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Research article

Extension of Chaplygin's existence and uniqueness method for fractal-fractional nonlinear differential equations

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Abstract: The existence and uniqueness of solutions to nonlinear ordinary differential equations with fractal-fractional derivatives, with Dirac-delta, exponential decay, power law, and generalized Mittag-Leffler kernels, have been the focus of this work. To do this, we used the Chaplygin approach, which entails creating two lower and upper sequences that converge to the solution of the equations under consideration. We have for each case provided the conditions under which these sequences are obtained and converge.

Keywords: fractal-fractional differentiation and integration; Chaplygin's method; existence and uniqueness
Mathematics Subject Classification: 34A08, 34A12

1. Introduction

Although they were just recently proposed, fractal-fractional differential and integral operators [1] with the Dirac-delta, exponential [2], power law [3], and generalized Mittag-Leffler kernels [4] have already caught the attention of several researchers working in both pure and applied mathematics [1–7]. This is mainly due to their unique property that one will recover the classical fractional differential operators when the fractal dimension is set to 1, but also when the fractional order is 1, we recover the fractal differential operator. In addition to this, one can view these differential operators as fractional differential operators within fractal geometry. However, when we treat both orders as one, we are able to obtain the classical derivative. Thus, one would expect that these differential operators will replicate more complex physical behaviors than their corresponding fractional derivatives. The fractal derivative [7] follows the same rules. Therefore, the defined fractal-fractional differential and integral operators can be included in the mathematical models properties like power law, fading memory, and crossover from stretched exponential to power law in fractal geometry. It is worth mentioning that the definition of fractal used here refers to a non-Newtonian generalization of the derivative that deals with the measurement of fractals, as described in fractal geometry, rather than a fractal sharp like the Julia set. This idea was developed to address anomalous diffusion, a problem where existing methods overlook the media's fractal character. The inspiration came from the fact that fractal characteristics are frequently seen in porous media, aquifers, turbulence and other types of media. Fractal media do not follow conventional diffusion or dispersion principles that are based on random travels in empty space [7]. Nonlocalities like power laws, memory loss, and crossover from stretched exponential to power laws are considered via fractional formulations. Within the confines of theory and applications, substantial findings have been obtained, but many more are still required. We will extend the Chaplygin sequential approach [8–12] in this study to show the existence and uniqueness of certain classes of nonlinear differential equations because the theory on existence and uniqueness is still being developed.

We now present some definitions for the fractional and fractal-fractional differential operators [1–4]. The Riemann-Liouville fractional derivative is defined by

$$ {}_{0}^{RL}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{0}^{t} f(\tau)(t-\tau)^{-\alpha} \, d\tau. \tag{1.1} $$

The corresponding integral is as follows [5]:

$$ {}_{0}^{RL}I_{t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} f(\tau)(t-\tau)^{\alpha-1} \, d\tau. \tag{1.2} $$

The Caputo fractional derivative [3] is as follows:

$$ {}_{0}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{d}{d\tau} f(\tau)(t-\tau)^{-\alpha} \, d\tau, \tag{1.3} $$

and the corresponding integral is defined by

$$ {}_{0}^{C}I_{t}^{\alpha} f(t) = f(0) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} f(\tau)(t-\tau)^{\alpha-1} \, d\tau. \tag{1.4} $$

The Caputo-Fabrizio fractional derivative [2] is defined by

$$ {}_{0}^{CF}D_{t}^{\alpha} f(t) = \frac{1}{1-\alpha} \int_{0}^{t} \frac{d}{d\tau} f(\tau) \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau, \tag{1.5} $$

and the Caputo-Fabrizio integral [2] is as follows:

$$ {}_{0}^{CF}I_{t}^{\alpha} f(t) = (1-\alpha) f(t) + \alpha \int_{0}^{t} f(\tau) \, d\tau. \tag{1.6} $$

The Atangana-Baleanu fractional derivative [6] is given as

$$ {}_{0}^{AB}D_{t}^{\alpha} f(t) = \frac{1}{1-\alpha} \int_{0}^{t} \frac{d}{d\tau} f(\tau) E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right] d\tau, \tag{1.7} $$

and the Atangana-Baleanu integral [6] is defined by

$$ {}_{0}^{AB}I_{t}^{\alpha} f(t) = (1-\alpha) f(t) + \frac{\alpha}{\Gamma(\alpha)} \int_{0}^{t} f(\tau)(t-\tau)^{\alpha-1} d\tau. \tag{1.8} $$

The definitions of fractal-fractional differentiation introduced in [1] will be now presented. A fractal-fractional derivative with the power-law kernel is given as

$$ {}_{0}^{FFP}D_{t}^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt^{\beta}} \int_{0}^{t} f(\tau)(t-\tau)^{-\alpha} d\tau, \tag{1.9} $$

where the fractal derivative [7] is as follows

$$ \frac{d}{dt^{\beta}} f(t) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{\beta} - t_1^{\beta}}. \tag{1.10} $$

where $\beta > 0$. A fractal-fractional derivative with the exponential decay kernel [1] is given as

$$ {}_{0}^{FFE}D_{t}^{\alpha,\beta} f(t) = \frac{1}{1-\alpha} \frac{d}{dt^{\beta}} \int_{0}^{t} f(\tau) \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau. \tag{1.11} $$

A fractal-fractional derivative with the Mittag-Leffler kernel [1] is given as

$$ {}_{0}^{FFM}D_{t}^{\alpha,\beta} f(t) = \frac{1}{1-\alpha} \frac{d}{dt^{\beta}} \int_{0}^{t} f(\tau) E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right] d\tau. \tag{1.12} $$

