Novel categories of spaces in the frame of fuzzy soft topologies

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Abstract: In the present paper, we introduce and discuss a new set of separation properties in fuzzy soft topological spaces called $FS\delta$-separation and $FS\delta$-regularity axioms by using fuzzy soft $\delta$-open sets and the quasi-coincident relation. We provide a comprehensive study of their properties with some supporting examples. Our analysis includes more characterizations, results, and theorems related to these notions, which contributes to a deeper understanding of fuzzy soft separability properties. We show that the $FS\delta$-separation and $FS\delta$-regularity axioms are harmonic and heredity property. Additionally, we examine the connections between $FS\delta^*$-compactness and $FS\delta$-separation axioms and explore the relationships between them. Overall, this work offers a new perspective on the theory of separation properties in fuzzy soft topological spaces, as well as provides a robust foundation for further research in the transmission of properties from fuzzy soft topologies to fuzzy and soft topologies and vice-versa by swapping between the membership function and characteristic function in the case of fuzzy topology and the set of parameters and a singleton set in the case of soft topology.

Keywords: fuzzy soft set; fuzzy soft $\delta$-open set; fuzzy soft topology; $FS\delta$-separation axioms; $FS\delta$-regularity axioms

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1. Introduction

Zadeh [50] was the first to come up with the unprecedented theory of fuzzy set ($F$-set) for dealing with some types of uncertainties where conventional tools fail. This theory brought a grand
paradigmatic change in mathematics and offered a convenient framework to model a huge number of empirical problems. On the other hand, this theory has its inherent difficulties, which are possibly attributed to the inadequacy of the parameterization tool and pre-requisite of membership function, as pointed out by Molodtsov in his pioneering work [41]. He introduced the concept of soft sets ($S$-sets) as a remarkable mathematical tool for coping with vagueness that is free from the aforementioned difficulties. Then, the $S$-set theory has been applied in many fields by many authors [16, 35, 36]. One of these fields that attracted a lot of attention is the abstract topological structures that were displayed by Shabir-Naz [48] and Çağman et al. [20]. Some divergences between classical and soft topologies were illuminated in [7].

Over time, complicated issues have appeared that need combining parameterization of $S$-sets with the membership degree of $F$-sets. To tackle such dilemmas, Maji et al. [37] put forward a new paradigm known as a fuzzy soft set ($FS$-set) and demonstrated how this paradigm is applied [38]. Since then, the $FS$-set theory and its applications have been studied by several intellectuals [5, 19, 24, 25]. To cover more situations and expand the range of applications, the concept of $FS$-set was generalized to $(a, b)$-Fuzzy soft sets by [11]. Kharal and Ahmad [34] defined the concept of mappings of $FS$-classes. Subsequently, the study of topological structure over the family of $FS$-sets was started by Tanay-Kandemir [49]. Mukherjee et al. [42] introduced the notions of $FS\delta$-open and $FS\delta$-closed sets, $FS\delta$-closure and $FS\delta$-interior operators, and $FS\delta$-continuity. Kandil et al. provided the concepts of fuzzy soft connected and fuzzy soft hyperconnected spaces in [31,32], respectively. Various concepts in fuzzy soft settings have been considered, such as disjoint union of fuzzy soft topological structures [6] and filters [26].

Since the importance of separation axioms in topological spaces, it was investigated topologies over the different types of uncertainty spaces. Kandil and El-Etiriby [29] structured separation axioms in the spaces of fuzzy topologies, then Kandil and El-Shafei [30] familiarized the axioms of regularity in fuzzy topologies and $FRr$-proximities. Saleh et al. [45] displayed stronger types of separation and regularity axioms in the spaces of fuzzy topologies using fuzzy pre-open sets. In fuzzy soft topological spaces, separation axioms have been presented and discussed by many authors; see, for example, [1, 2, 39, 40]. Kandil et al. [33] scrutinized the characterizations of separation axioms and regularity inspired by quasi-coincident and neighborhood systems. Recently, Saleh et al. [46, 47] have described another sorts of $FS$-separation axioms and regularity axioms. In soft setting, a wide class of separation axioms have been offered by Al-shami and his coauthors [12, 13, 17, 21–23]. They successfully exploited these axioms to address some real-life situations as given in [8, 9]. Alcantud [3] conducted an interesting work to describe the relationships between topological structures in soft and fuzzy settings.

To go along this line of research, we are writing this paper, which contributes to the understanding of fuzzy soft separability properties and produces some categories of fuzzy soft topological spaces. It is well known that the environment of the current work widens other known generalizations such as fuzzy topology and soft topology; this means the results and relationships obtained in these frames are special cases of their counterparts investigated herein. This is attributed to that the frameworks of soft and fuzzy topologies are produced by “fuzzy soft topology” by replacing the membership function with the characteristic function in the case of fuzzy topology and restricting the set of parameters by a singleton set in the case of soft topology. Hence, the paper enhances the body of knowledge and provides a comprehensive insight to study the properties and characteristics of topological structures.

After this introduction, the reader may pursue the content of this research as follows. In Section 2,
we requisition the definitions and findings that are needful to go along with the results obtained herein. In the next sections, Sections 3 and 4, we delve into the topic of separation properties in fuzzy soft topological spaces and propose a new set of axioms called \(FS\delta\)-separation (\(FS\delta T_i\), where \(i = 0, 1, 2, 3, 4\) and \(FS\delta\)-regularity (\(FS\delta R_i\), where \(i = 0, 1, 2, 3\)). These separations are structured by utilizing the ideas of fuzzy soft \(\delta\)-open sets and the quasi-coincident relation. We provide various characterizations of these properties and present a range of results, theorems, and relationships related to these notions. In Section 5, we look at the interplay between \(FS\delta\)-compactness and \(FS\delta\)-separation axioms and analyze the relationships between them. In the end, we outline the master contributions of this manuscript and suggest a road map for future direction in Section 6.

2. Some basic definitions

Here, we recall the basic definitions that will be needed in this sequel. In the present work, \(U\) refers to the universe set, \(E\) is the set of all parameters for \(U, I = [0, 1]\), and \(FS\) - refers to fuzzy soft.

Definition 2.1. [50] An \(F\)-set \(A\) of \(U\) is a mapping \(A : U \rightarrow I\). \(I^U\) refers to the set of all \(F\)-sets on \(U\). An \(F\)-point \(x_\lambda, \lambda \in (0, 1]\) is an \(F\)-set in \(U\) given by \(x_\lambda(y) = \lambda\) at \(x = y\) and \(x_\lambda(y) = 0\) for all \(y \in U\). For \(\alpha \in I, \alpha \in I^U\) refers to the \(F\)-constant function where \(\alpha(x) = \alpha, \forall x \in U\).

Definition 2.2. [37] An \(FS\) -set \(h_E = (f, E)\) on \(U\) is the set of ordered pairs \(h_E = \{(e, h(e)) : e \in E, h(e) \in I^U\}\).

In this content, \(FSS(U_E)\) refers to the set of all \(FS\)-sets on \(U\). \(\tilde{\alpha}_E \in FSS(U_E)\) defined by \(\tilde{\alpha}_E = \{(e, \alpha) : e \in E, \alpha \in I^U\}\) is called an \(FS\)-constant set.

Definition 2.3. [18, 43] An \(FS\)-point \(x^e_\alpha\) on \(U_E\) is an \(FS\)-set on \(U\) defined by \(x^e_\alpha(e') = x_\alpha\) if \(e' = e\) and \(x^e_\alpha(e') = 0\) if \(e' \in E - \{e\}\), where \(x_\alpha\) is the \(F\)-point in \(U\). \(FPS(U_E)\) refers to the set of all \(FS\)-points in \(U\). An \(FS\)-point \(x^e_\alpha \in f_E\) if \(\alpha \leq f(e)(x)\).

