AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 7111–7130.
DOI: 10.3934/math.2024347
Received: 07 December 2023
Revised: 28 January 2024
Accepted: 01 February 2024
Published: 19 February 2024

Research article

The multiplicative degree-Kirchhoff index and complexity of a class of linear networks

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Abstract: In this paper, we focus on the strong product of the pentagonal networks. Let R_n be a pentagonal network composed of $2n$ pentagons and n quadrilaterals. Let P_n^2 denote the graph formed by the strong product of R_n and its copy R_n'. By utilizing the decomposition theorem of the normalized Laplacian characteristics polynomial, we characterize the explicit formula of the multiplicative degree-Kirchhoff index completely. Moreover, the complexity of P_n^2 is determined.

Keywords: pentagonal network; strong product; the normalized Laplacian; multiplicative degree-Kirchhoff index; complexity
Mathematics Subject Classification: 05C50, 05C90

1. Introduction

Graph categories considered in this study are simple, finite, and linked. Allow the graph G to be made up of V_G and edge set E_G, i.e., $G = (V_G, E_G)$. For more graph notations, readers should refer to [1].

If and only if two neighbouring verticies i and j of G, the adjacency matrix $A(G) = (a_{ij})$ is a (0,1)-matrix, $a_{ij} = 1$. The diagonal degree matrix of G is

$$D_G = \mathrm{diag}(d_1, d_2, \cdots, d_n),$$

where d_i represents the degree of vertex i in G. The difference between the degree matrix D_G and adjacency matrix A_G of G gives rise to the Laplace matrix, denoted as L_G. The normalized Laplacian [2] is defined as

$$\mathcal{L}(G) = I - D(G)^{\frac{1}{2}}(D(G)^{-1}A(G))D(G)^{-\frac{1}{2}} = D(G)^{-\frac{1}{2}}L(G)D(G)^{-\frac{1}{2}}.$$

The (m,n)th-entry of $\mathcal{L}(G)$, which is designated

$$(\mathcal{L}(G))_{mn} = \begin{cases} 1, & m = n; \\ -\frac{1}{\sqrt{d_m d_n}}, & m \neq n\ ,\ v_m\ is\ adjacent\ to\ v_n; \\ 0, & otherwise. \end{cases}$$

The distance between both v_i and v_j, known as $d_{ij} = d_G(v_i, v_j)$, represents the length of the smallest path in question. Wiener and Dobrynin [3,4] introduced the Wiener index for the first time in 1947. In addition, the Wiener index is denoted as

$$W(G) = \sum_{i<j} d_{ij}.$$

For further information on the Wiener index, please refer to [5–9].

The Gutman index of a simple graph G is introduced [10] and denoted as

$$Gut(G) = \sum_{i<j} d_i d_j d_{ij},$$

taking into account the degree d_i of vertex v_i.

The Kirchhoff index [11, 12] characterizes graph G by summing the resistance distances between every pair of vertices, similar to the Wiener index, namely

$$Kf(G) = \sum_{i<j} r_{ij}.$$

The multiplicative degree-Kirchhoff index, initially introduced by Chen and Zhang [13] in 2007. It is an extension of the traditional Kirchhoff index, which is expressed as

$$Kf^*(G) = \sum_{i<j} d_i d_j r_{ij}.$$

The techniques of the Kirchhoff index and multiplication degree-Kirchhoff index can be found in [14–18]. The multiplication degree-Kirchhoff index has garnered significant attention due to its remarkable contributions in academia and practical applications in computer network science, epidemiology, social economics, and other fields. Further research results on the Kirchhoff index and multiplication degree-Kirchhoff index can be explored through [19–23].

The spanning tree of a graph G, also known as complexity, denoted as $\tau(G)$, refers to the number of subgraphs that encompass all vertices in G. This measure serves as a crucial indicator for network stability and plays a significant role in assessing the structural characteristics of graphs. For further insights into related topics such as the count of spanning trees, interested readers are encouraged to consult [24–26].

With the rapid advancement of scientific research and the successful application of topology in practical scenarios, topological theory has gained increasing recognition worldwide. The calculation problem concerning the phenylene Wiener index has been effectively resolved by Pavlović and Gutman [27]. Chen and Zhang [28] have developed a precise equation for predicting the Wiener index of random phenylene chains. Additionally, Liu et al. [29] have identified both the degree-Kirchhoff index and the number of spanning trees for L_n dicyclobutadieno derivatives of $[n]$ phenylenes.

Given two automorphic graphs S and K, we define the symbol $S \boxtimes K$ to represent the strong product of these two graphs with $V(S) \times V(K)$, which is commonly referred to as the strong product in graph theory literature. Readers can refer to [30] for more comprehensive definitions and concepts. Recently, Pan et al. [25] utilized the resistance distance of a strong prism formed by P_n and C_n to determine the Kirchhoff index. Similarly, Li et al. [31] derived graph invariants and spanning trees from the strong prism of a star S_n. Motivated by [30–33], we obtain the pentagonal network R_n and its strong product P_n^2. The pentagonal network consists of numerous adjacent pentagons and quadrilaterals with each quadrangle having a maximum of two non-adjacent pentagons, as shown in Figure 1. The P_n^2 is the strong product of R_n, as depicted in Figure 2. It obviously that

$$|V(P_n^2)| = 14n \text{ and } |E(P_n^2)| = 47n - 8.$$

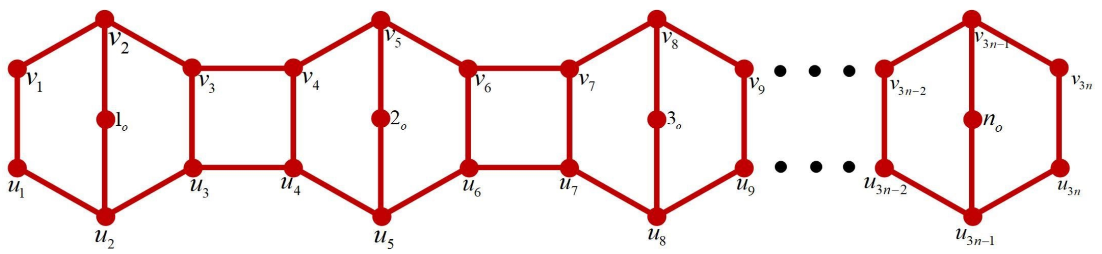

Figure 1. Linear pentagonal network R_n.