Fractal-fractional integrals with the power law, exponential decay and the Mittag-Leffler kernels, respectively are presented as follows:

$$ {}_{0}^{FFP}I_{t}^{\alpha,\beta} f(t) = \frac{\beta}{\Gamma(\alpha)} \int_{0}^{t} \tau^{\beta-1} f(\tau)(t-\tau)^{\alpha-1} d\tau, \tag{1.13} $$

$$ {}_{0}^{FFE}I_{t}^{\alpha,\beta} f(t) = \beta(1-\alpha) t^{\beta-1} f(t) + \alpha\beta \int_{0}^{t} \tau^{\beta-1} f(\tau) d\tau, \tag{1.14} $$

$$ {}_{0}^{FFM}I_{t}^{\alpha,\beta} f(t) = \beta(1-\alpha) t^{\beta-1} f(t) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{0}^{t} \tau^{\beta-1} f(\tau)(t-\tau)^{\alpha-1} d\tau. \tag{1.15} $$

Remark. It is worth noting for those who are not familiar with the theory of the concept of fractal-fractional that, in terms of a derivative, we have fractional kernels, including power law, exponential decay, and the generalized Mittag-Leffler function. However, when dealing with a fractal-fractional integral, one will have $t^{\beta-1}$ in the case of the classical fractal integral and the fractal-fractional integral obtained from the exponential decay case. In the case of the power law and the generalized Mittag-Leffler kernels, we have the following kernel:

$$ l^{\beta-1}(t-l)^{\alpha-1}. \tag{1.16} $$

Indeed, one can then interpret a fractal-fractional derivative as a fractional derivative in fractal geometry.

2. Chaplygin method for classical fractal nonlinear differential equations

In this part, we will first look at a general nonlinear ordinary differential equation with the fractal derivative as the differential operator. This class has been discovered to be suitable for simulating a subset of diffusion and flow issues in complex porous media by using fractal geometry. The nonlinear equation under consideration here is as follows:

$$\begin{cases} {}_{t_0}^{F}D_t^{\beta} y(t) = f(t, y(t)) \quad \text{if } t \in (t_0, t_0 + a], \\ y(t_0) = y_0, \end{cases} \tag{2.1}$$

where $t \in [t_0, t_0 + a], |y - y_0| < b$ and we also assume that $|f(t, y(t))| < M, \forall\, t \in [t_0, t_0 + a]$.

We shall show the existence of the lower and upper Chaplygin sequences $\{u_n\}$ and $\{v_n\}$.

Theorem 1. Let $f(t, y(t)) \in C[R_0, R]$, where

$$R_0 = \{(t, y) \mid t_0 \le t \le t_0 + a, |y - y_0| < b\}. \tag{2.2}$$

We assume that

$$|f(t, y(t))| < M \text{ on } R_0, \tag{2.3}$$

and

$$\lambda = \min\left\{a, \left(\frac{b}{M}\right)^{\frac{1}{\beta}}\right\}. \tag{2.4}$$

It is assumed that f_y, f_{yy} exist and $f_{yy} > 0$ in R_0. We consider $u_0 = u_0(t)$ and $v_0 = v_0(t)$ as two differentiable functions on $[t_0, t_0 + \lambda]$ with $(t, u_0(t))$ and $(t, v_0(t)) \in R_0$ and

$$\begin{aligned} {}_{t_0}^{F}D_t^{\beta} u_0(t) &< f(t, u_0(t)), u_0(t_0) = y_0, \\ {}_{t_0}^{F}D_t^{\beta} v_0(t) &> f(t, v_0(t)), v_0(t_0) = y_0. \end{aligned} \tag{2.5}$$

Then, we can find a Chaplygin sequence $\{u_n(t), v_n(t)\}$ such that

$$\begin{aligned} u_n(t) &< u_{n+1}(t) < y(t) < v_{n+1}(t) < v_n(t), \forall t \in (t_0, t_0 + \lambda], \\ v_n(t_0) &= y(t_0) = u_n(t_0), \end{aligned} \tag{2.6}$$

where the function $y(t)$ is the solution of the following equation in $[t_0, t_0 + \lambda]$:

$$y(t) = y(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, y(\tau))\, d\tau. \tag{2.7}$$

Additionally, $u_n(t)$ and $v_n(t)$ tends to $y(t)$ on $[t_0, t_0 + \lambda], n \to \infty$. For an appropriate $\gamma > 0$, the following is written:

$$0 \le v_0(t) - u_0(t) \le \gamma. \tag{2.8}$$

Then, $\forall n$ fixed and $t \in [t_0, t_0 + \lambda]$, the following is obtained:

$$|v_n(t) - u_n(t)| < \frac{2\gamma}{2^{2^n}}. \tag{2.9}$$

Proof. From the hypothesis of Theorem 1, we have that

$$ {}_{t_0}^{F}D_t^{\beta}u_0(t) < f(t,u_0(t)), \forall t \in (t_0, t_0+\lambda]. \tag{2.10}$$

Then, we write

$$\begin{aligned} u_0(t) &< u_0(t_0) + \beta\int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_0(\tau))\,d\tau \\ &< y(t_0) + \beta\int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_0(\tau))\,d\tau \\ &< y(t_0) + \beta\int_{t_0}^{t} \tau^{\beta-1} f(\tau, y(\tau))\,d\tau \\ &< y(t). \end{aligned} \tag{2.11}$$

On the other hand, we have

$$ {}_{t_0}^{F}D_t^{\beta}v_0(t) > f(t,v_0(t)), \forall t \in (t_0, t_0+\lambda]. \tag{2.12}$$

Then, we write

$$\begin{aligned} v_0(t) &> v_0(t_0) + \beta\int_{t_0}^{t} \tau^{\beta-1} f(\tau, v_0(\tau))\,d\tau \\ &> y(t_0) + \beta\int_{t_0}^{t} \tau^{\beta-1} f(\tau, v_0(\tau))\,d\tau \\ &> y(t_0) + \beta\int_{t_0}^{t} \tau^{\beta-1} f(\tau, y(\tau))\,d\tau \\ &> y(t). \end{aligned} \tag{2.13}$$

