{X}}(\cdot)$ has been a subject of considerable interest during recent decades. For good sources of references to the research literature in this area along with statistical applications, consult [11, 14, 19, 20, 24–26, 60, 65, 67, 71] and the references therein. The increased dimensionality of $\mathbf{X}$ can provide challenges when attempting to move beyond standard multiple linear models and pursue nonparametric estimation of $r(\mathbf{x})$. This strongly motivates the consideration of regression models that offer dimension reduction. Single index models are widely used to achieve this by assuming that the predictors' influence on the response can be simplified to a single index. This index represents a projection on a specified direction and is combined with a nonparametric link function. This simplifies the predictors to a single-variable index while still including important characteristics. Additionally, because the nonparametric link function only operates on a one-dimensional index, these models are not affected by the problem of having a high number of dimensions, known as the curse of dimensionality. The single index model extends the concept of linear regression by incorporating a link function that is equivalent to the identity function; the interested reader may refer to [6, 35, 41, 50, 69]. The statistical challenges associated with the analysis of functional random variables, which are variables that take values in an infinite-dimensional space, have gained increasing attention in the statistics literature in recent decades. The motivation behind the development of this research issue stems from the abundant data collected on a progressively more precise temporal/spatial grids. This is evident in fields such as meteorology, medicine, satellite images, and various other research domains. Consequently, the statistical analysis of this data, viewed as unpredictable functions, gave rise to numerous complex theoretical and computational research inquiries. To gain a comprehensive understanding of both the theoretical and practical aspects of functional data analysis, readers are encouraged to consult the monographs by [8] for linear models involving random variables in a Hilbert space, and by [64] for scalar-on-function and function-on-function linear models, functional principal component analysis, and parametric discriminant analysis. The work of [28] primarily emphasises nonparametric techniques, particularly kernel-based estimation, for scalar-on-function nonlinear regression models. They expanded the application of these technologies to include classification and discrimination analysis. The book by [42] explores the extension of many statistical concepts, including goodness-of-fit tests, portmanteau tests, and change detection, to the framework of functional data. Recent advances in functional data analysis have highlighted the necessity of developing models aiming to reduce dimensionality effects (see [22, 34, 51] recent surveys), see also [2, 12, 13], and semiparametric ideas are natural candidates for that purpose. In that way, [29] and [1] studied the Functional Single-Index Model (FSIM). [43] proposed functional single index composite quantile regression and estimated the unknown slope function and link function by using B-spline basis functions. [61] proposed a new compact functional single index model, in which the coefficient function is only nonzero in a subregion. [76] investigated the estimation of a general functional single index model, in which the conditional distribution of the response depends on the functional predictor via a functional single index structure. [70] developed a new estimation procedure that combines a functional principal component analysis of the functional predictors, B-spline model for the parameters, and profile estimation of the unknown parameters and functions in the model. [53, 54] investigated the estimation of the functional single index regression model with missing responses at random for strong mixing time series data. [27] introduced a new functional single-index varying coefficient model with the functional predictor being single-index part. By means of functional principal components analysis and basis function approximation, they obtain the estimators of slope function and coefficient functions, and propose an iterative estimating procedure. [62] developed a new automatic and location-adaptive procedure for estimating regression in FSIM that is based on k-Nearest Neighbours (kNN) ideas. Motivated by the analysis of imaging data, [47] proposed a novel functional varying-coefficient single-index model to carry out the regression analysis of functional response data on a set of covariates of interest. [4] investigated a functional Hilbertian regressor for a nonparametric estimation of the conditional cumulative distribution with a scalar response variable in the single index structure. [21] developed an alternative approach, where their methodology includes the multi-index case and does not anchor the true parameter on a prespecified sieve, and they provide a detailed theoretical analysis of a direct kernel-based estimation scheme which establishes a polynomial convergence rate. [3] considered the nonparametric estimation of the conditional density of a scalar response variable given a random variable taking values in separable Hilbert space in the single-index structure. [52] investigated the estimation of conditional density function based on the single-index model for functional time series data. [49] proposed a model that accommodates the existence of interaction between a single functional predictor and other covariates, which has usually been ignored in previous works. For that, the authors consider a semiparametric single-index structure to model the potential interaction between the functional predictor and other covariates, which provides a feasible two-stage procedure to address estimation and inference issues. To explore recent progress and relevant sources in semiparametric and/or functional data analysis, we refer to [39, 40, 46, 58, 75] which considered the kNN method in a single index regression model when the explanatory variable is valued in functional space in the setting of the quasi-association dependence condition. It should be noted that most of the contributions involved above are in the case of the samples being observed completely. Therefore, it is imperative to thoroughly examine the functioning statistical models within the context of censored data. From a pragmatic perspective, this type of data holds significance. Clinical trials commonly involve survival statistics or failure time data, which are frequently affected by censoring. More precisely, numerous statistical experiments provide incomplete samples, even when conducted under carefully regulated circumstances. For instance, clinical data pertaining to the survival of various diseases is typically obscured by other life-threatening dangers that ultimately lead to mortality. To be more precise, let $(Y_i^0)_{1\leq i\leq n}$ be the life (or survival) times supposed independent identically distributed (i.i.d.) sequence of random variables (r.v.s) with common unknown continuous distribution function (d.f.) $F(\cdot)$ and density $f(\cdot)$. In many practical situations, instead observing the lifetimes Y^0 we observe only censored lifetimes of the items. Further, we assume that $(C_i)_{1\leq i\leq n}$ is an i.i.d. sequence of censoring r.v.s with common continuous d.f. $G(\cdot)$ which is independent of $(Y_i^0)_{1\leq i\leq n}$, and we observe the n pairs $(Y_i, \delta_i)_{1\leq i\leq n}$ with $Y_i = \min(Y_i^0, C_i) =: Y_i^0 \wedge C_i$ and $\delta_i = \mathbb{1}_{\{Y_i^0 \leq C_i\}}$, where $\mathbb{1}_A$ is the indicator function of the set A (if $\delta_i = 1$, this means that $Y_i^0 \leq C_i$ and $Y_i = Y_i^0$, and the observed value is the true lifetime of subject i; otherwise,

$C_i < Y_i^0$ and hence $Y_i = C_i$ and so the actual lifetime Y_i^0 is not observed). Following the convention usually assumed, in right censored model, we suppose that $(C_i)_{1\le i\le n}$ and $(X_i, Y_i^0)_{1\le i\le n}$ are independent, for instance, see [14, 15, 45, 63].

The primary objective of this paper is to delineate the asymptotic properties of the estimator $\widehat{r_n}(\theta,\cdot)$, defined in (2.7) below, in a censorship setting where the variables exhibit a functional form and adhere to a single index structure. This includes investigating the rates of almost complete convergence (a.co.*). Additionally, we derive the asymptotic distribution under certain mild conditions.

The rest of this work is organized as follows: In Section 2, we describe the single index regression model for functional data and in the censored framework. In Section 3, we will establish our main results of the uniform almost complete convergence of the kernel estimators (in §3.1) and the asymptotic normality (in §3.2) under non-restrictive conditions. In Section 4, we discuss the impact of our contribution in practical application of our results for the construction of the confidence interval. In Section 5, we perform a simulation study to show that our proposed model works well for finite samples of simulated data and a real data example. We conclude this paper with some remarks and future works in Section 6. To avoid interrupting the flow of the presentation, all mathematical developments are relegated to Section 7.

2. Model

Let $\{(X_i, C_i, Y_i^0),\ 1 \le i \le n\}$ be n independent identical copies of a triple of random variables (X, C, Y^0), where X takes values in a separable real Hilbert space $\mathcal{H}$ endowed with the inner product $\langle\cdot,\cdot\rangle$ and its norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$, C is a censoring variable and Y^0 is the variable of interest, typically called a life time variable. We consider the semi-metric $d_\theta(\cdot,\cdot)$,† associated with the single-index θ, which is defined by $d_\theta(\chi_1,\chi_2) := |\langle\chi_1 - \chi_2, \theta\rangle|$. In the given topological structure and for any θ, our focus is to estimate nonparametrically the regression function, whenever it exists, of Y^0 given $\langle X,\theta\rangle$, denoted by

$$r_\theta(\chi) := \mathbb{E}(Y^0 \,|\, \langle\theta, X\rangle = \langle\theta,\chi\rangle). \tag{2.1}$$

Since the single functional index θ plays a crucial role in such a model, the identifiability of this model will be important. Clearly, the relation in (2.1) can be reformulated as

$$Y^0 = r(\langle X,\theta\rangle) + \epsilon,$$

where $\mathbb{E}(\epsilon\,|\,X) = 0$. To ensure the identifiability of the model θ (i.e., $\theta_1 = \theta_2$) we require that $r_j(\langle\chi,\theta_j\rangle)$ is differentiable with respect to (w.r.t) the variable $\langle\chi,\theta_j\rangle$, $j = 1, 2$, respectively, and that $\langle\theta, e_1\rangle = 1$,

*Let (z_n) for $n \in \mathbb{N}$, be a sequence of real r.v.'s. We say that (z_n) converges almost-completely (a.co.) toward zero if, and only if, for all

$$\epsilon > 0, \sum_{n=1}^{\infty} \mathbb{P}(|z_n| > \epsilon) < \infty.$$

Moreover, we say that the rate of the almost-complete convergence of (z_n) toward zero is of order u_n (with $u_n \to 0$) and we write $z_n = O_{a.co.}(u_n)$ if, and only if, there exists $\epsilon > 0$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}(|z_n| > \epsilon u_n) < \infty.$$

This kind of convergence implies both the almost-sure convergence and the convergence in probability.