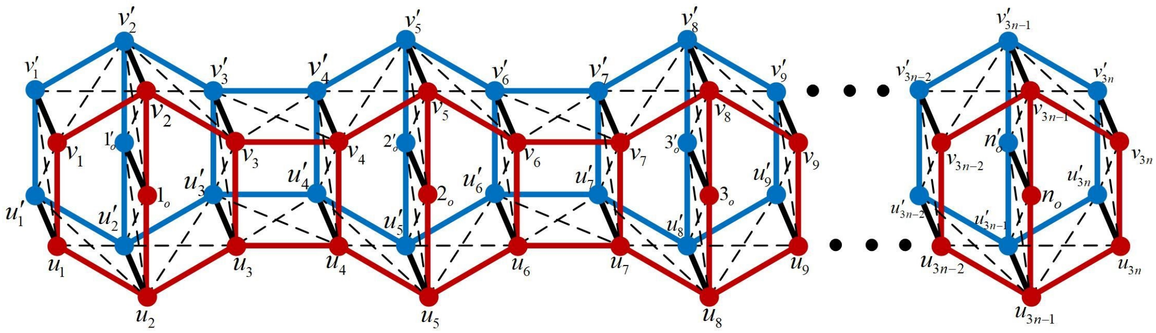

Figure 2. The strong product P_n^2 of the linear pentagonal.

In this paper, we focus on the strong product of pentagonal networks, specifically examining the graph P_n^2 with $n \geq 1$. The subsequent sections are organized as follows: Section 2 provides a comprehensive review of relevant research materials, presenting illustrations, concepts and lemmas. In Section 3, we derive the normalized Laplacian spectrum and present an explicit closed formula for the multiplicative degree-Kirchhoff index. Additionally, we calculate the complexity of P_n^2. In Section 4, we conclude the paper.

2. Preliminary results

In this section, let R_n represent the penagonal-quadrilateral networks, as illustrated in Figure 1. P_n^2 is composed of R_n and its copy R_n', positioned one in front and one behind, as shown in Figure 2. Moreover,

$$\Phi_A(x) = det(xI_n - A)$$

represents the characteristic polynomial of matrix A.

The fact that

$$\pi = (1_o, 2_o, \cdots, n_o)(1, 1')(2, 2') \cdots ((3n), (3n)')$$

is an automorphism deserves attention. Let

$$V_1 = \{1_o, 2_o, \cdots, n_o, u_1, u_2, \cdots, u_{3n}, v_1, \cdots, v_{3n}\},$$

$$V_2 = \{1_o', 2_o', \cdots, n_o', u_1', u_2', \cdots, u_{3n}', v_1', \cdots, v_{3n}'\},$$

$$|V(P_n^2)| = 14n \text{ and } |E(P_n^2)| = 47n - 8.$$

Subsequently, the normalized Laplacian matrix can be represented as a block matrix, that is

$$\mathcal{L}(P_n^2) = \begin{pmatrix} \mathcal{L}_{V_1V_1} & \mathcal{L}_{V_1V_2} \\ \mathcal{L}_{V_2V_1} & \mathcal{L}_{V_2V_2} \end{pmatrix},$$

in which

$$\mathcal{L}_{V_1V_1} = \mathcal{L}_{V_2V_2}, \quad \mathcal{L}_{V_1V_2} = \mathcal{L}_{V_2V_1}.$$

Let

$$W = \begin{pmatrix} \frac{1}{\sqrt{2}}I_{6n} & \frac{1}{\sqrt{2}}I_{6n} \\ \frac{1}{\sqrt{2}}I_{6n} & -\frac{1}{\sqrt{2}}I_{6n} \end{pmatrix},$$

then,

$$W\mathcal{L}(P_n^2)W' = \begin{pmatrix} \mathcal{L}_A & 0 \\ 0 & \mathcal{L}_S \end{pmatrix},$$

where

$$\mathcal{L}_A = \mathcal{L}_{V_1V_1} + \mathcal{L}_{V_1V_2} \text{ and } \mathcal{L}_S = \mathcal{L}_{V_1V_1} - \mathcal{L}_{V_1V_2}.$$

Observe that W' and W are transpose matrices of each other.

The characteristic polynomial of the matrix R, is denoted as

$$\Phi(R) := \det(xI - R).$$

The decomposition theorem process for P_n^2 is obtained in a similar manner to Pan and Li [34], thus we omit this proof and present it as follows:

Lemma 2.1. [35] *Assuming that the determination of $\mathcal{L}_A$ and $\mathcal{L}_S$ has been previously described, then*

$$\Phi_{\mathcal{L}(L_n)}(x) = \Phi_{\mathcal{L}_A}(x) \cdot \Phi_{\mathcal{L}_S}(x).$$

