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Research article

An ε-approximate solution of BVPs based on improved multiscale orthonormal basis

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Abstract: In the present paper, we construct a set of multiscale orthonormal basis based on Legendre polynomials. Using this orthonormal basis, a new algorithm is designed for solving the second-order boundary value problems. This algorithm is to find numerical solution by seeking ε-approximate solution. Moreover, we prove that the order of convergence depends on the boundedness of $u^{(m)}(x)$. In addition, third numerical examples are provided to validate the efciency and accuracy of the proposed method. Numerical results reveal that the present method yields extremely accurate approximation to the exact solution. Meanwhile, compared with the other algorithms, the results obtained demonstrate that our algorithm is remarkably effective and convenient.

Keywords: Legendre polynomials; ε-approximate solution; numerical solution
Mathematics Subject Classification: 65L10, 65L20

1. Introduction

The boundary value problems (BVPs) for differential equations have important applications in space science and engineering technology. A large number of mathematical models in the fields of engineering, astronomy, mechanics, economics, etc, are often described by differential BVPs [1–3]. Except for a few special types, the exact solution of the BVPs is difficult to express in analytical form. It is especially important to find an approximate solution to obtain its numerical solution. In [4], Sinc collocation method provided an exponential convergence rate for two-point BVPs. [5] constructed a simple collocation method by the Haar wavelets for the numerical solution of linear and nonlinear second-order BVPs with periodic boundary conditions. Erge [6] studied the quadratic/linear rational spline collocation method for linear BVPs. In [7], based on B-spline wavelets, the numerical solutions of nonlinear BVPs were derived. Pradip et al. used B-spline to Bratuis problem which is an important nonlinear BVPs in [8–10]. [11–16] solved BVPs by the reproducing kernel method. Based on the idea of least squares, Xu et al. [17–19] gave an effective algorithm in reproducing kernel space for solving fractional differential integral equations and interface problems.

It is a common technique to use orthogonal polynomials to solve differential equations. In [20–23], the authors used Chebyshev-Galerkin scheme for the time-fractional diffusion equation. In [24], the authors developed Jacobi rational operational approach for time-fractional sub-diffusion equation on a semi-infinite domain. [25–28] developed multiscale orthonormal basis to solve BVPs with various boundary conditions, and the stability and convergence order were also discussed. Legendre wavelet is widely used in various fields, such as signal system, because of its good properties. In this paper, a multiscale function is constructed by using Legendre polynomials to solve the approximate solution of differential equations. We use the multiscale fine ability of Legendre wavelet to construct multiwavelet, which has better approximation than single wavelet. In addition, we improve Legendre wavelet for specific problems, and the improved one still has compact support. We know that for functions with compact support, the better the tight support, the more concentrated the energy. Moreover, in the calculation process, the calculation speed can be enhanced, and the error accumulation is low.

The purpose of this paper is to construct a set of multiscale orthonormal basis with compact support based on Legendre wavelet to find the approximate solution of the boundary value problems:

$$\begin{cases} u''(x) + p(x)u'(x) + q(x)u(x) = F(x, u), \quad x \in (0, 1), \\ a_1u(0) + b_1u(1) + c_1u'(0) + d_1u'(1) = \alpha_1, \\ a_2u(0) + b_2u(1) + c_2u'(0) + d_2u'(1) = \alpha_2, \end{cases} \tag{1.1}$$

where $p(x)$ and $q(x)$ are both smooth. $a_i, b_i, c_i, d_i,\ i = 1, 2$ are constants. When F is just about the function of x, $F(x, u) = f(x)$, Eq (1.1) is linear boundary value problem. According to [21], the nonlinear boundary value problem can be transformed into a linear boundary value problem by using Quasi-Newton method. So this paper mainly studies the case of $F(x, u) = f(x)$, that is, the linear boundary value problem.

As we all know, if the basis function has good properties, the approximate solution of the boundary value problem has good convergence, stability and so on. In [25], the orthonormal basis on [0,1] was constructed by the compact support function to obtain the numerical solution of the boundary value problem. But the basis function is not compactly supported at [0,1], and the approximating solution is linearly convergent. In this paper, based on the idea of wavelet, a set of orthonormal bases with compact support is constructed by using Legendre polynomials, and the approximate solution of the boundary value problem is obtained by using these bases. Based on the constructed orthonormal basis, the proposed algorithm has convergence and stability, and the convergence order of the algorithm is more than 2 orders.

The purpose of this work is to deduce the numerical solutions of Eq (1.1). In Section 2, using wavelet theory, a set of multiscale orthonormal basis is presented by Legendre polynomials in $W_2^3[0, 1]$. The constructed basis is compactly supported. It is well known that the compact support performance generates sparse matrices during calculation, thus improving the convergence rate . The numerical method of ε-approximate solution is presented in Section 3. And Section 4 proves the convergence order of ε-approximate solution and stability. In Section 5, the proposed algorithm has been applied to some numerical experiments. Finally, we end with some conclusions in Section 6.

2. Basis functions in $W_2^3[0,1]$

Wu and Lin introduced the reproducing kernel space $W_2^1[0,1]$ and $W_2^3[0,1]$ [29]. Let

$$W_{2,0}^3[0,1] = \{u|u(0) = u'(0) = u''(0) = 0, \quad u \in W_2^3[0,1]\}.$$

Clearly, $W_{2,0}^3[0,1]$ is the closed subspace of $W_2^3[0,1]$.

