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Research article

Traveling wave solution of (3+1)-dimensional negative-order KdV-Calogero-Bogoyavlenskii-Schiff equation

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Abstract: We explored the (3+1)-dimensional negative-order Korteweg-de Vries-alogero-Bogoyavlenskii-Schiff (KdV-CBS) equation, which develops the classical Korteweg-de Vries (KdV) equation and extends the contents of nonlinear partial differential equations. A traveling wave transformation is employed to transform the partial differential equation into a system of ordinary differential equations linked with a cubic polynomial. Utilizing the complete discriminant system for polynomial method, the roots of the cubic polynomial were classified. Through this approach, a series of exact solutions for the KdV-CBS equation were derived, encompassing rational function solutions, Jacobi elliptic function solutions, hyperbolic function solutions, and trigonometric function solutions. These solutions not only simplified and expedited the process of solving the equation but also provide concrete and insightful expressions for phenomena such as optical solitons. Presenting these obtained solutions through 3D, 2D, and contour plots offers researchers a deeper understanding of the properties of the model and allows them to better grasp the physical characteristics associated with the studied model. This research not only provides a new perspective for the in-depth exploration of theoretical aspects but also offers valuable guidance for the practical application and advancement of related technologies.

Keywords: KdV-Calogero-Bogoyavlenskii-Schiff equation; traveling wave solution; complete discriminant system
Mathematics Subject Classification: 35C05, 35C07, 35R11

1. Introduction

Partial differential equations refer to equations that involve certain partial derivatives of an unknown function. The highest order of the partial derivatives of the unknown function appearing in the equation is referred to as the order of the equation. Second-order linear and nonlinear partial differential equations are consistently important subjects of study [1, 2]. These equations usually include elliptic, hyperbolic and parabolic, types. The calculation method for these equations is the main research problem. The developments of modern physics, mechanics, and engineering have given rise to many new nonlinear problems [3, 4], which lead to issues beyond the aforementioned equations, such as mixed-type equations, degenerate equations, and high-order partial differential equations. To date, these problems remain important research topics owing to their typically complex and challenging nature [5–8]. The discussion and resolution of partial differential equation problems often require the application of theories and methods from other branches of mathematics, such as functional analysis, algebra, topology, and differential geometry. On the other hand, the rapid development of electronic computers enables one to solve various equations numerically, which has revealed many important facts. Therefore, the research on numerical methods [9], building on the achievements already made, is expected to develop faster.

The study of solutions to the (3+1)-dimensional negative-order KdV-CBS equation offers several advantages [10, 11]. First, obtaining the exact solutions of the equation provides an in-depth understanding of its behavior and properties in terms of wave evolution, interaction, and stability. This, in turn, reveals the formation and evolution mechanisms of multi-dimensional nonlinear soliton waves [12–15]. Second, studying the solutions of the equation contributes to a deeper understanding of critical phenomena in the physical systems described by the equation, offering profound insights and guidance for practical issues and technological advancements. The use of complete discriminant system for polynomial method [16] is a robust mathematical tool capable of handling more complex equations and providing information on the existence and uniqueness of solutions. Furthermore, by comparing the obtained exact solutions with previous numerical simulation results, the accuracy of the numerical simulation can be validated, confirming the effectiveness of the numerical methods and the reliability of the equation model. Overall, this study provides a framework for exploring mathematical properties and their physical significance, offering valuable guidance for theoretical research and practical applications.

Previous research on the KdV equation and its generalized models have achieved significant results [17–23]. Researchers have explored the properties and behaviors of solutions to the KdV equation using various mathematical methods such as symmetry analysis [24–26], Bäcklund transformations [27, 28], and obtained variant solutions include singular solutions [29], interaction solutions [30], soliton solutions [31], and exact traveling wave solutions [32–36]. Additionally, some researchers have utilized methods like the singular manifold approach [37] and unified method to study the KdV equation, obtaining important numerical solutions and understanding the evolution of solutions. However, previous works on the (3+1)-dimensional negative-order KdV-CBS equation are relatively limited. Previous studies have focused on the numerical simulation and numerical solutions of the equation, lacking in-depth exploration of exact solutions for this model. Therefore, we aim to fill this research gap using the complete discriminant system for the polynomial method to obtain the exact solutions of the equation. Additionally, by comparing our obtained exact solutions with previous numerical simulation results, we aim to validate the accuracy of the solutions and investigate their properties and behavior. This will provide valuable guidance for addressing practical problems.