Lemma 2.2. [13] *The graph G is an undirected connected graph with n vertices and m edges. Then*

$$Kf^*(G) = 2m\sum_{k=2}^{n}\frac{1}{\lambda_k}.$$

Lemma 2.3. [2] *The number of spanning trees in G, referred to as the graph's complexity, can be considered a fundamental measure in graph theory. Then*

$$\tau(G) = \frac{1}{2m}\prod_{i=1}^{n} d_i \cdot \prod_{j=2}^{n}\lambda_j.$$

3. Main results

In this section, we explore the methodology for deriving an explicit analytical expression of the multiplicative Kirchhoff index by traversing the normalized Laplacian matrix. Meanwhile, we determine the computational complexity of P_n^2. Subsequently, employing the normalized Laplacian, we derive matrices of order $6n$ as

$$\mathcal{L}_{V_1V_1} = \begin{pmatrix}
1 & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{\sqrt{35}} & 1 & -\frac{1}{7} & \cdots & 0 & 0 & 0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{7} & 1 & \cdots & 0 & 0 & 0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{35}} & 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 \\
0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & 1 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\
-\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & 1 & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 & -\frac{1}{\sqrt{35}} & 1 & -\frac{1}{7} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 & 0 & -\frac{1}{7} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & 1
\end{pmatrix}$$

and

$$\mathcal{L}_{V_1V_2} = \begin{pmatrix}
-\frac{1}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{\sqrt{35}} & -\frac{1}{7} & -\frac{1}{7} & \cdots & 0 & 0 & 0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{7} & -\frac{1}{7} & \cdots & 0 & 0 & 0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{1}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 \\
0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\
-\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 & -\frac{1}{\sqrt{35}} & -\frac{1}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 & 0 & -\frac{1}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & -\frac{1}{5}
\end{pmatrix}.$$

Due to

$$\mathcal{L}_A = \mathcal{L}_{V_1V_1}(P_n^2) + \mathcal{L}_{V_1V_2}(P_n^2)$$

and

$$\mathcal{L}_S = \mathcal{L}_{V_1V_1}(P_n^2) - \mathcal{L}_{V_1V_2}(P_n^2),$$

it can be convincingly argued that

$$\mathcal{L}_A = 2\begin{pmatrix} \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{3}{7} & -\frac{1}{7} & \cdots & 0 & 0 & 0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{3}{7} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{3}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\ -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{3}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 & 0 & -\frac{1}{7} & \frac{3}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 & 0 & 0 & 0 & \cdots & \frac{3}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} \end{pmatrix}$$

and

$$\mathcal{L}_S = diag(\frac{6}{5}, \frac{8}{7}, \frac{8}{7}, \cdots, \frac{8}{7}, \frac{6}{5}, \frac{6}{5}, \frac{8}{7}, \frac{8}{7}, \cdots, \frac{8}{7}, \frac{6}{5}).$$

Utilizing Lemma 2.1, it is revealed that the P_n^2 normalized Laplacian spectrum consists of the eigenvalues from $\mathcal{L}_A$ and $\mathcal{L}_S$. It is established that the $\mathcal{L}_S$ possesses eigenvalues $\frac{6}{5}$ and $\frac{8}{7}$ with multiplicities of 4 and $(6n-4)$, respectively.

Let

$$M = \begin{pmatrix} \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{3}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{3}{7} & -\frac{1}{7} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{3}{7} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{3}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{3}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} \end{pmatrix}_{(3n)\times(3n)}$$

and

$$N = diag(-\frac{1}{5}, -\frac{1}{7}, -\frac{1}{7}, -\frac{1}{7}, -\frac{1}{7}, \cdots, -\frac{1}{7}, -\frac{1}{7}, -\frac{1}{5}),$$

where the matrices M and N are both of order $3n$.

The matrices M and N are combined to form a block matrix, denoted as $\frac{1}{2}\mathcal{L}_A$, in the following manner:

$$\frac{1}{2}\mathcal{L}_A = \begin{pmatrix} M & N \\ N & M \end{pmatrix}.$$

Suppose that

$$W = \begin{pmatrix} \frac{1}{\sqrt{2}}I_{3n} & \frac{1}{\sqrt{2}}I_{3n} \\ \frac{1}{\sqrt{2}}I_{3n} & -\frac{1}{\sqrt{2}}I_{3n} \end{pmatrix}$$

is a block matrix. Hence, we can obtain

$$W(\frac{1}{2}\mathcal{L}_A)W' = \begin{pmatrix} M+N & 0 \\ 0 & M-N \end{pmatrix}.$$

Let $J = M + N$ and $K = M - N$. Then,

$$J = \begin{pmatrix} \frac{1}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{4}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{35}} & \frac{3}{5} \end{pmatrix}_{(3n)\times(3n)}$$

and

$$K = \begin{pmatrix} \frac{3}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{4}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{4}{7} & -\frac{1}{7} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{4}{7} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{4}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{4}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{35}} & \frac{3}{5} \end{pmatrix}_{(3n)\times(3n)},$$

in which the diagonal elements are

$$(\frac{3}{5}, \frac{4}{7}, \frac{4}{7}, \frac{4}{7}, \cdots \frac{4}{7}, \frac{4}{7}, \frac{3}{5}).$$

Based on Lemma 2.1, it is evident and demonstrable that the eigenvalues of $\frac{1}{2}\mathcal{L}_A$ are identical to those of J and K. Assume that the eigenvalues of J and K are σ_i and ς_j $(i, j = 1, 2, \cdots, 3n)$, respectively, with

$$\sigma_1 \le \sigma_2 \le \sigma_3 \le \cdots \le \sigma_{3n}, \quad \varsigma_1 \le \varsigma_2 \le \varsigma_3 \le \cdots \le \varsigma_{3n}.$$

We verify $\sigma_1 \geq 0$ and $\varsigma_1 \geq 0$. In addition, it is easy to know that the normalized Laplacian spectrum of P_n^2 is $\{2\sigma_1, 2\sigma_2, \cdots, 2\sigma_{3n}, 2\varsigma_1, 2\varsigma_2, \cdots, 2\varsigma_{3n}\}$. Note that

$$|E(P_n^2)| = 47n - 8,$$

we can obtain Lemma 3.1 according to Lemma 2.2.