Legendre polynomials are mathematically important functions. This section constructs the orthonormal basis in $W_2^3[0,1]$ by Legendre polynomials. Legendre polynomials are known to be orthogonal on $L^2[-1,1]$. For convenience, we first compress Legendre's polynomials to $[0,1]$, and get the following four functions:

$$\varphi^0(x) = 1; \qquad \varphi^1(x) = \sqrt{3}(-1+2x);$$
$$\varphi^2(x) = \sqrt{5}(1-6x+6x^2); \quad \varphi^3(x) = \sqrt{7}(-1+12x-30x^2+20x^3).$$

By translating and weighting the above four functions, we can construct

$$\psi^l(x) = \sum_{j=0}^{3}(a_{lj}\varphi^j(2x) + b_{lj}\varphi^j(2x-1)), \quad l = 0,1,2,3. \tag{2.1}$$

In application, we hope $\psi^l(x)$ has good properties, for example, as many coefficients as zero and orthogonality, so $\psi^l(x)$ needs to meet the following conditions

$$\int_0^1 x^j\psi^l(x)dx = 0, \quad j = 0,1,2,\cdots,l+3, \tag{2.2}$$

$$\int_0^1 \psi^i(x)\psi^j(x)dx = \delta_{ij}, \quad i,j = 0,1,2,3. \tag{2.3}$$

The coefficients a_{lj}, b_{lj} can be get by Eqs (2.2) and (2.3), immediately $\psi^l(x)$ is as follows:

$$\psi^0(x) = \sqrt{\frac{15}{17}}\begin{cases} -3+56x-216x^2+224x^3, & x\in[0,\frac{1}{2}],\\ 61-296x+456x^2-224x^3, & x\in[\frac{1}{2},1].\end{cases} \tag{2.4}$$

$$\psi^1(x) = \sqrt{\frac{1}{21}}\begin{cases} -11+270x-1320x^2+1680x^3, & x\in[0,\frac{1}{2}],\\ -619+2670x-3720x^2+1680x^3, & x\in[\frac{1}{2},1].\end{cases} \tag{2.5}$$

$$\psi^2(x) = \sqrt{\frac{35}{17}}\begin{cases} -1+30x-174x^2+256x^3, & x\in[0,\frac{1}{2}],\\ 111-450x+594x^2-256x^3, & x\in[\frac{1}{2},1].\end{cases} \tag{2.6}$$

$$\psi^3(x) = \sqrt{\frac{5}{21}}\begin{cases} 1-36x+246x^2-420x^3, & x\in[0,\frac{1}{2}],\\ 209-804x+1014x^2-420x^3, & x\in[\frac{1}{2},1].\end{cases} \tag{2.7}$$

Through the ideas of the wavelet, scale transformation of the functions $\psi^l(x)$ gets Legendre wavelet

$$\psi_{ik}^l(x) = 2^{\frac{i-1}{2}}\psi^l(2^ix-k), \;\; l = 0,1,2,3; \;\; i = 1,2,\cdots; \;\; k = 0,1,\cdots,2^{i-1}-1.$$

Clearly, $\psi_{ik}^{l}(x)$ has compactly support in $[\frac{k}{2^{i-1}}, \frac{k+1}{2^{i-1}}]$. Let

$$W_i = span\{\psi_{ik}^{l}(x)\}_{l=0}^{3}, \quad i = 1, 2, \cdots; \quad k = 0, 1, \cdots, 2^{i-1} - 1.$$

Then,

$$L^2[0, 1] = V_0 \bigoplus_{i=1}^{\infty} W_i,$$

where

$$V_0 = \{\varphi^0(x), \varphi^1(x), \varphi^2(x), \varphi^3(x)\}.$$

According to the above analysis, we can get the following theorem.

Theorem 2.1.

$$\{\rho_j(x)\}_{j=1}^{\infty} = \{\varphi^0(x), \varphi^1(x), \varphi^2(x), \varphi^3(x), \psi_{10}^{0}(x), \psi_{10}^{1}(x), \psi_{10}^{2}(x), \psi_{10}^{3}(x), \cdots, \psi_{ik}^{0}(x), \psi_{ik}^{1}(x), \psi_{ik}^{2}(x), \psi_{ik}^{3}(x), \cdots\}$$

is the orthonormal basis in $L^2[0, 1]$.

Now we generate the orthonormal basis in $W_{2,0}^{3}[0, 1]$ from the basis in $L^2[0, 1]$. Note

$$J^3 u(x) = \frac{1}{2} \int_0^x (x - t)^2 u(t) dt. \tag{2.8}$$

Theorem 2.2. $\{J^3 \rho_j(x)\}_{j=1}^{\infty}$ *is the orthonormal basis in* $W_{2,0}^{3}[0, 1]$.

Proof. Only need to prove completeness and orthogonality. For $u \in W_{2,0}^{3}[0, 1]$, if

$$< u, J^3 \rho_j >_{W_{2,0}^{3}} = 0,$$

you can deduce $u \equiv 0$, then $\{J^3 \rho_j(x)\}_{j=1}^{\infty}$ are complete. In fact,

$$< u, J^3 \rho_j >_{W_{2,0}^{3}} = < u''', \rho_j >_{L^2} = \int_0^1 u''' \rho_j dx = 0. \tag{2.9}$$

From Theorem 2.1, $u''' \equiv 0$. Due to $u \in W_{2,0}^{3}[0, 1]$, $u(0) = u'(0) = u''(0) = 0$, then, $u \equiv 0$.