Furthermore, the study of soliton wave solutions extends beyond theoretical realms to play a crucial role in practical applications. In the field of optics, soliton wave solutions in optical fiber communication are particularly noteworthy. Optical fiber communication often faces challenges such as the dispersion and loss. Dispersion widens optical pulses, causing signal distortion, while the nonlinear nature of optical fibers results in compression effects, narrowing the pulses [38]. Soliton waves are precisely capable of balancing these challenges, maintaining the shape of optical pulses, and exhibiting outstanding performance. Researchers have successfully applied various mathematical methods, such as inverse scale transformation method and Darboux transformation method, to study, analyze, and solve the nonlinear Schrödinger equation (NLSE). These methods, each with its advantages and disadvantages, provide exact solutions for optical solitons in fiber communication. This research not only promotes the development of theoretical physics but also plays a crucial role in the advancement of optical communication technology. In conclusion, soliton wave solutions play a crucial role not only in theoretical research but also in practical applications, achieving significant results in addressing challenges in optical transmission.

The (3+1)-dimensional negative-order KdV-CBS equation is described as follows [39]

$$u_{xt} + u_{xxxy} + 4u_x u_{xy} + 2u_{xx}u_y + \lambda u_{xx} + \mu u_{xy} + \nu u_{xz} = 0, \tag{1.1}$$

where $u = u(t, x, y, z)$ is an unknown function, which stands for the water wave velocity on the surface of shallow water waves. λ, μ, and ν represent unspecified coefficients.

The paper is arranged as follows. The first section introduces the relevant background of the research topic. In Section 2, we conduct mathematical analysis and traveling wave transformations on this equation to make it conform to the form requirements of the complete discriminant system for the polynomial method. In Section 3, we classify all the solutions of the equation. In the following part, we draw a graph of the obtained solution. Finally, we provide a brief summary in Section 5.

2. Mathematical analysis

In this section, we first make traveling wave transformation:

$$u(t, x, y, z) = \Phi(\xi), \quad \xi = ax + by + cz - kt, \tag{2.1}$$

where a, b, c and k represent arbitrary constants.

Inserting (2.1) into (1.1), we can obtain the ordinary differential equation

$$a^2 b\Phi^{(4)} + 6ab\Phi'\Phi'' + (\lambda a + \mu b + \nu c - k)\Phi'' = 0. \tag{2.2}$$

Integrating both sides of Eq (2.2) yields

$$a^2 b\Phi''' + 3ab(\Phi')^2 + (\lambda a + \mu b + \nu c - k)\Phi' = c_1, \tag{2.3}$$

where c_1 is the integral constant.

Multiplying both sides of Eq (2.3) by Φ'' and integrating once yields

$$a^2 b(\Phi'')^2 + 2ab(\Phi')^3 + (\lambda a + \mu b + \nu c - k)(\Phi')^2 = 2c_1\Phi' + c_2, \tag{2.4}$$

where c_2 is the integral constant.

Let's assume $\Phi' = \phi$. Then, Eq (2.4) can be simplified as

$$(\phi')^2 = d_3\phi^3 + d_2\phi^2 + d_1\phi + d_0, \tag{2.5}$$

where $d_3 = -\frac{2}{a}$, $d_2 = \frac{k-\lambda a-\mu b-\nu c}{a^2 b}$, while d_1 and d_0 are arbitrary constants.

For the convenience of calculation, we make the following assumptions

$$V=(d_3)^{\frac{1}{3}}\phi,\ \xi_1=(d_3)^{\frac{1}{3}}\xi,\ b_2=d_2(d_3)^{-\frac{2}{3}},\ b_1=d_1(d_3)^{-\frac{1}{3}},\ b_0=d_0. \tag{2.6}$$

Then, Eq (2.5) can be rewritten as

$$(V'_{\xi_1})^2=V^3+b_2V^2+b_1V+b_0, \tag{2.7}$$

its integral expression can be recorded as

$$\pm(d_3)^{\frac{1}{3}}(\xi-\xi_0)=\int\frac{\mathrm{d}V}{\sqrt{V^3+b_2V^2+b_1V+b_0}}, \tag{2.8}$$

where ξ_0 is the integration constant.