Lemma 3.1. *Assume that P_n^2 is the strong product of the pentagonal network. Then,*

$$\begin{aligned} Kf^*(P_n^2) &= 2(47n-8)\left(2 \times \frac{5}{6} + (6n-2)\frac{7}{8} + \frac{1}{2}\sum_{i=2}^{3n}\frac{1}{\sigma_i} + \frac{1}{2}\sum_{j=1}^{3n}\frac{1}{\varsigma_j}\right) \\ &= (47n-8)\left(\frac{63n-1}{6} + \sum_{i=2}^{3n}\frac{1}{\sigma_i} + \sum_{j=1}^{3n}\frac{1}{\varsigma_j}\right). \end{aligned}$$

Subsequently, we partition the computation of the aforementioned equation into two distinct components and prioritize the initial calculation of $\sum_{i=2}^{3n}\frac{1}{\sigma_i}$.

Lemma 3.2. *Suppose that $\sigma_i (i = 1, 2, \cdots, 3n)$ is defined as described previously.*

$$\sum_{i=2}^{3n}\frac{1}{\sigma_i} = \frac{1035n^3 + 142n^2 + 617n}{2(81n + 490)}.$$

Proof. Suppose that

$$\Phi(J) = x^{3n} + a_1x^{3n-1} + \cdots + a_{3n}x^2 + a_{3n+1}x = x(x^{3n-1} + a_1x^{3n-2} + \cdots + a_{3n-1}x + a_{3n-2}).$$

Then $\sigma_2, \sigma_3, \cdots, \sigma_{3n}$ fulfil the following equation

$$x^{3n-1} + a_1x^{3n-2} + \cdots + a_{3n-2}x + a_{3n-1} = 0,$$

and we observe that $\frac{1}{\sigma_2}, \frac{1}{\sigma_3}, \cdots, \frac{1}{\sigma_{3n}}$ are the roots of the following equation

$$a_{3n-1}x^{3n-1} + a_{3n-2}x^{3n-2} + \cdots + a_1x + 1 = 0.$$

By Vieta's Theorem, one has

$$\sum_{i=2}^{3n}\frac{1}{\sigma_i} = \frac{(-1)^{3n-2}a_{3n-2}}{(-1)^{3n-1}a_{3n-1}}. \tag{3.1}$$

For each value of i from 1 to $3n + 1$, we consider J_i and assign j_i as the determinant of J_i. We will derive the formula for j_i , which can be utilized to calculate $(-1)^{3n-2}a_{3n-2}$ and $(-1)^{3n-1}a_{3n-1}$. Then one has

$$j_1 = \frac{1}{5},\ j_2 = \frac{1}{35},\ j_3 = \frac{1}{245},\ j_4 = \frac{1}{1715},\ j_5 = \frac{1}{12005},\ j_6 = \frac{1}{84035},\ j_7 = \frac{1}{588245},\ j_8 = \frac{1}{4117715},$$

and

$$\begin{cases} j_{3i} = \frac{2}{7} j_{3i-1} - \frac{1}{49} j_{3i-2}, & 1 \le i \le n; \\ j_{3i+1} = \frac{2}{7} j_{3i} - \frac{1}{49} j_{3i-1}, & 0 \le i \le n-1; \\ j_{3i+2} = \frac{2}{7} j_{3i+1} - \frac{1}{49} j_{3i}, & 0 \le i \le n-1. \end{cases}$$

Through a straightforward computation, one can derive the following general formulas:

$$\begin{cases} j_{3i} = \frac{7}{5} \cdot (\frac{1}{343})^i, & 1 \le i \le n; \\ j_{3i+1} = \frac{1}{5} \cdot (\frac{1}{343})^i, & 0 \le i \le n-1; \\ j_{3i+2} = \frac{1}{35} \cdot (\frac{1}{343})^i, & 0 \le i \le n-1. \end{cases} \tag{3.2}$$

According to Eq (3.1), we divide the numerator and denominator into two facts and reveal them later. For the sake of convenience, we represent the diagonal elements of J as l_{ii} in a simplified manner. □

Fact 3.3.

$$(-1)^{3n-1} a_{3n-1} = \frac{490 + 81n}{25} (\frac{1}{343})^n.$$

Proof. Given that J is a left-right symmetric matrix, the sum of all principal minors can be represented by the number $(-1)^{3n-1} a_{3n-1}$, where these minors correspond to rows and columns with indices equal to $(3n-1)$.

$$(-1)^{3n-1} a_{3n-1} = \sum_{i=1}^{3n} det\, \mathcal{L}_A[i] = \sum_{i=1}^{3n} det \begin{pmatrix} J_{i-1} & 0 \\ 0 & J_{3n-i} \end{pmatrix} = \sum_{i=1}^{3n} j_{i-1} \cdot j_{3n-i}, \tag{3.3}$$

where

$$J_{3n-i} = \begin{pmatrix} l_{i+1,i+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & l_{3n-1,3n-1} & -\frac{1}{\sqrt{35}} \\ 0 & \cdots & -\frac{1}{\sqrt{35}} & l_{3n,3n} \end{pmatrix}.$$

By Eqs (3.2) and (3.3), we have

$$\begin{aligned} (-1)^{3n-1} a_{3n-1} &= 2 j_{3n+1} + \sum_{l=1}^{n} j_{3(l-1)+2} \cdot j_{3(n-l)+2} + \sum_{l=1}^{n} j_{3l} \cdot j_{3(n-l)+1} + \sum_{l=0}^{n-1} j_{3l+1} \cdot j_{3(n-l)} \\ &= \frac{98}{5} \cdot \Big(\frac{1}{343}\Big)^n + \frac{343n}{35^2} \cdot \Big(\frac{1}{343}\Big)^n + \frac{7n}{25} \cdot \Big(\frac{1}{343}\Big)^n + n \cdot \Big(\frac{1}{343}\Big)^n \\ &= \frac{490 + 81n}{25} \Big(\frac{1}{343}\Big)^n. \end{aligned}$$

