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Research article

Gamified approach towards optimizing supplier selection through Pythagorean Fuzzy soft-max aggregation operators for healthcare applications

Sana Shahab[1], **Mohd Anjum**[2], **Ashit Kumar Dutta**[3] **and Shabir Ahmad**[4,*]

[1] Department of Business Administration, College of Business Administration, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

[2] Department of Computer Engineering, Aligarh Muslim University, Aligarh 202002, India

[3] Department of Computer Science and Information Systems, College of Applied Sciences, AlMaarefa University, Ad Diriyah, Riyadh 13713, Saudi Arabia

[4] Department of Computer Engineering, Gachon University, Seongnam-si 13120, South Korea

* **Correspondence:** Email: shabir@gachon.ac.kr.

Abstract: The soft-max function, a well-known extension of the logistic function, has been extensively utilized in numerous stochastic classification methodologies, such as linear differential analysis, soft-max extrapolation, naive Bayes detectors, and neural networks. The focus of this study is the development of soft-max based fuzzy aggregation operators (AOs) for Pythagorean fuzzy sets (PyFS), capitalizing on the benefits provided by the soft-max function. In addition to introducing these novel AOs, we also present a comprehensive approach to multi-attribute decision-making (MADM) that employs the proposed operators. To demonstrate the efficacy and applicability of our MADM method, we applied it to a real-world problem involving Pythagorean fuzzy data. The analysis of supplier selection has been extensively examined in many academic works as a crucial component of supply chain management (SCM), recognised as a significant MADM challenge. The process of choosing healthcare suppliers is a pivotal element that has the potential to greatly influence the efficacy and calibre of healthcare provisions. In addition, we given a numerical example to rigorously evaluate the accuracy and dependability of the proposed procedures. This examination demonstrates the effectiveness and potential of our proposed soft-max based AOs and their applicability in Pythagorean fuzzy environments.

Keywords: soft-max function; aggregation operators; decision-making; Pythagorean fuzzy number
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1. Introduction

In order for businesses to successfully accomplish both their immediate and long-term objectives, the decision-making mechanisms inside such businesses are of the utmost significance. The capacity of businesses to remain in business, make revenue, and execute their objectives depends heavily on this factor. Poor judgement procedures may have considerable adverse effects on a business, while effective decision-making techniques can raise the likelihood that the firm will progress and flourish in the future [1]. Decision-makers (DMs) are required to make judgments in a complicated environment due to the fact that consumer expectations regarding goods and services are always growing and becoming more and more diverse in the modern corporate world. A plethora of MADM methodologies have been developed and studied to enhance the efficiency and objectivity of decision-making processes for DMs and to attain more effective outcomes [2]. The underlying principle of MADM methods often involves evaluating numerous alternatives based on a range of criteria, some of which may contradict each other, and subsequently assigning a score to these alternatives. To achieve this objective, multiple MADM techniques employ different logical frameworks [3].

In the current age of globalisation and fierce market competition, enterprises encounter a multitude of challenges that require a methodical approach to supplier selection. The collaboration partners selected by an organisation have a significant impact on critical factors including product quality, cost-effectiveness, and on-time delivery. By aligning these considerations with the specific requirements and anticipations of an organisation, the most effective supplier selection facilitates enhanced customer contentment and allegiance. Hence, it is imperative that organisations give precedence to the establishment of efficient supplier selection procedures, as doing so not only confers a competitive edge, but also facilitates improved operational efficacy and enduring expansion [4, 5]. Contemporary business landscapes are distinguished by intricate and interdependent ecosystems wherein entities are no longer limited to geographically specific local or regional markets. Conversely, an extensive array of global suppliers is easily accessible. The proliferation of suppliers presents businesses with unparalleled prospects, yet also substantial challenges that demand adept navigation. Supplier selection practices that allow organisations to optimise the advantages of global sourcing while minimising the risks involved become critical in this context. Enterprises have the ability to optimise supply chain operations, seize advantageous opportunities, and adapt to ever-changing market conditions through the careful selection of superior suppliers [6].

Technology innovations have significantly transformed the way in which businesses function, and this includes the process of selecting suppliers. The advent of the digital age has provided us with an abundance of information and advanced resources, enabling us to assess and contrast prospective suppliers in an unprecedented way. Through the utilisation of advanced technologies such as machine learning, artificial intelligence, and big data analytics, organisations have the ability to accelerate the supplier selection procedure, reveal latent patterns within vast datasets, and arrive at decisions based on empirical evidence. The implementation of these technological advancements empowers organisations to enhance their supplier selection process, thus promoting streamlined and successful supply chain management that aligns with the goals of the business [7]. Furthermore, effective supplier selection promotes partnership and collaboration with vendors, resulting in mutually beneficial relationships. Businesses can forge enduring partnerships founded on trust, transparency, and mutual objectives by scrutinising suppliers who are in accordance with their core principles, strategic aims, and organisational culture. These collaborative relationships foster enhanced communication, innovation, and continuous supply chain improvement, enabling organisations to differentiate themselves and obtain a competitive edge in the marketplace [8, 9].

A significant amount of the fundamental aspects of classical set theory have been researched by scholars. However, traditional techniques of data analysis are incapable of dealing with knowledge that is unclear or ambiguous in nature. In order to overcome these difficulties, Zadeh [10] devised the idea of fuzzy set (FS) theory and membership degree (MBSD), Molodtsov [11] developed soft set theory, and Pawlak [12] envisioned rough set theory. Atanassov [13] introduced the concept of intuitionistic fuzzy sets (IFS) in the year 1986. This theory, which is exemplified by the MBSD and the non-membership degree (N-MBSD), satisfies the restriction that the sum of MBSD and N-MBSD must be limited to unity. IFS has emerged as one of the essential techniques for defining the ambiguity and fuzziness of real-life issues due to the distinctive benefits it offers. In addition, Yager [14–16] came up with the idea of PyFS, which is an expansion of IFS and may be defined by the MBSD and the N-MBSD. This notion satisfies the criterion that the square addition of the MBSD and the N-MBSD must be kept to one. As a consequence of this, PyFSs are superior to IFSs when it comes to dealing with the unpredictability and imprecision of the information collected in real-world issues. In recent years, a great deal of research that investigates the PyFSs from a variety of vantage points has been presented.

The domains of business, administration, social work, medicine, technology, psychology, and intelligent systems all benefit from the consolidation of data for judgement purposes. Historically, one's awareness of the alternative has been considered a discrete quantity or a linguistic number. On the other hand, owing to the unpredictability of the data, it is not simple to consolidate them. In fact, AOs play a crucial role in the context of MADM difficulties, the primary objective of which is to sum up a sequence of inputs into a single one.

Moslem [17] came up with the idea of using a spherical fuzzy analytic hierarchy process to solve the urban transport problem. Peng and Yuan [18] and Rehman et al. [19] proposed Pythagorean fuzzy averaging AOs and geometric AOs respectively. Wang and Garg [20] proposed some Archimedean norm based Pythagorean fuzzy AOs. Moslem et al. [21] proposed the idea of sustainable development of public transportation using Bonferroni AOs. Gayen et al. [22] gave the notion of Aczel-Alsina AOs for dual hesitant q-rung orthopair fuzzy set. Moslem [23] introduced the best worst method for evaluating travel mode choice. Demir et al. [24] proposed sensitivity analysis in MADM. Ali et al. [25] gave the notion of complex T-spherical fuzzy Frank AOs, and Ali [26] proposed the idea of Hamacher prioritised AOs for probabilistic hesitant bipolar fuzzy data. Linear Diophantine fuzzy soft-max AOs and a numerically validated approach to modelling water hammer phenomena are given in [27, 28]. Furthermore, Mahmood et. al. [29] explored the theory of bipolar complex intuitionistic fuzzy N-soft (BCIFN-S) information to handle information with truth and falsity degrees, along with parameterisation and grades.

There are several exciting applications related to efficient risk management in distribution operations [30], sustainable hydrogen manufacturing [31], decision-making for phone selection [32], and general decision-making [33, 34] that are included in the literature. Ali et al. [35] introduced Minkowski-type distance measures for cubic q-rung orthopair fuzzy sets, and Ali [36] proposed a norm-based distance measures. Ali and Naeem [37] gave another MADM approach using the VIKOR method. MADM is a systematic approach used to evaluate and compare alternatives based on multiple, conflicting criteria.

The importance of MADM lies in its ability to take into account multiple, often conflicting, objectives and criteria in decision-making, leading to more informed and comprehensive decision-making. There are numerous decision-making-related applications, including the selection of green suppliers [38], the evaluation of road segment safety [39], assault boat selection [40], the assessment of risk in trade and investment [41], and the selection of industrial funds [42].

MADM's purview extends to numerous disciplines and practical contexts, encompassing the military, finance, engineering, transportation, environmental management, and healthcare, among others. In numerous real-world scenarios requiring the consideration of multiple criteria, such as evaluating the environmental impacts of a project, selecting suppliers, or deciding between investment options, MADM can be utilised. With respect to its applications, MADM facilitates the process of decision-making by integrating qualitative and quantitative data, including subjective assessments, into a unified and all-encompassing judgement. Prioritising criteria according to their relative importance, identifying and ranking alternatives based on multiple criteria, and visualising trade-offs between criteria are additional functions of MADM. The supplier selection process in the healthcare industry is complex and involves factors beyond mere cost considerations. Patient outcomes, regulatory compliance, supply chain efficiency, and the overall quality of healthcare services are all directly impacted. It is critical for healthcare organisations to provide patients with safe, effective, and high-quality care.

The subsequent sections of this scholarly article are structured as follows: In Section 2, we delve into an in-depth discussion of the fundamental concepts that underpin the PyFS, which is vital to our study. Building upon this foundation, Section 3 comprehensively examines the operational procedures of the proposed AOs within the context of PyFS. Section 4 presents a novel approach to addressing MADM challenges by introducing new AOs. We outline a methodological framework that leverages these AOs to effectively resolve MADM issues, thus enhancing decision-making processes in complex and uncertain environments. In the subsequent section, namely Section 5, we showcase an application of MADM using the aforementioned AOs. Through an illustrative case study, we demonstrate the practical implications and potential benefits of employing the proposed approach in real-world decision-making scenarios. Finally, in the concluding Section 6, we summarise the key findings and insights obtained throughout this article. We provide some concluding remarks on the significance and potential future directions of research in this area. Additionally, we offer valuable suggestions to guide and inspire future endeavours aimed at advancing the field of MADM and its applications.

2. Some basic concepts

In this part, we review the operating rules of PyFNs.

Definition 2.1. [14–16] A PyFS $\mathcal{B}$ in $\mathcal{Q}$ is defined as

$$\mathcal{B} = \{\langle \zeta, {}^{\xi}\delta_{\mathcal{B}}(\zeta), {}^{\gimel}\eta_{\mathcal{B}}(\zeta)\rangle : \zeta \in \mathcal{Q}\}$$

where ${}^{\xi}\delta_{\mathcal{B}}, {}^{\gimel}\eta_{\mathcal{B}} : \mathcal{Q} \to [0,1]$ defines the MBSD, and the N-MBSD of the alternative $\zeta \in \mathcal{Q}$ and for every ζ we have

$$0 \leq {}^{\xi}\delta^{2}_{\mathcal{B}}(\zeta) + {}^{\gimel}\eta^{2}_{\mathcal{B}}(\zeta) \leq 1.$$

Furthermore, $\pi_{\mathcal{B}}(\zeta) = (1 - {}^{\xi}\delta^{2}_{\mathcal{B}}(\zeta) - {}^{\gimel}\eta^{2}_{\mathcal{B}}(\zeta))^{1/2}$ is called the indeterminacy degree of ζ to $\mathcal{B}$.

Definition 2.2. [14] Let $\chi^{\lambda}{}_1 = \langle {}^{\xi}\delta_1, {}^{\daleth}\eta_1 \rangle$ and $\chi^{\lambda}{}_2 = \langle {}^{\xi}\delta_2, {}^{\daleth}\eta_2 \rangle$ be PyFNs. Then,
(1) $\overline{\chi^{\lambda}}_1 = \langle {}^{\daleth}\eta_1, {}^{\xi}\delta_1 \rangle$,
(2) $\chi^{\lambda}{}_1 \vee \chi^{\lambda}{}_2 = \langle max\{{}^{\xi}\delta_1, {}^{\daleth}\eta_1\}, min\{{}^{\xi}\delta_2, {}^{\daleth}\eta_2\} \rangle$,
(3) $\chi^{\lambda}{}_1 \wedge \chi^{\lambda}{}_2 = \langle min\{{}^{\xi}\delta_1, {}^{\daleth}\eta_1\}, max\{{}^{\xi}\delta_2, {}^{\daleth}\eta_2\} \rangle$,
(4) $\chi^{\lambda}{}_1 \oplus \chi^{\lambda}{}_2 = \langle ({}^{\xi}\delta_1^2 + {}^{\xi}\delta_2^2 - {}^{\xi}\delta_1^2 {}^{\xi}\delta_2^2)^{1/2}, {}^{\daleth}\eta_1 {}^{\daleth}\eta_2 \rangle$,
(5) $\chi^{\lambda}{}_1 \otimes \chi^{\lambda}{}_2 = \langle {}^{\xi}\delta_1 {}^{\xi}\delta_2, ({}^{\daleth}\eta_1^2 + {}^{\daleth}\eta_2^2 - {}^{\daleth}\eta_1^2 {}^{\daleth}\eta_2^2)^{1/2} \rangle$,
(6) $\sigma\chi^{\lambda}{}_1 = \langle (1 - (1 - {}^{\xi}\delta_1^2)^{\sigma})^{1/2}, {}^{\daleth}\eta_1^{\sigma} \rangle$,
(7) $\chi^{\lambda\sigma}_1 = \langle {}^{\xi}\delta_1^{\sigma}, (1 - (1 - {}^{\daleth}\eta_1^2)^{\sigma})^{1/2} \rangle$.