Definition 2.4. [43, 49] The triplet \((U, \tau, E)\) is called a fuzzy soft topological space (briefly, an \(FSTS\)), where \(U\) is an initial universal set, \(E\) is a fixed set of parameters, and \(\tau\) is a family of \(FS\)-sets on \(U\) such that \(\tau\) is closed under arbitrary union and finite intersection and \(0_E, 1_E\) belong to \(\tau\). The elements in \(\tau\) are called fuzzy soft open sets (briefly, \(FSO\)-sets) and the complements of them are called fuzzy soft closed sets (briefly, \(FSC\)-sets).

Definition 2.5. [18] The \(FS\)-sets \(h_E\) and \(g_E\) are called quasi-coincident, denoted by \(f_Eqg_E\) if there are \(e \in E, u \in U\) such that \(h(e)(u) + g(e)(u) > 1\). If \(h_E\) is not quasi-coincident with \(g_E\), then we write \(h_E\tilde{q}g_E\).

Proposition 2.1. [18, 46] Let \(x^e_\alpha, y^e_\beta \in FPS(U_E), f_E, g_E, h_E \in FSS(U_E),\) and \(\{f_E : i \in J\} \subseteq FSS(U_E),\) then

(i) \(f_E\tilde{q}g_E \Leftrightarrow f_E \subseteq g_E^c\) and \(f_E\tilde{q}f_E^c\).

(ii) \(f_E \cap g_E = 0_E \Rightarrow f_E\tilde{q}g_E\).

(iii) \(f_E\tilde{q}g_E, h_E \subseteq g_E \Rightarrow f_E\tilde{q}h_E\).

(iv) \(f_E\tilde{q}g_E \Rightarrow x^e_\alpha q_g,\) for some \(x^e_\alpha \in f_E\).
respectively. Let $u$

Definition 2.7. [34] Let $S$ be an $S$-set $g$

(ii) $x \neq y \Rightarrow x_qy_l, \forall r, t \in I$.

(viii) $x_qy_l \Leftrightarrow x \neq y \text{ or } (x = y \text{ and } r + t \leq 1)$.

Definition 2.6. [42] An FS-set $h_E$ in $(U, \tau, E)$ is called $q$-neighborhood (briefly, $q$-nbd) of $x^*_o$ if there is an FSO-set $g_E$ such that $x^*_o \subseteq g_E$.

Definition 2.7. [34] Let $S(U_E)$ and $S(V_K)$ be two classes of all FS-sets over $U$ and $V$, respectively. Let $u : U \rightarrow V$ and $p : E \rightarrow K$ be two maps, then $f_{ap} : S(U_E) \rightarrow S(V_K)$ is called a fuzzy soft map (or an FS-map) for which:

(i) If $f_E \in S(U_E)$, then the image of $f_E$ denoted by $f_{ap}(f_E)$ is the FS-set on $V$ given by $f_{ap}(f_E)(k) = \sup(u(f(e)) : e \in p^{-1}(k))$ if $p^{-1}(k) \neq \emptyset$, and $f_{ap}(f_E)(k) = \emptyset$ otherwise, for all $k \in K$.

(ii) If $g_K \in S(V_K)$, then the preimage of $g_K$ denoted by $f_{ap}^{-1}(g_K)$ is the FS-set on $U$ defined by, $f_{ap}^{-1}(g_K)(e) = u^{-1}(g(p(e)))$ for all $e \in E$.

The FS-map $f_{ap}$ is called one-to-one (onto), if $u$ and $p$ are one-to-one (onto). For more details about the properties of FS-maps; see, [34].

Definition 2.8. [46] Let $(U, \tau, E)$ be an FSTS and $Y \subseteq U$. Let $h^\tau_E$ be an FS-set on $Y_E$ such that $h^\tau_E : E \rightarrow I^\tau$, $h^\tau_E(e) \in I^\tau$ and $h^\tau_E(e)(x) = 1$ if $x \in Y$, $h^\tau_E(e)(x) = 0$ if $x \in Y$. Let $\tau_Y = \{h^\tau_E \cap g_E : g_E \in \tau\}$, then $\tau_Y$ is a fuzzy soft topology (in short, FST) on $Y$ and $(Y, \tau_Y, E)$ is called an FS-subspace of $(U, \tau, E)$. If $h^\tau_E \in \tau$ (resp., $h^\tau_E \in \tau^c$), then $(Y, \tau_Y, E)$ is called an FS-open (resp., closed) subspace of $(U, \tau, E)$.

Definition 2.9. [18,42] For an FS-set $h_E$ in $(U, \tau, E)$, we have:

(i) The FS-closure $cl(h_E)$ of $h_E$ is the intersection of all FSC-sets containing $h_E$, and the FS-interior $int(h_E)$ of $h_E$ is the union of all FSO-sets contained in $h_E$.

(ii) $h_E$ is said to be a fuzzy soft regular open set (FSRO-set) if $h_E = int(cl(h_E))$. The complement of an FSRO-set is called a fuzzy closed regular set (FSRC-set). FSRO($U_E$) refers to the set of all FSRO-sets and FSRC($U_E$) refers to the set of all FSRC-sets.

(iii) $h_E$ is said to be a fuzzy soft $\delta$-neighborhood (briefly, FS$\delta$-nbd) of $x^*_o$ if and only if there is FSRO $q$-nbd $g_E$ of $x^*_o$ such that $g_E \subseteq f_E$.

Definition 2.10. [42] Let $h_E$ be an FS-set in $(U, \tau, E)$, then:

(i) An $FS$-point $x^*_o$ is called an $FS$-$\delta$-cluster point of $h_E$ if and only if every FSRO $q$-nbd $f_E$ of $x^*_o$, $f_E \subseteq g_E$. The set of all FS$\delta$-cluster points of $h_E$ is called the FS$\delta$-closure of $h_E$, denoted by $cl_\delta(h_E)$; that is, $cl_\delta(h_E) = \bigcap \{g_E \in FSRC(U_E) : h_E \subseteq g_E\}$.

(ii) An FS-set $h_E$ is called a fuzzy soft $\delta$-closed set (FS$\delta$C-set) if and only if $h_E = cl_\delta(h_E)$. The complement of an FS$\delta$C-set is called a fuzzy soft $\delta$-open set (FS$\delta$O-set). FS$\delta$C($U_E$) refers to the set of all FS$\delta$C-sets and FS$\delta$O($U_E$) refers to the set of all FS$\delta$O-sets.
(iii) The $FS\delta$-interior $\text{int}_\delta(h_E)$ of $h_E$ is defined by $\text{int}_\delta(h_E) = \bar{1}_E - \text{cl}_\delta(h_E^c)$; that is, $\text{int}_\delta(h_E) = \bigcup \{g_E \in FSRO(U_E) : g_E \subseteq h_E\}$. Consequently, $h_E$ is $FS\delta$-open if and only if $h_E = \text{int}_\delta(h_E)$.

Notation. For $x^c_\epsilon$ in $FSP(U_E)$, $O_{x^c_\epsilon}$ refers to an $FS\delta O$-set containing $x^c_\epsilon$. In general, $O_{h_E}$ refers to an $FS\delta O$-set containing an $FS$-set $h_E$.

Result 1. [42] Every $FSRO$-set is an $FS\delta O$-set and every $FS\delta O$-set is an $FS O$-set. Moreover, if $h_E$ is an $FS$-semi open set in $(U, \tau, E)$, then $cl(h_E) = cl_\delta(h_E)$.

Result 2. [42] If $h_E$ is an $FS O$-set in $(U, \tau, E)$, then $cl(h_E)$ is an $FSRC$-set; that is, $\{cl(g_E) : g_E \in \tau\} = \{h_E : h_E \in FSRC(U_E)\}$, and for any $FS$-set $h_E$ in $(U, \tau, E)$, $cl_\delta(h_E) = \cap \{cl(g_E) : h_E \subseteq g_E, g_E \in \tau\}$.

Theorem 2.1. [42] For any $FS$-sets $f_E$ and $g_E$ in $(U, \tau, E)$, we have:

(i) $cl_\delta(0_E) = 0_E$ and $cl_\delta(1_E) = 1_E$.

(ii) $cl_\delta(f_E)$ is an $FS\delta C$-set, that is, $cl_\delta(cl_\delta(f_E)) = cl_\delta(f_E)$.