We have now obtained that $\forall t \in (t_0, t_0+\lambda]$

$$u_0(t) < y(t) < v_0(t). \tag{2.14}$$

We now define the following functions

$$g_1(t,y;u_0,v_0) = f(t,u_0(t)) + f_y(t,u_0(t))(y-u_0(t)), \tag{2.15}$$

and

$$g_2(t,y;u_0,v_0) = f(t,u_0(t)) + \frac{f(t,u_0(t)) - f(t,v_0(t))}{u_0(t)-v_0(t)}(y-u_0(t)). \tag{2.16}$$

Note that, when $t = t_0$

$$\begin{aligned} g_1(t_0, y; u_0, v_0) &= f(t_0, u_0(t_0)) + f_y(t, u_0(t_0))(y(t_0) - y(t_0)) \\ &= f(t_0, u_0(t_0)) = f(t_0, y_0), \end{aligned} \tag{2.17}$$

and

$$g_2(t_0, y; u_0, v_0) = f(t_0, u_0(t_0)) = f(t_0, y_0). \tag{2.18}$$

Replacing in g_1, y by u_1 and v_1 in g_2 respectively, we get

$$\begin{aligned} {}_{t_0}^{F}D_t^{\beta} u_1(t) &= g_1(t_0, u_1(t); u_0, v_0), u_1(t_0) = y_0, \\ {}_{t_0}^{F}D_t^{\beta} v_1(t) &= g_2(t_0, v_1(t); u_0, v_0), v_1(t_0) = y_0. \end{aligned} \tag{2.19}$$

The above exists on $[t_0, t_0 + \lambda]$. By the hypothesis of Theorem 1, we have that

$$\begin{aligned} u_0(t) &< u_0(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_0(\tau)) \, d\tau \\ &= y(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} f(t_0, u_0(\tau)) \, d\tau \\ &< y(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} g_1(\tau, u_0(\tau); u_0, v_0) \, d\tau \\ &< u_1(t), \forall t \in (t_0, t_0 + \lambda]. \end{aligned} \tag{2.20}$$

We have on the other hand,

$$\begin{aligned} v_0(t) &> y(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} f(t_0, u_0(\tau)) \, d\tau \\ &> y(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} g_2(\tau, v_0(\tau); u_0, v_0) \, d\tau \\ &> v_1(t), \forall t \in (t_0, t_0 + \lambda]. \end{aligned} \tag{2.21}$$

We now show that the defined $u_1(t)$ and $v_1(t)$ verified the differential inequality

$${}_{t_0}^{F}D_t^{\beta} u_1(t) = g_1(t_0, u_1(t); u_0, v_0), \forall t \in [t_0, t_0 + \lambda]; \tag{2.22}$$

then

$$u_1(t) < u_1(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} g_1(\tau, u_1(\tau); u_0, v_0) \, d\tau \tag{2.23}$$

$$
\begin{aligned}
&< u_1(t_0) + \beta \int\limits_{t_0}^{t} \tau^{\beta-1} f(\tau, u_1(\tau))\, d\tau \\
&< y(t_0) + \beta \int\limits_{t_0}^{t} \tau^{\beta-1} f(\tau, u_1(\tau))\, d\tau.
\end{aligned}
$$

Also, we have that

$$
\begin{aligned}
u_0(t) &< u_0(t_0) + \beta \int\limits_{t_0}^{t} \tau^{\beta-1} f(\tau, u_0(\tau))\, d\tau \\
&= u_0(t_0) + \beta \int\limits_{t_0}^{t} \tau^{\beta-1} g_2(\tau, u_0(\tau); u_0, v_0)\, d\tau.
\end{aligned} \tag{2.24}
$$

The above shows that

$$
u_0(t) < u_1(t), \forall t \in (t_0, t_0 + \lambda]. \tag{2.25}
$$

We have in addition that $\forall t \in [t_0, t_0 + \lambda]$

$$
f_y(t, u_0(t)) < \frac{f(t, u_0(t)) - f(t, v_0(t))}{u_0(t) - v_0(t)}, \tag{2.26}
$$

and

$$
\begin{aligned}
f(t, v_1(t)) &= f(t, u_0(t)) + f_{v_1}(t, u_0(t))(v_1(t) - u_0(t)) \\
&+ \frac{1}{2} f_{v_1 v_1}(t, \xi)(v_1(t) - u_0(t))^2; u_0(t) < \xi < v_1(t).
\end{aligned} \tag{2.27}
$$

By a repetition of the mean value theorem and $f_{v_1 v_1}(t, \xi) > 0$, we end up with

$$
\begin{aligned}
v_1(t) &= v_1(t_0) + \beta \int\limits_{t_0}^{t} \tau^{\beta-1} g_2(\tau, v_1(\tau); u_0, v_0)\, d\tau \\
&> v_1(t_0) + \beta \int\limits_{t_0}^{t} \tau^{\beta-1} f(\tau, v_1(\tau))\, d\tau \\
&= y(t_0) + \beta \int\limits_{t_0}^{t} \tau^{\beta-1} f(\tau, y(\tau))\, d\tau.
\end{aligned} \tag{2.28}
$$

Therefore, by maximal and minimal solutions, we have that $\forall t \in (t_0, t_0 + \lambda]$

$$
u_1(t) < y(t) < v_1(t). \tag{2.29}
$$

Therefore

$$
u_0(t) < u_1(t) < y(t) < v_1(t) < v_0(t), \forall t \in (t_0, t_0 + \lambda]. \tag{2.30}
$$