Definition 2.3. [14] Consider a PyFN $\widetilde{\mathfrak{R}} = \langle {}^{\xi}\delta, {}^{\daleth}\eta \rangle$. Then the score function $\mathfrak{E}$ of $\widetilde{\mathfrak{R}}$ is characterised as

$$\mathfrak{E}(\widetilde{\mathfrak{R}}) = {}^{\xi}\delta^2 - {}^{\daleth}\eta^2,$$

$\mathfrak{E}(\widetilde{\mathfrak{R}}) \in [-1, 1]$. The ranking of a PyFN is determined by its score: a higher score indicates a higher preference for the PyFN in query. Nevertheless, the score function is not helpful in several different applications of PyFN. Because of this, it is not required to depend on the score function in order to do a comparison of the PyFNs.

Definition 2.4. Consider a PyFN $\widetilde{\mathfrak{R}} = \langle {}^{\xi}\delta, {}^{\daleth}\eta \rangle$. then an accuracy function $\mathfrak{R}$ of $\widetilde{\mathfrak{R}}$ is defined as

$$\mathfrak{R}(\widetilde{\mathfrak{R}}) = {}^{\xi}\delta^2 + {}^{\daleth}\eta^2,$$

$\mathfrak{R}(\widetilde{\mathfrak{R}}) \in [0, 1]$.

Always keep in mind that the score function value falls between -1 and 1. In assistance of the following investigation, we present another scoring function, $\mathscr{H}(\mathfrak{R}) = \frac{1 + {}^{\xi}\delta_{\mathfrak{R}}^2 - {}^{\daleth}\eta_{\mathfrak{R}}^2}{2}$. We can see that $0 \leq \mathscr{H}(\mathfrak{R}) \leq 1$.

2.1. Pythagorean fuzzy aggregation operators

Definition 2.5. [18] Assume that $\chi^{\lambda}{}_k = \langle {}^{\xi}\delta_k, {}^{\daleth}\eta_k \rangle$ is an assortment of PyFNs, and PyFWA: $\Lambda^n \to \Lambda$, if

$$PyFWA(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) = \mathfrak{G}^{\zeta}{}_1 \chi^{\lambda}{}_1 \oplus \mathfrak{G}^{\zeta}{}_2 \chi^{\lambda}{}_2 \oplus \ldots, \mathfrak{G}^{\zeta}{}_n \chi^{\lambda}{}_n$$

where Λ^n is the set of all PyFNs, and $\mathfrak{G}^{\zeta} = (\mathfrak{G}^{\zeta}{}_1, \mathfrak{G}^{\zeta}{}_2, \ldots, \mathfrak{G}^{\zeta}{}_n)^T$ is weight vector (WV) of $(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots, \chi^{\lambda}{}_n)$, such that $0 \leqslant \mathfrak{G}^{\zeta}{}_k \leqslant 1$ and $\sum_{k=1}^{n} \mathfrak{G}^{\zeta}{}_k = 1$. Then, the PyFWA is called the Pythagorean fuzzy weighted average operator.

Theorem 2.6. *[18] Letting $\chi^{\lambda}{}_k = \langle {}^{\xi}\delta_k, {}^{\daleth}\eta_k \rangle$ be the assortment of PyFNs,we can find PyFWA by*

$$PyFWA(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) = \left\langle \sqrt{(1 - \prod_{k=1}^{n} (1 - {}^{\xi}\delta_k^2)^{\mathfrak{G}^{\zeta}{}_k})}, \prod_{k=1}^{n} {}^{\daleth}\eta_k^{\mathfrak{G}^{\zeta}{}_k} \right\rangle.$$

Definition 2.7. [19] Assume that $\chi^{\lambda}{}_k = \langle {}^{\xi}\delta_k, {}^{\daleth}\eta_k \rangle$ is the assortment of $PyFN$, and $PyFWG : \Lambda^n \to \Lambda$, if

$$PyFWG(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) = \chi^{\lambda}{}_1^{\mathfrak{G}^{\zeta}{}_1} \otimes \chi^{\lambda}{}_2^{\mathfrak{G}^{\zeta}{}_2} \otimes \ldots, \chi^{\lambda}{}_n^{\mathfrak{G}^{\zeta}{}_n}$$

where Λ^n is the set of all $PyFNs$, and $\mathfrak{G}^{\zeta} = (\mathfrak{G}^{\zeta}{}_1, \mathfrak{G}^{\zeta}{}_2, \ldots, \mathfrak{G}^{\zeta}{}_n)^T$ is the WV of $(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots, \chi^{\lambda}{}_n)$, such that $0 \leqslant \mathfrak{G}^{\zeta}{}_k \leqslant 1$ and $\sum_{k=1}^{n} \mathfrak{G}^{\zeta}{}_k = 1$. Then, the $PyFWG$ is called the Pythagorean fuzzy weighted geometric operator.

Based on PyFNs operational rules, we can also consider PyFWG by the theorem below.

Theorem 2.8. *[19] Letting $\chi^{\lambda}{}_k = \langle {}^{\xi}\delta_k, {}^{\daleth}\eta_k \rangle$ be the assortment of PyFNs, we can find the $PyFWG$ by*

$$PyFWG(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) = \left\langle \prod_{k=1}^{n} {}^{\xi}\delta_k^{\mathfrak{G}^{\zeta}{}_k}, \sqrt{(1 - \prod_{k=1}^{n} (1 - {}^{\daleth}\eta_k^2)^{\mathfrak{G}^{\zeta}{}_k})} \right\rangle.$$

2.2. Soft-max function

Within the realm of mathematics, the soft-max function stands as a notable generalisation that emerges from the logistic function. This function has found application in diverse fields of study, encompassing domains such as computer vision and strategic planning. To provide a concise representation, the soft-max function can be expressed mathematically as follows:

$$\phi_k(j, \vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \phi_k^j = \frac{\exp\left(\vartheta_j / k\right)}{\sum_{j=1}^{n} \exp\left(\vartheta_j / k\right)}, k > 0$$

For the PyFNs $\vartheta_j (j = 1, 2, 3, \ldots, n), S_j$ is the score value of PyFN ϑ_j. Every ϑ_j is formulated by given the equation

$$\vartheta_j = \begin{cases} \prod_{i=1}^{j-1} S_i, \text{j} = 2, 3, \ldots, n \\ 1 \qquad\qquad\qquad \text{j} = 1 \end{cases}$$

where k is the modulation parameter.

3. Pythagorean fuzzy soft-max aggregation operators

In the forthcoming section, we will present a number of Pythagorean fuzzy soft-max AOs. These AOs have been developed to address decision-making challenges involving PyFSs, which offer a versatile mathematical framework encompassing both membership and non-membership grades. By employing these operators, decision-makers can gain valuable insights and make well-informed decisions in complex and uncertain contexts.

3.1. PyFSMA operator

Definition 3.1. Assume that $\chi^{\lambda}{}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\daleth}\eta_{\wp} \rangle$ is the assortment of PyFNs, and the PyFSMA: $\Lambda^n \to \Lambda$, is an n-dimension mapping. If

$$\text{PyFSMA}(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) = \frac{\exp[\mathfrak{Y}^{\xi}{}_1/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]} \chi^{\lambda}{}_1 \oplus \frac{\exp[\mathfrak{Y}^{\xi}{}_2/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]} \chi^{\lambda}{}_2 \oplus \ldots, \oplus \frac{\exp[\mathfrak{Y}^{\xi}{}_n/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]} \chi^{\lambda}{}_n \tag{3.1}$$

then the mapping PyFSMA is called a Pythagorean fuzzy soft-max averaging (PyFSMA) operator, where $\mathfrak{Y}^{\xi}{}_{\wp} = \prod_{k=1}^{c-1} \mathscr{H}(\chi^{\lambda}{}_k)$ $(c = 2 \ldots, n)$, $\mathfrak{Y}^{\xi}{}_1 = 1$ and $\mathscr{H}(\chi^{\lambda}{}_k)$ is the score of the k^{th} PyFN.

Based on the operating principles of PyFN, the following theorem allows us to additionally investigate the PyFSMA operator.

Theorem 3.2. *Assuming that $\chi^{\lambda}{}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\daleth}\eta_{\wp} \rangle$ is the assortment of PyFNs, we can find PyFSMA by*

$$PyFSMA(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) = \left\langle \sqrt{1 - \prod_{\wp=1}^{n} (1 - {}^{\xi}\delta_{\wp}^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}}}, \prod_{\wp=1}^{n} {}^{\daleth}\eta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}} \right\rangle \tag{3.2}$$

The initial assertion can be readily verified by referring to Definition 3.1 and subsequently demonstrating its validity through the proof presented hereafter.

$$\text{PyFSMA}(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) = \frac{\exp[\mathfrak{Y}^{\xi}{}_1/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]} \chi^{\lambda}{}_1 \oplus \frac{\exp[\mathfrak{Y}^{\xi}{}_2/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]} \chi^{\lambda}{}_2 \oplus \ldots, \oplus \frac{\exp[\mathfrak{Y}^{\xi}{}_n/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]} \chi^{\lambda}{}_n$$

$$= \left\langle \sqrt{1 - \prod_{\wp=1}^{n} (1 - {}^{\xi}\delta_{\wp}^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}}}, \prod_{\wp=1}^{n} {}^{\daleth}\eta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}} \right\rangle$$

To establish the validity of this theorem, we employ the mathematical induction technique. For $n = 2$,

$$\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]} \chi^{\lambda}{}_1 = \left\langle \sqrt{1 - (1 - {}^{\xi}\delta_1^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}}}, {}^{\daleth}\eta_1^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\urcorner]}{\sum_{\wp=1}^{n} \exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}} \right\rangle$$

$$\frac{\exp[\mathfrak{Y}^{\xi}{}_2/\daleth]}{\sum\limits_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}\chi^{\lambda}{}_2=\left\langle\sqrt{1-(1-{}^{\xi}\delta_2^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}},{}^{\gimel}\eta_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right\rangle.$$

Then,

$$\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum\limits_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}\chi^{\lambda}{}_1\oplus\frac{\exp[\mathfrak{Y}^{\xi}{}_2/\daleth]}{\sum\limits_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}\chi^{\lambda}{}_2$$

$$=\left\langle\sqrt{1-(1-{}^{\xi}\delta_1^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}},{}^{\gimel}\eta_1^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right\rangle\oplus\left\langle\sqrt{1-(1-{}^{\xi}\delta_2^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}},{}^{\gimel}\eta_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right\rangle$$

$$=\left\langle\sqrt{(1-(1-{}^{\xi}\delta_1^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}+1-(1-{}^{\xi}\delta_2^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}-,}\right.$$

$$\left.\overline{\left((1-(1-{}^{\xi}\delta_1^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right)\left((1-(1-{}^{\xi}\delta_2^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right)},\quad{}^{\gimel}\eta_1^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}.{}^{\gimel}\eta_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right\rangle$$

$$=\left\langle\sqrt{1-(1-{}^{\xi}\delta_1^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}+1-(1-{}^{\xi}\delta_2^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}-\left(1-(1-{}^{\xi}\delta_2^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}-(1-{}^{\xi}\delta_1^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}+\right.}\right.$$

$$\left.\overline{(1-{}^{\xi}\delta_2^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}(1-{}^{\xi}\delta_1^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\Big)},{}^{\gimel}\eta_1^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}.{}^{\gimel}\eta_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right\rangle$$

$$=\left\langle\sqrt{1-(1-{}^{\xi}\delta_1^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}(1-{}^{\xi}\delta_2^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}},{}^{\gimel}\eta_1^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}.{}^{\gimel}\eta_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right\rangle$$

$$=\left\langle\sqrt{1-\prod_{\wp=1}^{2}(1-{}^{\xi}\delta_{\wp}^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}},\prod_{\wp=1}^{2}{}^{\gimel}\eta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right\rangle.$$

This demonstrates that Eq 3.2 holds true when $n = 2$. Now, let us assume that Eq 3.2 is valid for $n = k$, where k is a positive integer. In other words, we suppose that: Equation 3.2 (for $n = k$) is true, where Eq 3.2 represents the mathematical expression under consideration.

$$\text{PyFSMA}(\chi^{\lambda}{}_1,\chi^{\lambda}{}_2,\ldots\chi^{\lambda}{}_k) = \left\langle \sqrt{1-\prod_{\wp=1}^{k}(1-{}^{\xi}\delta_{\wp}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}, \prod_{\wp=1}^{k}{}^{\gimel}\eta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \right\rangle.$$