(iii) $cl(f_E) \subseteq cl_\delta(f_E)$ and if $f_E \in \tau$, then $cl(f_E) = cl_\delta(f_E)$.

Result 3. [42] The $FS\delta$-closure operator on $(U, \tau, E)$ satisfies the Kuratowski closure axioms so that there is one topology on $U$. This topology is defined as follows:

The set of all $FS\delta O$-sets of $(U, \tau, E)$ forms an $FS$-topology, denoted by $\tau_\delta$. It is called an $FS\delta$-topology on $U$, and the triplet $(U, \tau_\delta, E)$ is called an $FS\delta$-topological space. Moreover, $\tau_\delta \subseteq \tau$.

Definition 2.11. [42] An $FS$-map $f_{up} : (U, \tau, E) \longrightarrow (V, \delta, K)$ is called:

(i) $FS\delta$-open if $f_{up}(h_E)$ is an $FS\delta O$-set in $V$ for all $FS\delta O$-sets $h_E$ in $U$.

(ii) $FS\delta$-closed if $f_{up}(g_E)$ is an $FS\delta C$-set in $V$ for all $FS\delta C$-sets $g_E$ in $U$.

Theorem 2.2. [42] Let $f_{up} : (U, \tau, E) \longrightarrow (V, \delta, K)$ be an $FS$-map, then the next items are equivalent:

(i) $f_{up}$ is $FS\delta$-continuous.

(ii) $f_{up}^{-1}(g_K)$ is an $FS\delta O$-set in $(U, \tau, E)$ for all $FS\delta O$-sets $g_K$ in $(V, \delta, K)$.

(iii) $f_{up}^{-1}(g_K)$ is an $FS\delta C$-set in $(U, \tau, E)$ for all $FS\delta C$-sets $g_K$ in $(V, \delta, K)$.

Definition 2.12. [46] An $FSTS$ $(U, \tau, E)$ is said to be:

(i) $FST_0$ if for any $x^e_\epsilon, y^e_\epsilon \in FSP(U_E)$ with $x^e_\epsilon \bar{\in} y^e_\epsilon\delta$, then $x^e_\epsilon \bar{\in} \text{cl}(y^e_\epsilon)$ or $\text{cl}(x^e_\epsilon) \bar{\in} y^e_\epsilon\delta$.

(ii) $FST_1$ if for any $x^e_\epsilon, y^e_\epsilon \in FSP(U_E)$ with $x^e_\epsilon \bar{\in} y^e_\epsilon\delta$, then $x^e_\epsilon \bar{\in} \text{cl}(y^e_\epsilon)$ and $\text{cl}(x^e_\epsilon) \bar{\in} y^e_\epsilon\delta$.

(iii) $FST_2$ if for any $x^e_\epsilon, y^e_\epsilon \in FSP(U_E)$ with $x^e_\epsilon \bar{\in} y^e_\epsilon\delta$, there are $FSO$-sets $O^e_\delta, O^e_\delta \subseteq \tau$ such that $O^e_\delta \bar{\in} \text{cl}(O^e_\delta\delta)$.

Definition 2.13. [44] For $FSTS(U, \tau, E)$ and $h_E \in FSS(U_E)$, then:

(i) A family $\mathcal{A} = \{l_E : i \in J\}$ of $FS\delta$-sets is called an $FS\delta^*$-open cover of $h_E$ if for all $x^c_\epsilon \bar{\in} h_E$ there is $l_0 \in J$ such that $x^c_\epsilon \bar{\in} f_{l_0,E}$.

(ii) $h_E$ is called an $FS\delta^*$-compact set if every $FS\delta^*$-open cover of $h_E$ has a finite $FS\delta^*$-open subcover.

In general, $(U, \tau, E)$ is $FS\delta^*$-compact if $1_E$ itself is $FS\delta^*$-compact.
3. Fuzzy soft $\delta$-separation axioms

Here, we are going to give the definitions of a new class of separation axioms called $FS\delta$-separation axioms (or $FS-\delta T_i$, $i = 0, 1, 2$) and study some their properties.

**Definition 3.1.** An $FSTS$ $(U, \tau, E)$ is said to be:

(i) $FS-\delta T_0$ if for any $x^\epsilon, y^\epsilon \in FS P(U_E)$ with $x^\epsilon \tilde{q} y^\epsilon$, there is an $FS\delta O$-set $O_{x^\epsilon}$ such that $O_{x^\epsilon} \tilde{q} y^\epsilon$, or there is an $FS\delta O$-set $O_{y^\epsilon}$ such that $O_{y^\epsilon} \tilde{q} x^\epsilon$.

(ii) $FS-\delta T_1$ if for any $x^\epsilon, y^\epsilon \in FS P(U_E)$ with $x^\epsilon \tilde{q} y^\epsilon$, there are $FS\delta O$-sets $O_{x^\epsilon}, O_{y^\epsilon}$ such that $y^\epsilon \tilde{q} O_{x^\epsilon}$ and $x^\epsilon \tilde{q} O_{y^\epsilon}$.

(iii) $FS-\delta T_2$ if for any $x^\epsilon, y^\epsilon \in FS P(U_E)$ with $x^\epsilon \tilde{q} y^\epsilon$, there are $FS\delta O$-sets $O_{x^\epsilon}, O_{y^\epsilon}$ such that $O_{x^\epsilon} \tilde{q} O_{y^\epsilon}$.

**Lemma 3.1.** For $FSTS$ $(U, \tau, E)$, $x^\epsilon \in FS P(U_E)$, and $h_E \in FSS(U_E)$, then:

(i) $x^\epsilon \in int(h_E) \Leftrightarrow$ there is an $FS\delta O$-set $O_{x^\epsilon}$ such that $O_{x^\epsilon} \subseteq h_E$.

(ii) $x^\epsilon \tilde{q} cl(h_E) \Leftrightarrow O_{x^\epsilon} \tilde{q} h_E$ for any $FS\delta O$-set $O_{x^\epsilon}$ in $(U, \tau, E)$.

(iii) $g_E \tilde{q} h_E \Rightarrow g_E \tilde{q} cl(h_E)$ for any $FS\delta O$-set $g_E$ in $(U, \tau, E)$.

In the next results, we give some characterizations of $FS-\delta T_i$ space, $i = 0, 1, 2$.

**Theorem 3.1.** An $FSTS$ $(U, \tau, E)$ is $FS-\delta T_0$ if and only if for any $x^\epsilon, y^\epsilon \in FS P(U_E)$ with $x^\epsilon \tilde{q} y^\epsilon$ implies $x^\epsilon \tilde{q} cl(h(y^\epsilon))$ or $cl(x^\epsilon) \tilde{q} y^\epsilon$.

**Proof.** Let $(U, \tau, E)$ be $FS-\delta T_0$ and $x^\epsilon \tilde{q} y^\epsilon$ for any $x^\epsilon, y^\epsilon \in FS P(U_E)$. Then there is an $FS\delta O$-set $O_{x^\epsilon}$ such that $y^\epsilon \tilde{q} O_{x^\epsilon}$ or there is an $FS\delta O$-set $O_{y^\epsilon}$ such that $x^\epsilon \tilde{q} O_{y^\epsilon}$. From (ii) of the above lemma, we get $x^\epsilon \tilde{q} cl(h(y^\epsilon))$ or $cl(x^\epsilon) \tilde{q} y^\epsilon$.

Conversely, let $x^\epsilon \tilde{q} y^\epsilon$. By given $x^\epsilon \tilde{q} cl(h(y^\epsilon))$ or $cl(x^\epsilon) \tilde{q} y^\epsilon$, and again from (ii) of the above lemma, there is an $FS\delta O$-set $O_{x^\epsilon}$ such that $O_{x^\epsilon} \tilde{q} y^\epsilon$ or there is $O_{y^\epsilon}$ with $O_{y^\epsilon} \tilde{q} x^\epsilon$. Hence, $(U, \tau, E)$ is $FS-\delta T_0$.