This is the completion of the proof. □

Fact 3.4.

$$(-1)^{3n-2}a_{3n-2} = \frac{1035n^3 + 142n^2 - 617n}{50}\Big(\frac{1}{343}\Big)^n.$$

Proof. We note that the sum of all principal minors of order $3n-2$ in J can be expressed as $(-1)^{3n-2}a_{3n-2}$. After that,

$$(-1)^{3n-2}a_{3n-2} \;=\; \sum_{1\le i<j}^{3n} \begin{vmatrix} J_{i-1} & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & J_{3n-j} \end{vmatrix}, \quad 1 \le i < j \le 3n-2,$$

where

$$Z \;=\; \begin{pmatrix} l_{i+1,i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_{j-1,j-1} \end{pmatrix}$$

and

$$J_{3n-j} \;=\; \begin{pmatrix} l_{j+1,j+1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & l_{3n-1,3n-1} & -\frac{1}{\sqrt{35}} \\ 0 & \cdots & -\frac{1}{\sqrt{35}} & l_{3n,3n} \end{pmatrix}.$$

Note that

$$(-1)^{3n-2}a_{3n-2} = \sum_{1\le i<j}^{3n} det\, J_{i-1} \cdot det\, Z \cdot det\, J_{3n-j} = \sum_{1\le i<j}^{3n} det\, Z \cdot s_{i-1} \cdot s_{3n-j}. \tag{3.4}$$

According to Eq (3.4), the determinant of Z varies depending on the values of i and j, as well as s and t. Consequently, we can categorize the primary scenarios into six distinct classifications.

Case 1. $i = 3s$, $j = 3t$ for $1 \le s < t \le n$, and

$$det\, Z_1 \;=\; \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{2}{7} \end{vmatrix}_{(3s-3t-1)} = 21(s-t)\Big(\frac{1}{343}\Big)^{s-t}.$$

Case 2. $i = 3s$, $j = 3t+1$ for $1 \le s \le t \le n$, and

$$det\, Z_2 \;=\; \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{2}{7} \end{vmatrix}_{(3s-3t)} = [3(s-t)+1]\Big(\frac{1}{343}\Big)^{s-t},$$

or $i = 3s + 2$, $j = 3t$ for $0 \le s < t \le n$, and

$$det\, Z_3 = \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{2}{7} \end{vmatrix}_{(3s-3t-3)} = (3s-3t-2)\Big(\frac{1}{245}\Big)^{s-t-1}.$$

Case 3. In the same way, $i = 3s$, $j = 3t + 2$ for $1 \le s \le t \le n$, and

$$det\, Z_4 = \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{1}{5} \end{vmatrix}_{(3s-3t+1)} = 49(3s-3t+2)\Big(\frac{1}{343}\Big)^{s-t+1},$$

or $i = 3s + 1$, $j = 3t$ for $0 \le s < t \le n$, and

$$det\, Z_5 = \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{2}{7} \end{vmatrix}_{(3s-3t-2)} = 49(3s-3t-1)\Big(\frac{1}{343}\Big)^{s-t-1}.$$

Case 4. Similarly, $i = 3s + 1$, $j = 3t + 1$ for $0 \le s < t \le n$, and

$$det\, Z_6 = \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{2}{7} \end{vmatrix}_{(3s-3t-1)} = 21(s-t)\Big(\frac{1}{343}\Big)^{s-t},$$

or $i = 3s + 2$, $j = 3t + 2$ for $0 \leq k < l \leq n$, and

$$det\, Z_7 = \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{1}{5} \end{vmatrix}_{(3s-3t-1)} = det\, Z_6.$$

Case 5. $i = 3s + 1$, $j = 3t + 2$ for $0 \leq s \leq t \leq n$, and

$$det\, Z_8 = \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(3s-3t)} = (3s - 3t + 1)\Big(\frac{1}{343}\Big)^{s-t}.$$

Case 6. $i = 3s + 2$, $j = 3t + 1$ for $0 \leq k < l \leq n$, and

$$det\, Z_9 = \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{2}{7} \end{vmatrix}_{(3s-3t-2)} = 49(3s - 3t - 1)\Big(\frac{1}{343}\Big)^{s-t}.$$

Therefore, we can obtain

$$(-1)^{3n-2} a_{3n-2} = \sum_{1 \leq i < j \leq 3n} det\, Z \cdot j_{k-1} \cdot j_{3n-l} = \wp_1 + \wp_2 + \wp_3, \tag{3.5}$$

where

$$\begin{aligned} \wp_1 = & \sum_{1 \leq s < t \leq n} det\, Z_1 \cdot j_{3s-1} \cdot j_{3n-3t} + \sum_{1 \leq s \leq t \leq n} det\, Z_2 \cdot j_{3s-1} \cdot j_{3n-3t-1} \\ & + \sum_{1 \leq s \leq t \leq n-1} det\, Z_4 \cdot j_{3s-1} \cdot j_{3n-3t} + \sum_{1 \leq s \leq n} det\, J[3s, 3n+2] \cdot j_{3s-1} \\ = & \frac{7n(n^2-1)}{50}\Big(\frac{1}{343}\Big)^n + \frac{(n^2+2n)(n+1)}{2450}\Big(\frac{1}{343}\Big)^n + \frac{n^2(n-1)}{2450}\Big(\frac{1}{343}\Big)^n + \frac{n(3n+1)}{490}\Big(\frac{1}{343}\Big)^n \\ = & \frac{345n^3 + 17n^2 - 336n}{50}\Big(\frac{1}{343}\Big)^n, \end{aligned}$$