†A semi-metric (sometimes called pseudo-metric) $d_\theta(\cdot,\cdot)$ is a metric which allows $d_\theta(x_1, x_2) = 0$ for some $x_1 \ne x_2$.

where e_1 is the first element of an orthonormal basis of the space $\mathcal{H}$. For more details on the proof of identifiability of the functional single model, see [29]. It is worth highlighting that when censored data is considered, the smoothing nonparametric regression estimator $\widehat{r}(\theta,\chi)$ of $r(\theta,\chi)$ can be viewed as a local least squares estimator, that is

$$\widehat{r}(\theta,\chi) \quad = \quad \arg\min_{t\in\mathbb{R}} \sum_{i=1}^{n} W_{i,n}(\theta,\chi)(Y_i - t)^2,$$

where $W_{i,n}(\theta,\chi)$ are the Nadaraya-Watson weights given by

$$W_{i,n}(\theta,\chi) \quad = \quad \frac{K(h_n^{-1}|\langle\chi - X_i,\theta\rangle|)}{\sum_{i=1}^{n} K(h_n^{-1}|\langle\chi - X_i,\theta\rangle|)}.$$

The [59, 73] type-estimator of $r(\theta,\chi)$ is defined as

$$\widehat{r}(\theta,\chi) = \sum_{i=1}^{n} Y_i^0 W_{i,n}(\theta,\chi) = \frac{\sum_{i=1}^{n} Y_i^0\, K(h_n^{-1}|\langle\chi - X_i,\theta\rangle|)}{\sum_{i=1}^{n} K(h_n^{-1}|\langle\chi - X_i,\theta\rangle|)}, \tag{2.2}$$

where $K(\cdot)$ is a kernel function, and $h_n := h_{n,K}$ is a sequence of bandwidths (positive real numbers) decreasing to zero as n goes to infinity. When working with censored data, the initial challenge is understanding and handling the censorship itself, which occurs when observations are incomplete or only partially obtained. One of the suggested approaches to tackle censored data is to apply the method of *synthetic* data, which allows to take into account the censoring effect on the lifetime distribution, see for instance: [16–18, 45]. Theoretically, the synthetic data transformation provides equal expected values for both variables. In this case, synthetic response variable (transformed variable) is given by $\dfrac{\delta_i\, Y_i}{\overline{G}(Y_i)}$, $1 \leq i \leq n$, where $\overline{G}(\cdot) = 1 - G(\cdot)$, and if we assume (Y_i^0, X_i) and (C_i) are independent (usually supposed for censored data), we get

$$\begin{aligned}
\mathbb{E}\left[\frac{\delta_i\, Y_i}{\overline{G}(Y_i)}\,\Big|\, X_i\right] &= \mathbb{E}\left\{\mathbb{E}\left[\frac{\delta_i\, Y_i}{\overline{G}(Y_i)}\,\Big|\, Y_i^0, X_i\right]\,\Big|\, X_i\right\} \\
&= \mathbb{E}\left\{\mathbb{E}\left[\frac{\mathbb{1}_{\{Y_i^0\leq C_i\}}\, Y_i}{\overline{G}(Y_i)}\,\Big|\, Y_i^0, X_i\right]\,\Big|\, X_i\right\} \\
&= \mathbb{E}\left\{\frac{Y_i^0}{\overline{G}(Y_i^0)}\mathbb{E}\left[\mathbb{1}_{\{Y_i^0\leq C_i\}}\,\Big|\, Y_i^0\right]\,\Big|\, X_i\right\} \\
&= \mathbb{E}\left[Y_i^0 \,|\, X_i\right].
\end{aligned} \tag{2.3}$$

In the same manner, we get by conditioning w.r.t $\langle\theta, X_i\rangle$

$$\mathbb{E}\left[\frac{\delta_i\, Y_i}{\overline{G}(Y_i)}\,\Big|\,\langle\theta, X_i\rangle\right] \quad = \quad \mathbb{E}\left[Y_i^0 \,|\, \langle\theta, X_i\rangle\right]. \tag{2.4}$$

Thus, for the infinite dimension case one can define the pseudo-estimator of $r_\theta(\chi)$ as:

$$\widetilde{r}(\theta,\chi) = \frac{\displaystyle\sum_{i=1}^{n} \frac{\delta_i Y_i}{\overline{G}(Y_i)} K(h_n^{-1}|\langle \chi - X_i, \theta\rangle|)}{\displaystyle\sum_{i=1}^{n} K(h_n^{-1}|\langle \chi - X_i, \theta\rangle|)} = \frac{\widetilde{r}_N(\theta,\chi)}{\widehat{r}_D(\theta,\chi)}, \tag{2.5}$$

where

$$\begin{aligned}
\widetilde{r}_N(\theta,\chi) &= \frac{1}{n\mathbb{E}[K_1(\theta,\chi)]} \sum_{i=1}^{n} \frac{\delta_i Y_i}{\overline{G}(Y_i)} K_i(\theta,\chi) \\
\widehat{r}_D(\theta,\chi) &= \frac{1}{n\mathbb{E}[K_1(\theta,\chi)]} \sum_{i=1}^{n} K_i(\theta,\chi) \\
K_i(\theta,\chi) &= K(h_n^{-1}|\langle \chi - X_i, \theta\rangle|) = K(h_n^{-1} d_\theta(\chi, X_i)).
\end{aligned}$$

In practice, since the values Y^0 are censored observations, the survival function $\overline{G}(\cdot) = 1 - G(\cdot)$ is unknown (because $G(\cdot)$ is too), so it is impossible to use the estimator (2.5). To complete the building of the good estimator of $r(\theta,\chi)$, we may can replace the function $\overline{G}(\cdot)$ by its corresponding estimator [44], $\overline{G}_n(\cdot) := 1 - G_n(\cdot)$ defined by

$$\overline{G}_n(t) = \begin{cases} \displaystyle\prod_{i=1}^{n} \left(1 - \frac{1-\delta_{(i)}}{n-i+1}\right)^{\mathbb{1}_{\{Y_{(i)} \le t\}}}, & \text{if } t < Y_{(n)}; \\ 0, & \text{if } t \ge Y_{(n)}. \end{cases} \tag{2.6}$$

Where $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}$ are the order statistics of Y_i and $\delta_{(i)}$ is the indicator of no censoring associated with $Y_{(i)}$. [23] presented some probabilistic strong functional limit theorems extending the results known for completely observable data. More precisely, for a continuous $G(\cdot)$, as $n \to \infty$, we have

$$\sup_{0 \le t \le Y_{(n)}} \left| \frac{1}{G_n(t)} - \frac{1}{G(t)} \right| = O\left(\left(\frac{\log_2 n}{n} \right)^{\frac{1}{2}} \right),$$

almost surely where $\log_2 u = \log_+ (\log_+ u), \log_+ u = \log(u \vee e)$. Such a result will be useful to establish the main results of this paper. Now, plugging the estimator (2.6) into (2.5), one can define an estimator of $r(\theta,\chi)$ as:

$$\widehat{r}_n(\theta,\chi) = \frac{\displaystyle\frac{1}{n\mathbb{E}[K_1(\theta,\chi)]} \sum_{i=1}^{n} \frac{\delta_i Y_i}{\overline{G}_n(Y_i)} K(h_n^{-1}|\langle \chi - X_i, \theta\rangle|)}{\displaystyle\frac{1}{n\mathbb{E}[K_1(\theta,\chi)]} \sum_{i=1}^{n} K(h_n^{-1}|\langle \chi - X_i, \theta\rangle|)} =: \frac{\widehat{r}_N(\theta,\chi)}{\widehat{r}_D(\theta,\chi)}. \tag{2.7}$$

We emphasize that this study represents the first exploration of semi-parametric regression for censored functional data. It is important to note that the model investigated here differs significantly from the conditional quantile examined in [38]. Additionally, the asymptotic findings presented in our paper diverge from those established in the aforementioned study. Specifically, our results not only confirm uniform consistency, but also demonstrate the asymptotic normality of the estimator we construct. Henceforth, let us consider Y^0 as having bounded values. While the assumption of boundedness for Y^0 can be substituted with a finite moment assumption, such a replacement would introduce significantly greater complexity into the proofs. Specifically, it necessitates the utilization of a truncation argument.

3. Main results

3.1. Uniform almost complete convergence

Throughout this paper, when no confusion will be possible, we will denote by C, C' or/and $C_{\theta,x}$ some strictly positive generic constants which may be different in each appearance. For any sequences (u_n) and (v_n) we write $u_n = O(v_n)$ if $|u_n| \leq C|v_n|, \forall n$ (if the property holds almost completely we use the symbol $O_{a.co.}$). We design by $\mathcal{N}_\chi$ a fixed neighborhood of a fixed point χ in $\mathcal{H}$, and by

$$B_\theta(\chi, r) := \{\chi_1 \in \mathcal{H} : \ |\langle \chi - \chi_1, \theta\rangle| < r\},$$

the ball centered at χ, with radius r. Let $\tau_F = \sup\{y : \overline{F}(y) > 0\}$ and $\tau_G = \sup\{y : \overline{G}(y) > 0\}$ be upper endpoints of $F(\cdot)$ and $G(\cdot)$, respectively. We assume that $\tau_F < \infty$, and $\overline{G}(\tau_F) > 0$, which implies that $\tau_F < \tau_G$. Let us consider the following coverings of the compacts $S_\mathcal{H}$ and $\Theta_\mathcal{H}$:

$$S_\mathcal{H} \subset \bigcup_{k=1}^{d_n^{S_\mathcal{H}}} B(\chi_k, l_n) \ \text{ and } \ \Theta_\mathcal{H} \subset \bigcup_{j=1}^{d_n^{\Theta_\mathcal{H}}} B(t_j, l_n),$$

with χ_k, $t_j \in \mathcal{H}$ and $\forall \chi \in \mathcal{H}$; $\forall \theta \in \Theta_\mathcal{H}$, one sets $k(\chi) = \arg\min_{k\in\{1,\ldots d_n^{S_\mathcal{H}}\}} \|\chi - \chi_k\|$ and $j(\theta) = \arg\min_{j\in\{1,\ldots d_n^{\Theta_\mathcal{H}}\}} \|\theta - t_j\|$. Note that this cover of the compact subset is necessary to derive our uniform consistency, and it is a key point to ensure the geometric link between the number $d_n^{S_\mathcal{H}}$, $d_n^{\Theta_\mathcal{H}}$ of balls and the sequence of radius l_n. In abstract semi-metric spaces, it is usually assumed that $d_n^{S_\mathcal{H}} l_n$ ($d_n^{\Theta_\mathcal{H}} l_n$) is bounded, see [1] for more discussion. We now list the assumptions needed to derive our first result.

(A0) (X, Y^0) and C are independent;

(A1) (i) $\mathbb{P}(X \in B_\theta(\chi, \epsilon)) =: \phi_{\theta,\chi}(\epsilon) > 0$, and $\forall \epsilon > 0$, $\lim_{\epsilon\to 0} \phi_{\theta,\chi}(\epsilon) = 0$,

(ii) There exists a differentiable function $\phi(\cdot)$, $\forall \chi \in S_\mathcal{H}$, $\theta \in \Theta_\mathcal{H}, 0 < C\phi(h_n) \leq \phi_{\theta,\chi}(h_n) \leq C'\phi(h_n) < \infty$, and $\exists \xi_0 > 0$, $\forall \xi < \xi_0$, $\phi'(\xi) < C$;

(A2) The function $r(\cdot, \cdot)$ satisfies the following Hölder condition: $\forall \theta \in \mathcal{H}$, $\forall(\chi_1, \chi_2) \in \mathcal{N}_\chi \times \mathcal{N}_\chi$:

$$|r(\theta, \chi_1) - r(\theta, \chi_2)| \leq C_{\theta,\chi} d_\theta(\chi_1, \chi_2)^\gamma, \ \gamma > 0;$$

(A3) $K(\cdot)$ is a bounded continuous function such that

(i) $0 < C\mathbb{1}_{[0,1]}(t) < K(t) < C'\mathbb{1}_{[0,1]}(t) < \infty$,

(ii) $\forall t_1, t_2 \in \mathbb{R}$, $|K(t_1) - K(t_2)| \leq C|t_1 - t_2|$;

(A4) For $l_n = O(\frac{\log n}{n})$, the sequences $d_n^{S_\mathcal{H}}$ and $d_n^{\Theta_\mathcal{H}}$ satisfy:

(i) $\dfrac{(\log n)^2}{n\phi(h_n)} < \log d_n^{S_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}} < \dfrac{n\phi(h_n)}{\log n}$,

(ii) $\sum_{n=1}^{\infty}(d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})^{1-\beta} < \infty$, for some $\beta > 2$,

(iii) $n\phi(h_n) = O((\log n)^2)$ and $\lim_{n\to\infty} n\phi(h_n) = \infty$.