**Remark 3.1.** Clearly, every $FS-\delta T_0$ is $FST_0$. The converse is not necessarily true.

**Example 3.1.** Let $U = [0, 1]$ and $E = \{e\}$, then the class $\tau = \{0_E, 1_E\} \cup \{f_i_E : i \in N\}$ is an $FST$ on $U$, where

$$f_i_E(e)(u) = \begin{cases} 1, & u = 0, \\ 1 - \frac{1}{i}, & 0 < u \leq \frac{1}{i}, \\ \frac{1}{i}, & \frac{1}{i} < u \leq 1. \end{cases}$$

One can check that $(U, \tau, E)$ is $FST_0$. On other hand, clearly $\{cl(h_E) : h_E \in \tau\} = \{g_E : g_E \in FS RC(U_E)\}$, and for any $l_E$ in $(U, \tau, E)$, we have $cl(l_E) = \bigcap \{g_E \in FS RC(U_E) : l_E \subseteq g_E\}$. Since $cl(f_i_E) = 1_E$ for all $i \in N$, $cl(1_E) = 1_E$, and $cl(0_E) = 0_E$, then $FS RC(U_E) = \{0_E, 1_E\}$. Therefore, $FS\delta C(U_E) = \{0_E, 1_E\} = FS\delta O(U_E)$. Thus, $(U, \tau, E)$ is not $FS-\delta T_0$.

**Theorem 3.2.** Let $f_{up} : (U, \tau, E) \longrightarrow (V, \delta, K)$ be one-to-one and $FS\delta$-continuous. If $(V, \delta, K)$ is $FS-\delta T_0$, then $(U, \tau, E)$ also is $FS-\delta T_0$. 

Proof. Let \( x^*_t \bar{y}^*_t \) for any \( x^*_t, y^*_t \in \text{FS}(P(U_E)). \) Since \( f_{up} \) is one-to-one, then \( f_{up}(x^*_t) \bar{q}f_{up}(y^*_t). \) Since \((V, \delta, K)\) is FS-\( \delta T_0\), there is an \( \text{FS} \, \delta \text{-O-set} \, O_{f_{up}(x^*_t)} \) such that \( f_{up}(y^*_t) \bar{q}O_{f_{up}(x^*_t)} \), or there is an \( \text{FS} \, \delta \text{-O-set} \, O_{f_{up}(y^*_t)} \) such that \( f_{up}(x^*_t) \bar{q}O_{f_{up}(y^*_t)} \). Since \( f_{up} \) is FS-\( \delta \)-continuous, we have \( f_{up}^{-1}(O_{f_{up}(x^*_t)}) \) as an \( \text{FS} \, \delta \text{-O-set} \) in \((U, \tau, E)\) with \( y^*_t \bar{q}f_{up}^{-1}(O_{f_{up}(x^*_t)}) \), or there is an \( \text{FS} \, \delta \text{-O-set} \, f_{up}^{-1}(O_{f_{up}(y^*_t)}) \) in \((U, \tau, E)\) with \( x^*_t \bar{q}f_{up}^{-1}(O_{f_{up}(y^*_t)}) \). Hence, \((U, \tau, E)\) is FS-\( \delta T_0\).

**Theorem 3.3.** An \( \text{FSTS} \) \((U, \tau, E)\) is FS-\( \delta T_1\) if and only if for any \( x^*_t, y^*_t \in \text{FS}(P(U_E)) \) with \( x^*_t \bar{y}^*_t \) implies \( x^*_t \bar{q}c_{\delta}(y^*_t) \) and \( c_{\delta}(x^*_t) \bar{q}y^*_t \).

**Proof.** By a similar way to that in Theorem 3.3.

**Theorem 3.4.** For an \( \text{FSTS} \) \((U, \tau, E)\), the next items are equivalent:

(i) \((U, \tau, E)\) is FS-\( \delta T_1\).

(ii) \( c_{\delta}(x^*_t) = x^*_t \) for all \( x^*_t \in \text{FS}(P(U_E)) \).

**Proof.** (i) \( \implies \) (ii). Let \((U, \tau, E)\) be FS-\( \delta T_1\) and \( x^*_t, y^*_t \in \text{FS}(P(U_E)) \) with \( x^*_t \bar{y}^*_t \), then there is an \( \text{FS} \, \delta \text{-O-set} \, O_{\bar{y}^*_t} \) such that \( x^*_t \bar{q}O_{\bar{y}^*_t} \). This implies \( O_{\bar{y}^*_t} \subseteq (x^*_t)^\beta \); thus, \((x^*_t)^\beta \) is an \( \text{FS} \, \delta \text{-O-set} \) and is, \( x^*_t \) is an \( \text{FS} \, \delta \text{-C-set} \) for all \( x^*_a \in \text{FS}(P(U_E)) \). Hence, \( c_{\delta}(x^*_t) = x^*_t \).

(ii) \( \implies \) (i). Let \( x^*_t, y^*_t \in \text{FS}(P(U_E)) \) with \( x^*_t \bar{y}^*_t \), then \( x^*_a \subseteq (y^*_b)^\beta \) and \( y^*_a \subseteq (x^*_b)^\beta \) (since \( \text{FS} \)-points \( x^*_a, y^*_b \) are \( \text{FS} \, \delta \text{-C-sets} \). Now, take \( O_{\bar{y}^*_t} = (y^*_b)^\beta \) and \( O_{\bar{x}^*_t} = (x^*_b)^\beta \). Thus, there are \( \text{FS} \, \delta \text{-O-sets} \, O_{\bar{y}^*_t} \) and \( O_{\bar{x}^*_t} \) such that \( x^*_t \bar{q}(x^*_t)^\beta = O_{\bar{x}^*_t} \) and \( y^*_t \bar{q}(y^*_t)^\beta = O_{\bar{y}^*_t} \). The result holds.

**Theorem 3.5.** If \((V, \delta, K)\) is an \( \text{FS} \, \delta T_1\) and \( f_{up} \) : \((U, \tau, E) \rightarrow (V, \delta, K)\) is one-to-one and \( \text{FS} \, \delta \text{-continuous} \), then so is \((U, \tau, E)\).

**Proof.** Let \((V, \delta, K)\) be FS-\( \delta T_1\) and \( x^*_t \bar{y}^*_t \) for any \( x^*_t, y^*_t \in \text{FSS}(U_E) \). Since \( f_{up} \) is one-to-one, we have \( f_{up}(x^*_t) \bar{q}f_{up}(y^*_t) \). Since \((V, \delta, K)\) is FS-\( \delta T_1\), there are \( \text{FS} \, \delta \text{-O-sets} \, O_{f_{up}(x^*_t)}, O_{f_{up}(y^*_t)} \in \delta \) such that \( f_{up}(y^*_t) \bar{q}O_{f_{up}(x^*_t)} \) and \( f_{up}(x^*_t) \bar{q}O_{f_{up}(y^*_t)} \). Since \( f_{up} \) is FS-\( \delta \)-continuous, we have \( f_{up}^{-1}(O_{f_{up}(x^*_t)}) \) and \( f_{up}^{-1}(O_{f_{up}(y^*_t)}) \) as \( \text{FS} \, \delta \text{-O-sets} \) in \((U, \tau, E)\) with \( y^*_t \bar{q}f_{up}^{-1}(O_{f_{up}(x^*_t)}) \) and \( x^*_t \bar{q}f_{up}^{-1}(O_{f_{up}(y^*_t)}) \). Hence, \((U, \tau, E)\) is FS-\( \delta T_1\).

**Theorem 3.6.** If \( \text{FSTS}(U, \tau, E) \) is FS-\( \delta T_2\), then \( x^*_t = \cap\{cl_{\delta}(h_{E}) : x^*_t \bar{e}h_{E}\} \).