$$
\begin{aligned}
\wp_2 =& \sum_{1\le s<t\le n} det\, Z_5 \cdot j_{3s} \cdot j_{3n-3t+2} + \sum_{1\le s<t\le n} det\, Z_6 \cdot j_{3s} \cdot j_{3n-3t+1} \\
&+ \sum_{1\le s\le t\le n-1} det\, Z_8 \cdot j_{3s} \cdot j_{3n-3t} + \sum_{1\le s\le n} det\, J[3s+1, 3n+2] \cdot j_{3s} \\
&+ \sum_{1\le s\le n} det\, J[1, 3t] \cdot j_{3n-3t+2} + \sum_{1\le t\le n} det\, S[1, 3t+1] \cdot s_{3n-3t+1} \\
&+ \sum_{0\le t\le n-1} det\, J[1, 3t+2] \cdot j_{3n-3t} + det\, J[1, 3n+2] \\
=& \frac{49}{50}(n^3 - n^2 - 2n + 2)\Big(\frac{1}{343}\Big)^{n-1} + \frac{49n(n^2-1)}{50}\Big(\frac{1}{343}\Big)^n + \frac{49n(n^2-1)}{50}\Big(\frac{1}{343}\Big)^n \\
&+ \frac{7n(3n-1)}{10}\Big(\frac{1}{343}\Big)^n + \frac{7n(3n+1)}{10}\Big(\frac{1}{343}\Big)^{n-1} + \frac{21n(n+1)}{10}\Big(\frac{1}{343}\Big)^n \\
&+ \frac{7n(3n-1)}{10}\Big(\frac{1}{343}\Big)^n + n(3n+1)\Big(\frac{1}{343}\Big)^n \\
=& \frac{345n^3 + 73n^2 - 92n}{50}\Big(\frac{1}{343}\Big)^n
\end{aligned}
$$

and

$$
\begin{aligned}
\wp_3 =& \sum_{0\le s<t\le n} det\, Z_3 \cdot j_{3s+1} \cdot j_{3n-3t+2} + \sum_{0\le s<t\le n-1} det\, Z_7 \cdot j_{3s+1} \cdot j_{3n-3t} \\
&+ \sum_{0\le s<t\le n} det\, Z_9 \cdot j_{3s+1} \cdot j_{3n-3t+1} + \sum_{0\le s\le n-1} det\, J[3s+2, 3n+2] \cdot j_{3s+1} \\
=& \frac{49n(n^2+n-4)}{50}\Big(\frac{1}{343}\Big)^{n-1} + \frac{n(n^2-1)}{50}\Big(\frac{1}{343}\Big)^n + \frac{49n(n^2+2n-1)}{50}\Big(\frac{1}{343}\Big)^n + \frac{21n(n+1)}{10}\Big(\frac{1}{343}\Big)^{n-1} \\
=& \frac{345n^3 + 52n^2 - 189n}{50}\Big(\frac{1}{343}\Big)^n.
\end{aligned}
$$

By substituting $\wp_1$, $\wp_2$, and $\wp_3$ into Eq (3.5), the desired outcome can be deduced.

$$
(-1)^{3n-2}a_{3n-2} = \wp_1 + \wp_2 + \wp_3 = \frac{1035n^3 + 142n^2 - 617n}{50}\Big(\frac{1}{343}\Big)^n.
$$

This is the completion of the proof. □

Let

$$
0 = \varsigma_1 < \varsigma_2 \le \varsigma_3 \le \cdots \le \varsigma_{3n}
$$

represent the eigenvalues of J. By Facts 3.3 and 3.4, we can further investigate Lemma 3.2. According to Eq (3.1), it is evident that

$$
\sum_{i=2}^{3n} \frac{1}{\sigma_i} = \frac{(-1)^{3n-2}a_{3n-2}}{(-1)^{3n-1}a_{3n-1}} = \frac{1035n^3 + 142n^2 + 617n}{2(490 + 81n)}.
$$

Considering Lemma 3.1, we will focus on the calculations of $\sum_{j=1}^{3n} \frac{1}{\varsigma_j}$. Let

$$
\delta(n) = \frac{10290n(11 + 2\sqrt{30}) + 3600 + 12430\sqrt{30}}{60000}
$$

and

$$\xi(n) = \frac{10290n(11-2\sqrt{30}) + 3600 - 12430\sqrt{30}}{60000}.$$

Lemma 3.5. The variable ς_j (where j ranges from 1 to $3n+2$) is assumed to be defined as previously described. One has

$$\sum_{j=1}^{3n} \frac{1}{\varsigma_j} = \frac{(-1)^{3n-1} b_{3n-1}}{det\ K},$$

where

$$det\ K = \frac{45+11\sqrt{30}}{125}\left(\frac{11+4\sqrt{30}}{343}\right)^n + \frac{45-11\sqrt{30}}{125}\left(\frac{11-4\sqrt{30}}{343}\right)^n$$

and

$$(-1)^{3n-1} b_{3n-1} = \delta(n)\left(\frac{11+4\sqrt{30}}{343}\right)^n - \xi(n)\left(\frac{11-4\sqrt{30}}{343}\right)^n.$$

Proof. The representation of $\Phi(K)$ can be expressed as

$$y^{3n} + b_1 y^{3n-1} + \cdots + b_{3n-2} y^2 + b_{3n-1} y = y(y^{3n+1} + b_1 y^{3n} + \cdots + b_{3n-2} y + b_{3n-1}),$$

where $\varsigma_1, \varsigma_2, \cdots, \varsigma_{3n}$ represent the roots of the equation.

$$y^{3n-1} + b_1 y^{3n-2} + \cdots + b_{3n-2} y + b_{3n-1} = 0,$$

and the equation is determined to possess $\frac{1}{\varsigma_1}, \frac{1}{\varsigma_2}, \cdots, \frac{1}{\varsigma_{3n}}$ as its solutions

$$b_{3n-1} y^{3n-1} + b_{3n-2} y^{3n-2} + \cdots + b_1 y + 1 = 0.$$

By Vieta's Theorem, one holds that

$$\sum_{j=1}^{3n} \frac{1}{\varsigma_j} = \frac{(-1)^{3n-1} b_{3n-1}}{det\ K}. \tag{3.6}$$