Comments on the assumptions

We first recall that our assumptions are not restrictive and may be considered as standard in the present context. Assumption (A0) was introduced in [18] and used in [10, 36, 45] which is plausible whenever the censoring is independent to the characteristics of the patients under study. This condition can be relaxed to conditional independence, but this will add extra complexity to the proofs. Note also that the assumption $\tau_F < \tau_G$ (implying $\overline{G}(Y^0) > \overline{G}(\tau_F)$) is classical for asymptotic normality results in the censorship framework. Assumption (A1) controls the behavior of the small ball probability around zero and is the usual condition on the small ball probability. Assumption (A2) imposes some smoothness of the regression operator required to control the convergence rate of bias. Assumption (A3) pertains to the choice of the kernel $K(\cdot)$, a common practice in nonparametric functional estimation. It is important to observe that the Parzen symmetric kernel is not suitable in this particular situation due to the fact that the random process $|\langle \chi - X_i, \theta\rangle|$ is always positive. As a result, we will instead use $K(\cdot)$ with a support range of $[0, 1]$. This is an extension of the often assumed condition on the kernel in the situation of multiple variables, where $K(\cdot)$ is expected to be a density function that exhibits spherical symmetry. Assumption (A4) serves a technical purpose, allowing for the evaluation of the asymptotic variance component. This assumption pertains to the Kolmogorov entropy of the functional subsets $\Theta_{\mathcal{H}}$ and $S_{\mathcal{H}}$. Such assumptions in functional data analysis were originally introduced by [9] and have since found application in functional single index models, as demonstrated by [5] and [38].

In the following theorem, we first present the uniform almost complete convergence rate of the estimator $\widehat{r}_n(\theta,\chi)$.

Theorem 3.1. *Assume that (A0)–(A4) hold. We then have, as $n \to \infty$,*

$$\sup_{\chi\in S_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\widehat{r}_n(\theta,\chi) - r(\theta,\chi)\right| = O(h_n^{\gamma}) + O_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right).$$

In the particular case where the functional single index is fixed we get the following result.

Corollary 3.2. *Under the Assumptions (A0)–(A4), we have, as $n \to \infty$,*

$$\sup_{\chi\in S_{\mathcal{H}}} \left|\widehat{r}_n(\theta,\chi) - r(\theta,\chi)\right| = O(h_n^{\gamma}) + O_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}}{n\phi(h_n)}}\right).$$

Proof of Theorem 3.1. We may consider the following decomposition

$$\begin{aligned}\widehat{r}_n(\theta,\chi) - r(\theta,\chi) &= \frac{\widehat{r}_N(\theta,\chi) - r(\theta,\chi)\widehat{r}_D(\theta,\chi)}{\widehat{r}_D(\theta,\chi)} \\ &= \frac{\widehat{r}_N(\theta,\chi) - \widetilde{r}_N(\theta,\chi)}{\widehat{r}_D(\theta,\chi)} + \frac{\widetilde{r}_N(\theta,\chi) - \mathbb{E}\left[\widetilde{r}_N(\theta,\chi)\right]}{\widehat{r}_D(\theta,\chi)}\end{aligned}$$

$$+\frac{\mathbb{E}\left[\widetilde{r}_N(\theta,\chi)\right]-r(\theta,\chi)}{\widehat{r}_D(\theta,\chi)}+r(\theta,\chi)\frac{(1-\widehat{r}_D(\theta,\chi))}{\widehat{r}_D(\theta,\chi)}. \tag{3.1}$$

Through this decomposition, the proof of Theorem 3.1 will rely on the following lemmas, the proofs of which are deferred to Section 7. □

Lemma 3.3. *Under the Assumptions (A1), (A3), and (A4), we have, as $n \to \infty$,*

$$\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi)-1\right| = O_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right).$$

Corollary 3.4. *Under the Assumptions (A1), (A3), and (A4), we have, as $n \to \infty$,*

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{\chi\in\mathcal{H}}\left|\widehat{r}_D(\theta,\chi)\right|<1/2\Big)<\infty. \tag{3.2}$$

Lemma 3.5. *Under the Assumptions (A0), (A1), (A3), and (A4), we have, as $n \to \infty$,*

$$\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_N(\theta,\chi)-\widetilde{r}_N(\theta,\chi)\right| = O_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right).$$

Lemma 3.6. *Under the Assumptions (A0), (A1-(i)), and (A2), we have, as $n \to \infty$,*

$$\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\mathbb{E}\Big[\widetilde{r}_N(\theta,\chi)\Big]-r(\theta,\chi)\right| = O(h_n^{\gamma}).$$

Lemma 3.7. *Under the Assumptions (A0), (A1), (A3), and (A4), we have, as $n \to \infty$,*

$$\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widetilde{r}_N(\theta,\chi)-\mathbb{E}\Big[\widetilde{r}_N(\theta,\chi)\Big]\right| = O_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right).$$

3.2. Asymptotic normality

Our second result consists of establishing the asymptotic normality of the estimator $\widehat{r}_n(\theta,\chi)$. We fix a point χ in $\mathcal{H}$, and we introduce the following assumptions:

(A1') The concentration property in (A1-(i)) holds. Moreover, there exists a function $\beta_{\theta,\chi}(\cdot)$ satisfying:

$$\forall s\in[0,1],\quad \lim_{n\to\infty}\frac{\phi_{\theta,\chi}(sh_n)}{\phi_{\theta,\chi}(h_n)}=\beta_{\theta,\chi}(s);$$

(A3') The kernel $K(\cdot)$ is a bounded continuous function with support $[0,1]$, differentiable, its derivative $K'(\cdot)$ exists, and is such that there exist two constants C and C' with $-\infty < C < K'(t) < C' < 0$, for all $t\in[0,1]$;

(A4') The bandwidth h_n satisfies :

(i) $\lim_{n\to\infty} h_n = 0$, and $\lim_{n\to\infty}\frac{1}{n\phi_{\theta,\chi}(h_n)}=0$,

(ii) $\lim_{n\to\infty} n\phi_{\theta,\chi}(h_n)h_n^{2\gamma}=0$.

Comments on the assumptions.

Again assumption (A1') controls the behavior of the small ball probability around zero and is the usual condition on the small ball probability. This approximately shows that the small ball probability can be written approximately as the product of two independent functions, refer to [56] for the diffusion process, [7] for a Gaussian measure, and [48] for a general Gaussian process, and these assumptions have been employed by [55] for strongly mixing processes. For example, the function $\phi(\cdot)$ can be expressed as $\phi(\epsilon) = \epsilon^{\delta} \exp(-C/\epsilon^{a})$ with $\delta \geq 0$ and $a \geq 0$, and it corresponds to the Ornstein-Uhlenbeck and general diffusion processes (for such processes, $a = 2$ and $\delta = 0$) and the fractal processes (for such processes, $\delta > 0$ and $a = 0$). We refer to the paper of [30] and for other examples [68]. As was discussed in [55], the assumption (A1') is consistent with the assumptions made by [33] in the context of density estimation for functional data. When $\mathcal{H}$ is a separable Hilbert space and is infinite dimensional, $\phi(h)$ could decrease to zero as $h \to 0$ exponentially fast, for instance, see [33]. This condition also permits us to present an explicitly asymptotic variance term. The condition (A3') is essential in the variance calculation as in [55]. The assumption (A4'-*ii*) will be used to remove the bias term in the asymptotic normality results.

Below, we write $Z \overset{\mathcal{D}}{\to} \mathcal{N}(\mu, \sigma^2)$ whenever the random variable Z follows a normal law with expectation μ and variance σ^2. Our main result of this section is summarized in the following theorem.

Theorem 3.8. *Assume that (A0)-(A1'),(A2), and (A3')-(A4') hold. We have, as $n \to \infty$,*

$$(n\phi_{\theta,\chi}(h_n))^{1/2}(\widehat{r}_n(\theta,\chi) - r(\theta,\chi)) \overset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, \sigma_r^2(\theta,\chi)),$$

where

$$\sigma_r^2(\theta,\chi) := \frac{C_2\,(r(\theta,\chi))^2}{C_1^2} \quad \textit{with}\ \ C_l = K^l(1) - \int_0^1 (K^l)'(s)\beta_{\theta,\chi}(s)\,ds,\ \textit{for}\ \ l = 1,\, 2.$$

Proof of Theorem 3.8. Considering the decomposition:

$$\begin{aligned}\widehat{r}_n(\theta,\chi) - r(\theta,\chi) &= \frac{\widehat{r}_N(\theta,\chi) - r(\theta,\chi)\widehat{r}_D(\theta,\chi)}{\widehat{r}_D(\theta,\chi)} \\ &= \frac{\widehat{r}_N(\theta,\chi) - \mathbb{E}\left[\widehat{r}_N(\theta,\chi)\right] + \mathbb{E}\left[\widehat{r}_N(\theta,\chi)\right] - r(\theta,\chi)\widehat{r}_D(\theta,\chi)}{\widehat{r}_D(\theta,\chi)}. \end{aligned} \tag{3.3}$$

Using Eq (3.3), we conclude that the remaining task is to demonstrate the convergence in probability of the numerator to 1, the vanishing of the last term of the denominator, and the asymptotic normality of the first term of the denominator. □

Finally, we will need the following lemmas to state Theorem 3.8:

Lemma 3.9. *Under Assumptions (A1'), (A3'), (A4'-(i)), we have, as $n \to \infty$,*

$$\widehat{r}_D(\theta,\chi) - 1 \quad \to \quad 0,\ \textit{in probability}.$$

Lemma 3.10. *Under Assumptions (A0)-(A1'), (A2) and (A3')-(A4'), we have, as $n \to \infty$,*

$$(n\phi_{\theta,\chi}(h_n))^{1/2}\Big(\mathbb{E}\left[\widehat{r}_N(\theta,\chi)\right] - r(\theta,\chi)\widehat{r}_D(\theta,\chi)\Big) \to 0.$$

Lemma 3.11. *Under Assumptions (A0)-(A1') and (A3')-(A4'), we have, as $n \to \infty$,*

$$(n\phi_{\theta,\chi}(h_n))^{1/2}\Big(\widehat{r}_N(\theta,\chi) - \mathbb{E}\Big[\widehat{r}_N(\theta,\chi)\Big]\Big) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_r^2(\theta,\chi)).$$

Remark 3.12. *We allow C to be defective, that is, such that $\mathbb{P}(C = \infty)$ is possibly positive, to cover the uncensored case corresponding to the particular case where $\mathbb{P}(C = \infty) = 1$. It is to be noted that the uncensored case corresponds to the case $G(x) = 0$ for all $x \in \mathbb{R}$.*