**Proof.** Let \((U, \tau, E)\) be FS-\( \delta T_2\) and \( x^*_t \in \text{FS}(P(U_E)) \), then for any \( x^*_t \bar{y}^*_t \), there are \( \text{FS} \, \delta \text{-O-sets} \, h_{E} = O_{\bar{x}^*_t} \) and \( O_{\bar{y}^*_t} \) such that \( h_{E} \bar{q}O_{\bar{y}^*_t} \). From (ii) of Lemma 3.2, we have \( y^*_t \bar{q}cl_{\delta}(h_{E}) \) and \( y^*_t \bar{q} \cap \{cl_{\delta}(h_{E}) : x^*_t \bar{e}h_{E}\} \). From (v) of Proposition 2.2, we have \( \cap\{cl_{\delta}(h_{E}) : x^*_t \bar{e}h_{E}\} \subseteq x^*_t \), but \( x^*_t \bar{e} \cap \{cl_{\delta}(h_{E}) : x^*_t \bar{e}h_{E}\} \). The result holds.

**Proposition 3.1.** Every \( \text{FS} \, \delta T_i \) is FS-\( \delta T_{i-1} \), \( i = 1, 2 \).

**Proof.** It is obvious.

The next example shows that the converse of the above proposition is not necessarily true.

**Example 3.2.** Let \( U = \{x, y\}, E = \{e\}, \) and \( \tau = \{0_E, 1_E\} \cup \{x^*_r : r \in (0, 1)\} \), then \( \tau \) is an \( \text{FST} \) on \( U \). It is easy to check that \((U, \tau, E)\) is FS-\( \delta T_0\). Indeed, all members in \( \tau \) are FS-\( RO \)-sets, so they are \( \text{FS} \, \delta \text{-O-sets} \). For any \( x^*_t, y^*_t \) with \( x^*_t \bar{y}^*_t \), there is an \( \text{FS} \, \delta \text{-O-set} \, O_{\bar{x}^*_t} = x^*_t \) such that \( O_{\bar{y}^*_t} = y^*_t \bar{q}x^*_t \). On the other hand, the unique \( \text{FS} \, \delta \text{-O-set} \) containing \( y^*_t \) is \( 1_E \), but \( 1_E \bar{q}x^*_t \). Therefore, \((U, \tau, E)\) is not FS-\( \delta T_1\).
Theorem 3.7. Let \((U, \tau, E)\) be FS-\(\delta T_1\) and \(h_E\) be any FS-\(\delta O\)-set. If \(h_E^c\) is also FS-\(\delta O\)-set in \((U, \tau, E)\), then \((U, \tau, E)\) is FS-\(\delta T_2\).

Proof. Let \(x_r^c, y_r^c \in FS(P(U_E))\). Since \((U, \tau, E)\) is FS-\(\delta T_1\), there is an FS-\(\delta O\)-set \(O_{x_r}^c\) such that \(O_{x_r}^c q_{x_r}^c\), or there is an FS-\(\delta O\)-set \(O_{y_r}^c\) such that \(O_{y_r}^c q_{y_r}^c\). Let us assume that \(O_{y_r}^c q_{y_r}^c\), then \(y_r^c \subseteq (O_{y_r}^c)^c\), which is an FS-\(\delta O\)-set by assumption, and \(O_{x_r}^c q_{x_r}^c\). This completes the proof.

Proposition 3.2. If every crisp FS-point \(x_1\) is FS-\(\delta O\)-set in \((U, \tau, E)\), then \((U, \tau, E)\) is FS-\(\delta T_2\).

Proof. It is obvious.

Theorem 3.8. If \((V, \delta, K)\) is FS-\(\delta T_2\) and \(f_{up} : (U, \tau, E) \rightarrow (V, \delta, K)\) is one-to-one and FS-\(\delta\)-continuous, then \((U, \tau, E)\) is FS-\(\delta T_2\).

Proof. It follows by using a similar way to that in Theorem 3.9.

Theorem 3.9. Every FS-subspace \((Y, r_y, E)\) of FS-\(\delta T_1(U, \tau, E)\) is FS-\(\delta T_1\), \(i = 0, 1, 2\).

Proof. As a sample, we prove the case \(i = 1\). The proof of the rest of the cases is similar. Let \(x_r^c, y_r^c \in FS(P(U_E))\), then also \(x_r^c, y_r^c \in FS(P(U_E))\), since \((U, \tau, E)\) is FS-\(\delta T_1\), there is an FS-\(\delta O\)-set \(O_{x_r}^c, O_{y_r}^c\) such that \(y_r^c q_{x_r}^c\) and \(x_r^c q_{y_r}^c\). Thus, \(O_{x_r}^c \cap h_E^c\) and \(O_{y_r}^c \cap h_E^c\) are FS-\(\delta O\)-sets in \((Y, r_y, E)\). Take \(O_{x_r}^c = O_{x_r}^c \cap h_E^c\) and \(O_{y_r}^c = O_{y_r}^c \cap h_E^c\), then \(y_r^c q_{x_r}^c\) and \(x_r^c q_{y_r}^c\). Hence, the result holds.

4. Fuzzy soft \(\delta\)-regularity axioms

Here, we introduce the definitions of a new class of regularity axioms, namely, FS-\(\delta\)-regularity axioms (or FS-\(\delta R_i\), \(i = 0, 1, 2, 3\)), and investigate some of their properties.

Definition 4.1. An FSSTS \((U, \tau, E)\) is said to be:

(i) FS-\(\delta R_0\) if for any \(x_r^c, y_r^c \in FS(P(U_E))\) with \(x_r^c q_{x_r}^c\), \(y_r^c q_{y_r}^c\) implies \(cl_0(x_r^c q_{y_r}^c)\).

(ii) FS-\(\delta R_1\) if for any \(x_r^c, y_r^c \in FS(P(U_E))\) with \(x_r^c q_{x_r}^c\), \(y_r^c q_{y_r}^c\), there are FS-\(\delta O\)-sets \(O_{x_r}^c\) and \(O_{y_r}^c\) such that \(O_{x_r}^c q_{O_{y_r}^c}\).

In the next results, some descriptions of FS-\(\delta R_i\) spaces for \(i = 0, 1\) are investigated.

Theorem 4.1. In an FSSTS \((U, \tau, E)\), the next items are equivalent:

(i) \((U, \tau, E)\) is FS-\(\delta R_0\).

(ii) \(cl_0(x_r^c) \subseteq O_{x_r}^c\) for any FS-\(\delta O\)-set \(O_{x_r}^c\).

(iii) \(cl_0(x_r^c) \subseteq \cap \{O_{x_r}^c : O_{x_r}^c \in FS-\delta OS(U_E)\}\) for all \(x_r^c \in FS(P(U_E))\).

Proof. (i) \(\implies\) (ii) Let \((U, \tau, E)\) be FS-\(\delta R_0\) and \(y_r^c q_{cl_0(x_r^c)}\), then \(x_r^c q_{cl_0(y_r^c)}\). From (ii) of Lemma 3.2, we have \(y_r^c q_{O_{x_r}^c}\), and by (v) of Proposition 2.2, we get \(cl_0(x_r^c) \subseteq O_{x_r}^c\) for any FS-\(\delta O\)-set \(O_{x_r}^c\). The result holds.

(ii) \(\implies\) (iii) It is clear.

(iii) \(\implies\) (i) Let \(cl_0(x_r^c) \subseteq \cap \{O_{x_r}^c : O_{x_r}^c \in FS-\delta O(U_E)\}\) for any \(O_{x_r}^c\) and let \(x_r^c, y_r^c \in FS(P(U_E))\) with \(x_r^c q_{cl_0(y_r^c)}\), then \(x_r^c \in \{cl_0(y_r^c)^c = O_{x_r}^c\}\), which is an FS-\(\delta O\)-set containing \(x_r^c\). So by hypothesis, \(cl_0(x_r^c) \subseteq O_{x_r}^c = [cl_0(y_r^c)^c = int_0 \{y_r^c\}^c \subseteq (y_r^c)^c\). Thus, \(cl_0(x_r^c) q_{y_r}^c\). Hence, \((U, \tau, E)\) is FS-\(\delta R_0\).
Theorem 4.2. An FSTS $(U, \tau, E)$ is FS-$\delta R_0$ if and only if $h_E$ is an FS-$\delta C$-set with $x_r \bar{q} h_E$, and there is an FS-$\delta O$-set $O_{h_E}$ containing $h_E$ such that $x_r \bar{q} O_{h_E}$.