According to Theorem 2.1 and Eq (2.9), orthonormal is obvious. □

Because of $W_{2,0}^{3}[0, 1] \subset W_2^3[0, 1]$ and three more conditions for $W_2^3[0, 1]$ than $W_{2,0}^{3}[0, 1]$. So the orthonormal basis for $W_2^3[0, 1]$ as follows:

Theorem 2.3.

$$\{J^3 g_j(x)\}_{j=1}^{\infty} = \{1, x, \frac{x^2}{2}\} \cup \{J^3 \rho_j(x)\}_{j=1}^{\infty}$$

are the orthonormal basis in $W_2^3[0, 1]$.

3. A multiscale algorithm for Eq (1.1)

Put L: $W_2^3[0,1] \to L^2[0,1]$,

$$Lu = u''(x) + p(x)u'(x) + q(x)u(x).$$

L is a linear bounded operator in [27]. Let B_i: $W_2^3[0,1] \to \mathbb{R}$, and

$$B_i u = a_i u(0) + b_i u(1) + c_i u'(0) + d_i u'(1), \quad i = 1, 2.$$

The Quasi-Newton method is used to transform Eq (1.1) into a linear boundary value problem, and its operator equation is as follows:

$$\begin{cases} Lu = f(x), \\ B_1 u = \alpha_1, \quad B_2 u = \alpha_2. \end{cases} \tag{3.1}$$

Definition 3.1. *u^ε is named ε-approximate solution for Eq (3.1), $\forall \varepsilon > 0$, if*

$$\|Lu^\varepsilon - f\|_{L^2}^2 + \sum_{i=1}^{2}(B_i u^\varepsilon - \alpha_i)^2 < \varepsilon^2.$$

In [27], it is shown that ε-approximate solution for Eq (3.1) exists by the following theorem.

Theorem 3.1. *Equation (3.1) exists ε-approximate solution*

$$u_n^\varepsilon(x) = \sum_{k=1}^{n} c_k^* J^3 g_k(x),$$

where n is a natural number determined by ε, and c_i^* satisfies

$$\|\sum_{k=1}^{n} c_k^* L J^3 g_k - Lu\|_{L^2}^2 + \sum_{l=1}^{2}(\sum_{k=1}^{n} c_k^* J^3 g_k - B_l u)^2 = \min_{c_k}\{\|\sum_{k=1}^{n} c_k L J^3 g_k - Lu\|_{L^2}^2 + \sum_{l=1}^{2}(\sum_{k=1}^{n} c_k J^3 g_k - B_l u)^2\}.$$

To seek the ε-approximate solution, we just need c_k^*. Let G be quadratic form about

$$\boldsymbol{c} = (c_1, \cdots, c_n)^T,$$

$$G(c_1, \cdots, c_n) = \|\sum_{k=1}^{n} c_k L J^3 g_k - Lu\|_{L^2}^2 + \sum_{l=1}^{2}(\sum_{k=1}^{n} c_k J^3 g_k - B_l u)^2. \tag{3.2}$$

From Theorem 3.1,

$$\boldsymbol{c}^* = (c_1^*, \cdots, c_n^*)^T$$

is the minimum point of $G(c_1, \cdots, c_n)$. If L is reversible, the minimum point of G exists and is unique.

In fact, the partial derivative of $G(c_1, \cdots, c_n)$ with respect to c_j:

$$\frac{\partial G}{\partial c_j} = 2\sum_{k=1}^{n} c_k \langle LJ^3 g_k, LJ^3 g_j\rangle_{L^2} - 2\langle LJ^3 g_j, Lu\rangle_{L^2} + 2\sum_{l=1}^{2}(\sum_{k=1}^{n} c_k J^3 g_k J^3 g_j - J^3 g_j B_l u).$$

Let

$$\frac{\partial}{\partial c_j}G(c_1,\cdots,c_n)=0,$$

so

$$\sum_{k=1}^{n}c_k\langle LJ^3g_k, LJ^3g_j\rangle_{L^2}+2\sum_{k=1}^{n}c_kJ^3g_kJ^3g_j=\langle LJ^3g_j, Lu\rangle_{L^2}+\sum_{l=1}^{2}J^3g_jB_lu. \tag{3.3}$$

Let $\boldsymbol{A_n}$ be the n-order matrix and $\boldsymbol{b_n}$ be the n-dimensional vector, i.e.,

$$\begin{aligned}\boldsymbol{A_n} &= \left(\ \langle LJ^3g_k, LJ^3g_j\rangle_{L^2}+2J^3g_kJ^3g_j\ \right)_{n\times n},\\ \boldsymbol{b_n} &= \left(\ \langle LJ^3g_k, Lu\rangle_{L^2}+\sum_{l=1}^{2}J^3g_jB_lu\ \right)_{n}.\end{aligned}$$

Then Eq (3.3) changes to

$$\boldsymbol{A_n c}=\boldsymbol{b_n}. \tag{3.4}$$

If L is invertible, Eq (3.4) has only one solution $\boldsymbol{c}^*$, and $\boldsymbol{c}^*$ is minimum point of G. Equation (3.4) has an unique solution is proved as follows.