3. Traveling wave solution of Eq (1.1)

In this section, we first assume that the triple-order polynomial is $f(V)=V^3+b_2V^2+b_1V+b_0$. Then its complete discrimination system is presented as

$$\Delta=-27(\frac{2b_2^3}{27}+b_0-\frac{b_1b_2}{3})^2-4(b_1-\frac{b_2^2}{3})^3,\ \ D_1=b_1-\frac{b_2^2}{3}. \tag{3.1}$$

Case 1. If $\Delta=0$ and $D_1<0$, we have $f(V)=(V-\sigma_1)^2(V-\sigma_2)$, where $\sigma_1\neq\sigma_2$. When $V>\sigma_2$, Eq (2.8) can be rewritten as

$$\pm(\xi_1-\xi_0)=\int\frac{dV}{(V-\sigma_1)\sqrt{V-\sigma_2}}=\begin{cases}\frac{1}{\sqrt{\sigma_1-\sigma_2}}\ln|\frac{\sqrt{V-\sigma_2}-\sqrt{\sigma_1-\sigma_2}}{\sqrt{V-\sigma_2}+\sqrt{\sigma_1+\sigma_2}}|, & \sigma_1>\sigma_2,\\ \frac{2}{\sqrt{\sigma_2-\sigma_1}}\arctan\sqrt{\frac{V-\sigma_2}{\sigma_2-\sigma_1}}, & \sigma_1<\sigma_2.\end{cases} \tag{3.2}$$

Next, we calculate Eq (3.2). Then, the solution of Eq (1.1) is

$$\begin{aligned}\phi_1(t,x,y,z)&=(-\frac{2}{a})^{-\frac{1}{3}}\{(\sigma_1-\sigma_2)\tanh^2[\frac{\sqrt{\sigma_1-\sigma_2}}{2}(-\frac{2}{a})^{\frac{1}{3}}(ax+by+cz-kt-\xi_0)]+\sigma_2\},\ \ \sigma_1>\sigma_2,\\ \phi_2(t,x,y,z)&=(-\frac{2}{a})^{-\frac{1}{3}}\{(\sigma_1-\sigma_2)\coth^2[\frac{\sqrt{\sigma_1-\sigma_2}}{2}(-\frac{2}{a})^{\frac{1}{3}}(ax+by+cz-kt-\xi_0)]+\sigma_2\},\ \ \sigma_1>\sigma_2,\\ \phi_3(t,x,y,z)&=(-\frac{2}{a})^{-\frac{1}{3}}\{(-\sigma_1+\sigma_2)\tan^2[\frac{\sqrt{-\sigma_1+\sigma_2}}{2}(-\frac{2}{a})^{\frac{1}{3}}(ax+by+cz-kt-\xi_0)]+\sigma_2\},\ \ \sigma_1<\sigma_2.\end{aligned} \tag{3.3}$$

Case 2. If $\Delta=0$ and $D_1=0$, we obtain $f(V)=(V-\sigma)^3$.
By substituting $f(V)=(V-\sigma)^3$ into Eq (2.8), the solutions of Eq (1.1) is obtained.

$$\phi_4(t,x,y,z)=(-\frac{a}{2})[4(ax+by+cz-kt-\xi_0)^{-2}-\frac{k-\lambda a-\mu b-\nu c}{3a^2b}]. \tag{3.4}$$

Case 3. If $\Delta>0$ and $D_1<0$, then we have $f(V)=(V-\sigma_1)(V-\sigma_2)(V-\sigma_3)$. It is assumed that $\sigma_1<\sigma_2<\sigma_3$.

If $\sigma_1 < V < \sigma_3$, we make the transformation $V = \sigma_1 + (\sigma_2 - \sigma_1)\sin^2\vartheta$. Through Eq (2.8), we can get

$$\begin{aligned}\pm(\xi_1 - \xi_0) &= \int \frac{dV}{\sqrt{f(V)}} = \int \frac{2(\sigma_2 - \sigma_1)\sin\vartheta\cos\vartheta d\vartheta}{\sqrt{\sigma_3 - \sigma_1}(\sigma_2 - \sigma_1)\sin\vartheta\cos\vartheta\sqrt{1 - \chi^2\sin^2\vartheta}} \\ &= \frac{2}{\sqrt{\sigma_3 - \sigma_1}} \int \frac{d\vartheta}{\sqrt{1 - \chi^2\sin^2\vartheta}},\end{aligned} \tag{3.5}$$

where $\chi^2 = \frac{\sigma_2 - \sigma_1}{\sigma_3 - \sigma_1}$.