4. Methodology for estimating of a single functional index

Recall that in practical applications, the functional index is typically unknown in practice and needs to be estimated. In this context, two important rules exist for carrying out the estimation. The initial one is the Least Squares Cross-Validation (LSCV) rule defined by

$$\widehat{\theta} = \arg\min_{\theta\in\Theta} \frac{1}{n}\sum_{i=1}^{n}(Y_i - \widehat{r}_n^{-i}(\theta, X_i))^2, \tag{4.1}$$

where $\widehat{r}_n^{-i}(\theta,\chi)$ is the leave-one-out estimator of $\widehat{r}_n(\theta,\chi)$. The second one is the Maximum Likelihood Cross-Validation (MLCV) rule expressed by

$$\hat{\theta} = \arg\min_{\theta\in\Theta} \frac{1}{n}\sum_{i=1}^{n}\log \widehat{f}(Y_i \mid \widehat{r}_n^{-i}(\theta, X_i)), \tag{4.2}$$

where $\widehat{f}(\cdot \mid \cdot)$ is the estimator of the conditional density of Y given $\langle\theta, X\rangle$. Of course, in practice we must optimize these rule over finite subset Θ of indices. According to [1], we propose to select the optimal index from the following subset

$$\Theta = \Theta_n = \left\{\theta \in \mathcal{H}, \theta = \sum_{i=1}^{k} c_i e_i,\ \|\theta\| = 1, \text{ and } \exists j \in [1,k] \text{ such that } \langle\theta, e_j\rangle > 0\right\}, \tag{4.3}$$

where $(e_i)_{i=1,\dots,k}$ are finite basis functions of the Hilbert subspace spanned by the covariates $(X_i)_i$, and $(c_i)_i$ some real calibrated constants allowing to ensure the model's identifiability. The common way is to choose the $(c_i)_i$ with calibration from the subset $\{-1, 0, 1\}$. In practice, it is shown that one can estimate any parameter by the cross-validation method. Another alternative, which will be adopted in this section, consists of selecting $\theta(t)$ among the eigenfunctions of the covariance operator $\mathbb{E}\left[(X' - \mathbb{E}(X'))\langle X', \cdot\rangle_{\mathcal{H}}\right]$, where $X(t)$ is, for instance, a diffusion-type process defined on a real interval $[a, b]$ and $X'(t)$ its first derivative see, for instance, [4]. Given a training sample $\mathcal{L}$, the covariance operator can be estimated by its empirical version $(1/|\mathcal{L}|)\sum_{i\in\mathcal{L}}\left(X_i' - \mathbb{E}X'\right)^t\left(X_i' - \mathbb{E}X'\right)$. Consequently, one can obtain a discretized version of the eigenfunctions $\theta_i(t)$ by applying the principle component analysis method, as in [37]. Let θ^* be the first eigenfunction corresponding to the highest eigenvalue of the empirical covariance operator, which will replace θ in the simulation steps to calculate the regression estimator. Lastly, it should be noted that there are numerous more concepts in vectorial statistics that can be extended to the functional case. We refer to the local linear approach and jump-preserving approaches proposed by [39]. The primary characteristic of this latest version is the concurrent estimate of the link function and the single index through the utilization of distinct scoring functions. Therefore, implementing this latest approach in functional data analysis has great promise in the current study. In addition to this approach, we can also utilize the concepts proposed by [72] to estimate the linear component of this model. We refer back to [74] for other alternative estimating methods in the linear model. Nevertheless, in our work we have concentrated on the estimation of the nonparametric function and have established the uniform consistency over a single index. We point out that the obtained uniform consistency constitutes a mathematical support for the convergence for any estimator of a single index (see, [1]).

As previously mentioned, the asymptotic variance $\sigma_r^2(\theta,\chi)$ given in Theorem 3.8 depends on the functions $C_1(\theta,\chi)$, $C_2(\theta,\chi)$, and $r(\theta,\chi)$. So, the estimation of $\sigma_r^2(\theta,\chi)$ will be basically obtained by estimating these functions, which are precisely based on the estimation of the functional parameter θ. Let us denote by θ^* the first eigenfunction corresponding to the first higher eigenvalue, and the estimator $\widehat{C}_l(\theta^*,\chi)$ defined as in [28, p. 44] by

$$\widehat{C}_l(\theta^*,\chi) = \frac{1}{n\widehat{\phi_{\theta^*}}(\chi, h_n)} \sum_{i=1}^{n} K_i^l(\theta^*,\chi),\ l = 1, 2\,, \tag{4.4}$$

where

$$\widehat{\phi_{\theta^*}}(\chi, h_n) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{|\langle \chi - X_i, \theta^* \rangle| < h_n\}}.$$

By applying the kernel estimator of $r(\theta^*,\chi)$ given above, the quantity $\sigma_r^2(\theta^*,\chi)$ can be estimated finally by:

$$\widehat{\sigma}_r^2(\theta^*,\chi) = \frac{\widehat{C}_2(\theta^*,\chi)\left(\widehat{r}(\theta^*,\chi)\right)^2}{\widehat{C}_1^{\,2}(\theta^*,\chi)}. \tag{4.5}$$

Corollary 4.1. *Under the Assumptions of Theorem 3.8, we have*

$$\sqrt{\frac{n\widehat{\phi_{\theta^*}}(\chi, h_n)}{\widehat{\sigma}_r^2(\theta^*,\chi)}}\left(\widehat{r}_n(\theta^*,\chi) - r(\theta^*,\chi)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1). \tag{4.6}$$

Thus, following Corollary 4.1, we can approximate $(1 - \zeta)$ confidence interval of $r(\theta^*,\chi)$ by

$$\widehat{r}_n(\theta^*,\chi) \pm t_{\zeta/2} \times \frac{\widehat{\sigma}(\theta^*,\chi)}{\sqrt{n\widehat{\phi_{\theta^*}}(\chi, h_n)}}, \tag{4.7}$$

where $t_{\zeta/2}$ is the upper $\zeta/2$ quantile of the standard normal variable $\mathcal{N}(0, 1)$. Using approximated constants directly from the limiting law to generate confidence bands might sometimes result in inadequate convergence qualities. It should be noted that this drawback can be circumvented by employing the functional variant of the conventional wild bootstrap technique. Within the context of censored functional data, the existing methods in the literature for regression estimations on functional data, such as the one proposed by [31], can be regarded in a similar manner. Nevertheless, it is necessary to provide the theoretical justification.

5. Monte Carlo experiments

5.1. A simulation study

In this section, we show the feasibility and the efficiency of the constructed estimator through a Monte Carlo experiment. Specifically, through this simulated data analysis we aim to:

(1) Examine the easy implementation of the constructed estimator;
(2) Evaluate the effect of the censorship as well as the regularity of the conditional distribution on the performance of the estimator.

For this empirical analysis, we generate a functional random variable by taking

$$X_i(t) = b_i \cos(\pi t) + \eta_i \ \text{ for all } t \in (0, 1) \text{ and for } i = 1, 2, \ldots, n = 150,$$

where b_i is generated according to the normal distribution $\mathcal{N}(1, 2)$ and η_i is drawn from the standard normal distribution $\mathcal{N}(2, 1)$. The curves are discretized on a grid generated from 100 equispaced measurements in $(0, 1)$.

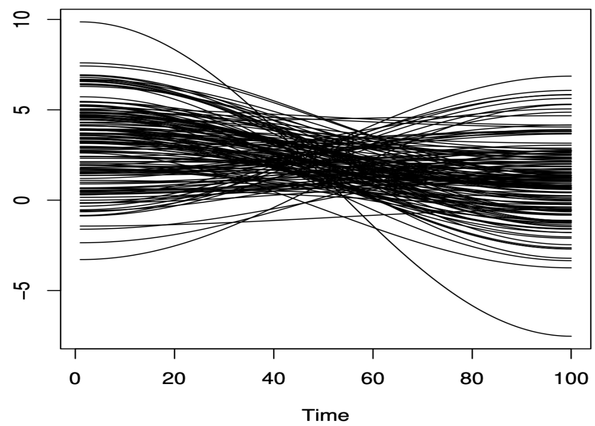

Figure 1. A sample of 150 curves.

Next, the scalar response Y is generated by a single index model as follows:

$$Y_i = r(\langle \theta, X_i \rangle) + \epsilon_i, \quad r(z) = \frac{1}{1 + z^2}, \quad \text{for} \quad i = 1, \ldots, n = 150, \tag{5.1}$$

where the errors (ϵ_i) are assumed to be independent of (X_i). As the conditional distribution of Y given $X = x$ is explicitly given by the law of the error ϵ shifted by $r(\langle \theta, X_i \rangle)$, we consider four types of errors which are generated from the following normal mixture distributions (cf. [57] for more details on the normal mixture distributions).

Law	Distribution function
Standard normal distribution (S.N.D.)	$\mathcal{N}(0, 1)$
Skewed bimodal distribution (C.B.D.)	$\frac{3}{4}\mathcal{N}(0, 1) + \frac{1}{4}\mathcal{N}(\frac{3}{2}, \frac{1}{9})$

Now, concerning the functional index scalar θ, we put $\theta = e_1$ as the first element of the Karhunen-Loève basis functions. Explicitly, θ is the eigenfunction associated to the first eigenvalue of the covariance operator of the process $(X_i)_i$. Clearly this index is an eligible functionals index because it belongs in the same Hilbert subspace of the functional variable and is an element of Θ_n (see the previous section). In practice, if this index is unknown, then we estimated it by using the rule (4.1 for $J = 5$. The second feature of our study concerns the censorship aspect. The latter is modeled by considering a censoring variable C distributed as an Exponential distribution $\text{Exp}(1/\lambda)$. Typically, the censoring rate is evaluated by the parameter λ. We choose three values of λ that are $\lambda = 2, 0.5$. Such values allowing to generate data with two censoring percentage 60% and 10%. Now the computationablity of the estimator as well as the effect of different parameters involved in its efficiency, such as the conditional model, the choice single index, the choice of the smoothing parameter, and the censorship are examined by drawing m independent n-samples of random pairs (X_i, Y_i) and we compute, for each sample, empirical values of the quantities

$$NT(\chi_0, \theta) = \sqrt{\frac{n\widehat{\phi_\theta}(\chi_0, h_n)}{\widehat{\sigma}_r^2(\hat{\theta}, \chi_0)}} \left(\widehat{r}_n(\hat{\theta}, \chi_0) - r(\theta, \chi_0)\right),$$

where χ_0 is an arbitrary conditioning curve X_{i_0} and $\widehat{\sigma}_r^2(\theta, \chi_0)$ is the plug-in estimator estimating the asymptotic variance expressed by (4.5). In this quantity, we select the optimal smoothing parameter by the cross-validation rule based on the Asymmetric Least Squares Error defined by

$$h_{CV} = \arg\min_{h_n \in H_n} \frac{1}{n} \sum_{i=1}^{n} (Y_i - \widehat{r}_n^{-i}(\theta, X_i))^2,$$

where H_n is the set of positive real numbers, h_n, such that, the ball centered at x with radius h_n contains exactly k neighbors of x. k is selected from the subset $H_n = \{5, 15, 25, \ldots, 55\}$. Moreover, we point out that the quantity $NT(\chi_0, \theta)$ is obtained by using the β-kernel. Finally, we highlight the impact of the choice of the single index in the estimation quality, we compare this result to an arbitrary choice of θ. Indeed, We repeat the same algorithm where $\hat{\theta} = e_3$ the third, of element of the Karhunen-Loève basis functions. After computing the m independent values of $(Z_i = NT_i(\chi_0, \hat{\theta}_{opt}, h_{opt}))_{i=1,\ldots,m}$, for all these situations we compare in Figures 2–5 the Q-Q plot of these quantities.