Theorem 3.2. *If L is invertible, Eq (3.3) has only one solution.*

Proof. The homogeneous linear equation of Eq (3.4) is

$$\sum_{k=1}^{n}c_k\langle LJ^3g_k, LJ^3g_j\rangle_{L^2}+2\sum_{k=1}^{n}c_kJ^3g_kJ^3g_j=0.$$

Just prove that the above equation has an unique solution. Let $c_j(j=1,2,\cdots,n)$ multiply to both sides of the equation, and add all equations together so that

$$\langle\sum_{k=1}^{n}c_kLJ^3g_k, \sum_{j=1}^{n}c_jLJ^3g_j\rangle_{L^2}+2\sum_{k=1}^{n}c_kJ^3g_k\sum_{j=1}^{n}c_jJ^3g_j=0.$$

That is

$$\|\sum_{k=1}^{n}c_kLJ^3g_k\|_{L^2}+2(\sum_{k=1}^{n}c_kJ^3g_k)^2=0.$$

Clearly,

$$\|\sum_{k=1}^{n}c_kLJ^3g_k\|_{L^2}^2=0,\qquad(\sum_{k=1}^{n}c_kJ^3g_k)^2=0.$$

Because J^3g_k is orthonormal basis, if L is invertible, $c_k=0$. So Eq (3.3) has only one solution. □

4. Analysis of convergence and stability

Convergence and stability are important properties of algorithms. This section deals with the convergence and stability.

4.1. Analysis of convergence

In order to discuss the convergence, Theorem 4.1 is given as follows:

Theorem 4.1. *$J^3\psi_{ik}^l(x)$ is compactly supported in $[\frac{k}{2^{i-1}}, \frac{k+1}{2^{i-1}}]$.*

Proof. When

$$x < \frac{k}{2^{i-1}}, \quad J^3\psi_{ik}^l(x) = 0.$$

When $x > \frac{k+1}{2^{i-1}}$, because of $\psi_{ik}^l(x)$ with compact support, then,

$$\begin{aligned} J^3\psi_{ik}^l(x) &= \frac{1}{2}\int_0^x (x-t)^2\psi_{ik}^l(t)dt = \frac{1}{2}\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} (x-t)^2\psi_{ik}^l(t)dt \\ &= 2^{\frac{i-3}{2}}\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} (x-t)^2\psi^l(2^{i-1}t-k)dt \\ &= 2^{-\frac{5(i-1)}{2}}\int_0^1 (s-2^{i-1}x-k)^2\psi^l(s)ds, \quad s = 2^{i-1}t-k. \end{aligned} \tag{4.1}$$

According to Eq (2.2), $J^3\psi_{ik}^l(x) = 0$. So $J^3\psi_{ik}^l(x)$ has compactly support in $[\frac{k}{2^{i-1}}, \frac{k+1}{2^{i-1}}]$. □

Note

$$(J^3\psi_{i,k}^l(x))' = J^2\psi_{i,k}^l(x), \quad (J^3\psi_{i,k}^l(x))'' = J^1\psi_{i,k}^l(x).$$

By referring to the proof of Theorem 4.1, $J^1\psi_{ik}^l(x)$ and $J^2\psi_{ik}^l(x)$ are compactly supported in $[\frac{k}{2^{i-1}}, \frac{k+1}{2^{i-1}}]$.

The order of convergence will proceed below. Assume

$$u(x) = \sum_{j=0}^{2} c_j\frac{x^j}{j!} + \sum_{j=0}^{3} d_jJ^3\varphi^j(x) + \sum_{i=1}^{\infty}\sum_{k=0}^{2^{i-1}-1}\sum_{l=0}^{3}\left(c_{i,k}^{(l)}J^3\psi_{i,k}^l\right), \tag{4.2}$$

where

$$c_j =< u, \frac{x^j}{j!} >_{W_2^3}, \quad d_j =< u, \varphi^j(x) >_{W_2^3},$$

and

$$c_{i,k}^{(l)} =< u, \quad J^3\psi_{i,k}^l(x) >_{W_2^3} .$$

And

$$u_n(x) = \sum_{j=0}^{2} c_j\frac{x^j}{j!} + \sum_{j=0}^{3} d_jJ^3\varphi^j(x) + \sum_{i=1}^{n}\sum_{k=0}^{2^{i-1}-1}\sum_{l=0}^{3}\left(c_{i,k}^{(l)}J^3\psi_{i,k}^l\right).$$

Theorem 4.2. *Assume $u_n^\varepsilon(x)$ is the ε-approximate solution of Eq (3.1). If $u^{(m)}(x)$ is bounded in $[0,1]$, $m \in \mathbb{N}, 3 \le m \le 7$, then,*

$$|u(x) - u_n^\varepsilon(x)| \le 2^{-(m-2)n}M,$$

here M is a constant.