Proof. Let $(U, \tau, E)$ be FS-$\delta R_0$ and $h_E \in FS \delta C(U_E)$ with $x_r \bar{q} h_E$, then $x_r \in h_E^t = O_{h_E}$. From (ii) of Theorem 4.2, we have $cl_d(x_r^c) \subseteq h_E^t = O_{h_E}$, and $h_E \subseteq [cl_d(x_r^c)]^t = h_E$. Since $x_r^c \subseteq cl_d(x_r^c)$, then $[cl_d(x_r^c)]^t \subseteq (x_r^c)^c$. Therefore, $x_r^c \bar{q}[cl_d(x_r^c)]^t = O_{h_E}$. The result holds.

The converse part is obvious.

Theorem 4.3. In an FSTS $(U, \tau, E)$, the next properties are equivalent:

(i) $(U, \tau, E)$ is FS-$\delta R_0$.

(ii) If $g_E$ is FS-$\delta C$-set with $x_r \bar{q} g_E$, then $cl_d(x_r) \bar{q} g_E$.

(iii) If $x_r^c \bar{q} cl_d(y_r^c)$, then $cl_d(x_r) \bar{q} cl_d(y_r)$.

Proof. (i) $\implies$ (ii) Let $g_E$ be an FS-$\delta C$-set with $x_r \bar{q} g_E$. Since $(U, \tau, E)$ is FS-$\delta R_0$, then by the above theorem there is an FS-$\delta O$-set $O_{g_E}$ such that $x_r \bar{q} O_{g_E}$. From (ii) of Lemma 3.2, we have $cl_d(x_r^c) \bar{q} g_E$.

(ii) $\implies$ (iii) It is obvious.

(iii) $\implies$ (i) Let $x_r^c, y_r^c \in FSP(U_E)$ with $x_r^c \bar{q} cl_d(y_r^c)$. By given $cl_d(x_r^c) \bar{q} cl_d(y_r^c)$, since $y_r^c \subseteq cl_d(y_r^c)$, we have $cl_d(x_r^c) \bar{q} y_r^c$. Thus $(U, \tau, E)$ is FS-$\delta R_0$.

Proposition 4.1. An FSTS $(U, \tau, E)$ is FS-$\delta R_1$ if and only if for any $x_r^c, y_r^c \in FSP(U_E)$ with $x_r^c \bar{q} cl_d(y_r^c)$, there are FS-$\delta O$-sets $O_{cl_d(x_r^c)}$ and $O_{cl_d(y_r^c)}$ such that $O_{cl_d(x_r^c)} \bar{q} O_{cl_d(y_r^c)}$.

Proof. It follows from that of the above theorem and from (ii) of Theorem 4.2.

Theorem 4.4. Every FS-subspace $(Y, \tau_Y, E)$ of FS-$\delta R_i$ is FS-$\delta R_i$, $i = 0, 1$.

Proof. As a sample, we prove the case $i = 1$. The proof of the rest case is similar.

Let $x_r^c, y_r^c \in FS$-points in $(Y_E)$ with $x_r^c \bar{q} cl_d(y_r^c)$, then also $x_r^c, y_r^c \in FSP(U_E)$ and $x_r^c \bar{q} cl_d(y_r^c)$. Since $(U, \tau, E)$ is FS-$\delta R_1$, there are FS-$\delta O$-sets $O_{x_r^c}, O_{y_r^c}$ such that $O_{x_r^c} \bar{q} O_{y_r^c}$. Take $O_{x_r^c} = O_{y_r^c} \cap h_E^t$ and $O_{y_r^c} = O_{y_r^c} \cap h_E^t$, then $O_{x_r^c}, O_{y_r^c}$ are FS-$\delta O$-sets in $(Y, \tau_Y, E)$ and $O_{y_r^c} \bar{q} O_{x_r^c}$. Hence, $(Y, \delta_Y E)$ is FS-$\delta R_1$.

Proposition 4.2. For FSTS $(U, \tau, E)$, every FS-$\delta T_i$ is FS-$\delta R_{i-1}$, $i = 1, 2$.

Proof. It is obvious.

The next example shows that the converse of the above proposition is not necessarily true.

Example 4.1. Let $U\{u\}$ and $E = \{e_1, e_2\}$. The family $\tau = \{0_E, 1_E, h_E = \{(e_1, u_{0,5}), (e_2, u_{0,5})\}$ is an FST on $U$. One can check that $(U, \tau, E)$ is FS-$\delta R_0$, but is not FS-$\delta T_0$. Indeed, for $x_{0,7}^c \bar{q} x_{0,2}^c$, the unique FS-$\delta O$-set containing $u_{0,5}^{e_1}$ is $1_E$, but $1_E \bar{q} u_{0,2}^{e_2}$.

Theorem 4.5. An FSTS $(U, \tau, E)$ is FS-$\delta T_i$ if and only if it is both FS-$\delta T_{i-1}$ and FS-$\delta R_{i-1}, i = 1, 2$.

Proof. As a sample, we prove the case $i = 2$. The proof of the rest case is similar. Necessity follows from the Proposition 3.11 and 4.7.

Conversely, let $(U, \tau, E)$ be FS-$\delta T_1$ and FS-$\delta R_1$, and let $x_r^c, y_r^c \in FSP(U_E)$ with $x_r^c \bar{q} y_r^c$. By Theorem 3.7, we have $x_r^c \bar{q} cl_d(y_r^c)$. Since $(U, \tau, E)$ is FS-$\delta R_1$, there are FS-$\delta O$-sets $O_{x_r^c}, O_{y_r^c}$ such that $O_{x_r^c} \bar{q} O_{y_r^c}$. Therefore, $(U, \tau, E)$ is FS-$\delta T_2$. 

Definition 4.2. An FSTS $(U, \tau, E)$ is said to be:

(i) $FS\delta$-regular(or $FS\delta$-$R_2$) if for any $FS\delta$-C-set $h_E$ and any $FS\delta$-point $x'$, with $x'\in h_E$, there are $FS\delta$-sets $O_{x'}$ and $O_{hE}$ such that $O_{x'} \cap O_{hE}$.

(ii) $FS\delta$-normal(or $FS\delta$-$R_3$) if for any $FS\delta$-C-sets $h_E$ and $g_E$ with $h_E \cap g_E$, there are $FS\delta$-sets $O_{hE}$ and $O_{gE}$ such that $O_{hE} \cap O_{gE}$.

(iii) $FS\delta$-$T_3$(resp., $FS\delta$-$T_4$) if it is $FS\delta$-$R_2$(resp., $FS\delta$-$R_3$) and $FS\delta$-$T_1$.

Theorem 4.6. For an FSTS $(U, \tau, E)$, the next items are equivalent:

(i) $(U, \tau, E)$ is $FS\delta$-$R_2$.

(ii) For any $x' \in FS\delta P(U_E)$ and any $FS\delta$-C-set $O_{x'}$, there is an $FS\delta$-C-set $O_{x'}$ containing $x'$ such that $cl(O_{x'}) \subseteq O_{x'}$.

Proof. (i) $\implies$ (ii) Let $x' \in FS\delta P(U_E)$ and $O_{x'}$ be any $FS\delta$-C-set containing $x'$, then $O_{x'} = h_E$ is an $FS\delta$-C-set. Clearly, $O_{hE} \cap O_{x'}$ and $O_{hE} \cap O_{x'}$. Since $(U, \tau, E)$ is $FS\delta$-$R_2$, there are $FS\delta$-sets $O_{x'}$, $O_{x'}$ such that $O_{x'} \cap O_{x'} = O_{hE}$, then $O_{x'} \subseteq O_{x'}$. Clearly, $O_{x'} \subseteq O_{x'} = O_{hE}$, so $O_{x'} \subseteq O_{x'}$. Therefore, $cl(O_{x'}) \subseteq O_{x'}$.