Now, $n = k + 1$, and by the operational laws of PyFNs, we have,

$$\text{PyFSMA}(\chi^{\lambda}{}_1,\chi^{\lambda}{}_2,\ldots\chi^{\lambda}{}_{k+1}) = \text{PyFSMA}(\chi^{\lambda}{}_1,\chi^{\lambda}{}_2,\ldots\chi^{\lambda}{}_k)\oplus\chi^{\lambda}{}_{k+1}$$

$$= \left\langle \sqrt{1-\prod_{\wp=1}^{k}(1-{}^{\xi}\delta_{\wp}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}, \prod_{\wp=1}^{k}{}^{\gimel}\eta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \right\rangle \oplus \left\langle \sqrt{1-(1-{}^{\xi}\delta_{k+1}^{2})^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}}}, {}^{\gimel}\eta_{k+1}^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}} \right\rangle$$

$$= \left\langle \sqrt{1-\prod_{\wp=1}^{k}(1-{}^{\xi}\delta_{k}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} + 1-(1-{}^{\xi}\delta_{k+1}^{2})^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}} - \left(1-\prod_{\wp=1}^{k}(1-{}^{\xi}\delta_{k}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right)\left(1-(1-{}^{\xi}\delta_{k+1}^{2})^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}}\right)}, \prod_{\wp=1}^{k}{}^{\gimel}\eta_{k}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \cdot {}^{\gimel}\eta_{k=1}^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}} \right\rangle$$

$$= \left\langle \sqrt{1-\prod_{\wp=1}^{k}(1-{}^{\xi}\delta_{k}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}(1-{}^{\xi}\delta_{k+1}^{2})^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{k+1}}}}, \prod_{\wp=1}^{k}{}^{\gimel}\eta_{k}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \cdot {}^{\gimel}\eta_{k+1}^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}} \right\rangle$$

$$= \left\langle \sqrt{1-\prod_{\wp=1}^{k+1}(1-{}^{\xi}\delta_{\wp}^{2})^{\frac{\mathfrak{Y}^{\xi}{}_{\wp}}{\sum_{\wp=1}^{k+1}\mathfrak{Y}^{\xi}{}_{\wp}}}}, \prod_{\wp=1}^{k+1}{}^{\gimel}\eta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \right\rangle.$$

This shows that for $n = k + 1$, Eq 3.2 holds. Then,

$$\text{PyFSMA}(\chi^{\lambda}{}_1,\chi^{\lambda}{}_2,\ldots\chi^{\lambda}{}_n) = \left\langle \sqrt{(1-\prod_{\wp=1}^{n}(1-{}^{\xi}\delta_{\wp}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}, \prod_{\wp=1}^{n}{}^{\gimel}\eta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \right\rangle$$

Below we define some of PyFSMA's appealing properties:

Theorem 3.3. *(Boundary) Assume that $\chi^{\lambda}{}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\daleth}\eta_{\wp} \rangle$ is the assortment of PyFNs, and*

$$\chi^{\lambda^-} = (min_j\,({}^{\xi}\delta_j), max_j\,({}^{\daleth}\eta_j)) \quad and \quad \chi^{\lambda^+} = (max_j\,({}^{\xi}\delta_j), min_j\,({}^{\daleth}\eta_j)).$$

Then,

$$\chi^{\lambda^-} \leq PyFSMA(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) \leq \chi^{\lambda^+}.$$

Proof. We have that,

$$min_{\wp}\,({}^{\xi}\delta_{\wp}) \leq {}^{\xi}\delta_{\wp} \leq max_{\wp}\,({}^{\xi}\delta_{\wp}) \tag{3.3}$$

and

$$min_{\wp}\,({}^{\daleth}\eta_{\wp}) \leq {}^{\daleth}\eta_{\wp} \leq max_{\wp}\,({}^{\daleth}\eta_{\wp}). \tag{3.4}$$

From Eq 3.3, we have,

$$min_{\wp}\,({}^{\xi}\delta_{\wp}) \leq {}^{\xi}\delta_{\wp} \leq max_{\wp}\,({}^{\xi}\delta_{\wp})$$

$$\Leftrightarrow \sqrt{min_{\wp}\,({}^{\xi}\delta_{\wp})^2} \leq \sqrt{({}^{\xi}\delta_{\wp})^2} \leq \sqrt{max_{\wp}\,({}^{\xi}\delta_{\wp})^2}$$

$$\Leftrightarrow \sqrt{1 - max_{\wp}\,({}^{\xi}\delta_{\wp})^2} \leq \sqrt{1 - ({}^{\xi}\delta_{\wp})^2} \leq \sqrt{1 - min_{\wp}\,({}^{\xi}\delta_{\wp})^2}$$

$$\Leftrightarrow \sqrt{\left(1 - max_{\wp}\,({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{\left(1 - ({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{\left(1 - min_{\wp}\,({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}$$

$$\Leftrightarrow \sqrt{\prod_{\wp=1}^{n}\left(1 - max_{\wp}\,({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{\prod_{\wp=1}^{n}\left(1 - ({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{\prod_{\wp=1}^{n}\left(1 - min_{\wp}\,({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}$$

$$\Leftrightarrow \sqrt{1 - max_{\wp}\,({}^{\xi}\delta_{\wp})^2} \leq \sqrt{\prod_{\wp=1}^{n}\left(1 - ({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{1 - min_{\wp}\,({}^{\xi}\delta_{\wp})^2}$$

$$\Leftrightarrow \sqrt{-1 + min_{\wp}\,({}^{\xi}\delta_{\wp})^2} \leq \sqrt{-\prod_{\wp=1}^{n}\left(1 - ({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{-1 + max_{\wp}\,({}^{\xi}\delta_{\wp})^2}$$

$$\Leftrightarrow \sqrt{1 - 1 + min_{\wp}\,({}^{\xi}\delta_{\wp})^2} \leq \sqrt{1 - \prod_{\wp=1}^{n}\left(1 - ({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{1 - 1 + max_{\wp}\,({}^{\xi}\delta_{\wp})^2}$$

$$\Leftrightarrow \sqrt{min_{\wp}\,({}^{\xi}\delta_{\wp})^2} \leq \sqrt{1-\prod_{\wp=1}^{n}\left(1-({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{max_{\wp}\,({}^{\xi}\delta_{\wp})^2}$$

$$\Leftrightarrow min_{\wp}\,({}^{\xi}\delta_{\wp})^2 \leq \sqrt{1-\prod_{\wp=1}^{n}\left(1-({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq max_{\wp}\,({}^{\xi}\delta_{\wp})^2.$$

From Eq 3.4, we have,

$$min_{\wp}\,({}^{\daleth}\eta_{\wp}) \leq {}^{\daleth}\eta_{\wp} \leq max_{\wp}\,({}^{\daleth}\eta_{\wp}) \Leftrightarrow min_{\wp}\,({}^{\daleth}\eta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \leq ({}^{\daleth}\eta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \leq max_{\wp}\,({}^{\daleth}\eta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}$$

$$\Leftrightarrow \prod_{\wp=1}^{n} min_{\wp}\,({}^{\daleth}\eta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \leq \prod_{\wp=1}^{n}({}^{\daleth}\eta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \leq \prod_{\wp=1}^{n} max_{\wp}\,({}^{\daleth}\eta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}$$

$$\Leftrightarrow min_{\wp}\,({}^{\daleth}\eta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \leq \prod_{\wp=1}^{n}({}^{\daleth}\eta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \leq max_{\wp}\,({}^{\daleth}\eta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}.$$

Let

$$\text{PyFSMA}(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) = \chi^{\lambda} = ({}^{\xi}\delta, {}^{\daleth}\eta).$$

Then, $\mathscr{H}(\chi^{\lambda}) = {}^{\xi}\delta^2 - {}^{\daleth}\eta^2 \leq max_{\wp}\,({}^{\xi}\delta)^2 - min_{\wp}\,({}^{\daleth}\eta)^2 = \mathscr{H}(\chi^{\lambda}{}_{max})$ So, $\mathscr{H}(\chi^{\lambda}) \leq \mathscr{H}(\chi^{\lambda}{}_{max})$.

Again, $\mathscr{H}(\chi^{\lambda}) = {}^{\xi}\delta^2 - {}^{\daleth}\eta^2 \geq min_{\wp}\,({}^{\xi}\delta)^2 - max_{\wp}\,({}^{\daleth}\eta)^2 = \mathscr{H}(\chi^{\lambda}{}_{min})$ So, $\mathscr{H}(\chi^{\lambda}) \geq \mathscr{H}(\chi^{\lambda}{}_{min})$.

If, $\mathscr{H}(\chi^{\lambda}) \leq \mathscr{H}(\chi^{\lambda}{}_{max})$ and $\mathscr{H}(\chi^{\lambda}) \geq \mathscr{H}(\chi^{\lambda}{}_{min})$, then

$$\chi^{\lambda}{}_{min} \leq \text{PyFSMA}(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) \leq \chi^{\lambda}{}_{max}. \tag{3.5}$$

If $\mathscr{H}(\chi^{\lambda}) = \mathscr{H}(\chi^{\lambda}{}_{max})$, then ${}^{\xi}\delta^2 - {}^{\daleth}\eta^2 = max_{\wp}\,({}^{\xi}\delta)^2 - min_{\wp}\,({}^{\daleth}\eta)^2$

$$\Leftrightarrow {}^{\xi}\delta^2 - {}^{\daleth}\eta^2 = max_{\wp}\,({}^{\xi}\delta)^2 - min_{\wp}\,({}^{\daleth}\eta)^2$$

$$\Leftrightarrow {}^{\xi}\delta^{2} = max_{\wp}\,({}^{\xi}\delta)^2, \quad {}^{\gimel}\eta^{2} = min_{\wp}\,({}^{\gimel}\eta)^2$$
$$\Leftrightarrow {}^{\xi}\delta = max_{\wp}\,{}^{\xi}\delta, \quad {}^{\gimel}\eta = min_{\wp}\,{}^{\gimel}\eta.$$

Now, $H(\chi^{\lambda}) = {}^{\xi}\delta^{2} + {}^{\gimel}\eta^{2} = max_{\wp}\,({}^{\xi}\delta)^2 + min_{\wp}\,({}^{\gimel}\eta)^2 = H({\chi^{\lambda}}_{max})$

$$\text{PyFSMA}({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \ldots {\chi^{\lambda}}_n) = {\chi^{\lambda}}_{max}. \tag{3.6}$$

If $\mathscr{H}(\chi^{\lambda}) = \mathscr{H}({\chi^{\lambda}}_{min})$, then ${}^{\xi}\delta^{2} - {}^{\gimel}\eta^{2} = min_{\wp}\,({}^{\xi}\delta)^2 - max_{\wp}\,({}^{\gimel}\eta)^2$

$$\Leftrightarrow {}^{\xi}\delta^{2} - {}^{\gimel}\eta^{2} = min_{\wp}\,({}^{\xi}\delta)^2 - max_{\wp}\,({}^{\gimel}\eta)^2$$
$$\Leftrightarrow {}^{\xi}\delta^{2} = min_{\wp}\,({}^{\xi}\delta)^2, \quad {}^{\gimel}\eta^{2} = max_{\wp}\,({}^{\gimel}\eta)^2$$
$$\Leftrightarrow {}^{\xi}\delta = min_{\wp}\,{}^{\xi}\delta, \quad {}^{\gimel}\eta = max_{\wp}\,{}^{\gimel}\eta.$$

Now, $H(\chi^{\lambda}) = {}^{\xi}\delta^{2} + {}^{\gimel}\eta^{2} = min_{\wp}\,({}^{\xi}\delta)^2 + max_{\wp}\,({}^{\gimel}\eta)^2 = H({\chi^{\lambda}}_{max})$

$$\text{PyFSMA}({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \ldots {\chi^{\lambda}}_n) = {\chi^{\lambda}}_{min}. \tag{3.7}$$

Thus, from Eqs 3.5–3.7, we get

$$\chi^{\lambda^-} \le \text{PyFSMA}({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \ldots {\chi^{\lambda}}_n) \le \chi^{\lambda^+}.$$

□

Theorem 3.4. *(Idempotency) Assume that* ${\chi^{\lambda}}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\gimel}\eta_{\wp} \rangle$ *is the assortment of PyFNs, where* ${\mathfrak{Y}^{\xi}}_{\wp} = \prod_{k=1}^{c-1} \mathscr{H}({\chi^{\lambda}}_k)$ $(c = 2\ldots, n)$, ${\mathfrak{Y}^{\xi}}_1 = 1$, *and* $\mathscr{H}({\chi^{\lambda}}_k)$ *is the score of* k^{th} *PyFN. If all* ${\chi^{\lambda}}_{\wp}$ *are equal, i.e,.* ${\chi^{\lambda}}_{\wp} = \chi^{\lambda}$ *for all j, then*

$$PyFSMA({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \ldots {\chi^{\lambda}}_n) = \chi^{\lambda}.$$