Proof. From Definition 3.1 and Theorem 3.1, we get

$$\begin{aligned}|u(x)-u_n^{\varepsilon}(x)| &\le M_0\|u-u_n^{\varepsilon}\|_{W_2^3} \le M_0\|L^{-1}\|\|L(u-u_n^{\varepsilon})\|_{L^2}\\ &\le M_0\|L^{-1}\|(\|L(u-u_n^{\varepsilon})\|_{L^2}+|B_1(u-u_n^{\varepsilon})|+|B_2(u-u_n^{\varepsilon})|)\\ &\le M_0\|L^{-1}\|(\|L(u-u_n)\|_{L^2}+\|B_1(u-u_n)\|+\|B_2(u-u_n)\|).\end{aligned}$$

Obviously,

$$\|B_1(u-u_n)\|=0,\quad \|B_2(u-u_n)\|=0.$$

That is

$$\begin{aligned}|u(x)-u_n^{\varepsilon}(x)| &\le M_0\|L^{-1}\|\|L(u-u_n)\|_{L^2} \le M_0\|L^{-1}\|(\int_0^1 (L(u-u_n))^2dx)^{\frac{1}{2}}\\ &\le M_0\|L^{-1}(\|\max_{x\in[0,1]}\{|L(u-u_n)|^2\})^{\frac{1}{2}}\\ &\le 3M_0\|L^{-1}\|M_1\max_{x\in[0,1]}\{|u-u_n|,|u'-u_n'|,|u''-u_n''|\},\end{aligned}$$

where

$$M_1=\max_{x\in[0,1]}\{1,|p(x)|,|q(x)|\}.$$

We know

$$|u-u_n|=|\sum_{i=n+1}^{\infty}\sum_{k=0}^{2^{i-1}-1}\sum_{l=0}^{3}c_{i,k}^{(l)}J^3\psi_{i,k}^l(x)|\le\sum_{i=n+1}^{\infty}\sum_{k=0}^{2^{i-1}-1}\sum_{l=0}^{3}|c_{i,k}^{(l)}||J^3\psi_{i,k}^l(x)|.$$

By the compactly support of $J^p\psi_{ik}^l(x),\ p=1,2,3$, fixed i, then $J^p\psi_{ik}^l(x)\neq 0$ only in $[\frac{k}{2^{i-1}},\frac{k+1}{2^{i-1}}]$,

$$|u-u_n|\le\sum_{i=n+1}^{\infty}\sum_{l=0}^{3}|c_{i,k}^{(l)}||J^3\psi_{i,k}^l(x)|.$$

Similarly,

$$|u'-u_n'|\le\sum_{i=n+1}^{\infty}\sum_{l=0}^{3}|c_{i,k}^{(l)}||J^2\psi_{i,k}^l(x)|$$

and

$$|u''-u_n''|\le\sum_{i=n+1}^{\infty}\sum_{l=0}^{3}|c_{i,k}^{(l)}||J^1\psi_{i,k}^l(x)|.$$

Through $J^1\psi_{i,k}^l(x)$, $J^2\psi_{i,k}^l(x)$ and $J^3\psi_{i,k}^l(x)$, you can get

$$|u(x)-u_n^{\varepsilon}(x)|\le 3M_0M_1\|L^{-1}\||u''-u_n''|.$$

As $|u''-u_n''|$, $|c_{i,k}^{(l)}|$ and $|J^1\psi_{i,k}^l(x)|$ will be discussed below. We can get that $|c_{i,k}^{(l)}|$ is related to $u^{(m)}(x)$. In fact,

$$|c_{i,k}^{(l)}|=|<u,J^3\psi_{i,k}^l(x)>_{W_2^3}|=|\int_0^1(u'''(x))\psi_{i,k}^l(x)dx|=|\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}}u'''(x)\psi_{i,k}^l(x)dx|. \tag{4.3}$$

Taylor's expansion of $u'''(x)$ at $\frac{k}{2^{i-1}}$ is

$$u'''(x)=\sum_{j=3}^{m-1}\frac{u^{(j)}(\frac{k}{2^{i-1}})}{(j-3)!}(x-\frac{k}{2^{i-1}})^{j-3}+\frac{u^{(m)}(\xi)}{(m-3)!}(x-\frac{k}{2^{i-1}})^{m-3},\quad \xi\in[\frac{k}{2^{i-1}},\frac{k+1}{2^{i-1}}].$$

Equation (4.3) is changed to

$$\begin{aligned}|c_{i,k}^{(l)}|&=\left|\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}}\left(\sum_{j=3}^{m-1}\frac{u^{(j)}(\frac{k}{2^{i-1}})}{(j-3)!}(x-\frac{k}{2^{i-1}})^{j-3}+\frac{u^{(m)}(\xi)}{(m-3)!}(x-\frac{k}{2^{i-1}})^{m-3}\right)\psi_{i,k}^{l}(x)dx\right|\\&=\left|\sum_{j=3}^{m-1}\frac{u^{(j)}(\frac{k}{2^{i-1}})}{(j-3)!}\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}}(x-\frac{k}{2^{i-1}})^{j-3}\psi_{ik}^{l}(x)dx\right|+\left|\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}}\frac{u^{(m)}(\xi)}{(m-3)!}(x-\frac{k}{2^{i-1}})^{m-3})\psi_{i,k}^{l}(x)dx\right|,\end{aligned}$$

where

$$\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}}(x-\frac{k}{2^{i-1}})^{j-3}\psi_{ik}^{l}(x)dx=2^{\frac{i-1}{2}}\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}}(x-\frac{k}{2^{i-1}})^{j-3}\psi^{l}(2^{i-1}x-k)dx\xlongequal{t=2^{i-1}x-k}2^{\frac{(-3-2j)(i-1)}{2}}\int_0^1(t)^{j-3}\psi^{l}(t)dt.$$