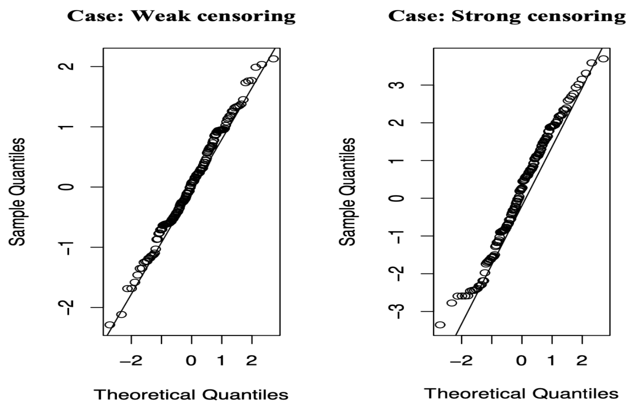

Figure 2. Optimal θ S.N.D. model.

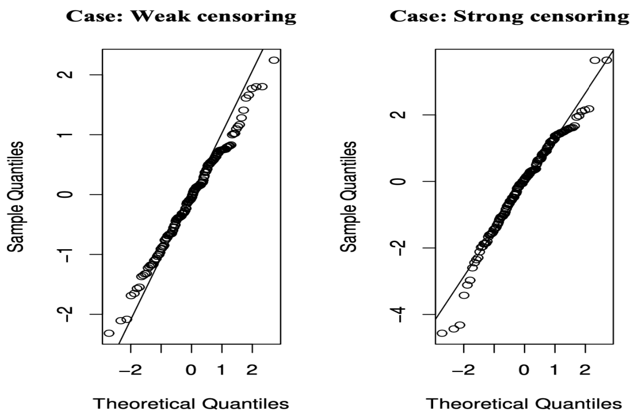

Figure 3. Optimal θ C.B.D. model.

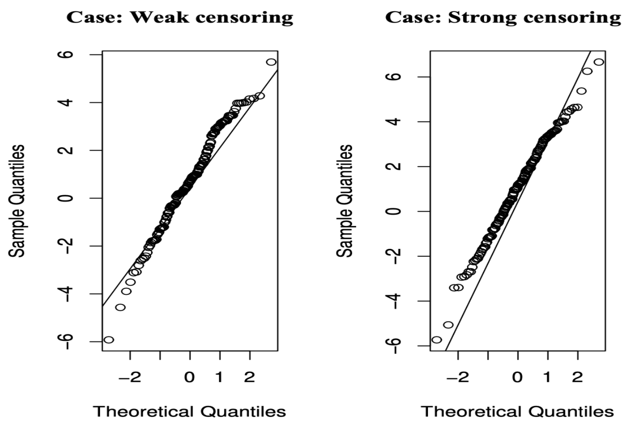

Figure 4. Arbitrary θ S.N.D. model.

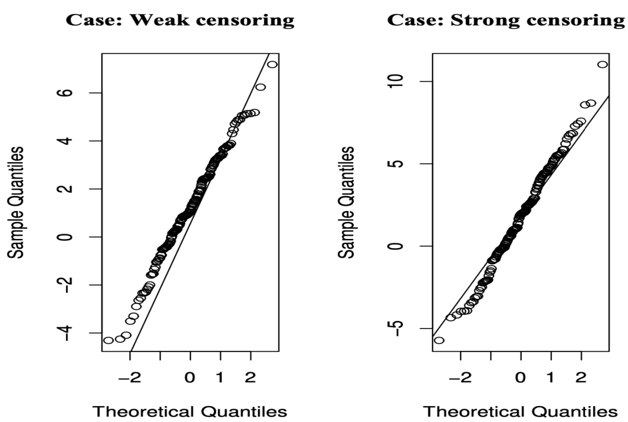

Figure 5. Arbitrary θ C.B.D. model.

Finally, we see clearly that the asymptotic behavior of the distribution of $\mathrm{NT}(\chi_0, \theta)$ is strongly affected by the choice of θ, the censorship rate, and the conditional distribution type even if the effect of the last is not significant compared to the the censoring character and the index choice.

5.2. Real data example

The objective of this section is to assess the performance of the censored functional version of the single index structure. This analysis complements the discussion in the previous section, where we evaluated the ease of implementing the constructed estimator for different levels of censoring. In this section, we return to emphasizing the importance of the functional single index model in practical applications. It is well-documented that the econometrics field is the most suitable domain for this type of semiparametric model (refer to [1] for further discussion). To illustrate the practical relevance, we consider financial data in the form of the intraday return of the Nikkei stock index during the period from January 1, 1984, to December 31, 2021. Given the nature of time-varying data, it can be treated as censored data, as trading in stock markets typically halts whenever the index falls below a certain threshold, especially during crisis periods like the Covid-19 pandemic. Considering the numerous stoppages during this period, the censoring rate of this data is highly significant. The data under study was obtained from the website `https://fred.stlouisfed.org/series/NIKKEI225`. To ensure a representative sample of independent functional data, we selected three separate months: January, May, and September. This selection resulted in a total of 111 functional observations. To address the heteroscedastic nature of the data, we opted for a transformed version obtained through logarithmic differencing. Typically, we proceed with

$$\text{For each selected month} \quad X_i(d) = -100 \log\left(\frac{s(d)}{s(d-1)}\right), \quad d \text{ is the days of the month,}$$

where $s(t)$ is the daily return. Therefore, our objective is to forecast the variation of $X(t)$ one month ahead based on the historical trajectory of the preceding month. Typically, we frame this forecasting problem by defining Y as $X(d_0) - X(d_{30})$. We then assess our prediction approach in comparison to the nonparametric regression model defined by

$$\widetilde{r}(x) = \sum_{i=1}^{n} \frac{\sum_{i=1}^{n} K(h^{-1}d(x, X_i))Y_i}{\sum_{j=1}^{n} K(h^{-1}d(x, X_j))}.$$

In conclusion, it is worth noting that we maintain consistent selection methods from the previous section, employing the same kernel. Nevertheless, due to the discontinuity of the functional variable, the selection of the PCA-metric becomes imperative. Additionally, the feasibility of both methods is assessed using the Mean Squared Error (MSE) defined by

$$MSE = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \widehat{R}_n^{-i}(X_i))^2,$$

where $\widehat{R}_n^{-i}(X_i)$ means either $\widehat{r}_n^{-i}(\theta, X_i)$ or $\widetilde{r}(x)$. Ultimately, we partition the data into two sets: a learning sample comprising 80 observations, and a testing sample with 31 observations. This observation split is repeated multiple times (50 times), and for each iteration, we compute the Mean Squared Error (MSE) for both estimators. The resulting errors are depicted in the boxplot shown in Figure 6.

As expected, the semiparametric approach outperforms the nonparametric one, a conclusion supported by the lower MSE values.

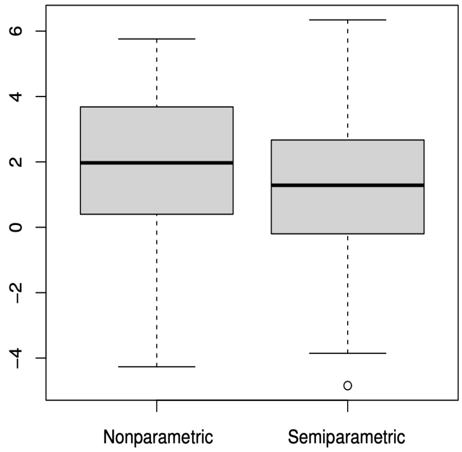

Figure 6. MSE of both estimator.

6. Concluding remarks

In this study, we investigated the nonparametric estimation challenge associated with the regression function in the Single Functional Index Model within the framework of random censoring. We focus on the kernel-type estimator for regression operator, achieving the strong almost-complete consistency along with its rate and the asymptotic normality. We apply our findings to construct confidence intervals. Finally, the usefulness of our proposed methodology is illustrated through both finite sample results and the analysis of real data. Multiple avenues exist for developing our method further. Observe that mixing is a type of asymptotic independence assumption that is commonly used in the pursuit of simplicity, but can be unrealistic in situations where the data are highly dependent. Extending non-parametric functional ideas to a general dependence structure is a discipline that is still in the beginning stages. Notably, the ergodic framework eschews the commonly employed strong mixing condition and its variants for measuring dependence, as well as the extremely involved probabilistic calculations that this condition necessitates. It would be intriguing to extend our work to the case of functional ergodic data, which requires nontrivial mathematics; however, this is well outside the scope of this paper. As a perspective on this work, one can relax the stationarity assumption we made in this paper and investigate similar limit theorems for local stationary functional processes. In this case, we allow the Single Functional Index estimator to change smoothly over time. It will be useful to establish similar results in this paper for data-driven bandwidth. The proof of such a statement, however, should require a different methodology than that used in the present paper, and we leave this problem open for future research. In a similar direction, it will be natural to consider in a future investigation the functional kNN local linear approach expectile regression estimators to obtain an alternative estimator which benefits from advantages of both methods, the local linear method and the kNN approach.

7. Proofs of theorems

This section is dedicated to establishing the proof of our main result. The notation introduced earlier will persist throughout the subsequent discussion.