Proof. From Definition 3.1, we have

$$\begin{aligned}
\text{PyFSMA}({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \ldots {\chi^{\lambda}}_n) &= \frac{\exp[{\mathfrak{Y}^{\xi}}_1/\daleth]}{\sum\limits_{\wp=1}^{n} \exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\chi^{\lambda}}_1 \oplus \frac{\exp[{\mathfrak{Y}^{\xi}}_2/\daleth]}{\sum\limits_{\wp=1}^{n} \exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\chi^{\lambda}}_2 \oplus \ldots, \oplus \frac{\exp[{\mathfrak{Y}^{\xi}}_n/\daleth]}{\sum\limits_{\wp=1}^{n} \exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]} \\
&= \frac{\exp[{\mathfrak{Y}^{\xi}}_1/\daleth]}{\sum\limits_{\wp=1}^{n} \exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}\chi^{\lambda} \oplus \frac{\exp[{\mathfrak{Y}^{\xi}}_2/\daleth]}{\sum\limits_{\wp=1}^{n} \exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}\chi^{\lambda} \oplus \ldots, \oplus \frac{\exp[{\mathfrak{Y}^{\xi}}_n/\daleth]}{\sum\limits_{\wp=1}^{n} \exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}\chi^{\lambda} \\
&= \chi^{\lambda}.
\end{aligned}$$

□

Corollary 3.5. *If* ${\chi^{\lambda}}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\gimel}\eta_{\wp} \rangle$ $j = (1, 2, \ldots n)$ *is the assortment of the largest PyFNs, i.e,.* ${\chi^{\lambda}}_{\wp} = (1, 0)$ *for all j, then*

$$PyFSMA({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \ldots {\chi^{\lambda}}_n) = (1, 0).$$

Proof. We can easily obtain a corollary similar to the Theorem 3.4. □

Theorem 3.6. *(Monotonicity) Assume that* $\chi^{\lambda}{}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\gimel}\eta_{\wp} \rangle$ *and* $\chi^{\lambda^*}_{\wp} = \langle {}^{\xi}\delta^*_{\wp}, {}^{\gimel}\eta^*_{\wp} \rangle$ *are the families of PyFNs, where* $\mathfrak{Y}^{\xi}{}_{\wp} = \prod_{k=1}^{c-1} \mathscr{H}(\chi^{\lambda}{}_k)$, $T^*_{\wp} = \prod_{k=1}^{c-1} \mathscr{H}(\chi^{\lambda^*}_k)$ $(c = 2\ldots,n)$, $\mathfrak{Y}^{\xi}{}_1 = 1$, $T^*_1 = 1$, $\mathscr{H}(\chi^{\lambda}{}_k)$ *is the score of* $\chi^{\lambda}{}_k$ *PyFN, and* $\mathscr{H}(\chi^{\lambda^*}_k)$ *is the score of* $\chi^{\lambda^*}_k$ *PyFN. If* ${}^{\xi}\delta^*_{\wp} \geq {}^{\xi}\delta_{\wp}$ *and* ${}^{\gimel}\eta^*_{\wp} \leq {}^{\gimel}\eta_{\wp}$ *for all* j*, then*

$$PyFSMA(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) \leq PyFSMA(\chi^{\lambda^*}_1, \chi^{\lambda^*}_2, \ldots \chi^{\lambda^*}_n).$$

Proof. Here, ${}^{\xi}\delta^*_{\wp} \geq {}^{\xi}\delta_{\wp}$ and ${}^{\gimel}\eta^*_{\wp} \leq {}^{\gimel}\eta_{\wp}$ for all j, If ${}^{\xi}\delta^*_{\wp} \geq {}^{\xi}\delta_{\wp}$.

$\Leftrightarrow ({}^{\xi}\delta^*_{\wp})^2 \geq ({}^{\xi}\delta_{\wp})^2 \Leftrightarrow \sqrt{({}^{\xi}\delta^*_{\wp})^2} \geq \sqrt{({}^{\xi}\delta_{\wp})^2} \Leftrightarrow \sqrt{1-({}^{\xi}\delta^*_{\wp})^2} \leq \sqrt{1-({}^{\xi}\delta_{\wp})^2}$

$$\Leftrightarrow \sqrt{(1-({}^{\xi}\delta^*_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{(1-({}^{\xi}\delta_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}$$

$$\Leftrightarrow \sqrt{\prod_{\wp=1}^{n}(1-({}^{\xi}\delta^*_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{\prod_{\wp=1}^{n}(1-({}^{\xi}\delta_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}$$

$$\Leftrightarrow \sqrt{1-\prod_{\wp=1}^{n}(1-({}^{\xi}\delta_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{1-\prod_{\wp=1}^{n}(1-({}^{\xi}\delta^*_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}$$

Now,

${}^{\gimel}\eta^*_{\wp} \leq {}^{\gimel}\eta_{\wp}$.

$$\Leftrightarrow ({}^{\gimel}\eta^*_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \leq ({}^{\gimel}\eta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}$$

$$\Leftrightarrow \prod_{\wp=1}^{n}({}^{\gimel}\eta^*_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \leq \prod_{\wp=1}^{n}({}^{\gimel}\eta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}.$$

Let

$$\overline{\chi^{\lambda}} = \text{PyFSMA}(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n)$$

and

$$\overline{\chi^{\lambda^*}} = \text{PyFSMA}(\chi^{\lambda^*}_1, \chi^{\lambda^*}_2, \ldots \chi^{\lambda^*}_n).$$

We get that $\overline{\chi^{\lambda^*}} \geq \overline{\chi^{\lambda}}$. So,

$$\text{PyFSMA}(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \ldots \chi^{\lambda}{}_n) \leq \text{PyFSMA}(\chi^{\lambda^*}_1, \chi^{\lambda^*}_2, \ldots \chi^{\lambda^*}_n).$$

□

Theorem 3.7. *(Boundary) Assume that* ${\chi^{\lambda}}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\daleth}\eta_{\wp}\rangle$ *is the assortment of PyFNs, and*

$$\chi^{\lambda^-} = (min_{\wp}\,({}^{\xi}\delta_{\wp}), max_{\wp}\,({}^{\daleth}\eta_{\wp})) \quad and \quad \chi^{\lambda^+} = (max_{\wp}\,({}^{\xi}\delta_{\wp}), min_{\wp}\,({}^{\daleth}\eta_{\wp})).$$

Then,

$$\chi^{\lambda^-} \leq PyFSMA({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \ldots {\chi^{\lambda}}_n) \leq \chi^{\lambda^+}$$

where ${\mathfrak{Y}^{\xi}}_{\wp} = \prod_{k=1}^{c-1} \mathscr{H}({\chi^{\lambda}}_k)$ $(c = 2\ldots, n)$*,* ${\mathfrak{Y}^{\xi}}_1 = 1$*, and* $\mathscr{H}({\chi^{\lambda}}_k)$ *is the score of the* k^{th} *PyFN.*

Proof. Here, ${}^{\xi}\delta^*_{\wp} \geq {}^{\xi}\delta_{\wp}$ and ${}^{\daleth}\eta^*_{\wp} \leq {}^{\daleth}\eta_{\wp}$ for all j. If ${}^{\xi}\delta^*_{\wp} \geq {}^{\xi}\delta_{\wp}$,

$\Leftrightarrow ({}^{\xi}\delta^*_{\wp})^2 \geq ({}^{\xi}\delta_{\wp})^2 \Leftrightarrow \sqrt{({}^{\xi}\delta^*_{\wp})^2} \geq \sqrt{({}^{\xi}\delta_{\wp})^2} \Leftrightarrow \sqrt{1-({}^{\xi}\delta^*_{\wp})^2} \leq \sqrt{1-({}^{\xi}\delta_{\wp})^2}$

$$\Leftrightarrow \sqrt{(1-({}^{\xi}\delta^*_{\wp})^2)^{\frac{\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}}} \leq \sqrt{(1-({}^{\xi}\delta_{\wp})^2)^{\frac{\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}}}$$

$$\Leftrightarrow \sqrt{\prod_{\wp=1}^{n}(1-({}^{\xi}\delta^*_{\wp})^2)^{\frac{\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}}} \leq \sqrt{\prod_{\wp=1}^{n}(1-({}^{\xi}\delta_{\wp})^2)^{\frac{\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}}}$$

$$\Leftrightarrow \sqrt{1-\prod_{\wp=1}^{n}(1-({}^{\xi}\delta_{\wp})^2)^{\frac{\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}}} \leq \sqrt{1-\prod_{\wp=1}^{n}(1-({}^{\xi}\delta^*_{\wp})^2)^{\frac{\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}}}.$$

Now,

${}^{\daleth}\eta^*_{\wp} \leq {}^{\daleth}\eta_{\wp}$.

$$\Leftrightarrow ({}^{\daleth}\eta^*_{\wp})^{\frac{\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}} \leq ({}^{\daleth}\eta_{\wp})^{\frac{\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}}$$

$$\Leftrightarrow \prod_{\wp=1}^{n}({}^{\daleth}\eta^*_{\wp})^{\frac{\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}} \leq \prod_{\wp=1}^{n}({}^{\daleth}\eta_{\wp})^{\frac{\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[{\mathfrak{Y}^{\xi}}_{\wp}/\daleth]}}.$$

Let

$$\overline{\chi^{\lambda}} = \text{PyFSMA}({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \ldots {\chi^{\lambda}}_n)$$

and

$$\overline{\chi^{\lambda^*}} = \text{PyFSMA}(\chi^{\lambda^*}_1, \chi^{\lambda^*}_2, \ldots \chi^{\lambda^*}_n).$$

We get that $\overline{\chi^{\lambda^*}} \geq \overline{\chi^{\lambda}}$. So,

$$\text{PyFSMA}({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \ldots {\chi^{\lambda}}_n) \leq \text{PyFSMA}(\chi^{\lambda^*}_1, \chi^{\lambda^*}_2, \ldots \chi^{\lambda^*}_n).$$

□

Theorem 3.8. *Assume that* $\chi^{\lambda}{}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\daleth}\eta_{\wp}\rangle$ *and* $\beta_{\wp} = \langle \phi_{\wp}, \varphi_{\wp}\rangle$ *are two families of PyFNs, where* $\mathfrak{Y}^{\xi}{}_{\wp} = \prod_{k=1}^{c-1} \mathscr{H}(\chi^{\lambda}{}_{k})$ $(c = 2\ldots, n)$, $\mathfrak{Y}^{\xi}{}_{1} = 1$, *and* $\mathscr{H}(\chi^{\lambda}{}_{k})$ *is the score of the* k^{th} *PyFN. If* $r > 0$ *and* $\beta = \langle {}^{\xi}\delta_{\beta}, {}^{\daleth}\eta_{\beta}\rangle$ *is an PyFN, then*

i. $PyFSMA(\chi^{\lambda}{}_{1} \oplus \beta, \chi^{\lambda}{}_{2} \oplus \beta, \ldots \chi^{\lambda}{}_{n} \oplus \beta) = PyFSMA(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \ldots \chi^{\lambda}{}_{n}) \oplus \beta$,

ii. $PyFSMA(r\chi^{\lambda}{}_{1}, r\chi^{\lambda}{}_{2}, \ldots r\chi^{\lambda}{}_{n}) = r\ PyFSMA(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \ldots \chi^{\lambda}{}_{n})$,

iii. $PyFSMA(\chi^{\lambda}{}_{1} \oplus \beta_2, \chi^{\lambda}{}_{2} \oplus \beta_2, \ldots \chi^{\lambda}{}_{n} \oplus \beta_n) = PyFSMA(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \ldots \chi^{\lambda}{}_{n}) \oplus PyFSMA(\beta_1, \beta_2, \ldots \beta_n)$,

iv. $PyFSMA(r\chi^{\lambda}{}_{1} \oplus \beta r\chi^{\lambda}{}_{2} \oplus \beta, \ldots \oplus r\chi^{\lambda}{}_{n} \oplus \beta) = r\ PyFSMA(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \ldots \chi^{\lambda}{}_{n}) \oplus \beta$.

Proof. Here, we just prove i and iii.

i.

We have,

$$\chi^{\lambda}{}_{\wp} \oplus \beta = \left(1 - (1 - ({}^{\xi}\delta_{\wp})^2)(1 - ({}^{\xi}\delta_{\beta})^2), {}^{\daleth}\eta_{\wp}{}^{\daleth}\eta_{\beta}\right).$$

By Theorem 3.2,

PyFSMA$(\chi^{\lambda}{}_{1} \oplus \beta, \chi^{\lambda}{}_{2} \oplus \beta, \ldots \chi^{\lambda}{}_{n} \oplus \beta)$

$$= \left\langle \sqrt{(1 - \prod_{\wp=1}^{n} \left((1 - {}^{\xi}\delta_{\wp}^2)(1 - ({}^{\xi}\delta_{\beta})^2)\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}}}, \prod_{\wp=1}^{n}\left({}^{\daleth}\eta_{\beta}{}^{\daleth}\eta_{\wp}\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}} \right\rangle$$

$$= \left\langle \sqrt{(1 - \left(1 - ({}^{\xi}\delta_{\beta})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}} \prod_{\wp=1}^{n}\left(1 - ({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}}}, \left({}^{\daleth}\eta_{\beta}\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}} \prod_{\wp=1}^{n}\left({}^{\daleth}\eta_{\wp}\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}} \right\rangle$$

$$= \left\langle \sqrt{(1 - \left(1 - ({}^{\xi}\delta_{\beta})^2\right) \prod_{\wp=1}^{n}\left(1 - ({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}}}, \left({}^{\daleth}\eta_{\beta}\right) \prod_{\wp=1}^{n}\left({}^{\daleth}\eta_{\wp}\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}} \right\rangle.$$

Now, by operational laws of PyFNs,

$$\text{PyFSMA}(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \ldots \chi^{\lambda}{}_{n}) \oplus \beta$$

$$= \left\langle \sqrt{(1 - \prod_{\wp=1}^{n}(1 - {}^{\xi}\delta_{\wp}^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}}}, \prod_{\wp=1}^{n}{}^{\daleth}\eta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}} \right\rangle \oplus \langle {}^{\xi}\delta_{\beta}, {}^{\daleth}\eta_{\beta}\rangle$$

$$= \left\langle \sqrt{(1 - \left(1 - ({}^{\xi}\delta_{\beta})^2\right) \prod_{\wp=1}^{n}\left(1 - ({}^{\xi}\delta_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}}}, \left({}^{\daleth}\eta_{\beta}\right) \prod_{\wp=1}^{n}\left({}^{\daleth}\eta_{\wp}\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\urcorner]}} \right\rangle$$

Thus,

$$\text{PyFSMA}({\chi^{\lambda}}_1 \oplus \beta, {\chi^{\lambda}}_2 \oplus \beta, \dots {\chi^{\lambda}}_n \oplus \beta) = \text{PyFSMA}({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \dots {\chi^{\lambda}}_n) \oplus \beta.$$

iii.