According to Eq (2.2),

$$\sum_{j=3}^{m-1}\frac{u^{(j)}(\frac{k}{2^{i-1}})}{(j-3)!}\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}}(x-\frac{k}{2^{i-1}})^{j-3}\psi_{ik}^{l}(x)dx=0,$$

so

$$\begin{aligned}|c_{i,k}^{(l)}|&=\left|\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}}\frac{u^{(m)}(\xi)}{(m-3)!}(x-\frac{k}{2^{i-1}})^{m-3}\psi_{i,k}^{l}(x)dx\right|\\&\leq\left|\frac{u^{(m)}(\xi)}{(m-3)!}\right|\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}}|x-\frac{k}{2^{i-1}}|^{m-3}|\psi_{i,k}^{l}(x)|dx\\&\leq\left|\frac{u^{(m)}(\xi)}{(m-3)!}\right|2^{-(m-3)(i-1)}\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}}|\psi_{i,k}^{l}(x)|dx\\&\leq\left|\frac{u^{(m)}(\xi)}{(m-3)!}\right|2^{-(m-3)(i-1)}2^{-\frac{i-1}{2}}.\end{aligned}$$

Because $u^{(m)}(x)$ is bounded,

$$|\frac{u^{(m)}(\xi)}{(m-3)!}|\leq M_3,$$

then,

$$|c_{i,k}^{(l)}|\leq 2^{-\frac{(2m-5)(i-1)}{2}}M_3. \tag{4.4}$$

By the compactly support of $J^1\psi_{ik}^{l}(x)$,

$$|J^1\psi_{i,k}^{l}(x)|=\left|\int_0^x\psi_{i,k}^{l}(t)dt\right|\leq\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}}|\psi_{i,k}^{l}(t)|dt\leq 2^{-\frac{i-1}{2}}.$$

According to the above analysis,

$$|u'' - u_n''| \leq \sum_{i=n+1}^{\infty} 4M_3 2^{-\frac{(2m-5)(i-1)}{2}} 2^{-\frac{(i-1)}{2}} = 4M_3 2^{-(m-2)n}.$$

That is

$$|u(x) - u_n^{\varepsilon}(x)| \leq 2^{-(m-2)n} M,$$

where M is a constant. □

4.2. Analysis of stability

Stability analysis is conducted below. According to the third section, the stability of the algorithm is related to the stability of Eq (3.4). By the following Property 4.1, the stability of the algorithm can be discussed by the number of conditions of the matrix $\boldsymbol{A}$.

Property 4.1. *If the matrix* $\boldsymbol{A}$ *is symmetric and reversible, then*

$$cond(A) = \left|\frac{\lambda_{max}}{\lambda_{min}}\right|,$$

where λ_{max} *and* λ_{min} *are the largest and smallest eigenvalues of* $\boldsymbol{A}$ *respectively.*

In this paper,

$$\boldsymbol{A_n} = \begin{pmatrix} a_{ij} \end{pmatrix}_{n\times n} = \begin{pmatrix} \langle LJ^3 g_i, LJ^3 g_j\rangle_{L^2} + 2J^3 g_i J^3 g_j \end{pmatrix}_{n\times n}.$$

Clearly, $\boldsymbol{A_n}$ is symmetric. From Theorem 3.2, $\boldsymbol{A_n}$ is reversible. In order to discuss the stability of the algorithm, only the eigenvalues of matrix $\boldsymbol{A_n}$ need to be discussed.

Theorem 4.3. *Assume* $u \in W_2^3$ *and* $\|u\|_{W_2^3} = 1$*. If L is an invertible differential operator, then,*

$$\|Lu\|_{L^2} \geq \frac{1}{\|L^{-1}\|}.$$

Proof. Since L is an invertible, assume $Lu = v$, then $u = L^{-1}v$. Moreover

$$1 = \|u\|_{W_2^3} = \|L^{-1}v\|_{W_2^3} \leq \|L^{-1}\|\|v\|_{L^2}.$$

Then,

$$\|v\|_{L^2} \geq \frac{1}{L^{-1}}.$$

That is,

$$\|Lu\|_{L^2} \geq \frac{1}{\|L^{-1}\|}.$$

□

Theorem 4.4. *Let* λ *be the eigenvalues of matrix* $\boldsymbol{A}$ *of Eq (3.4),* $\boldsymbol{x} = (x_1, \cdots, x_n)^T$ *is related eigenvalue of* λ *and* $\|\boldsymbol{x}\| = 1$*, then,*

$$\lambda \leq \|L\|^2 + 2.$$

Proof. By $\boldsymbol{Ax} = \lambda \boldsymbol{x}$,

$$
\begin{aligned}
\lambda x_i &= \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} (\langle LJ^3 g_i, LJ^3 g_j \rangle_{L^2} + 2J^3 g_i J^3 g_j) x_j \\
&= \langle LJ^3 g_i, \sum_{j=1}^{n} x_j LJ^3 g_j \rangle_{L^2} + 2J^3 g_i \sum_{j=1}^{n} (J^3 g_j x_j), \quad i = 1, \cdots, n.
\end{aligned} \tag{4.5}
$$