(i) $\implies$ (i) Let $x' \in FS\delta P(U_E)$ and $g_E$ be any $FS\delta$-C-set with $x' \cap g_E$, then $x' \cap g_E = O_{x'}$ which is an $FS\delta$-C-set containing $x'$. So there is an $FS\delta$-C-set $O_{x'}$ such that $cl(O_{x'}) \subseteq O_{x'} = g_E$, which implies $cl(O_{x'}) \subseteq O_{x'}$. Clearly, $cl(O_{x'}) \cap cl(O_{x'}) \subseteq O_{hE}$ and $O_{x'} \cap O_{x'}$. Thus, the result holds.

Theorem 4.7. An FSTS $(U, \tau, E)$ is $FS\delta$-$R_2$ if and only if for any $FS\delta$-C-set $h_E$ with $x' \in h_E$, there are $FS\delta$-C-sets $O_{x'}$, $O_{hE}$ such that $cl(O_{x'}) \subseteq O_{hE}$.

Proof. Let $x' \in FS\delta P(U_E)$ and $h_E$ be an $FS\delta$-C-set with $x' \cap h_E$. Since $(U, \tau, E)$ is $FS\delta$-$R_2$, there are $FS\delta$-C-sets $O_{x'}$, $O_{hE}$ such that $O_{hE} \cap O_{x'}$. From (iii) of Lemma 3.2, we obtain $cl(O_{hE}) \subseteq O_{x'}$, that is, $cl(O_{hE}) \cap O_{x'}$. Again $(U, \tau, E)$ is $FS\delta$-$R_2$, and there are $FS\delta$-C-sets $O_{x'}$, $O_{hE}$ such that $O_{x'} \cap O_{hE}$. By (iii) of Lemma 3.2, we have $cl(O_{x'}) \subseteq O_{x'}$. Now, put $O_{x'} = O_{x'} \cap O_{x'}$. Since $(U, \tau, E)$ is $FS\delta$-$R_2$ and $O_{x'}$ is an $FS\delta$-C-set, then by the above theorem, there is an $FS\delta$-C-set $O_{x'}$ such that $cl(O_{x'}) \subseteq O_{x'}$, that is, $cl(O_{x'}) \subseteq O_{x'}$. Since $cl(O_{hE}) \subseteq O_{x'}$, then $cl(O_{hE}) \subseteq O_{x'}$. Conversely, it follows from hypothesis.

Theorem 4.8. For an FSTS $(U, \tau, E)$, the next items are equivalent:

(i) $(U, \tau, E)$ is $FS\delta$-$R_3$.

(ii) For any $FS\delta$-C-set $h_E$ and any $FS\delta$-C-set $O_{hE}$, there is an $FS\delta$-C-set $O_{hE}$ containing $h_E$ such that $cl(O_{hE}) \subseteq O_{hE}$.

Proof. Let $(U, \tau, E)$ be $FS\delta$-$R_3$, $h_E$ be an $FS\delta$-C-set, and $O_{hE}$ be any $FS\delta$-C-set containing $h_E$, then $O_{hE}$ is an $FS\delta$-C-set. Since $O_{hE} \cap O_{hE}$, that is, $h_E \cap O_{hE}$, $(U, \tau, E)$ is $FS\delta$-$R_3$, there are $FS\delta$-C-sets $O_{hE}$, $O_{hE}$ such that $O_{hE} \cap O_{hE}$, then $O_{hE} \subseteq O_{hE}$, $O_{hE} \subseteq O_{hE}$. Since $O_{hE} \subseteq O_{hE}$, then $O_{hE} \subseteq O_{hE}$, and $cl(O_{hE}) \subseteq O_{hE}$. The result holds.

Conversely, let $f_E, g_E$ be two $FS\delta$-C-sets with $f_E \cap g_E$, then $f_E \subseteq O_{fE}$ which is an $FS\delta$-C-set containing $f_E$. By hypothesis, there is an $FS\delta$-C-set $O_{fE}$ such that $cl(O_{fE}) \subseteq g_E$, then $g_E \subseteq cl(O_{fE})$. The result holds.
Theorem 4.9. An FSTS \((U, \tau, E)\) is FS-\(\delta R_3\) if and only if for any two FS \(\delta C\)-sets \(h_E, g_E\) with \(h_E \bar{q} g_E\), there are FS \(\delta O\)-sets \(O_{h_E}, O_{g_E}\) such that \(\text{cl}(O_{h_E}) \bar{q} \text{cl}(O_{g_E})\).

Proof. It is analogous to that in Theorem 4.12.

5. More characterizations and relations

Saleh et. al [44] introduced and studied a new type of FS-compactness, namely, FS-\(\delta^*\)-compactness. In this section, we study more properties and investigate the relations between FS-\(\delta^*\)-compact and FS-\(\delta\)-separation axioms, which are introduced in this work.

To begin we show that the axioms FS-\(\delta R_i\), \(i = 1, 2, 3\) and FS-\(\delta T_i\), \(i = 1, 2, 3, 4\) are harmonic.

Theorem 5.1. For an FSTS \((U, \tau, E)\), we have:

\[FS - \delta R_3 \land FS - \delta R_0 \Rightarrow FS - \delta R_2 \Rightarrow FS - \delta R_1 \Rightarrow FS - \delta R_0.\]

Proof. Let \((U, \tau, E)\) be FS-\(\delta R_3\), FS-\(\delta R_0\), \(x'_r \in \text{FSP}(U_E)\) for any FS-\(\delta C\)-set \(f_E\), with \(x'_r \bar{q} h_E\).

Since \((U, \tau, E)\) is FS-\(\delta R_0\), then \(\text{cl}_{q} (x'_r) \bar{q} f_E\) where \(\text{cl}_{q} (x'_r), h_E\) are FS-\(\delta C\)-sets. Again, \((U, \tau, E)\) is FS-\(\delta R_3\), so there are FS-\(\delta O\)-sets \(O_{\text{cl}_{q} (x'_r)}, O_{h_E}\) such that \(O_{\text{cl}_{q} (x'_r)} \bar{q} O_{h_E}\). Put \(O_{\text{cl}_{q} (x'_r)} = O_{x'_r}\), and we have \(O_{x'_r} \bar{q} O_{h_E}\). Thus, \((U, \tau, E)\) is FS-\(\delta R_2\).

The proof for the rest of the cases is obvious.

Theorem 5.2. For an FSTS \((U, \tau, E)\), we have:

\[FS - \delta T_4 \Rightarrow FS - \delta T_3 \Rightarrow FS - \delta T_2 \Rightarrow FS - \delta T_1 \Rightarrow FS - \delta T_0.\]

Proof. Let \((U, \tau, E)\) be FS-\(\delta T_4\), then it is both FS-\(\delta R_3\) and FS-\(\delta T_1\). From Proposition 4.7, we have \((U, \tau, E)\) is FS-\(R_0\). Let us assume that \(x'_r \in \text{FSP}(U_E), h_E\) is an FS-\(\delta C\)-set with \(x'_r \bar{q} h_E\), then by Theorem 4.4, \(\text{cl}_{q} (x'_r) \bar{q} h_E\), where \(\text{cl}_{q} (x'_r), h_E\) are FS-\(\delta C\)-sets. Since \((U, \tau, E)\) is FS-\(\delta R_3\), there are FS-\(\delta O\)-sets \(O_{\text{cl}_{q} (x'_r)}, O_{h_E}\) such that \(O_{\text{cl}_{q} (x'_r)} \bar{q} O_{h_E}\). Take \(O_{\text{cl}_{q} (x'_r)} = O_{x'_r}\), and we have \(O_{x'_r} \bar{q} O_{h_E}\). Thus, \((U, \tau, E)\) is FS-\(\delta R_2\). Hence, we obtain the result.

The proof of the rest of the cases follows from the above theorem and Proposition 3.11.

From the above theorems, Definition 4.10, and Proposition 4.7, we obtain the next result.