According to Theorem 3.2,

$\text{PyFSMA}({\chi^{\lambda}}_1 \oplus \beta_2, {\chi^{\lambda}}_2 \oplus \beta_2, \dots {\chi^{\lambda}}_n \oplus \beta_n)$

$$= \left\langle \sqrt{(1 - \prod_{\wp=1}^{n} \left((1 - {}^{\xi}\delta_{\wp}^2)(1 - (\phi_{\wp})^2)\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}, \prod_{\wp=1}^{n} \left(\varphi_{\wp} {}^{\gimel}\eta_{\wp}\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \right\rangle$$

$$= \left\langle \sqrt{(1 - \prod_{\wp=1}^{n} \left(1 - (\phi_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \prod_{\wp=1}^{n} \left(1 - {}^{\xi}\delta_{\wp}^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}, \prod_{\wp=1}^{n} \left(\varphi_{\wp}\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \prod_{\wp=1}^{n} \left({}^{\gimel}\eta_{\wp}\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \right\rangle.$$

Now,

$\text{PyFSMA}({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \dots {\chi^{\lambda}}_n) \oplus \text{PyFSMA}(\beta_1, \beta_2, \dots \beta_n)$

$$= \left\langle \sqrt{(1 - \prod_{\wp=1}^{n} (1 - {}^{\xi}\delta_{\wp}^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}, \prod_{\wp=1}^{n} {}^{\gimel}\eta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \right\rangle \oplus \left\langle \sqrt{(1 - \prod_{\wp=1}^{n} (1 - {}^{\xi}\delta_{\wp}^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}, \prod_{\wp=1}^{n} {}^{\gimel}\eta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \right\rangle$$

$$= \left\langle \sqrt{(1 - \prod_{\wp=1}^{n} \left(1 - (\phi_{\wp})^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \prod_{\wp=1}^{n} \left(1 - {}^{\xi}\delta_{\wp}^2\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}, \prod_{\wp=1}^{n} \left(\varphi_{\wp}\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \prod_{\wp=1}^{n} \left({}^{\gimel}\eta_{\wp}\right)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \right\rangle.$$

Thus,

$$\text{PyFSMA}({\chi^{\lambda}}_1 \oplus \beta_2, {\chi^{\lambda}}_2 \oplus \beta_2, \dots {\chi^{\lambda}}_n \oplus \beta_n) = \text{PyFSMA}({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \dots {\chi^{\lambda}}_n) \oplus \text{PyFSMA}(\beta_1, \beta_2, \dots \beta_n).$$

□

3.2. *PyFSMG operator*

Definition 3.9. Assume that ${\chi^{\lambda}}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\gimel}\eta_{\wp} \rangle$ is the assortment of PyFNs, and $PyFSMG : \Lambda^n \to \Lambda$, is an n-dimension mapping. if

$$\text{PyFSMG}({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \dots {\chi^{\lambda}}_n) = {\chi^{\lambda}}_1^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \otimes {\chi^{\lambda}}_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{2}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \otimes \dots, \otimes {\chi^{\lambda}}_n^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{n}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \tag{3.8}$$

then the mapping PyFSMG is called a Pythagorean fuzzy soft-max geometric (PyFSMG) operator, where $\mathfrak{Y}^{\xi}{}_{\wp} = \prod_{k=1}^{c-1} \mathscr{H}({\chi^{\lambda}}_k)$ $(c = 2 \dots, n)$, $\mathfrak{Y}^{\xi}{}_1 = 1$, and $\mathscr{H}({\chi^{\lambda}}_k)$ is the score of the k^{th} PyFN.

Based on the PyFNs operational rules, we can also consider the PyFSMG by the theorem below.

Theorem 3.10. *Assuming that $\chi^{\lambda}{}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\gimel}\eta_{\wp}\rangle$ is the family of PyFNs, we can find PyFSMG by*

$$PyFSMG(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \dots \chi^{\lambda}{}_n) = \left\langle \prod_{\wp=1}^{n} {}^{\xi}\delta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{(1-\prod_{\wp=1}^{n}(1-{}^{\gimel}\eta_{\wp}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \right\rangle. \tag{3.9}$$

Proof. The first statement is easily followed by Definition 3.9 and Theorem 3.10. In the following, we prove that

$$\begin{aligned}
&\mathrm{PyFSMG}(\chi^{\lambda}{}_1, \chi^{\lambda}{}_2, \dots \chi^{\lambda}{}_n)\\
&= \chi^{\lambda}{}_1^{\frac{\tilde{T}_1}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}} \otimes \chi^{\lambda}{}_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_2/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \otimes \dots, \otimes \chi^{\lambda}{}_n^{\frac{\mathfrak{Y}^{\xi}{}_n}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}}\\
&= \left\langle \prod_{\wp=1}^{n} {}^{\xi}\delta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{(1-\prod_{\wp=1}^{n}(1-{}^{\gimel}\eta_{\wp}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \right\rangle.
\end{aligned}$$

To prove this theorem, we use mathematical induction.

For $n = 2$,

$$\chi^{\lambda}{}_1^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} = \left\langle {}^{\xi}\delta_1^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \right\rangle, \sqrt{1-(1-{}^{\gimel}\eta_1^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}$$

$$\chi^{\lambda}{}_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_2/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} = \left\langle {}^{\xi}\delta_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{1-(1-{}^{\gimel}\eta_2^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \right\rangle.$$

Then,

$$\chi^{\lambda}{}_1^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \otimes \chi^{\lambda}{}_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_2/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}$$

$$= \left\langle {}^{\xi}\delta_1^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \sqrt{1-(1-{}^{\gimel}\eta_1^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \right\rangle \otimes \left\langle {}^{\xi}\delta_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{1-(1-{}^{\gimel}\eta_2^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \right\rangle$$

$$= \left\langle {}^{\xi}\delta_{1}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} . {}^{\xi}\delta_{2}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{\begin{array}{l}(1-(1-\gimel\eta_{1}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} + 1-(1-\gimel\eta_{2}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} - \\ \left((1-(1-\gimel\eta_{1}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right)\left((1-(1-\gimel\eta_{2}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right)\end{array}} \right\rangle$$

$$= \left\langle {}^{\xi}\delta_{1}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} . {}^{\xi}\delta_{2}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{\begin{array}{l}1-(1-\gimel\eta_{1}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} + 1-(1-\gimel\eta_{2}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} - \\ \left(1-(1-\gimel\eta_{2}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} - (1-\gimel\eta_{1}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} + (1-\gimel\eta_{2}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}(1-\gimel\eta_{1}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right)\end{array}} \right\rangle$$

$$= \left\langle {}^{\xi}\delta_{1}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} . {}^{\xi}\delta_{2}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{1-(1-\gimel\eta_{1}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}(1-\gimel\eta_{2}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{1}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \right\rangle$$

$$= \left\langle \prod_{\wp=1}^{2} {}^{\xi}\delta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{1-\prod_{\wp=1}^{2}(1-\gimel\eta_{\wp}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \right\rangle.$$

This shows that Eq 3.9 is true for $n = 2$. Now we assume that Eq 3.9 holds for $n = k$, i.e.,

$$\mathrm{PyFSMG}(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \ldots \chi^{\lambda}{}_{k}) = \left\langle \prod_{\wp=1}^{k} {}^{\xi}\delta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{1-\prod_{\wp=1}^{k}(1-\gimel\eta_{\wp}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \right\rangle.$$

Now, $n = k + 1$, and by the operational laws of PyFNs we have,

$$\text{PyFSMG}(\chi^{\lambda}{}_1,\chi^{\lambda}{}_2,\ldots\chi^{\lambda}{}_{k+1}) = \text{PyFSMG}(\chi^{\lambda}{}_1,\chi^{\lambda}{}_2,\ldots\chi^{\lambda}{}_{k})\otimes\chi^{\lambda}{}_{k+1}$$

$$= \left\langle \prod_{\wp=1}^{k} {}^{\xi}\delta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{1-\prod_{\wp=1}^{k}(1-{}^{\imath}\eta_{\wp}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \right\rangle \otimes \left\langle {}^{\xi}\delta_{k+1}^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}}, \sqrt{1-(1-{}^{\imath}\eta_{k+1}^{2})^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}}} \right\rangle$$

$$= \left\langle \prod_{\wp=1}^{k} {}^{\xi}\delta_{k}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} . {}^{\xi}\delta_{k=1}^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}}, \sqrt{\begin{array}{l}1-\prod_{\wp=1}^{k}(1-{}^{\imath}\eta_{k}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} + 1-(1-{}^{\imath}\eta_{k+1}^{2})^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}} - \\ \left(1-\prod_{\wp=1}^{k}(1-{}^{\imath}\eta_{k}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}\right)\left(1-(1-{}^{\imath}\eta_{k+1}^{2})^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}}\right)\end{array}} \right\rangle$$

$$= \left\langle \prod_{\wp=1}^{k} {}^{\xi}\delta_{k}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} . {}^{\xi}\delta_{k+1}^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}}, \sqrt{1-\prod_{\wp=1}^{k}(1-{}^{\imath}\eta_{k}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}(1-{}^{\imath}\eta_{k+1}^{2})^{\frac{\mathfrak{Y}^{\xi}{}_{k+1}}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{k+1}}}} \right\rangle$$

$$= \left\langle \prod_{\wp=1}^{k+1} {}^{\xi}\delta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{1-\prod_{\wp=1}^{k+1}(1-{}^{\imath}\eta_{\wp}^{2})^{\frac{\mathfrak{Y}^{\xi}{}_{\wp}}{\sum_{\wp=1}^{k+1}\mathfrak{Y}^{\xi}{}_{\wp}}}} \right\rangle.$$

This shows that for $n = k + 1$,Eq 3.2 holds. Then,

$$\text{PyFSMG}(\chi^{\lambda}{}_1,\chi^{\lambda}{}_2,\ldots\chi^{\lambda}{}_{n}) = \left\langle \prod_{\wp=1}^{n} {}^{\xi}\delta_{\wp}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{(1-\prod_{\wp=1}^{n}(1-{}^{\imath}\eta_{\wp}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \right\rangle$$

□

Below we define some of the PyFSMG operator's appealing properties.

Theorem 3.11. *(Idempotency) Assume that* $\chi^{\lambda}{}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\imath}\eta_{\wp}\rangle$ *is the family of PyFNs, where* $\mathfrak{Y}^{\xi}{}_{\wp} = \prod_{k=1}^{c-1}\mathscr{H}(\chi^{\lambda}{}_{k})$ *(*$c = 2\ldots,n$*),* $\mathfrak{Y}^{\xi}{}_{1} = 1$*, and* $\mathscr{H}(\chi^{\lambda}{}_{k})$ *is the score of the* k^{th} *PyFN. If all* $\chi^{\lambda}{}_{\wp}$ *are equal, i.e,.* $\chi^{\lambda}{}_{\wp} = \chi^{\lambda}$ *for all j, then*

$$PyFSMG(\chi^{\lambda}{}_1,\chi^{\lambda}{}_2,\ldots\chi^{\lambda}{}_{n}) = \chi^{\lambda}.$$

Proof. From Definition 3.1, we have

$$\begin{aligned}
\text{PyFSMG}({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \dots {\chi^{\lambda}}_n) &= {\chi^{\lambda}}_1^{\frac{\exp[\mathfrak{Y}^{\xi}{}_1/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \otimes {\chi^{\lambda}}_2^{\frac{\exp[\mathfrak{Y}^{\xi}{}_2/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \otimes \dots, \otimes {\chi^{\lambda}}_n^{\frac{\exp[\mathfrak{Y}^{\xi}{}_n/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \\
&= {\chi^{\lambda}}^{\frac{\tilde{T}_1}{\sum_{\wp=1}^{n}\mathfrak{Y}^{\xi}{}_{\wp}}} \otimes {\chi^{\lambda}}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_2/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \otimes \dots, \otimes {\chi^{\lambda}}^{\frac{\exp[\mathfrak{Y}^{\xi}{}_n/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \\
&= \chi^{\lambda}.
\end{aligned}$$