Proof of Lemma 3.3. Using the fact that

$$\mathbb{E}\left[\widehat{r}_D(\theta,\chi)\right] = \mathbb{E}\left[\widehat{r}_D(t_{j(\theta)},\chi_{k(\chi)})\right] = 1,$$

we can write

$$\begin{aligned}\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi) - \mathbb{E}\left[\widehat{r}_D(\theta,\chi)\right]\right| &\leq \sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi) - \widehat{r}_D(\theta,\chi_{k(\chi)})\right| \\ &+ \sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi_{k(\chi)}) - \widehat{r}_D(t_{j(\theta)},\chi_{k(\chi)})\right| \\ &+ \sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(t_{j(\theta)},\chi_{k(\chi)}) - 1\right|. \end{aligned} \tag{7.1}$$

For the first and the second terms of the right-hand side of the inequality (7.1), we use the Hölder continuity condition of $K(\cdot)$ (in view of (A3-(ii))) and the Cauchy-Schwartz's inequality to get

$$\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi) - \widehat{r}_D(\theta,\chi_{k(\chi)})\right| = O\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right), \tag{7.2}$$

and

$$\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi_{k(\chi)}) - \widehat{r}_D(t_{j(\theta)},\chi_{k(\chi)})\right| = O\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right). \tag{7.3}$$

Clearly, we have by the Lipschitz property of $K(\cdot)$ imposed in (A3-(ii)) and the Cauchy-Schwartz's inequality, we can write

$$\begin{aligned}\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi) - \widehat{r}_D(\theta,\chi_{k(\chi)})\right| &\leq \frac{C}{\phi_{\theta,\chi}(h_n)}\sup_{\theta\in\Theta_{\mathcal{H}}}\frac{1}{n}\sum_{i=1}^{n}\left|K\left(\frac{d_\theta(\chi,X_i)}{h_n}\right) - K\left(\frac{d_\theta(\chi_{k(\chi)},X_i)}{h_n}\right)\right| \\ &\leq \frac{C}{h_n\phi_{\theta,\chi}(h_n)}\sup_{\theta\in\Theta_{\mathcal{H}}}\frac{1}{n}\sum_{i=1}^{n}\left|\left|\langle\theta,\chi - X_i\rangle\right| - \left|\langle\theta,\chi_{k(\chi)} - X_i\rangle\right|\right| \\ &\leq C\sup_{\theta\in\Theta_{\mathcal{H}}}\frac{\|\theta\|\cdot\|\chi - \chi_{k(\chi)}\|}{h_n\phi(h_n)} \\ &\leq C\frac{l_n}{\phi(h_n)}.\end{aligned}$$

By the fact that $l_n = O(\frac{\log n}{n})$, for n sufficiently large, we obtain

$$\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi)-\widehat{r}_D(\theta,\chi_{k(\chi)})\right| = O\left(\frac{\log n}{n\phi(h_n)}\right) = o\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right). \tag{7.4}$$

Similarly, following the same steps leading to (7.4), we derive

$$\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi_{k(\chi)})-\widehat{r}_D(t_{j(\theta)},\chi_{k(\chi)})\right| = O\left(\frac{\log n}{n\phi(h_n)}\right) = o\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right). \tag{7.5}$$

For the last term of the right-hand side of (7.1), we obtain

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(t_{j(\theta)},\chi_{k(\chi)})-1\right| > \eta\left(\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}\right)^{1/2}\right)\\
&\quad\leq\ \mathbb{P}\left(\max_{k=1,\ldots,d_n^{S_{\mathcal{H}}}}\ \max_{j=1,\ldots,d_n^{\Theta_{\mathcal{H}}}}\left|\widehat{r}_D(t_{j(\theta)},\chi_{k(\chi)})-1\right| > \eta\left(\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}\right)^{1/2}\right)\\
&\quad\leq\ C d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}\,\mathbb{P}\left(\left|\widehat{r}_D(t_{j(\theta)},\chi_{k(\chi)})-1\right| > \eta\left(\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}\right)^{1/2}\right).
\end{aligned}$$

We highlight that

$$\begin{aligned}
\widehat{r}_D(t_{j(\theta)},\chi_{k(\chi)})-1 &= \frac{1}{n}\sum_{i=1}^{n}\frac{K_i(t_{j(\theta)},\chi_{k(\chi)})-\mathbb{E}\left[K_1(t_{j(\theta)},\chi_{k(\chi)})\right]}{\mathbb{E}\left[K_1(t_{j(\theta)},\chi_{k(\chi)})\right]}\\
&= \frac{1}{n}\sum_{i=1}^{n}\Psi_i,
\end{aligned}$$

where

$$\Psi_i := \frac{K_i(t_{j(\theta)},\chi_{k(\chi)})-\mathbb{E}\left[K_1(t_{j(\theta)},\chi_{k(\chi)})\right]}{\mathbb{E}\left[K_1(t_{j(\theta)},\chi_{k(\chi)})\right]}.$$

By combining (A1) and (A3-(i)), for all $1 \leq i \leq n$, we readily obtain

$$|\Psi_i| \leq C/\phi(h_n) \quad\text{and}\quad \mathbb{E}\left[|\Psi_i|^2\right] \leq C'/\phi(h_n). \tag{7.6}$$

Applying the Bernstein-type inequality, for instance, see [66], p. 855, we get

$$\mathbb{P}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\Psi_i\right| > \eta\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right) \leq C'\exp\{-C\eta^2\log(d_n^{S_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})\}.$$

This readily implies that

$$\mathbb{P}\left(\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(t_{j(\theta)},\chi_{k(\chi)})-1\right| > \eta\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right) \ \leq\ C'(d_n^{S_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})^{1-C\eta^2}.$$

Choosing η large enough in such a way that $\beta = C\eta^2 > 2$ and using the condition (A5-*iii*) in combination with the Borel-Cantelli's lemma, we infer that

$$\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(t_{j(\theta)},\chi_{k(\chi)})-1\right| = O\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right). \tag{7.7}$$

This last result completes the proof of the lemma together with results (7.2) and (7.3). □

Proof of Corollary 3.4. For each $\chi \in S_{\mathcal{H}}$ and $\theta \in \Theta_{\mathcal{H}}$, we have

$$\inf_{\chi\in S_{\mathcal{H}}}\inf_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi)\right| \le \frac{1}{2} \quad\Rightarrow\quad \exists\chi\in S_{\mathcal{H}}\,\exists\theta\in\Theta_{\mathcal{H}} \text{ such that } \sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|1-\widehat{r}_D(\theta,\chi)\right| > \frac{1}{2}.$$

Consequently, we have

$$\mathbb{P}\left(\inf_{\chi\in S_{\mathcal{H}}}\inf_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi)\right| \le \frac{1}{2}\right) \le \mathbb{P}\left(\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|1-\widehat{r}_D(\theta,\chi)\right| > \frac{1}{2}\right).$$

Lemma 3.3 conclude the proof of the corollary. □

Proof of Lemma 3.5. We first notice that we have

$$\begin{aligned}
\left|\widehat{r}_N(\theta,\chi)-\widetilde{r}_N(\theta,\chi)\right| &= \frac{1}{n\mathbb{E}[K_1(\theta,\chi)]}\sum_{i=1}^n\left|\left(\frac{\delta_iY_i}{\overline{G}_n(Y_i)}-\frac{\delta_iY_i}{\overline{G}(Y_i)}\right)K_i(\theta,\chi)\right| \\
&= \frac{1}{n\mathbb{E}[K_1(\theta,\chi)]}\sum_{i=1}^n\left|\left(\frac{\mathbb{1}_{\{Y_i^0\le C_i\}}Y_i^0}{\overline{G}_n(Y_i^0)}-\frac{\mathbb{1}_{\{Y_i^0\le C_i\}}Y_i^0}{\overline{G}(Y_i^0)}\right)K_i(\theta,\chi)\right| \\
&\le \frac{1}{n\mathbb{E}[K_1(\theta,\chi)]}\sum_{i=1}^n|Y_i^0K_i(\theta,\chi)|\left|\left(\frac{\overline{G}_n(Y_i^0)-\overline{G}(Y_i^0)}{\overline{G}_n(Y_i^0)\overline{G}(Y_i^0)}\right)\right| \\
&\le \frac{\tau_F}{\overline{G}_n(\tau_F)\overline{G}(\tau_F)}\sup_{t<\tau_F}\left|\overline{G}_n(t)-\overline{G}(t)\right|\frac{1}{n\mathbb{E}[K_1(\theta,\chi)]}\sum_{i=1}^n K_i(\theta,\chi) \\
&\le \widehat{r}_D(\theta,\chi)\frac{\tau_F}{\overline{G}_n(\tau_F)\overline{G}(\tau_F)}\sup_{t<\tau_F}\left|\overline{G}_n(t)-\overline{G}(t)\right| \\
&\le (\widehat{r}_D(\theta,\chi)-1)\frac{\tau_F}{\overline{G}_n(\tau_F)\overline{G}(\tau_F)}\sup_{t<\tau_F}\left|\overline{G}_n(t)-\overline{G}(t)\right| \\
&\quad +\frac{\tau_F}{\overline{G}_n(\tau_F)\overline{G}(\tau_F)}\sup_{t<\tau_F}\left|\overline{G}_n(t)-\overline{G}(t)\right|.
\end{aligned}$$

This readily implies that

$$\begin{aligned}
&\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_N(\theta,\chi)-\widetilde{r}_N(\theta,\chi)\right| \\
&\le \sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_D(\theta,\chi)-1\right|\frac{\tau_F}{\overline{G}_n(\tau_F)\overline{G}(\tau_F)}\sup_{t<\tau_F}\left|\overline{G}_n(t)-\overline{G}(t)\right| \\
&\quad +\frac{\tau_F}{\overline{G}_n(\tau_F)\overline{G}(\tau_F)}\sup_{t<\tau_F}\left|\overline{G}_n(t)-\overline{G}(t)\right|.
\end{aligned}$$

By Lemma 3.3 in conjunction with result of [32] or [23], that is :

$$\sup_{t<\tau_F}\left|\overline{G}_n(t)-\overline{G}(t)\right| = O\left(\left(\frac{\log_2 n}{n}\right)^{1/2}\right),$$

which implies as $n\to\infty$, $\overline{G}_n(\tau_F)\overline{G}(\tau_F)\sim(\overline{G}(\tau_F))^2$. Since

$$\frac{\log_2 n}{n} = o\left(\frac{\log n}{n\phi(h_n)}\right),$$

by (A4-(ii)), we get

$$\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widehat{r}_N(\theta,\chi)-\widetilde{r}_N(\theta,\chi)\right| = o\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right).$$

Hence, the proof is complete. □

Proof of Lemma 3.3. By stationarity, conditioning w.r.t $\langle\theta,X_1\rangle$ and the fact $\mathbb{E}\left[\mathbb{1}_{\{Y_1^0\le C_1\}}|Y_1^0\right]=\overline{G}(Y_1^0)$, we readily infer

$$\begin{aligned}
&\mathbb{E}\left[\widetilde{r}_N(\theta,\chi)\right]-r(\theta,\chi)\\
&= \frac{1}{\mathbb{E}[K_1(\theta,\chi)]}\mathbb{E}\left[\frac{\delta_1 Y_1}{\overline{G}(Y_1)}K_1(\theta,\chi)\right]-r(\theta,\chi)\\
&= \mathbb{E}\left[\frac{\mathbb{E}\left[\frac{\delta_1 Y_1}{\overline{G}(Y_1)}\,\Big|\langle\theta,X_1\rangle\right]K_1(\theta,\chi)-r(\theta,\chi)\mathbb{E}[K_1(\theta,\chi)]}{\mathbb{E}[K_1(\theta,\chi)]}\right]\\
&= \mathbb{E}\left[\frac{\mathbb{E}\left[\frac{Y_1^0}{\overline{G}(Y_1^0)}\mathbb{E}\left\{\mathbb{1}_{\{Y_1^0\le C_1\}}|Y_1^0\right\}\,\Big|\langle\theta,X_1\rangle\right]K_1(\theta,\chi)-r(\theta,\chi)\mathbb{E}[K_1(\theta,\chi)]}{\mathbb{E}[K_1(\theta,\chi)]}\right]\\
&= \mathbb{E}\left[\frac{r(\theta,X_1)K_1(\theta,\chi)-r(\theta,\chi)\mathbb{E}[K_1(\theta,\chi)]}{\mathbb{E}[K_1(\theta,\chi)]}\right]\\
&= \frac{1}{\mathbb{E}[K_1(\theta,\chi)]}\mathbb{E}\left[K_1(\theta,\chi)\mathbb{1}_{\{B_\theta(\chi,h_n)\}}(X_1)\left(r(\theta,X_1)-r(\theta,\chi)\right)\right].
\end{aligned}$$