Corollary 5.1. For an FSTS \((U, \tau, E)\), the next implications hold.

\[FS - \delta T_4 \quad \Rightarrow \quad FS - \delta T_3 \quad \Rightarrow \quad FS - \delta T_2 \quad \Rightarrow \quad FS - \delta T_1 \quad \Rightarrow \quad FS - \delta T_0\]

\[\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[FS - \delta R_3 \land FS - \delta R_0 \Rightarrow FS - \delta R_2 \Rightarrow FS - \delta R_1 \Rightarrow FS - \delta R_0.\]

Theorem 5.3. Let \((U, \tau, E)\) be FS-\(\delta T_3\) and \(g_E\) be an FS-\(\delta^*\)-compact set, then for any FS-\(\delta C\)-set \(h_E\) with \(h_E \bar{q} g_E\), there are FS-\(\delta O\)-sets \(O_{h_E}, O_{g_E}\) such that \(O_{h_E} \bar{q} O_{g_E}\).

Proof. Let \((U, \tau, E)\) be FS-\(\delta T_3\) and \(g_E\) be an FS-\(\delta^*\)-compact set, then for any FS-\(\delta C\)-set \(h_E\) with \(h_E \bar{q} g_E\), we have for any \(y'_r \in g_E\), there are FS-\(\delta O\)-sets \(O_{y'_r}, O_{h_E}\) such that \(O_{y'_r} \bar{q} O_{h_E}\). Clearly, \(\{O_{y'_r} : y'_r \in g_E\}\) is FS-\(\delta^*\)-open cover of \(g_E\). Since \(g_E\) is FS-\(\delta^*\)-compact, there is a finite FS-\(\delta^*\)-open subcover of \(g_E\), say, \(\{O_{y'_i} : i = 1, 2, \ldots, n\}\). One can verify that \(O_{g_E} = \bigcup_{i=1}^{n} O_{y'_i}\) and \(O_{h_E} = \bigcap_{i=1}^{n} O_{h_E}\) have the required property.
Theorem 5.4. Let \((U, \tau, E)\) be \(FS-\delta T_2\), \(x^*_r \in FSP(U_E)\) and \(g_E\) be an \(FS\delta^*\)-compact set with \(x^*_r \hat{q} g_E\), then there are \(FS\delta O\)-sets \(O_{\hat{q}}\) and \(O_{g_E}\) such that \(O_{\hat{q}} O_{g_E}\).

Moreover, if \(h_E, g_E\) are \(FS\delta^*\)-compact sets with \(h_E \hat{q} g_E\), then there are \(FS\delta O\)-sets \(O_{h_E}, O_{g_E}\) such that \(O_{h_E} O_{g_E}\).

Proof. It follows by a similar way to that in the above theorem.

Theorem 5.5. Every \(FS\delta^*\)-compact set in an \(FS-\delta T_2\) space is an \(FS\delta C\)-set.

Proof. Let \((U, \tau, E)\) be \(FS-\delta T_2\) and \(g_E\) be an \(FS\delta^*\)-compact set. From the above theorem for any \(x^*_r \in FSP(U_E)\) with \(x^*_r \hat{q} g_E\), there is \(FS\delta O\)-set \(O_{\hat{q}}\) such that \(O_{\hat{q}} O_{g_E}\); that is, for any \(x^*_r \in g_E\), there is \(FS\delta O\)-set \(O_{\hat{q}}\) such that \(O_{\hat{q}} \subseteq g_E\). Therefore, \(g_E\) is an \(FS\delta O\)-set in \((U, \tau, E)\). Thus, \(g_E\) is an \(FS\delta C\)-set.

Theorem 5.6. Let \((U, \tau, E)\) be \(FS-\delta R_1\), then \((U, \tau, E)\) is \(FS-\delta T_2\) if and only if every \(FS\delta^*\)-compact set is an \(FS\delta C\)-set.

Proof. The necessary parts follows directly from the above theorem. Conversely, if any \(FS\delta^*\)-compact set is an \(FS\delta C\)-set, then \((U, \tau, E)\) is an \(FS-\delta T_1\) space. Since \((U, \tau, E)\) is \(FS-\delta R_1\) and \(FS-\delta T_1\), then by Theorem 4.9, we obtain that \((U, \tau, E)\) is \(FS-\delta T_2\).

Theorem 5.7. For \(FSTS(U, \tau, E)\), every \(FS\delta^*\)-compact \(FS-\delta R_1\) space is \(FS-\delta R_2\) (\(FS-\delta R_3\)).

Proof. Let \((U, \tau, E)\) be an \(FS\delta^*\)-compact, \(FS-\delta R_1\) space and let \(h_E\) be an \(FS\delta C\)-set with \(x^*_r \hat{q} h_E\), then for any \(FS\)-point \(y^*_r \in h_E\), we have \(x^*_r \hat{q} cl(y^*_r)\). Since \((U, \tau, E)\) is \(FS-\delta R_1\), there are \(FS\delta O\)-sets \(O_{\hat{q}}, O_{\bar{g}}\) such that \(O_{\hat{q}} \subseteq \bar{g}\) so that the family \(\{O_{\hat{q}} : y^*_r \in h_E\}\) is an \(FS\delta^*\)-open cover of \(h_E\). Since \((U, \tau, E)\) is \(FS\delta^*\)-compact, \(h_E\) is \(FS\delta^*\)-compact and there is a finite \(FS\delta^*\)-open subcover of \(h_E\), say, \(\{O_{\hat{q}} : y^*_r \in h_E, i = 1, 2, \ldots, n\}\). Take \(O_{\hat{q}} = \bigcap_{i=1}^{n} O_{\hat{q}}\) and \(O_{h_E} = \bigcup_{i=1}^{n} O_{\hat{q}}\), then \(O_{\hat{q}}, O_{h_E}\) are \(FS\delta O\)-sets with \(O_{\hat{q}} \subseteq \bar{g}\). The result holds.

The proof of the rest case is analogous.

Corollary 5.2. For \(FS\delta^*\)-compact space \((U, \tau, E)\), the next items are equivalent:

(i) \((U, \tau, E)\) is \(FS-\delta R_1\).

(ii) \((U, \tau, E)\) is \(FS-\delta R_2\).

(iii) \((U, \tau, E)\) is \(FS-\delta R_0\) and \(FS-\delta R_3\).

Proof. It is obvious.

Theorem 5.8. \((U, \tau, E)\) is \(FS-\delta T_i \iff (U, \tau_{\delta}, E)\) is \(FST_i\), \(i = 0, 1, 2\).

Proof. It follows directly from Result 3 and Definition 3.1.

6. Concluding remarks and future work

It is well known that separation axioms provide some categories for topological spaces and help to prove some interesting properties of compactness and connectedness. Therefore, we have written this article to shed light on the properties of separability in the framework of fuzzy soft topologies.
We have defined and studied a new set of separation properties in fuzzy soft topological spaces, namely, $FS\delta$-separation and regularity properties via $FS\delta O$-sets by using quasi-coincident relation for $FS$-points. Several basic desirable properties, relations, and results have been obtained with some necessary examples. The relationships between $FS\delta^\ast$-compact spaces and $FS\delta$-separation have been investigated as well. We have shown that the implications $FS\delta T_4 \Rightarrow FS\delta T_3 \Rightarrow FS\delta T_2 \Rightarrow FS\delta T_1 \Rightarrow FS\delta T_0$ hold true, but we cannot get examples to show that the converse in these implications may not be true in general, except the case $FS\delta T_0 \Rightarrow FS\delta T_1$.

By and large, the results obtained in the manuscript frame “fuzzy soft topology” represent a wider view than that inspired by the frameworks of fuzzy and soft topologies, since these frames are created by replacing the membership function with the characteristic function in the case of fuzzy topology and restricting the set of parameters by a singleton set in the case of soft topology. The present results elucidate that the perspective on the theory of separation axioms adopted in this paper is very useful and will open up the door for further contributions. We plan in upcoming studies to generate fuzzy soft topologies by hybridizing $F$-set with the recent types of $F$-set like complemental fuzzy sets [4], (2, 1)-fuzzy sets [10], $(m, n)$-fuzzy sets [15], $n^{th}$ power root fuzzy sets [14, 27], and $k^m_n$-Rung picture fuzzy sets [28]. One may examine the current concepts and the previous ones in these hybridizations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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