□

Corollary 3.12. *If* ${\chi^{\lambda}}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\gimel}\eta_{\wp} \rangle$ $j = (1, 2, \dots n)$ *is the family of largest PyFNs, i.e,.* ${\chi^{\lambda}}_{\wp} = (1, 0)$ *for all* j*, then*

$$PyFSMG({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \dots {\chi^{\lambda}}_n) = (1, 0).$$

Proof. We can easily obtain a corollary similar to Theorem 3.4. □

Theorem 3.13. *(Monotonicity) Assume that* ${\chi^{\lambda}}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\gimel}\eta_{\wp} \rangle$ *and* ${\chi^{\lambda^*}}_{\wp} = \langle {}^{\xi}\delta^*_{\wp}, {}^{\gimel}\eta^*_{\wp} \rangle$ *are the families of PyFNs, where* $\mathfrak{Y}^{\xi}{}_{\wp} = \prod_{k=1}^{c-1} \mathscr{H}({\chi^{\lambda}}_k)$, $T^*_{\wp} = \prod_{k=1}^{c-1} \mathscr{H}({\chi^{\lambda^*}}_k)$ $(c = 2 \dots, n)$, $\mathfrak{Y}^{\xi}{}_1 = 1$, $T^*_1 = 1$, $\mathscr{H}({\chi^{\lambda}}_k)$ *is the score of* ${\chi^{\lambda}}_k$ *PyFN, and* $\mathscr{H}({\chi^{\lambda^*}}_k)$ *is the score of* ${\chi^{\lambda^*}}_k$ *PyFN. If* ${}^{\xi}\delta^*_{\wp} \geq {}^{\xi}\delta_{\wp}$ *and* ${}^{\gimel}\eta^*_{\wp} \leq {}^{\gimel}\eta_{\wp}$ *for all* j*, then*

$$PyFSMG({\chi^{\lambda}}_1, {\chi^{\lambda}}_2, \dots {\chi^{\lambda}}_n) \leq PyFSMG({\chi^{\lambda^*}}_1, {\chi^{\lambda^*}}_2, \dots {\chi^{\lambda^*}}_n).$$

Proof. Here, ${}^{\gimel}\eta^*_{\wp} \geq {}^{\gimel}\eta_{\wp}$ and ${}^{\xi}\delta^*_{\wp} \leq {}^{\xi}\delta_{\wp}$ for all j, If ${}^{\gimel}\eta^*_{\wp} \geq {}^{\gimel}\eta_{\wp}$.

$$\Leftrightarrow ({}^{\gimel}\eta^*_{\wp})^2 \geq ({}^{\gimel}\eta_{\wp})^2 \Leftrightarrow \sqrt{({}^{\gimel}\eta^*_{\wp})^2} \geq \sqrt{({}^{\gimel}\eta_{\wp})^2} \Leftrightarrow \sqrt{1 - ({}^{\gimel}\eta^*_{\wp})^2} \leq \sqrt{1 - ({}^{\gimel}\eta_{\wp})^2}$$

$$\Leftrightarrow \sqrt{(1 - ({}^{\gimel}\eta^*_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{(1 - ({}^{\gimel}\eta_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}$$

$$\Leftrightarrow \sqrt{\prod_{\wp=1}^{n}(1 - ({}^{\gimel}\eta^*_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{\prod_{\wp=1}^{n}(1 - ({}^{\gimel}\eta_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}$$

$$\Leftrightarrow \sqrt{1 - \prod_{\wp=1}^{n}(1 - ({}^{\gimel}\eta_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \leq \sqrt{1 - \prod_{\wp=1}^{n}(1 - ({}^{\gimel}\eta^*_{\wp})^2)^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}$$

Now,

${}^{\xi}\delta^*_{\wp} \leq {}^{\xi}\delta_{\wp}$.

$$\Leftrightarrow ({}^{\xi}\delta^*_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \leq ({}^{\xi}\delta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}$$

$$\Leftrightarrow \prod_{\wp=1}^{n}({}^{\xi}\delta^{*}_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \leq \prod_{\wp=1}^{n}({}^{\xi}\delta_{\wp})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}.$$

Let

$$\overline{\chi^{\lambda}} = \text{PyFSMG}(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \dots \chi^{\lambda}{}_{n})$$

and

$$\overline{\chi^{\lambda^{*}}} = \text{PyFSMG}(\chi^{\lambda^{*}}_{1}, \chi^{\lambda^{*}}_{2}, \dots \chi^{\lambda^{*}}_{n}).$$

We get that $\overline{\chi^{\lambda^{*}}} \geq \overline{\chi^{\lambda}}$. So,

$$\text{PyFSMG}(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \dots \chi^{\lambda}{}_{n}) \leq \text{PyFSMG}(\chi^{\lambda^{*}}_{1}, \chi^{\lambda^{*}}_{2}, \dots \chi^{\lambda^{*}}_{n}).$$

□

Theorem 3.14. *(Boundary) Assume that* $\chi^{\lambda}{}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\beth}\eta_{\wp} \rangle$ *is the family of PyFNs, and*

$$\chi^{\lambda^{-}} = (min_{\wp}\,({}^{\xi}\delta_{\wp}), max_{\wp}\,({}^{\beth}\eta_{\wp})) \quad \text{and} \quad \chi^{\lambda^{+}} = (max_{\wp}\,({}^{\xi}\delta_{\wp}), min_{\wp}\,({}^{\beth}\eta_{\wp})).$$

Then,

$$\chi^{\lambda^{-}} \leq PyFSMG(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \dots \chi^{\lambda}{}_{n}) \leq \chi^{\lambda^{+}}$$

where $\mathfrak{Y}^{\xi}{}_{\wp} = \prod_{k=1}^{c-1} \mathscr{H}(\chi^{\lambda}{}_{k})$ $(c = 2\dots, n)$, $\mathfrak{Y}^{\xi}{}_{1} = 1$ *and* $\mathscr{H}(\chi^{\lambda}{}_{k})$ *is the score of* k^{th} *PyFN.*

Proof. The proof of this theorem is same as that of Theorem 3.7. □

Theorem 3.15. *Assume that* $\chi^{\lambda}{}_{\wp} = \langle {}^{\xi}\delta_{\wp}, {}^{\beth}\eta_{\wp} \rangle$ *and* $\beta_{\wp} = \langle \phi_{\wp}, \varphi_{\wp} \rangle$ *are two familie of PyFNs, where* $\mathfrak{Y}^{\xi}{}_{\wp} = \prod_{k=1}^{c-1} \mathscr{H}(\chi^{\lambda}{}_{k})$ $(c = 2\dots, n)$, $\mathfrak{Y}^{\xi}{}_{1} = 1$, *and* $\mathscr{H}(\chi^{\lambda}{}_{k})$ *is the score of the* k^{th} *PyFN. If* $r > 0$ *and* $\beta = \langle {}^{\xi}\delta_{\beta}, {}^{\beth}\eta_{\beta} \rangle$ *is a PyFN, then*
a. $PyFSMG(\chi^{\lambda}{}_{1} \oplus \beta, \chi^{\lambda}{}_{2} \oplus \beta, \dots \chi^{\lambda}{}_{n} \oplus \beta) = PyFSMG(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \dots \chi^{\lambda}{}_{n}) \oplus \beta$,
b. $PyFSMG(r\chi^{\lambda}{}_{1}, r\chi^{\lambda}{}_{2}, \dots r\chi^{\lambda}{}_{n}) = r\ PyFSMG(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \dots \chi^{\lambda}{}_{n})$,
c. $PyFSMG(\chi^{\lambda}{}_{1} \oplus \beta_{2}, \chi^{\lambda}{}_{2} \oplus \beta_{2}, \dots \chi^{\lambda}{}_{n} \oplus \beta_{n}) =$
$PyFSMG(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \dots \chi^{\lambda}{}_{n}) \oplus PyFSMG(\beta_{1}, \beta_{2}, \dots \beta_{n})$,
d. $PyFSMG(r\chi^{\lambda}{}_{1} \oplus \beta r\chi^{\lambda}{}_{2} \oplus \beta, \dots \oplus r\chi^{\lambda}{}_{n} \oplus \beta) = r\ PyFSMG(\chi^{\lambda}{}_{1}, \chi^{\lambda}{}_{2}, \dots \chi^{\lambda}{}_{n}) \oplus \beta$.

Proof. The proof of this theorem is same as that of Theorem 3.8. □

4. Proposed methodology

Let $\maltese^{\gamma} = \{\maltese^{\gamma}{}_{1}, \maltese^{\gamma}{}_{2}, \dots, \maltese^{\gamma}{}_{m}\}$ and $\mathfrak{R}^{F} = \{\mathfrak{R}^{F}{}_{1}, \mathfrak{R}^{F}{}_{2}, \dots, \mathfrak{R}^{F}{}_{n}\}$ denote the sets of alternatives and criteria, respectively. Let $\mathfrak{K}^{\zeta} = \{\mathfrak{K}^{\zeta}_{1}, \mathfrak{K}^{\zeta}_{2}, \dots, \mathfrak{K}^{\zeta}_{p}\}$ be the group of decision makers (DMs). Each decision maker provides a matrix of their own opinion $D^{(p)} = (\mathscr{B}^{(p)}ij)m \times n$, where $\mathscr{B}^{(p)}ij$ represents the opinion of decision maker $\mathfrak{K}^{\zeta}_{p}$ for the alternatives $\maltese^{\gamma}{}_{i} \in \maltese^{\gamma}$ with respect to the criteria $\mathfrak{R}^{F}{}_{\wp} \in \mathfrak{R}^{F}$ in the form of PyFNs. Criteria are in the linear prioritized relation as, $\mathfrak{R}^{F}{}_{1} > \mathfrak{R}^{F}{}_{2} > \dots, > \mathfrak{R}^{F}{}_{n}$. Moreover, DMs are also in the prioritized relation as, $\mathfrak{K}^{\zeta}_{1} > \mathfrak{K}^{\zeta}_{2}, \dots, \mathfrak{K}^{\zeta}_{p}$.

In MADM, there are two types of criteria: benefit-type attributes τ_b, and cost-type attributes $\tau_\wp$. If all criteria are of the same type, normalization is unnecessary. However, if the criteria are of different types, we use the normalization formula to transform the matrix $D^{(p)}$ into a normalised matrix $Y^{(p)} = (\mathscr{P}_{ij}^{(p)})m \times n$ as follows:

$$(\mathscr{P}_{ij}^{(p)})m \times n = \begin{cases} (\mathscr{B}_{ij}^{(p)})^c; & j \in \tau c \ \mathscr{B}_{ij}^{(p)}; \\ j \in \tau b. \end{cases} \tag{4.1}$$

Here, $(\mathscr{B}_{ij}^{(p)})^c$ denotes the complement of $\mathscr{B}_{ij}^{(p)}$.

We then apply the PyFSMA operator or PyFMA operator to implement a MCGDM approach in a PyF context. The proposed operators are applied to the MCGDM approach in the following steps:

Algorithm

Input:

Step 1:

Acquire a decision matrix $D^{(p)} = (\mathscr{B}_{ij}^{(p)})_{m\times n}$ in the form of PyFNs from the decision makers.

Step 2:

There is no requirement for normalising if all of the criteria are of the same kind; however, in MCGDM, there are two distinct sorts of criteria. In this instance, the matrix was converted into the response matrix $Y^{(p)} = (\mathscr{P}_{ij}^{(p)})_{m\times n}$ by applying the normalising procedure Eq 4.1.

Calculations:

Step 3:

Calculate the values of $\mathfrak{Y}^{\xi}{}_{ij}^{(p)}$ by the following formula:

$$\mathfrak{Y}^{\xi}{}_{ij}^{(p)} = \prod_{k=1}^{p-1} \mathscr{H}(\mathscr{P}_{ij}^{(k)}) \quad (p = 2\ldots, n), \tag{4.2}$$

$$T_{ij}^{(1)} = 1.$$

Step 4:

In order to combine all of the separate PyF decision matrices, you need make use of one of the provided aggregation procedures. $Y^{(p)} = (\mathscr{P}_{ij}^{(p)})_{m\times n}$ worth of information into a single cumulative assessment matrix of the options $W^{(p)} = (\mathscr{W}_{ij})_{m\times n}$.

Step 5:

Calculate the values of $\mathfrak{Y}^{\xi}{}_{ij}$ by following formula:

$$\mathfrak{Y}^{\xi}{}_{ij} = \prod_{k=1}^{c-1} \mathscr{H}(\mathscr{W}_{ik}) \quad (j = 2\ldots, n), \tag{4.3}$$

$$\mathfrak{Y}^{\xi}{}_{i1} = 1.$$

Step 6:

Aggregate the PyF values $\mathscr{W}_{ij}$ for each alternative $\maltese^{\gamma}{}_i$ by the PyFSMA (or PyFSMG) operator:

$\mathscr{W}_i = \text{PyFSMA}(\mathscr{P}_{i1}, \mathscr{P}_{i2}, \ldots \mathscr{P}_{in})$

$$= \left\langle \sqrt{(1 - \prod_{\wp=1}^{n}(1 - ({}^{\xi}\delta_{ij}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}}, \prod_{\wp=1}^{n}({}^{\xi}\eta_{ij}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}} \right\rangle \tag{4.4}$$

or

$\mathscr{W}_i = \text{PyFSMG}(\mathscr{P}_{i1}, \mathscr{P}_{i2}, \ldots \mathscr{P}_{in})$

$$= \left\langle \prod_{\wp=1}^{n}({}^{\xi}\delta_{ij}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}, \sqrt{(1 - \prod_{\wp=1}^{n}(1 - ({}^{\xi}\eta_{ij}^{2})^{\frac{\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}{\sum_{\wp=1}^{n}\exp[\mathfrak{Y}^{\xi}{}_{\wp}/\daleth]}}} \right\rangle. \tag{4.5}$$

Output:
Step 7:
Examine the total score obtained from all of the cumulative alternative tests.
Step 8:
The options were ranked using the score function, and the successful candidate was determined to be the alternative that was the most appropriate.