To simplify the analysis, let R_p denote the p-th order principal minors of matrix K and k_p represent the determinant of R_p. We will derive an equation for k_p that can be utilized to calculate $(-1)^{3n-1} b_{3n-1}$ and $det\ K$ for values of p ranging from 1 to $3n-1$. Subsequently, we arrive at

$$k_1 = \frac{1}{5},\ k_2 = \frac{1}{35},\ k_3 = \frac{1}{245},\ k_4 = \frac{1}{1715},\ k_5 = \frac{1}{12005},\ k_6 = \frac{1}{84035},\ k_7 = \frac{1}{588245},\ k_8 = \frac{1}{4117715}$$

and

$$\begin{cases} k_{3p} = \frac{2}{7} k_{3p-1} - \frac{1}{49} k_{3p-2}, & 1 \le p \le n; \\ k_{3p+1} = \frac{2}{7} k_{3p} - \frac{1}{49} k_{3p-1}, & 1 \le p \le n; \\ k_{3p+2} = \frac{2}{7} k_{3p+1} - \frac{1}{49} k_{3p}, & 0 \le p \le n-1. \end{cases}$$

After a straightforward computation, the following general formulas can be derived

$$\begin{cases} k_{3p} = \frac{105+14\sqrt{30}}{150} \cdot (\frac{11+4\sqrt{30}}{343})^p + \frac{105-14\sqrt{30}}{150} \cdot (\frac{11-4\sqrt{30}}{343})^p, & 1 \le p \le n; \\ k_{3p+1} = \frac{45+8\sqrt{30}}{150} \cdot (\frac{11+4\sqrt{30}}{343})^p + \frac{45-8\sqrt{30}}{150} \cdot (\frac{11-4\sqrt{30}}{343})^p, & 1 \le p \le n; \\ k_{3p+2} = \frac{11+2\sqrt{30}}{70} \cdot (\frac{11+4\sqrt{30}}{343})^p + \frac{11-2\sqrt{30}}{70} \cdot (\frac{11-4\sqrt{30}}{343})^p, & 0 \le p \le n-1. \end{cases}$$

□

Subsequently, we proceed by examining the following Facts:

Fact 3.6.

$$det\ K = \frac{45+11\sqrt{30}}{125}(\frac{11+4\sqrt{30}}{343})^n + \frac{45-11\sqrt{30}}{125}(\frac{11-4\sqrt{30}}{343})^n.$$

Proof. By expanding $det\ K$ with respect to its last row, we obtain

$$det\ K = \frac{3}{5}k_{3n+1} - \frac{1}{35}k_{3n} = \frac{45+11\sqrt{30}}{125}(\frac{11+4\sqrt{30}}{343})^n + \frac{45-11\sqrt{30}}{125}(\frac{11-4\sqrt{30}}{343})^n.$$

The desired outcome has been achieved. □

Fact 3.7.

$$(-1)^{3n-1}b_{3n-1} = \delta(n)(\frac{11+4\sqrt{30}}{343})^n - \xi(n)(\frac{11-4\sqrt{30}}{343})^n,$$

where

$$\delta(n) = \frac{10290n(11+4\sqrt{30}) + 3600 + 12430\sqrt{30}}{60000}$$

and

$$\xi(n) = \frac{10290n(11-4\sqrt{30}) + 3600 - 12430\sqrt{30}}{60000}.$$

Proof. Considering that K has a $(3n-1)$-row and $(3n-1)$-column structure, the sum of all principal minors can be expressed as $(-1)^{3n-1}b_{3n-1}$. Here, l_{ii} represents the diagonal entries of K. It is noteworthy that K exhibits bilateral symmetry, which enables us to derive specific information.

$$(-1)^{3n-1}b_{3n-1} = \sum_{i=1}^{3n} det\ K[i] = \sum_{i=1}^{3n} det\begin{pmatrix} R_{i-1} & 0 \\ 0 & R_{3n-i} \end{pmatrix} = \sum_{i=1}^{3n} r_{i-1} \cdot r_{3n-i}, \tag{3.7}$$

where

$$R_{3n-i} = \begin{pmatrix} g_{i+1,i+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & g_{3n-1,3n-1} & -\frac{1}{\sqrt{35}} \\ 0 & \cdots & -\frac{1}{\sqrt{35}} & g_{3n,3n} \end{pmatrix}.$$

In line with Eq (3.7), we have

$$\begin{aligned}(-1)^{3n-1}b_{3n-1} =&2k_{3n-1} + \sum_{l=1}^{n} det\ K[3l] + \sum_{l=0}^{n-1} det\ K[3l+1] + \sum_{l=0}^{n-1} det\ K[3l+2] \\ =&2k_{3n-1} + \sum_{l=1}^{n} k_{3(l-1)+2} \cdot k_{3(n-l)+2} + \sum_{l=0}^{n-1} k_{3l} \cdot k_{3(n-l)+1} + \sum_{l=0}^{n-1} k_{3l+1} \cdot k_{3(n-l)}.\end{aligned} \tag{3.8}$$