We then define a mapping Λ such that $\forall t \in [t_0, t_0 + \lambda]$

$$\Lambda(u_0(t), v_0(t)) = (u_1(t), v_1(t)).$$

By repetition, we have that

$$\begin{aligned} \Lambda(u_1(t), v_1(t)) &= (u_2(t), v_2(t)) \\ &\vdots \\ \Lambda(u_n(t), v_n(t)) &= (u_{n+1}(t), v_{n+1}(t)). \end{aligned} \tag{2.31}$$

These functions satisfy the following:

(1)

$$\begin{cases} u_n(t) < u_n(t_0) + \beta \int\limits_{t_0}^{t} \tau^{\beta-1} f(\tau, u_n(\tau))\, d\tau, \\ u_n(t_0) = y_0. \end{cases} \tag{2.32}$$

(2)

$$\begin{cases} v_n(t) > v_n(t_0) + \beta \int\limits_{t_0}^{t} \tau^{\beta-1} f(\tau, v_n(\tau))\, d\tau, \\ v_n(t_0) = y_0. \end{cases} \tag{2.33}$$

(3)

$$u_n(t) < u_{n+1}(t) < y(t) < v_{n+1}(t) < v_n(t), \forall t \in (t_0, t_0 + \lambda]. \tag{2.34}$$

(4)

$$u_{n+1}(t) = u_{n+1}(t_0) + \beta \int\limits_{t_0}^{t} \tau^{\beta-1} g_1(\tau, u_{n+1}(\tau); u_n(\tau), v_n(\tau))\, d\tau. \tag{2.35}$$

(5)

$$v_{n+1}(t) = v_{n+1}(t_0) + \beta \int\limits_{t_0}^{t} \tau^{\beta-1} g_2(\tau, v_{n+1}(\tau); u_n(\tau), v_n(\tau))\, d\tau. \tag{2.36}$$

Indeed, $\{u_n\}$ and $\{v_n\}$ are bounded in $[t_0, t_0 + \lambda]$ and monotonic. They are also equicontinuous since they constitute the solution of the linear equation. By the Arzela-Ascoli theorem, there exist two subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$ of $\{u_n\}$ and $\{v_n\}$ that converge uniformly toward $y(t)$. To continue, we shall let

$$\begin{aligned} \Omega_1 &= \sup_{\substack{u_0(t) \le y \le v_0(t) \\ t \in [t_0, t_0+\lambda]}} \left| f_y(t, y) \right|, \\ \Omega_2 &= \sup_{\substack{u_0(t) \le y \le v_0(t) \\ t \in [t_0, t_0+\lambda]}} \left| f_{yy}(t, y) \right|. \end{aligned} \tag{2.37}$$

We have that

$$
\begin{aligned}
v_0(t) &> v_0(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, v_0(\tau))\, d\tau, \qquad (2.38)\\
u_0(t) &< u_0(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_0(\tau))\, d\tau.
\end{aligned}
$$

Substracting these two inequalities yields

$$
\begin{aligned}
v_0(t) - u_0(t) &> \beta \int_{t_0}^{t} \tau^{\beta-1} \left[f(\tau, v_0(\tau)) - f(\tau, u_0(\tau)) \right] d\tau \qquad (2.39)\\
&\geq \varepsilon + \beta \int_{t_0}^{t} \tau^{\beta-1} \left[f(\tau, v_0(\tau)) - f(\tau, u_0(\tau)) \right] d\tau\\
&\geq \varepsilon + \beta \int_{t_0}^{t} \tau^{\beta-1} f_y(\tau, \eta) \left[v_0(\tau) - u_0(\tau) \right] d\tau,
\end{aligned}
$$

where $u_0(\tau) < \eta < v_0(\tau), \forall t \in [t_0, t_0 + \lambda]$ and using the mean value theorem, the above is arranged as follows

$$
\begin{aligned}
v_0(t) - u_0(t) &\geq \varepsilon + \beta\Omega_1 \int_{t_0}^{t} \tau^{\beta-1} \left[v_0(\tau) - u_0(\tau) \right] d\tau \qquad (2.40)\\
&\geq \varepsilon \exp\left[\Omega_1 \left(t^{\beta} - t_0^{\beta} \right) \right].
\end{aligned}
$$

By the Gronwall inequality, we write

$$
v_0(t) - u_0(t) \geq \varepsilon \exp\left[\Omega_1 \lambda^{\beta} \right]. \qquad (2.41)
$$

We therefore assume that

$$
0 \leq v_0(t) - u_0(t) \leq \gamma.
$$

The assertion is correct when $n = 0$. We assume that for any fixed n, we have

$$
|v_n(t) - u_n(t)| \leq \frac{2\gamma}{2^{2^n}}. \qquad (2.42)
$$

We have the following from u_{n+1} and v_{n+1} and the mean value theorem

$$
v_{n+1}(t) - u_{n+1}(t) = \beta \int_{t_0}^{t} \tau^{\beta-1} \begin{bmatrix} \frac{f(\tau, u_n(\tau)) - f(\tau, v_n(\tau))}{u_n(\tau) - v_n(\tau)} \left(v_{n+1}(\tau) - u_n(\tau) \right) \\ -f_y(\tau, u_n(\tau)) \left(u_{n+1}(\tau) - u_n(\tau) \right) \end{bmatrix} d\tau \qquad (2.43)
$$

$$= \beta \int_{t_0}^{t} \tau^{\beta-1} \left[\begin{array}{c} f_y(\tau,\eta)(v_{n+1}(\tau) - u_{n+1}(\tau)) \\ + \left[f_y(\tau,\eta) - f_y(\tau, u_n(\tau)) \right] (u_{n+1}(\tau) - u_n(\tau)) \end{array} \right] d\tau.$$