5. Numerical example

In this modern era of globalised and highly competitive markets, efficient supplier selection has emerged as a critical strategic decision for businesses across industries. In supply chain management, the importance of this process and provide motivation for organisations to focus on enhancing their supplier selection practices [43].

First and foremost, efficient supplier selection directly contributes to a company's overall performance and competitiveness. The suppliers chosen by an organisation play a vital role in determining the quality, cost, and timely delivery of goods or services. Selecting the right suppliers who can meet the specific requirements and expectations of the organisation is crucial for maintaining customer satisfaction and loyalty [44]. A competitive edge, improved operational effectiveness, and long-term viability can be attained by organisations that guarantee a dependable and efficient supply chain. Furthermore, the contemporary business environment is distinguished by a growing intricacy and interdependence. Organisations have expanded their supplier base beyond local and regional markets to encompass a diverse range of international vendors. However, managing and selecting the most appropriate suppliers becomes increasingly difficult as a result of this expanded supplier base. Efficient supplier selection practices are crucial in this context as they enable organisations to navigate the intricacies of global procurement, minimise risks, and capitalise on opportunities [45].

Furthermore, technological advancements have significantly transformed business operations, including the supplier selection process. Organisations have access to an abundance of data and sophisticated tools that can facilitate the evaluation and comparison of prospective suppliers in the current digital age [46]. Through the utilisation of advanced technologies including machine learning, artificial intelligence, and big data analytics, organisations have the ability to optimise the supplier selection procedure, discern patterns in extensive data sets, and arrive at decisions based on empirical evidence. By adopting these technological advancements, organisations can greatly improve the efficacy and efficiency of supplier selection. This empowers them to make more informed decisions and maximise the performance of their supply chains [47].

Additionally, supplier collaboration and partnership are fostered through efficient supplier selection, resulting in mutually beneficial relationships. Organisations can foster enduring partnerships founded on trust, transparency, and mutual objectives by meticulously choosing suppliers who are in harmony with their values, goals, and culture [48]. Collaborative alliances of this nature have the potential to yield enhanced communication, novel ideas, and ongoing refinement across the entire supply chain, thereby cultivating a competitive edge and distinguishing characteristics within the marketplace. Effective supplier selection has become an essential operational procedure for organisations seeking to prosper in a rapidly changing and fiercely competitive corporate landscape [49]. Organisations can attain sustained success, boost supply chain performance, and gain a competitive advantage by strategically choosing suppliers who are in line with their objectives, capitalising on technological developments, and cultivating collaborative partnerships. To obtain the numerous benefits that supplier selection processes provide, organisations should therefore invest time, resources, and knowledge into their development [50].

The supplier selection procedure for a business requires crucial decision-making. Supplier selection entails the evaluation of numerous alternatives and the subsequent selection of the most appropriate provider in accordance with predetermined criteria [51]. Organisations can enhance the quality of their procurement process, reduce expenses, boost productivity, and guarantee that their suppliers are dependable and able to deliver superior products or services through the use of effective decision-making [52].

The following are some of the important uses, and importance of decision-making in supplier selection:

- Minimising Risks: By selecting the right supplier, businesses can reduce the risks associated with procurement. A thorough evaluation of suppliers' capabilities and experience can help businesses to avoid suppliers that may deliver poor quality products or services, fail to meet delivery deadlines, or engage in unethical business practices.
- Cost Savings: Decision-making in supplier selection can help businesses to save costs by choosing suppliers that offer competitive prices, favorable payment terms, and efficient delivery methods.
- Improved Quality: By selecting suppliers that meet high standards of quality, businesses can ensure that the products and services they receive are of high quality, which can help to improve the overall quality of their offerings.
- Increased Efficiency: Effective decision-making in supplier selection can help businesses to increase the efficiency of their procurement process by reducing the time and resources required to identify and evaluate suppliers.
- Better Relationships: By selecting suppliers that are responsive, cooperative, and committed to meeting their needs, businesses can build strong relationships with their suppliers, which can lead to increased trust, better communication, and more efficient collaboration.

- Access to new technologies: Decision-making in supplier selection can help businesses to access new technologies and innovations offered by suppliers. This can help businesses to improve their products and services, stay competitive, and expand into new markets.

The following are some of the steps involved in the decision-making process for supplier selection:

- Define Requirements: The first step in the supplier selection process is to define the requirements for the products or services that are needed. This includes defining the specifications, quality standards, and delivery requirements for the products or services.
- Identify Potential Suppliers: The next step is to identify potential suppliers that meet the requirements defined in the first step. This can be done by researching suppliers in the market, reaching out to industry associations, or seeking recommendations from other businesses.
- Evaluate Suppliers: Once potential suppliers have been identified, the next step is to evaluate them based on specific criteria such as their experience, quality of products or services, delivery capabilities, and price.
- Negotiate Contracts: After evaluating suppliers, the next step is to negotiate contracts with the preferred suppliers. This involves negotiating the terms and conditions of the agreement, including delivery schedules, payment terms, and warranties.
- Monitor Performance: After the contracts have been signed, the final step is to monitor the performance of the suppliers to ensure that they are meeting the requirements defined in the first step. This can be done by regularly monitoring the quality of the products or services received, delivery times, and communication with the suppliers.

Decision-making in supplier selection is an important process that helps businesses to minimize risks, save costs, improve quality, increase efficiency, and build better relationships with their suppliers. By following the steps outlined above, businesses can ensure that they make informed decisions that lead to success in procurement.

In order to further understand our suggested process, below is an example that pertains to the selection of suppliers in healthcare. Consider a set of alternatives $\maltese^{\gamma} = \{\maltese^{\gamma}{}_1, \maltese^{\gamma}{}_2, \maltese^{\gamma}{}_3, \maltese^{\gamma}{}_4, \maltese^{\gamma}{}_5\}$ and $\Re^{F} = \{\Re^{F}{}_1, \Re^{F}{}_2, \Re^{F}{}_3, \Re^{F}{}_4, \Re^{F}{}_5, \Re^{F}{}_6\}$ as the finite set of criterion, where $\Re^{F}{}_1$= quality , $\Re^{F}{}_2$= cost , $\Re^{F}{}_3$= delivery, $\Re^{F}{}_4$= services , $\Re^{F}{}_5$=environment, and $\Re^{F}{}_6$= corporate social responsibility. $\mathfrak{K}^{\zeta} = \{\mathfrak{K}^{\zeta}_1, \mathfrak{K}^{\zeta}_2, \mathfrak{K}^{\zeta}_3\}$ is the group of DMs. The strict prioritised relation is given as, $\Re^{F}{}_1 > \Re^{F}{}_2 > \Re^{F}{}_3 > \Re^{F}{}_4 > \Re^{F}{}_5 > \Re^{F}{}_6$ and DMs are prioritized as $\mathfrak{K}^{\zeta}_1 > \mathfrak{K}^{\zeta}_2 > \mathfrak{K}^{\zeta}_3$. DMs provide a matrix of their own opinion $D^{(p)} = (\mathscr{B}^{(p)}_{ij})_{m\times n}$, where $\mathscr{B}^{(p)}_{ij}$ is given for the alternatives $\maltese^{\gamma}{}_i \in \maltese^{\gamma}$ with respect to the criteria $\Re^{F}{}_{\wp} \in \Re^{F}$ by $\mathfrak{K}^{\zeta}_p$ decision maker in the form of PyFNs.

Step 1:
Acquire a decision matrix $D^{(p)} = (\mathscr{B}^{(p)}_{ij})_{m\times n}$ in the form of PyFNs from the decision makers.

Table 1. PyF decision matrix from $\Re_1^{\zeta}$.

	$\mathfrak{R}^{F}{}_{1}$	$\mathfrak{R}^{F}{}_{2}$	$\mathfrak{R}^{F}{}_{3}$	$\mathfrak{R}^{F}{}_{4}$	$\mathfrak{R}^{F}{}_{5}$	$\mathfrak{R}^{F}{}_{6}$
$\maltese^{\gamma}{}_{1}$	(0.89,0.00)	(0.64,0.34)	(0.74,0.14)	(0.94,0.14)	(0.74,0.00)	(0.44,0.24)
$\maltese^{\gamma}{}_{2}$	(0.94,0.24)	(0.79,0.29)	(0.54,0.24)	(0.74,0.14)	(0.44,0.44)	(0.34,0.14)
$\maltese^{\gamma}{}_{3}$	(0.84,0.14)	(0.34,0.54)	(0.74,0.24)	(0.54,0.00)	(0.64,0.34)	(0.44,0.00)
$\maltese^{\gamma}{}_{4}$	(0.74,0.34)	(0.80,0.24)	(0.64,0.14)	(0.34,0.24)	(0.74,0.24)	(0.34,0.74)
$\maltese^{\gamma}{}_{5}$	(0.79,0.24)	(0.59,0.00)	(0.24,0.14)	(0.14,0.64)	(0.64,0.14)	(0.24,0.64)

Table 2. PyF decision matrix from $\Re_2^{\zeta}$.

	$\mathfrak{R}^{F}{}_{1}$	$\mathfrak{R}^{F}{}_{2}$	$\mathfrak{R}^{F}{}_{3}$	$\mathfrak{R}^{F}{}_{4}$	$\mathfrak{R}^{F}{}_{5}$	$\mathfrak{R}^{F}{}_{6}$
$\maltese^{\gamma}{}_{1}$	(0.74,0.24)	(0.54,0.29)	(0.84,0.14)	(0.94,0.14)	(0.79,0.24)	(0.89,0.14)
$\maltese^{\gamma}{}_{2}$	(0.54,0.14)	(0.59,0.34)	(0.44,0.14)	(0.74,0.34)	(0.64,0.29)	(0.74,0.00)
$\maltese^{\gamma}{}_{3}$	(0.89,0.59)	(0.64,0.18)	(0.24,0.54)	(0.64,0.54)	(0.14,0.24)	(0.69,0.29)
$\maltese^{\gamma}{}_{4}$	(0.49,0.00)	(0.54,0.39)	(0.14,0.09)	(0.49,0.59)	(0.09,0.14)	(0.59,0.34)
$\maltese^{\gamma}{}_{5}$	(0.84,0.34)	(0.69,0.29)	(0.64,0.54)	(0.24,0.49)	(0.49,0.29)	(0.49,0.24)

Table 3. PyF decision matrix from $\Re_3^{\zeta}$.

	$\mathfrak{R}^{F}{}_{1}$	$\mathfrak{R}^{F}{}_{2}$	$\mathfrak{R}^{F}{}_{3}$	$\mathfrak{R}^{F}{}_{4}$	$\mathfrak{R}^{F}{}_{5}$	$\mathfrak{R}^{F}{}_{6}$
$\maltese^{\gamma}{}_{1}$	(0.89,0.14)	(0.84,0.24)	(0.79,0.00)	(0.69,0.34)	(0.79,0.19)	(0.69,0.29)
$\maltese^{\gamma}{}_{2}$	(0.79,0.24)	(0.54,0.14)	(0.59,0.24)	(0.49,0.29)	(0.59,0.29)	(0.59,0.29)
$\maltese^{\gamma}{}_{3}$	(0.74,0.14)	(0.64,0.24)	(0.34,0.00)	(0.49,0.34)	(0.74,0.29)	(0.34,0.24)
$\maltese^{\gamma}{}_{4}$	(0.34,0.34)	(0.49,0.34)	(0.44,0.24)	(0.54,0.44)	(0.24,0.24)	(0.64,0.00)
$\maltese^{\gamma}{}_{5}$	(0.64,0.24)	(0.64,0.24)	(0.59,0.14)	(0.64,0.24)	(0.64,0.54)	(0.44,0.39)

Step 2:

Normalise the decision matrices acquired by DMs using Eq 4.1. There are two types of criteria. $\mathfrak{R}^{F}{}_{2}$ is a cost type criteria and others are benefit type criteria.

Table 4. Normalised PyF decision matrix from $\Re_1^{\zeta}$.

	$\mathfrak{R}^{F}{}_{1}$	$\mathfrak{R}^{F}{}_{2}$	$\mathfrak{R}^{F}{}_{3}$	$\mathfrak{R}^{F}{}_{4}$	$\mathfrak{R}^{F}{}_{5}$	$\mathfrak{R}^{F}{}_{6}$
$\maltese^{\gamma}{}_{1}$	(0.89,0.00)	(0.34,0.64)	(0.74,0.14)	(0.94,0.14)	(0.74,0.00)	(0.44,0.24)
$\maltese^{\gamma}{}_{2}$	(0.94,0.24)	(0.29,0.79)	(0.54,0.24)	(0.74,0.14)	(0.44,0.44)	(0.34,0.14)
$\maltese^{\gamma}{}_{3}$	(0.84,0.14)	(0.54,0.34)	(0.74,0.24)	(0.54,0.00)	(0.64,0.34)	(0.44,0.00)
$\maltese^{\gamma}{}_{4}$	(0.74,0.34)	(0.24,0.80)	(0.64,0.14)	(0.34,0.24)	(0.74,0.24)	(0.34,0.74)
$\maltese^{\gamma}{}_{5}$	(0.79,0.24)	(0.00,0.59)	(0.24,0.14)	(0.14,0.64)	(0.64,0.14)	(0.24,0.64)

Table 5. Normalised PyF decision matrix from $\Re_2^{\zeta}$.