From Eq (2.5), the solution of Eq (2.1) is obtained as

$$V(\xi_1) = \sigma_1 + (\sigma_2 - \sigma_1)\mathbf{sn}^2(\frac{\sqrt{\sigma_3 - \sigma_1}}{2}(\xi_1 - \xi_0), \chi). \tag{3.6}$$

The solutions of Eq (1.1) is obtained as

$$\phi_5(t, x, y, z) = (-\frac{2}{a})^{-\frac{1}{3}}[\sigma_1 + (\sigma_2 - \sigma_1)\mathbf{sn}^2(\frac{\sqrt{\sigma_3 - \sigma_1}}{2}(-\frac{2}{a})^{\frac{1}{3}}(ax + by + cz - kt - \xi_0), \chi)]. \tag{3.7}$$

If $V > \sigma_3$, consider the transformation $V = \frac{-\sigma_2\sin^2\vartheta + \sigma_3}{\cos^2\vartheta}$. The solutions of Eq (1.1) is

$$\phi_6(t, x, y, z) = (-\frac{2}{a})^{-\frac{1}{3}}[\frac{\sigma_3 - \sigma_2\mathbf{sn}^2(\frac{\sqrt{\sigma_3 - \sigma_1}}{2}(-\frac{2}{a})^{\frac{1}{3}}(ax + by + cz - kt - \xi_0), \rho)}{\mathbf{cn}^2(\frac{\sqrt{\sigma_3 - \sigma_1}}{2}(-\frac{2}{a})^{\frac{1}{3}}(ax + by + cz - kt - \xi_0), \rho)}], \tag{3.8}$$

where $\rho^2 = \frac{\sigma_2 - \sigma_1}{\sigma_3 - \sigma_1}$.

Case 4. If $\Delta < 0$, we have $f(V) = (V - \sigma)(V^2 + pV + q)$, where $p^2 - 4q < 0$, σ is the only real root of $f(V) = 0$.

If $V > \sigma$, we consider the transformation $V = \sigma + \sqrt{\sigma^2 + p\sigma + q}\tan^2\frac{\vartheta}{2}$. From Eq (2.8), we can obtain

$$\begin{aligned}\xi_1 - \xi_0 &= \int \frac{dV}{\sqrt{(V - \vartheta)(V^2 + p\Phi + q)}} = \int \frac{\sqrt{\sigma^2 + p\sigma + q}\frac{\tan\frac{\vartheta}{2}}{\cos^2\frac{\vartheta}{2}}d\vartheta}{(\sigma^2 + p\sigma + q)^{\frac{3}{4}}\frac{\tan\frac{\vartheta}{2}}{\cos^2\frac{\vartheta}{2}}\sqrt{1 - \varrho^2\sin^2\vartheta}} \\ &= \frac{1}{(\sigma^2 + p\sigma + q)^{\frac{1}{4}}} \int \frac{d\vartheta}{\sqrt{1 - \varrho^2\sin^2\vartheta}},\end{aligned} \tag{3.9}$$

where $\varrho^2 = \frac{1}{2}(1 - \frac{\sigma + \frac{p}{2}}{\sqrt{\sigma^2 + p\sigma + q}})$.

Let $\mathbf{cn}((\sigma^2 + p\sigma + q)^{\frac{1}{4}}(\xi_1 - \xi_0), \varrho) = \cos\vartheta$,

$$\cos\vartheta = \frac{2\sqrt{\sigma^2 + p\sigma + q}}{V - \sigma + \sqrt{\sigma^2 + p\sigma + q}} - 1. \tag{3.10}$$