Due to the Lipschitz condition of the function $r(\cdot,\cdot)$ given in assumption (A2), we get

$$\begin{aligned}
\mathbb{1}_{B_\theta(\chi,h_n)}(X_1)|r(\theta,X_1)-r(\theta,\chi)| &\le Cd_\theta(\chi,X_1)^\gamma\\
&\le Ch_n^\gamma.
\end{aligned}$$

Finally, uniformly on χ and θ, one can write

$$\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\mathbb{E}\left[\widetilde{r}_N(\theta,\chi)\right]-r(\theta,\chi)\right| = O(h_n^\gamma). \tag{7.8}$$

This completes the proof. □

Proof of Lemma 3.5. Consider the following decomposition:

$$
\begin{aligned}
&\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widetilde{r}_N(\theta,\chi)-\mathbb{E}\Big[\widetilde{r}_N(\theta,\chi)\Big]\right|\\
&\leq \underbrace{\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widetilde{r}_N(\theta,\chi)-\widetilde{r}_N(\theta,\chi_{k(\chi)})\right|}_{F_1}\\
&\quad+\underbrace{\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widetilde{r}_N(\theta,\chi_{k(\chi)})-\widetilde{r}_N(t_{j(\theta)},\chi_{k(\chi)})\right|}_{F_2}\\
&\quad+\underbrace{\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widetilde{r}_N(t_{j(\theta)},\chi_{k(\chi)})-\mathbb{E}\left[\widetilde{r}_N(t_{j(\theta)},\chi_{k(\chi)})\right]\right|}_{F_3}\\
&\quad+\underbrace{\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\mathbb{E}\left[\widetilde{r}_N(t_{j(\theta)},\chi_{k(\chi)})\right]-\mathbb{E}\left[\widetilde{r}_N(\theta,\chi_{k(\chi)})\right]\right|}_{F_4}\\
&\quad+\underbrace{\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\mathbb{E}\left[\widetilde{r}_N(\theta,\chi_{k(\chi)})\right]-\mathbb{E}\Big[\widetilde{r}_N(\theta,\chi)\Big]\right|}_{F_5}.
\end{aligned}
$$

As both F_1 and F_5 can be treated in a similar way, we only deal with F_1. Indeed, by (A3-(ii)) and the Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
&\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widetilde{r}_N(\theta,\chi)-\widetilde{r}_N(\theta,\chi_{k(\chi)})\right|\\
&\leq \frac{1}{n}\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\sum_{i=1}^{n}\frac{\delta_i|Y_i|}{\overline{G}(Y_i)}\left|\frac{K_i(\theta,\chi)}{\mathbb{E}\left[K_1(\theta,\chi)\right]}-\frac{K_i(\theta,\chi_{k(\chi)})}{\mathbb{E}\left[K_1(\theta,\chi_{k(\chi)})\right]}\right|\\
&\leq \frac{\tau_F}{n\phi(h_n)}\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\sum_{i=1}^{n}\frac{\mathbb{1}_{\{Y_i^0\leq C_i\}}|Y_i^0|}{\overline{G}(Y_i^0)}\left|K_i(\theta,\chi)-K_i(\theta,\chi_{k(\chi)})\right|\\
&\leq \frac{C\tau_F}{\overline{G}(\tau_F)}\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\frac{\|\theta\|\cdot\|\chi-\chi_{k(\chi)}\|}{h_n\phi(h_n)}\\
&\leq \frac{C\tau_F}{\overline{G}(\tau_F)}\frac{l_n}{\phi(h_n)}.
\end{aligned}
$$

Thus, by assumption (A4-(ii)), since $l_n=O\left(\frac{\log n}{n}\right)$, by Jensen's inequality, as n tends to infinity, we get

$$
F_5\leq F_1=O\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right). \tag{7.9}
$$

Analogously, dealing with F_2 and F_4 in the same way, one may easily obtain

$$
\sup_{\chi\in S_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\left|\widetilde{r}_N(\theta,\chi)-\widetilde{r}_N(\theta,\chi_{k(\chi)})\right|
$$

$$
\begin{aligned}
&\leq \frac{C}{n\phi(h_n)} \sup_{\chi \in S_{\mathcal{H}}} \sup_{\theta \in \Theta_{\mathcal{H}}} \sum_{i=1}^{n} \frac{\mathbb{1}_{\{Y_i^0 \leq C_i\}} |Y_i^0|}{\overline{G}(Y_i^0)} \left| K_i(\theta, \chi) - K_i(t_{j(\theta)}, \chi) \right| \\
&\leq \frac{C\tau_F}{\overline{G}(\tau_F)} \sup_{\chi \in S_{\mathcal{H}}} \sup_{\theta \in \Theta_{\mathcal{H}}} \frac{\|\chi\| \cdot \|\theta - t_{j(\theta)}\|}{h_n \phi(h_n)} \\
&\leq \frac{C\tau_F}{\overline{G}(\tau_F)} \frac{l_n}{\phi(h_n)} \\
&= O\left(\frac{\log n}{n\phi(h_n)} \right).
\end{aligned}
$$

Hence, we conclude that

$$
F_4 \leq F_2 = O\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}} \right). \tag{7.10}
$$

Now, our focus turns to the upper bound of the term F_3. On the one hand, we express

$$
\begin{aligned}
\mathbb{P}(F_3 > \epsilon) &\leq d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}} \max_{1 \leq k \leq d_n^{S_{\mathcal{H}}}} \max_{1 \leq j \leq d_n^{\Theta_{\mathcal{H}}}} \mathbb{P}\left(\left| \widetilde{r}_N(t_{j(\theta)}, \chi_{k(\chi)}) - \mathbb{E}\left[\widetilde{r}_N(t_{j(\theta)}, \chi_{k(\chi)}) \right] \right| > \epsilon \right) \\
&\leq d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}} \max_{1 \leq k \leq d_n^{S_{\mathcal{H}}}} \max_{1 \leq j \leq d_n^{\Theta_{\mathcal{H}}}} \mathbb{P}\left(\left| \sum_{i=1}^{n} \widetilde{\Psi}_i \right| > n\epsilon \right),
\end{aligned}
$$

with

$$
\widetilde{\Psi}_i := \frac{\delta_i \, Y_i K_i(t_{j(\theta)}, \chi_{k(\chi)}) - \mathbb{E}\left[\delta_i \, Y_i K_i(t_{j(\theta)}, \chi_{k(\chi)}) \right]}{\overline{G}(Y_i) \mathbb{E}\left[K_1(t_{j(\theta)}, \chi_{k(\chi)}) \right]}.
$$

On the other hand, in order to apply the Bernstein-type exponential inequality we need to bound the two quantities $\left| \widetilde{\Psi}_i \right|$ and $\mathbb{E}\left| \widetilde{\Psi}_i \right|^2$, for each $i = 1, \ldots, n$. Indeed, because of (A1) and (A3-(i)), by the fact that

$$
\mathbb{E}\left[\mathbb{1}_{\{Y_i^0 \leq C_i\}} | Y_i^0 \right] = \overline{G}(Y_i^0),
$$

we have

$$
\begin{aligned}
\left| \widetilde{\Psi}_i \right| &= \left| \frac{\mathbb{1}_{\{Y_i^0 \leq C_i\}} Y_i^0 K_i(t_{j(\theta)}, \chi_{k(\chi)}) - \mathbb{E}\left[\mathbb{1}_{\{Y_i^0 \leq C_i\}} Y_i^0 K_i(t_{j(\theta)}, \chi_{k(\chi)}) \right]}{\overline{G}(Y_i^0) \mathbb{E}\left[K_1(t_{j(\theta)}, \chi_{k(\chi)}) \right]} \right| \\
&= \left| \frac{Y_i^0 K_i(t_{j(\theta)}, \chi_{k(\chi)}) - \mathbb{E}\left[Y_i^0 K_i(t_{j(\theta)}, \chi_{k(\chi)}) \right]}{\mathbb{E}\left[K_1(t_{j(\theta)}, \chi_{k(\chi)}) \right]} \right| \\
&\leq \tau_F \left(\left| \frac{K_i(t_{j(\theta)}, \chi_{k(\chi)})}{\mathbb{E}\left[K_1(t_{j(\theta)}, \chi_{k(\chi)}) \right]} \right| + 1 \right) \\
&= O\left(\frac{1}{\phi(h_n)} \right).
\end{aligned}
$$

By assumption (A1-(ii)) and the fact that, for each θ, χ and i, as n goes to ∞,

$$\mathbb{E}\left[K_i^l(\theta,\chi)\right] \to C_l\phi_{\theta,\chi}(h_n),\ l \in \{1,2\}, \tag{7.11}$$

for details see [30]. This readily implies that

$$\begin{aligned}
\mathbb{E}\left|\widetilde{\Psi}_i\right|^2 &= \mathbb{E}\left|\frac{\mathbb{1}_{\{Y_i^0\le C_i\}}Y_i^0K_i(t_{j(\theta)},\chi_{k(\chi)}) - \mathbb{E}\left[\mathbb{1}_{\{Y_i^0\le C_i\}}Y_i^0K_i(t_{j(\theta)},\chi_{k(\chi)})\right]}{\overline{G}(Y_i^0)\mathbb{E}\left[K_1(t_{j(\theta)},\chi_{k(\chi)})\right]}\right|^2 \\
&\le \mathbb{E}\left[\frac{\left(\frac{\mathbb{1}_{\{Y_i^0\le C_i\}}Y_i^0}{\overline{G}(Y_i^0)}\right)^2 K_i^2(t_{j(\theta)},\chi_{k(\chi)})}{\mathbb{E}^2\left[K_1(t_{j(\theta)},\chi_{k(\chi)})\right]}\right] \\
&\le \mathbb{E}\left[\frac{\left(\frac{\mathbb{E}\left[\mathbb{1}_{\{Y_i^0\le C_i\}}|Y_i^0\right]Y_i^0}{\overline{G}(Y_i^0)}\right)^2 K_i^2(t_{j(\theta)},\chi_{k(\chi)})}{\mathbb{E}^2\left[K_1(t_{j(\theta)},\chi_{k(\chi)})\right]}\right] \\
&\le \left(\frac{Y_i^0}{\mathbb{E}\left[K_1(t_{j(\theta)},\chi_{k(\chi)})\right]}\right)^2 \mathbb{E}\left[K_1^2(t_{j(\theta)},\chi_{k(\chi)})\right] \\
&\le \left(\frac{\tau_F}{C_1\phi_{\theta,\chi}(h_n)}\right)^2\left(C_2\phi_{\theta,\chi}(h_n)\right) \\
&= O\left(\frac{1}{\phi(h_n)}\right).
\end{aligned}$$