Let x_i multiply to both sides of Eq (4.5), and then add the equations from $j = 1$ to $j = n$ together so that

$$
\begin{aligned}
\lambda = \lambda x_i^2 &= \langle \sum_{i=1}^{n} x_i LJ^3 g_i, \sum_{j=1}^{n} x_j LJ^3 g_j \rangle_{L^2} + 2 \sum_{i=1}^{n} (J^3 g_i x_i) \sum_{j=1}^{n} (J^3 g_j x_j) \\
&= \Big\| \sum_{i=1}^{n} x_i LJ^3 g_i \Big\|_{L^2}^2 + 2(\sum_{i=1}^{n} (J^3 g_i x_i))^2 \\
&\leq \|L\|^2 \sum_{i=1}^{n} x_i^2 + 2 \sum_{i=1}^{n} x_i^2 \\
&= (\|L\|^2 + 2)\|\boldsymbol{x}\|.
\end{aligned} \tag{4.6}
$$

Since

$$
\|\boldsymbol{x}\| = 1, \quad \lambda \leq \|L\|^2 + 2.
$$

□

From Theorem 4.3 and Eq (4.6), we can get

$$
\lambda \geq \Big\| \sum_{i=1}^{n} x_i LJ^3 g_i \Big\|_{L^2} = \Big\| L(\sum_{i=1}^{n} x_i J^3 g_i) \Big\|_{L^2} \geq \frac{1}{L^{-1}}.
$$

Then,

$$
cond(\boldsymbol{A}) = \Big| \frac{\lambda_{max}}{\lambda_{min}} \Big| \leq \frac{\|L\|^2 + 2}{\frac{1}{\|L^{-1}\|}} = (\|L\|^2 + 2)\|L^{-1}\|.
$$

That is the condition number of $\boldsymbol{A}$ is bounded, so the presented method is stable.

5. Numerical examples

This section discusses numerical examples to reveal the accuracy of the proposed algorithm. Examples 5.1 and 5.3 are linear and nonlinear BVPs respectively. Example 5.2 shows that our method also applies to Eq (1.1) with other linear boundary value conditions. In this paper, N is the number of bases, and

$$
N = 7 + 4 * (2^n - 1), \quad n = 1, 2, \cdots .
$$

$e_N(x)$ is the absolute errors. *C.R.* and *cond* represent the convergence order and the condition number respectively. For convenience, we denote

$$
e_N(x) = |u(x) - u_N(x)|
$$

and

$$C.R. = \log_2 \frac{\max|e_N(x)|}{\max|e_{N+1}(x)|}.$$

Example 5.1. *Consider the test problem suggested in [28, 30]*

$$\begin{cases} u'' = u' + 2u + 4x - 2e^x, & x \in (0,1), \\ u(0) = 2, u(1) = e - 1, \end{cases}$$

where the exact solution is $u(x) = e^x - 2x + 1$. The numerical results are shown in Table 1. It is clear from Table 1 that the present method produces a converging solution for different values. In addition, the results of the proposed algorithm in Table 1 are compared with those in [28, 30]. Obviously, the proposed algorithm is better. Table 2 shows $e_N(x)$, C.R., cond and CPU time. The unit of CPU time is second, expressed as s.

Table 1. $e_N(x)$ of Example 5.1.

x	$e_N(x)$ of [30]	$e_{66}(x)$ of [28]	$e_{35}(x)$	$e_{67}(x)$
0	0	5.67e-9	9.94e-14	8.88e-16
0.1	1.19e-5	3.35e-9	2.04e-13	2.22e-16
0.2	4.18e-5	3.93e-10	2.16e-13	1.55e-15
0.3	4.96e-5	1.33e-9	1.42e-13	8.88e-16
0.4	6.04e-5	1.40e-9	1.68e-14	1.33e-15
0.5	6.33e-5	1.82e-9	1.82e-13	4.44e-16
0.6	6.23e-5	5.96e-9	1.52e-13	2.66e-15
0.7	5.76e-5	1.14e-8	1.66e-13	8.88e-16
0.8	4.23e-5	1.52e-8	4.36e-13	4.44e-15
0.9	2.15e-5	1.66e-8	4.93e-13	4.44e-16
1.0	0	1.90e-8	2.67e-13	4.44e-15

Table 2. $e_N(x)$, $C.R.$ and *cond* of Example 5.1.

n	N	$\max e_N(x)$	$C.R.$	*cond*	CPU(s)
1	11	1.66e-8		274.262	2.57
2	19	6.85e-11	7.92	274.262	7.89
3	35	6.55e-13	6.71	274.262	24.42
4	67	7.93e-15	6.40	274.262	82.73

Example 5.2. *Consider the problem suggested in [25,28].*

$$\begin{cases} u'' + u' + xu = f(x), & x \in (0,1), \\ u(0) = 2, \quad u(1) + u(\frac{1}{2}) = \sin\frac{1}{2} + \sin 1. \end{cases}$$

The exact solution is $u(x) = \sin x$, and $f(x) = \cos x - \sin x + x \sin x$. This problem is the boundary value problem with the multipoint boundary value conditions. Table 3 shows maximum absolute error ME_n, C.R. and cond., which compared with the other algorithms, the results obtained demonstrate that our algorithm is remarkably effective. The numerical errors are provided in Figures 1 and 2, also show a good accuracy.

Table 3. ME_n, C.R. and *cond* of Example 5.2.