Applying the absolute value on both sides leads to

$$|v_{n+1}(t) - u_{n+1}(t)| \leq \beta \int_{t_0}^{t} \tau^{\beta-1} \left[\begin{array}{c} \Omega_1 |v_{n+1}(\tau) - u_{n+1}(\tau)| \\ +\Omega_2 |\eta - u_n(\tau)| |u_{n+1}(\tau) - u_n(\tau)| \end{array} \right] d\tau. \tag{2.44}$$

But

$$\begin{aligned} |\eta - u_n(t)| &\leq |u_n(t) - v_n(t)|, \\ |u_{n+1}(t) - u_n(t)| &\leq |u_n(t) - v_n(t)|. \end{aligned} \tag{2.45}$$

Therefore

$$\begin{aligned} |v_{n+1}(t) - u_{n+1}(t)| &\leq \beta \int_{t_0}^{t} \tau^{\beta-1} \left[\Omega_1 |v_{n+1}(\tau) - u_{n+1}(\tau)| + \Omega_2 |u_n(\tau) - v_n(\tau)|^2 \right] d\tau \\ &\leq \beta\Omega_1 \int_{t_0}^{t} \tau^{\beta-1} |v_{n+1}(\tau) - u_{n+1}(\tau)| \, d\tau + \beta\Omega_2 \int_{t_0}^{t} \tau^{\beta-1} |u_n(\tau) - v_n(\tau)|^2 \, d\tau \\ &\leq \beta\Omega_1 \int_{t_0}^{t} \tau^{\beta-1} |v_{n+1}(\tau) - u_{n+1}(\tau)| \, d\tau + \beta\Omega_2 \int_{t_0}^{t} \tau^{\beta-1} \frac{2^2\gamma^2}{2^{2^{n+1}}} d\tau. \end{aligned} \tag{2.46}$$

By inductive hypothesis, therefore

$$\begin{aligned} |v_{n+1}(t) - u_{n+1}(t)| &\leq \beta\Omega_1 \int_{t_0}^{t} \tau^{\beta-1} |v_{n+1}(\tau) - u_{n+1}(\tau)| \, d\tau + \Omega_2 \frac{2^2\gamma^2}{2^{2^{n+1}}} \left(t^\beta - t_0^\beta \right) \\ &\leq \beta\Omega_1 \int_{t_0}^{t} \tau^{\beta-1} |v_{n+1}(\tau) - u_{n+1}(\tau)| \, d\tau + \Omega_2 \frac{2^2\gamma^2}{2^{2^{n+1}}} \lambda^\beta. \end{aligned} \tag{2.47}$$

By the Gronwall inequality, we obtain

$$|v_{n+1}(t) - u_{n+1}(t)| \leq \Omega_2 \frac{2^2\gamma^2}{2^{2^{n+1}}} \lambda^\beta \exp\left[\Omega_1 \lambda^\beta \right]. \tag{2.48}$$

By then choosing

$$\gamma = \left(2\Omega_2 \lambda^\beta \exp\left[\Omega_1 \lambda^\beta \right] \right)^{-1}, \tag{2.49}$$

we obtain

$$|u_{n+1}(t) - v_{n+1}(t)| \leq \frac{2\gamma}{2^{2^{n+1}}}, \tag{2.50}$$

which completes the proof. Therefore

$$\begin{aligned} |y(t) - u_n(t)| &\leq \frac{2\gamma}{2^{2^n}}, \\ |y(t) - v_n(t)| &\leq \frac{2\gamma}{2^{2^n}}. \end{aligned} \tag{2.51}$$

We now consider the fractal nonlinear equation and show the lower and upper Chaplygin sequences $\{u_n\}$ and $\{v_n\}$. This will be shown in Theorem 2.

Theorem 2. Let $f(t, y(t)) \in C[R_0, R]$, where

$$R_0 = \{(t, y) \mid t_0 \leq t \leq t_0 + a, |y - y_0| < b\}. \tag{2.52}$$

It is assumed that $f(t, y(t))$ is quasi-monotonically nondecreasing in $y, \forall t \in [t_0, t_0 + a]$; additionally, we assume that $\frac{\partial f}{\partial y}(t, y)$ exists and is continuous on R_0. Consider $u_0(t)$ as a continuous differentiable function on $[t_0, t_0 + \lambda]$ with

$$\lambda = \min\left\{a, \left(\frac{b}{M}\right)^{\frac{1}{\beta}}\right\}, \tag{2.53}$$

$(t, u_0(t)) \in R_0$ and

$${}_{t_0}^{F}D_t^{\beta} y(t) < f(t, u_0(t)), u_0(t_0) = y_0. \tag{2.54}$$

In addition, we consider

$$f(t, y) + f_y(t, y)(y - z) < f(t, z) \text{ if } y < z.$$

Then, there exists a Chaplygin sequence $\{u_n\}$ such that

$$\begin{aligned} u_n(t_0) &= y_0, \\ u_n(t) &< u_{n+1}(t) < y(t), \forall t \in (t_0, t_0 + \lambda], \end{aligned} \tag{2.55}$$

where

$$y(t) \leq y(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, y(\tau))\, d\tau, \tag{2.56}$$

on $[t_0, t_0 + \lambda]$ and

$$\lim_{n\to\infty} u_n(t) = y(t) \tag{2.57}$$

uniformly on $[t_0, t_0 + \lambda]$.