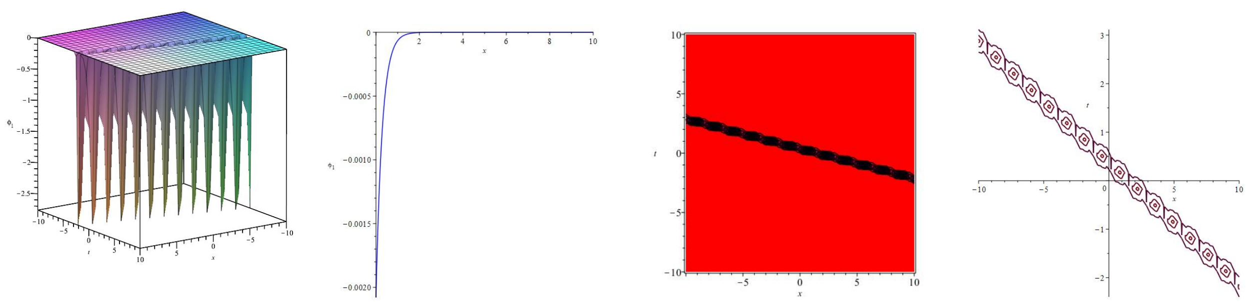
The solutions of Eq (1.1) is obtained as

$$\phi_7(t, x, y, z) = \Big(-\frac{2}{a}\Big)^{-\frac{1}{3}}\Big[\sigma + \frac{2\sqrt{\sigma^2 + p\sigma + q}}{1 + \mathbf{cn}((\sigma^2 + p\sigma + q)^{\frac{1}{4}}\big(-\frac{2}{a}\big)^{\frac{1}{3}}(ax + by + cz - kt - \xi_0), \varrho)} - \sqrt{\sigma^2 + p\sigma + q}\Big]. \tag{3.11}$$

Remark 3.1. In this article, we have constructed the solution to Eq (2.5). Next, we can obtain all solutions to Eq (1.1) using the following relationship. $u_i(t,x,y,z) = \Phi_i(\xi) = \int_{c_0}^{\xi} \phi_i(\xi)\mathrm{d}\xi$, for $i = 1,\ldots,7$.

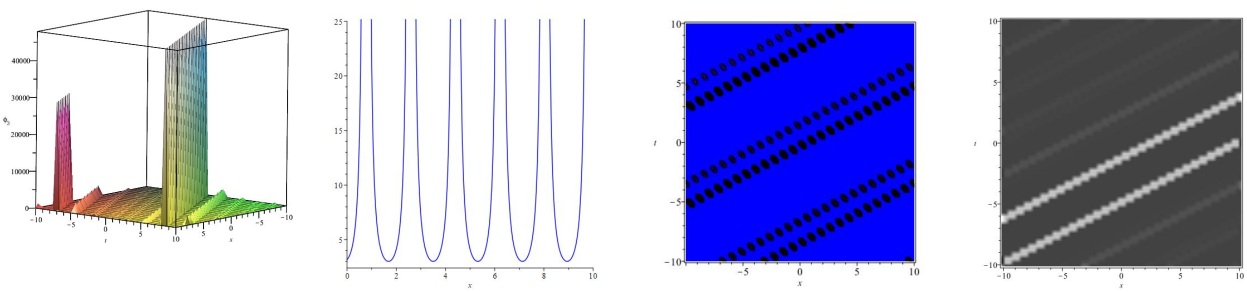
4. Graphical illustrations

In this section, the hyperbolic function solution is shown in Figure 1 by setting these parameter values. Moreover, the trigonometric function solution of Eq (1.1) is shown in Figure 2.



(a) 3D surface (b) 2D surface (c) Contour plot (d) Density plot

Figure 1. The graphics of $\phi_1(t,x,y,z)$ at $a = -2, b = 1, \mu = 2, v = 1, c = 2, \lambda = 4, k = 8, \xi_0 = 0, y = 1, z = 1, d_1 = 0, d_0 = 0$.



(a) 3D surface (b) 2D surface (c) Contour plot (d) Density plot

Figure 2. The graphics of $\phi_3(t,x,y,z)$ at $a = -2, b = 1, \mu = 2, v = 1, c = 2, \lambda = -4, k = -4, \xi_0 = 0, y = 1, z = 1, d_1 = 0, d_0 = 0$.

5. Conclusions

In general, we successfully solve Eq (1.1) using the complete discriminant system for the polynomial method. By classifying the roots of third-order polynomials, we can effectively classify the traveling wave solutions of Eq (1.1). We provide a series of corresponding solution expressions. By choosing the parameters properly, we clearly show the availability of these solutions. Compared with previous studies, our work is richer in (3+1)-dimensional traveling wave solutions, and also involves spatiotemporal fractional derivatives in KdV-CBS equations. In the future, we plan to explore higher-order and more complex fractional partial differential equations for wider application to a variety of complex engineering and physics problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest.

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