	The present method				[25]				[28]		
n	ME_n	C.R.	*cond*	n	ME_n	C.R.	*cond*	n	ME_n	C.R.	*cond*
11	3.73e-9		195.05	10	6.34e-6	3.96	182.06	11	1.88e-4		
19	2.97e-11	6.97	195.05	18	4.04e-7	3.98	182.06	19	5.99e-5	1.65	1.49×10^{6}
35	2.13e-13	7.12	195.05	34	2.54e-8	4.06	182.06	35	1.84e-5	1.70	3.74×10^{8}
67	2.77e-15	6.27	195.05	66	1.59e-9	3.95	182.06	67	4.62e-6	1.99	8.87×10^{10}

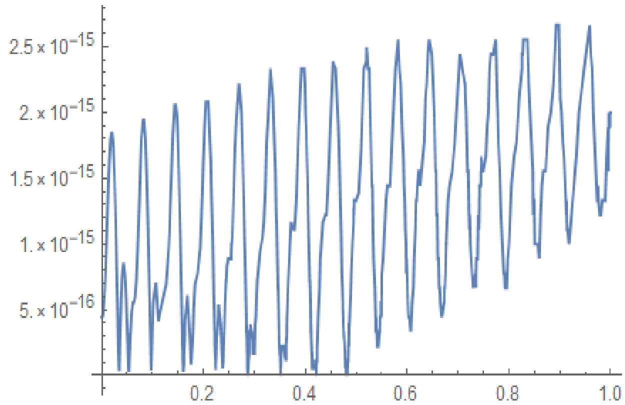

Figure 1. $e_N(x)$ of Example 5.3 (n=35).

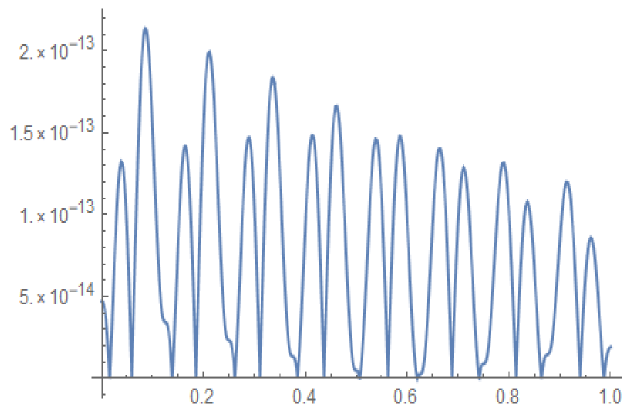

Figure 2. $e_N(x)$ of Example 5.3 (n=67).

Example 5.3. *Consider a nonlinear problem suggested in* [7, 9]

$$\begin{cases} u'' + \lambda e^u = 0, & x \in (0, 1) \\ u(0) = 0, u(1) = 0, \end{cases}$$

where

$$u(x) = -2ln(\cosh((x - \frac{1}{2})(\theta/2))/\cosh(\theta/4)),$$

and θ satisfies

$$\theta - \sqrt{2\lambda}\cosh(\theta/4) = 0.$$

This is the second-order nonlinear Bratu problem. Bratu equation is widely used in engineering fields, such as spark discharge, semiconductor manufacturing, etc. In the field of physics, the Bratu equation is used to describe the physical properties of microcrystalline silica gel solar energy. In the biological field, the Bratu equation is used to describe the kinetic model of some biochemical reactions in living organisms. To this problem, taking $u_0(x) = x(1-x)$, $k = 3$, where k is the number of iterations of the algorithm mentioned in [27]. when $\lambda = 1, \lambda = 2$, $e_N(x)$ are listed in Tables 4 and 5, respectively.

Table 4. $e_N(x)$ of Example 5.3 ($\lambda = 1$).

x	$e_N(x)$ of [8]	$e_N(x)$ of [9]	$e_N(x)$
0	0	0	$4.4959e-11$
0.2	$1.4958e-9$	$2.4390e-5$	$4.1096e-11$
0.4	$2.7218e-9$	$4.2096e-5$	$7.1502e-12$
0.6	$2.7218e-9$	$4.2096e-5$	$7.1483e-12$
0.8	$1.4958e-9$	$2.4390e-5$	$4.1104e-11$

Table 5. $e_N(x)$ of Example 5.3 ($\lambda = 2$).

x	$e_N(x)$ of [7]	$e_N(x)$ of [9]	$e_N(x)$
0	$5.8988e-26$	0	$1.1801e-12$
0.2	$1.3070e-7$	$6.9297e-5$	$2.5646e-10$
0.4	$1.4681e-7$	$1.0775e-4$	$1.6666e-9$
0.6	$1.4681e-7$	$1.0775e-4$	$1.6666e-9$
0.8	$1.3070e-7$	$6.9297e-5$	$2.5646e-10$

6. Conclusions

In this paper, based on Legendre's polynomials, we construct orthonormal basis in $L^2[0, 1]$ and $W_2^3[0, 1]$, respectively. It proves that this group of bases is orthonormal and compactly supported. According to the orthogonality of the basis, we present an algorithm to obtain the approximate solution of the boundary value problems. Using the compact support of the basis, we prove that the convergence order of the presented method related to the boundedness of $u^{(m)}(x)$. Finally, three numerical examples show that the absolute error and convergence order of the algorithm are better than other methods.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflicts of interest to declare.

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