Proof. By assumption, we have that $\frac{\partial f}{\partial y}(t, y) > 0$ since f is quasi-monotone. Moreover,

$${}_{t_0}^{F}D_t^{\beta} u_0(t) < f(t, u_0(t)). \tag{2.58}$$

Then, we write

$$u_0(t) < u_0(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_0(\tau))\, d\tau \tag{2.59}$$

$$< \quad y(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_0(\tau))\, d\tau.$$

But, we have that

$$y(t) = y(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, y(\tau))\, d\tau. \tag{2.60}$$

It follows therefore, that

$$u_0(t) < y(t), \forall t \in (t_0, t_0 + \lambda]. \tag{2.61}$$

From the maximal and minimal solution principle, the associated linear equation given by

$${}_{t_0}^{F}D_t^{\beta} z(t) = f(t, u_0(t)) + \frac{\partial f}{\partial z}(t, u_0(t))(z - u_0(t)) = g(t, z; u_0(t)), z(t_0) = y_0. \tag{2.62}$$

Replacing z by u, we first have that $g(t, z; u_0(t))$ is quasi-monotone on y since $\frac{\partial f}{\partial z} > 0$. Thus, we write

$$\begin{aligned} u_1(t) &= u_1(t_0) + \beta \int_{t_0}^{t} \tau^{\beta-1} g(\tau, u_1; u_0(\tau))\, d\tau \\ &< y_0 + \beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_1(\tau))\, d\tau \\ &< y(t). \end{aligned} \tag{2.63}$$

On the other hand, we have that

$$u_0(t) < u_n(t). \tag{2.64}$$

Thus, we have established that

$$u_0(t) < u_1(t) < y(t). \tag{2.65}$$

We can now define a mapping

$$\begin{aligned} \Lambda[u_0(t)] &= u_1(t), \\ \Lambda[u_1(t)] &= u_2(t), \\ \Lambda[u_2(t)] &= u_3(t), \\ &\vdots \\ \Lambda[u_n(t)] &= u_{n+1}(t), \end{aligned} \tag{2.66}$$

where

$$u_0(t) < u_1(t) < u_2(t) < ... < u_{n+1}(t) < y(t). \tag{2.67}$$

A similar routine can be used to obtain the upper sequence of Chaplygin.

3. Chaplygin method for fractal-fractional nonlinear differential equations with the exponential kernel

In this section, we shall present a detailed analysis of an extended version of the Chaplygin method for nonlinear differential equations with a fractal-fractional derivative with the exponential decay kernel. The nonlinear equation under investigation here is as follows:

$$\begin{cases} {}^{FFE}_{t_0}D_t^{\alpha,\beta} y(t) = f(t, y(t)), & \text{if } t \in (t_0, t_0 + a], \\ y(t_0) = y_0, & \text{if } t = t_0. \end{cases} \tag{3.1}$$

Applying the corresponding integral yields

$$y(t) = (1-\alpha)\beta t^{\beta-1} f(t, y(t)) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, y(\tau))\, d\tau. \tag{3.2}$$

We can find λ. We wish to have that $t \in [t_0, t_0 + a]$ and $|y(t)| < b$; thus

$$\begin{aligned} |y(t)| &\leq (1-\alpha)\beta t^{\beta-1} |f(t, y(t))| \\ &\quad + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} |f(\tau, y(\tau))|\, d\tau \\ &\leq (1-\alpha)\beta t^{\beta} |f(t, y(t))| \\ &\quad + \alpha\beta M \int_{t_0}^{t} \tau^{\beta-1} d\tau \\ &\leq (1-\alpha)\beta a^{\beta} M + \alpha M \left(t^{\beta} - t_0^{\beta}\right) \\ &\leq (1-\alpha)\beta a^{\beta} M + \alpha M a^{\beta} < b. \end{aligned} \tag{3.3}$$

Then, we get

$$a < \left(\frac{b}{((1-\alpha)\beta + \alpha) M}\right)^{\frac{1}{\beta}}. \tag{3.4}$$

Therefore, we have

$$\lambda = \min\left\{a, \left(\frac{b}{((1-\alpha)\beta + \alpha) M}\right)^{\frac{1}{\beta}}\right\}. \tag{3.5}$$

On one hand, we have that

$${}^{FFE}_{t_0}D_t^{\alpha,\beta} u_0(t) < f(t, u_0(t)). \tag{3.6}$$

Thus, we write

$$u_0(t) < (1-\alpha)\beta t^{\beta-1} f(t, u_0(t)) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_0(\tau))\, d\tau \tag{3.7}$$

$$
\begin{aligned}
&< (1-\alpha)\beta t^{\beta-1} g_1(t, u_0(t); u_0, v_0) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} g_1(\tau, u_0(\tau); u_0, v_0)\, d\tau \\
&< u_1(t), \forall t \in (t_0, t_0+\lambda].
\end{aligned}
$$

On the other hand, we have

$$
{}_{t_0}^{FFE}D_t^{\alpha,\beta} v_0(t) > f(t, v_0(t)). \tag{3.8}
$$

Thus, we have

$$
\begin{aligned}
v_0(t) &> (1-\alpha)\beta t^{\beta-1} f(t, v_0(t)) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, v_0(\tau))\, d\tau \qquad (3.9)\\
&> (1-\alpha)\beta t^{\beta-1} g_2(t, v_0(t); u_0, v_0) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} g_1(\tau, v_0(\tau); u_0, v_0)\, d\tau \\
&> v_1(t), \forall t \in (t_0, t_0+\lambda].
\end{aligned}
$$

But also, we have to show that

$$
\begin{aligned}
{}_{t_0}^{FFE}D_t^{\alpha,\beta} u_1(t) &= g_1(t_0, u_1(t); u_0, v_0), u_1(t_0) = y_0, \qquad (3.10)\\
{}_{t_0}^{FFE}D_t^{\alpha,\beta} v_1(t) &= g_2(t_0, v_1(t); u_0, v_0), v_1(t_0) = y_0.
\end{aligned}
$$

Then, we have that

$$
\begin{aligned}
u_1(t) &= (1-\alpha)\beta t^{\beta-1} g_1(t, u_1(t); u_0, v_0) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} g_1(\tau, u_1(\tau); u_0, v_0)\, d\tau \qquad (3.11)\\
&< (1-\alpha)\beta t^{\beta-1} f(t, u_1(t)) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_1(\tau))\, d\tau.
\end{aligned}
$$