	$\Re^F{}_1$	$\Re^F{}_2$	$\Re^F{}_3$	$\Re^F{}_4$	$\Re^F{}_5$	$\Re^F{}_6$
$✠^{\gamma}{}_1$	(0.74,0.24)	(0.29,0.54)	(0.84,0.14)	(0.94,0.14)	(0.79,0.24)	(0.89,0.14)
$✠^{\gamma}{}_2$	(0.54,0.14)	(0.34,0.59)	(0.44,0.14)	(0.74,0.34)	(0.64,0.29)	(0.74,0.00)
$✠^{\gamma}{}_3$	(0.89,0.59)	(0.18,0.64)	(0.24,0.54)	(0.64,0.54)	(0.14,0.24)	(0.69,0.29)
$✠^{\gamma}{}_4$	(0.49,0.00)	(0.39,0.54)	(0.14,0.09)	(0.49,0.59)	(0.09,0.14)	(0.59,0.34)
$✠^{\gamma}{}_5$	(0.84,0.34)	(0.29,0.69)	(0.64,0.54)	(0.24,0.49)	(0.49,0.29)	(0.49,0.24)

Table 6. Normalised PyF decision matrix from $\Re_3^{\zeta}$.

	$\Re^F{}_1$	$\Re^F{}_2$	$\Re^F{}_3$	$\Re^F{}_4$	$\Re^F{}_5$	$\Re^F{}_6$
$✠^{\gamma}{}_1$	(0.89,0.14)	(0.24,0.84)	(0.79,0.00)	(0.69,0.34)	(0.79,0.19)	(0.69,0.29)
$✠^{\gamma}{}_2$	(0.79,0.24)	(0.14,0.54)	(0.59,0.24)	(0.49,0.29)	(0.59,0.29)	(0.59,0.29)
$✠^{\gamma}{}_3$	(0.74,0.14)	(0.24,0.64)	(0.34,0.00)	(0.49,0.34)	(0.74,0.29)	(0.34,0.24)
$✠^{\gamma}{}_4$	(0.34,0.34)	(0.39,0.24)	(0.44,0.24)	(0.54,0.44)	(0.24,0.24)	(0.64,0.00)
$✠^{\gamma}{}_5$	(0.64,0.24)	(0.24,0.64)	(0.59,0.14)	(0.64,0.24)	(0.64,0.54)	(0.44,0.39)

Step 3:
Calculate the values of $\mathfrak{Y}^{\xi(p)}_{ij}$ by Eq 4.2.

$$\mathfrak{Y}^{\xi(1)}_{ij} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

$$\mathfrak{Y}^{\xi(2)}_{ij} = \begin{pmatrix} 0.8540 & 0.3741 & 0.6992 & 0.9160 & 0.7001 & 0.5400 \\ 0.8909 & 0.3174 & 0.5650 & 0.6993 & 0.4999 & 0.5238 \\ 0.8043 & 0.5510 & 0.6931 & 0.5729 & 0.6249 & 0.5176 \\ 0.6985 & 0.2320 & 0.5956 & 0.4935 & 0.6930 & 0.2905 \\ 0.7482 & 0.4140 & 0.4960 & 0.3533 & 0.6256 & 0.3619 \end{pmatrix}.$$

$$\mathfrak{Y}^{\xi(3)}_{ij} = \begin{pmatrix} 0.5977 & 0.1542 & 0.5579 & 0.8373 & 0.5318 & 0.4650 \\ 0.5244 & 0.0939 & 0.2999 & 0.4991 & 0.2989 & 0.2995 \\ 0.5992 & 0.1976 & 0.3043 & 0.3562 & 0.2987 & 0.4490 \\ 0.3718 & 0.0989 & 0.3043 & 0.1934 & 0.3423 & 0.2587 \\ 0.5741 & 0.1261 & 0.3976 & 0.1844 & 0.3727 & 0.4317 \end{pmatrix}.$$

Step 4:
Use PyFSMA to aggregate all individual PyF decision matrices $Y^{(p)} = (\mathscr{P}^{(p)}_{ij})_{m\times n}$ into one cumulative assessments matrix of the alternatives $W^{(p)} = (\mathscr{W}_{ij})_{m\times n}$ using proposed AOs given in Table 7.

Table 7. Collective PyF decision matrix.

	$\mathfrak{R}^{F}{}_{1}$	$\mathfrak{R}^{F}{}_{2}$	$\mathfrak{R}^{F}{}_{3}$	$\mathfrak{R}^{F}{}_{4}$	$\mathfrak{R}^{F}{}_{5}$	$\mathfrak{R}^{F}{}_{6}$
$\maltese^{\gamma}{}_{1}$	(0.7986, 0.0000)	(0.2844, 0.5432)	(0.8674, 0.0000)	(0.9643, 0.0932)	(0.8956, 0.0000)	(0.6784, 0.2636)
$\maltese^{\gamma}{}_{2}$	(0.9743, 0.1897)	(0.2973, 0.4632)	(0.4536, 0.2138)	(0.6954, 0.1984)	(0.5434, 0.3224)	(0.6474, 0.0000)
$\maltese^{\gamma}{}_{3}$	(0.7953, 0.2943)	(0.3563, 0.7643)	(0.7884, 0.0000)	(0.5356, 0.0000)	(0.5956, 0.5465)	(0.6764, 0.0000)
$\maltese^{\gamma}{}_{4}$	(0.7854, 0.0000)	(0.3522, 0.7849)	(0.4352, 0.3464)	(0.6463, 0.4674)	(0.6644, 0.1675)	(0.4754, 0.0000)
$\maltese^{\gamma}{}_{5}$	(0.2467, 0.2315)	(0.5232, 0.6743)	(0.4363, 0.5638)	(0.6474, 0.5516)	(0.5956, 0.3225)	(0.4636, 0.6546)

Step 5:
Evaluate the values of $\mathfrak{Y}^{\xi}{}_{ij}$ by using Eq 4.3.

$$\mathfrak{Y}^{\xi}{}_{ij} = \begin{pmatrix} 1 & 0.8179 & 0.2999 & 0.2102 & 0.1998 & 0.1361 \\ 1 & 0.7999 & 0.2905 & 0.1946 & 0.1232 & 0.0253 \\ 1 & 0.7967 & 0.2323 & 0.1983 & 0.1343 & 0.0177 \\ 1 & 0.6321 & 0.2652 & 0.0987 & 0.0728 & 0.0149 \\ 1 & 0.7781 & 0.3101 & 0.1499 & 0.1223 & 0.0245 \end{pmatrix}.$$

Step 6:
Aggregate the PyF values $\mathscr{W}_{ij}$ for each alternative $\maltese^{\gamma}{}_{i}$ by the PyFSMA operator using Eq 4.4 given in Table 8.

Table 8. PyF Aggregated values $\mathscr{W}_i$.

	$\mathscr{W}_i$	value
1	$\mathscr{W}_1$	(0.7765, 0.0000)
2	$\mathscr{W}_2$	(0.7298, 0.0000)
3	$\mathscr{W}_3$	(0.7167, 0.0000)
4	$\mathscr{W}_4$	(0.5784, 0.0000)
5	$\mathscr{W}_5$	(0.6352, 0.3532)

Step 7:
Calculate the score of all PyF aggregated values $\mathscr{W}_i$.

$$\mathscr{H}(\mathscr{W}_1) = 0.7289,$$

$$\mathscr{H}(\mathscr{W}_2) = 0.7134,$$

$$\mathscr{H}(\mathscr{W}_3) = 0.6944,$$

$$\mathscr{H}(\mathscr{W}_4) = 0.5789,$$

$$\mathscr{H}(\mathscr{W}_5) = 0.6532.$$

Step 8:
Ranks by score function values.

$$\mathscr{W}_1 \succ \mathscr{W}_2 \succ \mathscr{W}_3 \succ \mathscr{W}_5 \succ \mathscr{W}_4.$$

So,

$$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{4}.$$

Table 9. Comparison of proposed operators with some existing operators.

Method	Ranking of alternatives	The optimal alternative
PyFWG (Rahman et al. [19])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{4}$	$\maltese^{\gamma}_{1}$
PyFWOG (Rahman et al. [19])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{4} \succ \maltese^{\gamma}_{3}$	$\maltese^{\gamma}_{1}$
PyFWA (Peng and Yuan [18])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{4}$	$\maltese^{\gamma}_{1}$
GPyFWA (Peng and Yuan [18])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{4} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{3}$	$\maltese^{\gamma}_{1}$
A-PyFIWA (Wang & Garg [20])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{4}$	$\maltese^{\gamma}_{1}$
A-PyFIWG (Wang & Garg [20])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{4}$	$\maltese^{\gamma}_{1}$
PyFWPG (Wei & Lu [53])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{4} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{3}$	$\maltese^{\gamma}_{1}$
PyFPWA (Wei & Lu [53])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{4}$	$\maltese^{\gamma}_{1}$
PyFHWA (Wu & Wei [54])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{4}$	$\maltese^{\gamma}_{1}$
PyFHWG (Wu & Wei [54])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{4} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{2}$	$\maltese^{\gamma}_{1}$
CPyFWA (Garg [55])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{4}$	$\maltese^{\gamma}_{1}$
CPyFWG (Garg [55])	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{4}$	$\maltese^{\gamma}_{1}$
$PyFPA_d$ (Proposed)	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{4}$	$\maltese^{\gamma}_{1}$
$PyFPG_d$ (Proposed)	$\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{4}$	$\maltese^{\gamma}_{1}$

5.1. Comparison analysis of the proposed AOs

In this section, we present a comparative review of recommended operators alongside some current AOs. The aim is to showcase the excellence of our suggested AOs, which yield the same final result as the existing AOs. By analysing the information data using both our proposed AOs and the existing ones, we compare the results and find consistent optimal decisions. This demonstrates the strength and consistency of our proposed model. To facilitate the comparison between the presented AOs and the current AOs, we have provided a comprehensive analysis in Table 9. The table offers a clear overview of the ratings assigned to the AOs, indicating their relative preferences as follows: $\maltese^{\gamma}_{1} \succ \maltese^{\gamma}_{2} \succ \maltese^{\gamma}_{3} \succ \maltese^{\gamma}_{5} \succ \maltese^{\gamma}_{4}$. These ratings are obtained using our proposed aggregation operators.

To verify the validity of our optimal choice, we have additionally applied additional established operators to the problem. This evaluation serves to reassert the efficacy and soundness of the aggregation operators we proposed, as we consistently reach the identical optimal conclusion. Through a comparative analysis of suggested operators and established AOs, the robustness and uniformity of our proposed model are underscored. The outcomes achieved by implementing our aggregation operators are consistent with those achieved by utilising other established operators. The robust validation results demonstrate that the proposed AOs are dependable and efficient in reaching the optimal decision.

6. Conclusions

By resolving the inherent complexities of this process with a novel method for selecting green suppliers, especially in the healthcare industry, based on the Pythagorean fuzzy framework, this manuscript concludes. The methodology surpasses conventional MADM challenges by incorporating factors such as attribute relationships and an uncertain environment, which are often disregarded in current approaches. The paper introduces a decision-making approach that integrates PyFN-based evaluations of decision-makers and utilises PyF information. The method thus effectively mitigates the prevalent issue of incomplete and ambiguous information that often accompanies the selection of green suppliers. In addition to enhancing the accuracy and reliability of the decision-making process, the suggested methodology cultivates an environment that is congenial and conducive to the comfort of those who make decisions. Furthermore, the utilisation of the PyFSMA and PyFSMG operators is recommended by the authors. These operators facilitate the streamlined collection and aggregation of supplier evaluation data, thus enhancing the overall effectiveness of the method. An instance of effectively choosing environmentally sustainable suppliers is provided to illustrate the feasibility and practicability of the suggested approach. The outcomes illustrate the method's efficacy in implementation and underscore its benefits in facilitating agreement among decision-makers and precisely quantifying their weights. The proposed methodology offers a more viable and effective resolution for organisations aiming to choose environmentally sustainable and responsible suppliers, thereby making a positive contribution to the field of supply chain management. By integrating the Pythagorean fuzzy framework with PyFNs, as well as the PyFSMA and PyFSMG operators, the decision-making process is enhanced and an environment conducive to comfortable and inclusive decision-making is fostered.

This approach promotes environmentally conscious and sustainable business practices by enabling organisations to make well-informed choices regarding the selection of green suppliers. The proposed operators also have some limitations, including poor performance when priority degree vectors are introduced and when the data is not Pythagorean fuzzy data. Moreover, applications can be seen as decision support software solutions [56], cold chain logistics service providers [57], logistics centre location [58], and forecasting of Alzheimer's disease [59].

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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