Now, letting $\epsilon = \eta\sqrt{\dfrac{\log d_n^{S_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\phi(h_n)}}$, and applying Bernstein's inequality, we get

$$\begin{aligned}
\mathbb{P}\left(F_3 > \eta\sqrt{\frac{\log d_n^{S_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\phi(h_n)}}\right) &\le C' d_n^{S_\mathcal{H}} d_n^{\Theta_\mathcal{H}} \exp\left\{-C\eta^2\log(d_n^{S_\mathcal{H}} d_n^{\Theta_\mathcal{H}})\right\} \\
&\le C'\left(d_n^{S_\mathcal{H}} d_n^{\Theta_\mathcal{H}}\right)^{1-C\eta^2}.
\end{aligned}$$

By choosing $\beta = C\eta^2$ and using (A4-(ii)), we get

$$F_3 = O_{a.co.}\left(\sqrt{\frac{\log d_n^{S_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\phi(h_n)}}\right). \tag{7.12}$$

Therefore, Lemma 3.5 follows from results (7.9), (7.10) and (7.12). □

Proof of Lemma 3.9. It is clear that the result of Lemma 3.7 allows us to write

$$\widehat{r}_D(\theta,\chi) - 1 = O\left(\left(\frac{\log n}{n\phi_{\theta,\chi}(h_n)}\right)^{1/2}\right),$$

almost completely, which implies that $\widehat{r}_D(\theta,\chi)$ converges to 1, in probability as n goes to ∞. □

Proof of Lemma 3.10. By the result of Lemma 3.9, and the fact that $\overline{G}_n(t)$ converges asymptotically to $\overline{G}(t)$, for all $t < \tau_F$, and thanks to the result (7.8), under assumption (A4'-*ii*), it follows that

$$\left(n\phi_{\theta,\chi}(h_n)\right)^{1/2}\left(\mathbb{E}\left[\widehat{r}_n(\theta,\chi)\right] - \widehat{r}_D(\theta,\chi)r(\theta,\chi)\right) \quad = \quad O\left(\sqrt{n\phi_{\theta,\chi}(h_n)h_n^{2\gamma}}\right) \to 0, \text{ as } n \to \infty.$$

This completes the proof of the lemma. □

Proof of Lemma 3.11. For all θ, $\chi \in \mathcal{H}$, we define

$$Q_i(\theta,\chi) = \frac{\sqrt{\phi_{\theta,\chi}(h_n)}}{\sqrt{n}\mathbb{E}\left[K_1(\theta,\chi)\right]}\left(\frac{\delta_i Y_i}{\overline{G}_n(Y_i)}K_i(\theta,\chi) - \mathbb{E}\left[\frac{\delta_1 Y_1}{\overline{G}_n(Y_1)}K_1(\theta,\chi)\right]\right),$$

and

$$S_n := \sum_{i=1}^{n} Q_i(\theta,\chi).$$

We have

$$S_n = \sqrt{n\phi_{\theta,\chi}(h_n)}\left(\widehat{r}_N(\theta,\chi) - \mathbb{E}\left[\widehat{r}_N(\theta,\chi)\right]\right).$$

So, our claimed result is now

$$S_n \to \mathcal{N}(0, \sigma_r^2(\theta,\chi)). \tag{7.13}$$

Therefore, we have

$$\begin{aligned}
\operatorname{Var}(S_n) &= n\phi_{\theta,\chi}(h_n)\operatorname{Var}\left(\widehat{r}_N(\theta,\chi) - \mathbb{E}\left[\widehat{r}_N(\theta,\chi)\right]\right) \\
&= n\phi_{\theta,\chi}(h_n)\operatorname{Var}\left(\widehat{r}_N(\theta,\chi)\right).
\end{aligned}$$

Now, we have by conditioning w.r.t $\langle\theta, X_1\rangle$, and taking into account the stationarity property and the fact that $\overline{G}_n(t) \to \overline{G}(t)$, we have

$$\begin{aligned}
\operatorname{Var}(\widehat{r}_N(\theta,\chi)) &= \frac{1}{(n\mathbb{E}\left[K_1(\theta,\chi)\right])^2}\operatorname{Var}\left(\sum_{i=1}^{n}\frac{\delta_i Y_i}{\overline{G}_n(Y_i)}K_i(\theta,\chi)\right) \\
&= \frac{\left(\mathbb{E}\left[\frac{\delta_1 Y_1}{\overline{G}(Y_1)}\,\Big|\,\langle\theta, X_1\rangle\right]\right)^2}{(n\mathbb{E}\left[K_1(\theta,\chi)\right])^2}\operatorname{Var}\left(\sum_{i=1}^{n} K_i(\theta,\chi)\right) \\
&= (r(\theta,\chi))^2\left\{\frac{\mathbb{E}\left[K_1^2(\theta,\chi)\right]}{n(\mathbb{E}\left[K_1(\theta,\chi)\right])^2} - \frac{1}{n}\right\}.
\end{aligned}$$

Thanks to the result (7.11), we have

$$\begin{aligned}
\mathrm{Var}(\widehat{r_N}(\theta,\chi)) &= (r(\theta,\chi))^2\left[\frac{C_2\phi_{\theta,\chi}(h_n)}{n\left(C_1\phi_{\theta,\chi}(h_n)\right)^2}-\frac{1}{n}\right]\\
&= (r(\theta,\chi))^2\left[\frac{C_2}{nC_1^2\phi_{\theta,\chi}(h_n)}+o\left(\frac{1}{n\phi_{\theta,\chi}(h_n)}\right)\right]\\
&= (r(\theta,\chi))^2\left[\frac{C_2}{C_1^2}+o\left(1\right)\right]\frac{1}{n\phi_{\theta,\chi}(h_n)}, \qquad (7.14)
\end{aligned}$$

and finally

$$\begin{aligned}
\mathrm{Var}(S_n) &= n\phi_{\theta,\chi}(h_n)\,(r(\theta,\chi))^2\left[\frac{C_2}{C_1^2}+o\left(1\right)\right]\frac{1}{n\phi_{\theta,\chi}(h_n)}\\
&= (r(\theta,\chi))^2\left[\frac{C_2}{C_1^2}+o\left(1\right)\right]\\
&\to C_2/C_1^2\,(r(\theta,\chi))^2 =: \sigma_r^2(\theta,\chi),\ \text{as } n\to\infty.
\end{aligned}$$

Now, with $Z_i(\theta,\chi)=\dfrac{1}{n\mathbb{E}[K_1(\theta,\chi)]}\dfrac{\delta_iY_i}{\overline{G}_n(Y_i)}K_i(\theta,\chi)$, the only remaining task is, for some $l>0$, to prove

$$\frac{\sum_{i=1}^{n}\mathbb{E}\left[\left|Z_i(\theta,\chi)-\mathbb{E}Z_i(\theta,\chi)\right|^{2+l}\right]}{\left(\mathrm{Var}\left[\sum_{i=1}^{n}Z_i(\theta,\chi)\right]\right)^{(2+l)/2}}\longrightarrow 0,\ \text{ as } n\to\infty. \qquad (7.15)$$

Thanks to result (7.14), it is clear that

$$n\phi_{\theta,\chi}(h_n)\mathrm{Var}\left[\sum_{i=1}^{n}Z_i(\theta,\chi)\right]=n\phi_{\theta,\chi}(h_n)\mathrm{Var}\left(\widehat{r_N}(\theta,\chi)\right),$$

converges to $\frac{C_2}{C_1^2}\,(r(\theta,\chi))^2=\sigma_r^2(\theta,\chi)$, as $n\to\infty$. So, it remains to prove that, as $n\to\infty$:

$$(n\phi_{\theta,\chi}(h_n))^{(2+l)/2}\sum_{i=1}^{n}\mathbb{E}\left[|Z_i(\theta,\chi)-\mathbb{E}Z_i(\theta,\chi)|^{2+l}\right]\to 0.$$

Since $\left|\frac{\delta_iY_i}{\overline{G}_n(Y_i)}\right|<\tau_F$, and by the use of the elementary inequality : $(a+b)^p\le 2^{p-1}(a^p+b^p)$, $\forall a,b\ge 0$, $\forall p\ge 1$, we arrive at

$$\begin{aligned}
&(n\phi_{\theta,\chi}(h_n))^{(2+l)/2}\sum_{i=1}^{n}\mathbb{E}\left[|Z_i(\theta,\chi)-\mathbb{E}Z_i(\theta,\chi)|^{2+l}\right]\\
&\le (n\phi_{\theta,\chi}(h_n))^{(2+l)/2}\left(\frac{\tau_F}{C_1n\phi_{\theta,\chi}(h_n)}\right)^{2+l}\sum_{i=1}^{n}\mathbb{E}\left[\left(\left|K_i(\theta,\chi)\right|+\left|\mathbb{E}\left[K_i(\theta,\chi)\right]\right|\right)^{2+l}\right]
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\frac{\tau_F}{C_1}\right)^{2+l} \frac{1}{(n\phi_{\theta,\chi}(h_n))^{(2+l)/2}} \sum_{i=1}^{n} \mathbb{E}\left[\left(\left|K_i(\theta,\chi)\right| + \left|\mathbb{E}\left[K_i(\theta,\chi)\right]\right|\right)^{2+l}\right] \\
&\le 2^{1+l}\left(\frac{\tau_F}{C_1}\right)^{2+l} \frac{1}{(n\phi_{\theta,\chi}(h_n))^{(2+l)/2}} \sum_{i=1}^{n} \mathbb{E}\left[\left(\left|K_i(\theta,\chi)\right|^{2+l} + \left|\mathbb{E}\left[K_i(\theta,\chi)\right]\right|^{2+l}\right)\right] \\
&\le 2^{1+l}\left(\frac{\tau_F}{C_1}\right)^{2+l} \frac{1}{(n\phi_{\theta,\chi}(h_n))^{(2+l)/2}} n\left(\mathbb{E}\left[\left|K_1(\theta,\chi)\right|^{2+l}\right] + \left|\mathbb{E}\left[K_1(\theta,\chi)\right]\right|^{2+l}\right) \\
&\le 2^{1+l}\left(\frac{\tau_F}{C_1}\right)^{2+l} \frac{1}{(n\phi_{\theta,\chi}(h_n))^{(2+l)/2}} n\left(C\phi_{\theta,\chi}(h_n) + \left(\phi_{\theta,\chi}(h_n)\right)^{2+l}\right) \\
&\le 2^{1+l}\left(\frac{\tau_F}{C_1}\right)^{2+l} \frac{1}{(n\phi_{\theta,\chi}(h_n))^{(2+l)/2}} n\phi_{\theta,\chi}(h_n)\left(C + o(1)\right) \\
&= O\left(\frac{1}{(n\phi_{\theta,\chi}(h_n))^{l/2}}\right) \longrightarrow 0.
\end{aligned}
$$

The last convergence is a consequence of A4'(i). This finishes the proof of the lemma. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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