But also

$$
\begin{aligned}
u_0(t) &< (1-\alpha)\beta t^{\beta-1} f(t, u_0(t)) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_0(\tau))\, d\tau \qquad (3.12)\\
&= (1-\alpha)\beta t^{\beta-1} g_1(t, u_0(t); u_0, v_0) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} g_1(\tau, u_0(\tau); u_0, v_0)\, d\tau \\
&< u_1(t), \forall t \in (t_0, t_0+\lambda].
\end{aligned}
$$

On the other hand, we have that

$$
u_1(t) = (1-\alpha)\beta t^{\beta-1} g_1(t, u_1(t); u_0, v_0) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} g_1(\tau, u_1(\tau); u_0, v_0)\, d\tau. \tag{3.13}
$$

Notice that

$$f_y(t, u_0(t)) < \frac{f(t, u_0(t)) - f(t, v_0(t))}{u_0(t) - v_0(t)}. \tag{3.14}$$

Therefore, given the above, we have

$$\begin{aligned} u_1(t) &< (1-\alpha)\beta t^{\beta-1}\left\{f(t, u_0(t)) + \frac{f(t, u_0(t)) - f(t, v_0(t))}{u_0(t) - v_0(t)}(v_1(t) - u_0(t))\right\} \\ &\quad +\alpha\beta\int_{t_0}^{t}\tau^{\beta-1}\left\{f(\tau, u_0(\tau)) + \frac{f(\tau, u_0(\tau)) - f(\tau, v_0(\tau))}{u_0(\tau) - v_0(\tau)}(v_1(\tau) - u_0(\tau))\right\}d\tau \\ &< (1-\alpha)\beta t^{\beta-1}g_1(t, v_1(t); u_0, v_0) + \alpha\beta\int_{t_0}^{t}\tau^{\beta-1}g_1(\tau, v_1(\tau); u_0, v_0)\,d\tau \\ &< v_1(t), \forall t \in (t_0, t_0+\lambda], \end{aligned} \tag{3.15}$$

On the other hand, we have

$$v_1(t) = (1-\alpha)\beta t^{\beta-1}g_2(t, v_1(t); u_0, v_0) + \alpha\beta\int_{t_0}^{t}\tau^{\beta-1}g_2(\tau, v_1(\tau); u_0, v_0)\,d\tau. \tag{3.16}$$

Then, we write

$$\begin{aligned} v_1(t) &= (1-\alpha)\beta t^{\beta-1}\left\{f(t, u_0(t)) + \frac{f(t, u_0(t)) - f(t, v_0(t))}{u_0(t) - v_0(t)}(v_1(t) - u_0(t))\right\} \\ &\quad +\alpha\beta\int_{t_0}^{t}\tau^{\beta-1}\left\{f(\tau, u_0(\tau)) + \frac{f(\tau, u_0(\tau)) - f(\tau, v_0(\tau))}{u_0(\tau) - v_0(\tau)}(v_1(\tau) - u_0(\tau))\right\}d\tau. \end{aligned} \tag{3.17}$$

Using the mean value theorem and the monotonic property of $f_y(\cdot,\cdot)$ with respect to the second variable leads to

$$\begin{aligned} v_1(t) &> (1-\alpha)\beta t^{\beta-1}f(t, v_1(t)) + \alpha\beta\int_{t_0}^{t}\tau^{\beta-1}f(\tau, v_1(\tau))\,d\tau \\ &> (1-\alpha)\beta t^{\beta-1}f(t, y(t)) + \alpha\beta\int_{t_0}^{t}\tau^{\beta-1}f(\tau, y(\tau))\,d\tau \\ &= y(t). \end{aligned} \tag{3.18}$$

We have in general that

$$u_0(t) < u_1(t) < y(t) < v_1(t) < v_0(t), \forall t \in (t_0, t_0+\lambda]. \tag{3.19}$$

From here, as presented before, we can now have the transformation operator Λ such that

$$(u_{n+1}, v_{n+1}) = \Lambda(u_n, v_n)$$

for the function with the following relations

$$u_n(t) < (1-\alpha)\beta t^{\beta-1} f(t, u_n(t)) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_n(\tau))\, d\tau, u_n(t_0) = y_0, \tag{3.20}$$

$$v_n(t) > (1-\alpha)\beta t^{\beta-1} f(t, v_n(t)) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, v_n(\tau))\, d\tau, v_n(t_0) = y_0.$$

We have that

$$u_n(t) < u_{n+1}(t) < y(t) < v_{n+1}(t) < v_1(t), \forall t \in (t_0, t_0 + \lambda]. \tag{3.21}$$

Then, we write

$$\begin{aligned} u_{n+1}(t) &= (1-\alpha)\beta t^{\beta-1} g_1(t, u_{n+1}(t); u_n(t), v_n(t)) \\ &\quad + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} g_1(\tau, u_{n+1}(\tau); u_n(\tau), v_n(\tau))\, d\tau, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} v_{n+1}(t) &= (1-\alpha)\beta t^{\beta-1} g_2(t, v_{n+1}(t); u_n(t), v_n(t)) \\ &\quad + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} g_2(\tau, v_{n+1}(\tau); u_n(\tau), v_n(\tau))\, d\tau. \end{aligned} \tag{3.23}$$

Within $[t_0, t_0 + \lambda]$, u_{n+1} and v_{n+1} are monotonic, bounded uniformly and equicontinuous. For the second part, we will have Ω_1 and Ω_2 as before

$$v_0(t) > (1-\alpha)\beta t^{\beta-1} f(t, v_0(t)) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, v_0(\tau))\, d\tau, \tag{3.24}$$

$$u_0(t) < (1-\alpha)\beta t^{\beta-1} f(t, u_0(t)) + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} f(\tau, u_0(\tau))\, d\tau.$$