Immediately, we can get

$$\begin{aligned}\sum_{l=1}^{n} k_{3(l-1)+2} \cdot k_{3(n-l)+2} =&\frac{245n}{20}(\frac{11+4\sqrt{30}}{343})^{n+1} - (\frac{11-4\sqrt{30}}{343})^{n+1} \\ &+ \frac{\sqrt{30}}{600}(\frac{11+4\sqrt{30}}{343})^{n} - \frac{\sqrt{30}}{600}(\frac{11-4\sqrt{30}}{343})^{n},\end{aligned} \tag{3.9}$$

$$\begin{aligned}\sum_{l=0}^{n} k_{3l} \cdot k_{3(n-l)+1} =&\frac{2391n}{300}(\frac{11+4\sqrt{30}}{343})^{n+1} - (\frac{11-4\sqrt{30}}{343})^{n+1} \\ &+ \frac{137\sqrt{30}}{60000}(\frac{11+4\sqrt{30}}{343})^{n} - \frac{137\sqrt{30}}{60000}(\frac{11-4\sqrt{30}}{343})^{n},\end{aligned} \tag{3.10}$$

$$\begin{aligned}\sum_{l=0}^{n-1} k_{3l+1} \cdot k_{3(n-l)} =&\frac{2391n}{300}(\frac{11+4\sqrt{30}}{343})^{n+1} - (\frac{11-4\sqrt{30}}{343})^{n+1} \\ &+ \frac{1271\sqrt{30}}{60000}(\frac{11+4\sqrt{30}}{343})^{n} - \frac{1271\sqrt{30}}{60000}(\frac{11-4\sqrt{30}}{343})^{n}\end{aligned} \tag{3.11}$$

and

$$2k_{3n-1} = \frac{90+16\sqrt{30}}{75}(\frac{11+4\sqrt{30}}{343})^{n} + \frac{90-16\sqrt{30}}{75}(\frac{11-4\sqrt{30}}{343})^{n}. \tag{3.12}$$

By incorporating Eqs (3.9)–(3.12) into Eq (3.8), we can achieve Lemma 3.7. Utilizing Eq (3.6), in conjunction with Facts 3.6 and 3.7, Lemma 3.5 can be promptly derived. □

By integrating Lemmas 3.1, 3.2 and 3.5, we can readily deduce the Theorems 3.8 and 3.9.

Theorem 3.8. Assume that P_n^2 is the strong product of the pentagonal network. One has

$$Kf^*(P_n^2) = \frac{1029n^3 + 5736n^2 + 4275n + 850}{3} + (57n+30)\frac{(-1)^{3n-1}b_{3n-1}}{detK},$$

where

$$\begin{aligned}(-1)^{3n-1}b_{3n-1} &= \delta(n)(\frac{11+4\sqrt{30}}{343})^{n} - \xi(n)(\frac{11-4\sqrt{30}}{343})^{n}, \\ det\ K &= \frac{45+11\sqrt{30}}{125}(\frac{11+4\sqrt{30}}{343})^{n} + \frac{45-11\sqrt{30}}{125}(\frac{11-4\sqrt{30}}{343})^{n},\end{aligned}$$

additionally,

$$\delta(n) = \frac{10290n(11 + 2\sqrt{30}) + 3600 + 12430\sqrt{30}}{60000}$$

and

$$\xi(n) = \frac{10290n(11 - 2\sqrt{30}) + 3600 - 12430\sqrt{30}}{60000}.$$

Theorem 3.9. Let P_n^2 be the strong product of pentagonal network. Then

$$\tau(P_n^2) = \frac{3^5 \cdot 2^{32n+7}}{5}\Big((45 + 11\sqrt{30})(11 + 4\sqrt{30})^n + (45 - 11\sqrt{30})(11 - 4\sqrt{30})^n\Big).$$

Proof. Based on the proof of Lemma 2.2, it is evident that $\sigma_1, \sigma_2, \cdots \sigma_{2n+1}$ constitute the roots of the equation

$$x^{2n} + a_1 x^{2n-1} + \cdots + a_{2n-1}x + a_{2n} = 0.$$

Accordingly, one has

$$\prod_{i=2}^{3n} \sigma_i = (-1)^{3n-1} a^{3n-1}.$$

By Fact 3.3, we have

$$\prod_{i=2}^{3n} \sigma_i = \frac{12 + 23n}{25}\Big(\frac{1}{343}\Big)^n.$$

By the same method,

$$\prod_{j=1}^{3n} \varsigma_j = det\ K = \frac{45 + 11\sqrt{30}}{375}\Big(\frac{11 + 4\sqrt{30}}{343}\Big)^n + \frac{45 - 11\sqrt{30}}{375}\Big(\frac{(11 - 4\sqrt{30})}{343}\Big)^n.$$

Note that

$$\prod_{v \in V_{P_n^2}} d(P_n^2) = 5^4 \cdot 7^{16n-4}$$

and

$$|E(P_n^2)| = 48n - 7.$$

In conjunction with Lemma 2.3, we get

$$\begin{aligned}
\tau(P_n^2) &= \frac{1}{2|E(P_n^2)|}\Big((\frac{6}{5})^2 \cdot (\frac{8}{7})^{6n-2} \cdot \prod_{i=2}^{3n} 2\sigma_i \cdot \prod_{j=1}^{3n} 2\varsigma_j \cdot \prod_{v \in V_{P_n^2}} d(P_n^2)\Big) \\
&= \frac{3^5 \cdot 2^{32n+7}}{5}\Big((45 + 11\sqrt{30})(11 + 4\sqrt{30})^n + (45 - 11\sqrt{30})(11 - 4\sqrt{30})^n\Big).
\end{aligned}$$

This is the completion of the proof. □

4. Conclusions

In this study, we have derived explicit expressions for the multiplicative degree-Kirchhoff index and complexity of P_n^2 based on the spectrum of the Laplacian matrix, where $P_n^2 = R_n \boxtimes R'_n$. These two fundamental calculations serve as simple yet reliable graph invariants that effectively capture the stability of diverse networks. Future research should focus on applying our methodology to determine spectra for strong products of automorphic and symmetric networks.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to express their sincere gratitude to the editor and anonymous referees for providing valuable suggestions, which significantly contributed to the substantial improvement of the original manuscript. This work was supported by Anhui Jianzhu University Funding under Grant KJ212005, and by Natural Science Fund of Education Department of Anhui Province under Grant KJ2020A0478.

Conflict of interest

No potential confilcts of interest were reported by the authors.

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