Therefore,

$$\begin{aligned} 0 &\leq v_0(t) - u_0(t) \\ &\leq (1-\alpha)\beta t^{\beta-1} \left(f(t, v_0(t)) - f(t, u_0(t)) \right) \\ &\quad + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} \left[f(\tau, v_0(\tau)) - f(\tau, u_0(\tau)) \right] d\tau. \end{aligned} \tag{3.25}$$

We shall use the differentiation of f and the mean value theorem to obtain

$$\left(f(t, v_0(t)) - f(t, u_0(t)) \right) = f_y(t, \xi) \left(v_0(t) - u_0(t) \right), \tag{3.26}$$

$$u_0(t) \;<\; \xi < v_0(t).$$

Therefore,

$$\begin{aligned}
0 &\geq v_0(t) - u_0(t) \\
&\geq (1-\alpha)\beta t^{\beta-1} f_y(t,\xi)(v_0(t) - u_0(t)) \\
&\quad + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} f_y(\tau,\xi)(v_0(\tau) - u_0(\tau))\,d\tau
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
&\geq (1-\alpha)\beta t^{\beta} \min_{t\in[t_0,t_0+\lambda]} \left|f_y(t,\xi)\right| (v_0(t) - u_0(t)) \\
&\quad + \alpha\beta \int_{t_0}^{t} \tau^{\beta-1} \min_{l\in[t_0,\tau]} \left|f_y(l,\xi)\right| (v_0(\tau) - u_0(\tau))\,d\tau \\
&\geq (1-\alpha)\beta a^{\beta} M_f (v_0(t) - u_0(t)) \\
&\quad + \alpha\beta M_{f_y} \int_{t_0}^{t} \tau^{\beta-1} (v_0(\tau) - u_0(\tau))\,d\tau.
\end{aligned}$$

Under the condition that

$$1 + (\alpha - 1)\beta a^{\beta} M_f > 0, \tag{3.28}$$

then

$$\begin{aligned}
v_0(t) - u_0(t) &\geq \frac{\xi}{1 + (\alpha-1)\beta a^{\beta} M_f} \\
&\quad + \frac{\alpha\beta\xi}{1 + (\alpha-1)\beta a^{\beta} M_f} \int_{t_0}^{t} \tau^{\beta-1} (v_0(\tau) - u_0(\tau))\,d\tau.
\end{aligned} \tag{3.29}$$

By the Gronwall inequality, we have

$$\begin{aligned}
v_0(t) - u_0(t) &\geq \frac{\xi}{1 + (\alpha-1)\beta a^{\beta} M_f} \exp\left[\frac{\alpha M_f}{1 + (\alpha-1)\beta a^{\beta} M_f}\left(t^{\beta} - t_0^{\beta}\right)\right] \\
&\geq \frac{\xi}{1 + (\alpha-1)\beta a^{\beta} M_f} \exp\left[\frac{\alpha M_f}{1 + (\alpha-1)\beta a^{\beta} M_f} a^{\beta}\right].
\end{aligned} \tag{3.30}$$

On the other hand, we assume that

$$v_0(t) - u_0(t) \leq \gamma. \tag{3.31}$$

Therefore, when $n = 0$, we have the inequality (3.31), we assume that such inequality (3.31) is true for any fixed n that is

$$|u_n(t) - v_n(t)| \leq \frac{2\gamma}{2^{2^n}}. \tag{3.32}$$

Then,

$$v_{n+1}(t)-u_{n+1}(t) = (1-\alpha)\beta t^{\beta-1}\left[\begin{array}{c}\frac{f(t,u_n(t))-f(t,v_n(t))}{u_n(t)-v_n(t)}(v_{n+1}(t)-u_n(t)) \\ -f_y(t,u_n(t))(u_{n+1}(t)-u_n(t))\end{array}\right] + \alpha\beta\int_{t_0}^{t}\tau^{\beta-1}\left[\begin{array}{c}\frac{f(\tau,u_n(\tau))-f(\tau,v_n(\tau))}{u_n(\tau)-v_n(\tau)}(v_{n+1}(\tau)-u_n(\tau)) \\ -f_y(\tau,u_n(\tau))(u_{n+1}(\tau)-u_n(\tau))\end{array}\right]d\tau. \tag{3.33}$$

Again, using the mean value theorem and the monotonic property of f_y and f_{yy}, we have

$$\begin{aligned}|v_{n+1}(t)-u_{n+1}(t)| &\le (1-\alpha)\beta t^{\beta-1}\left[\begin{array}{c}f_y(t,\xi)(v_{n+1}(t)-u_{n+1}(t)) \\ +\left[f_y(t,\xi)-f_y(t,u_n(t))\right](u_{n+1}(t)-u_n(t))\end{array}\right] \\ &\quad +\alpha\beta\int_{t_0}^{t}\tau^{\beta-1}\left[\begin{array}{c}f_y(\tau,\xi)(v_{n+1}(\tau)-u_{n+1}(\tau)) \\ +\left[f_y(\tau,\xi)-f_y(\tau,u_n(\tau))\right](u_{n+1}(\tau)-u_n(\tau))\end{array}\right]d\tau \\ &\le (1-\alpha)\beta t^{\beta-1}\left[\begin{array}{c}f_y(t,\xi)(v_{n+1}(t)-u_{n+1}(t)) \\ +f_{yy}(t,\eta)|\xi-u_n|(u_{n+1}(t)-u_n(t))\end{array}\right] \\ &\quad +\alpha\beta\int_{t_0}^{t}\tau^{\beta-1}\left[\begin{array}{c}f_y(\tau,\xi)(v_{n+1}(\tau)-u_{n+1}(\tau)) \\ +f_{yy}(\tau,\eta)|\xi-u_n|(u_{n+1}(\tau)-u_n(\tau))\end{array}\right]d\tau. \\ &\le (1-\alpha)\beta a^{\beta}\left[\begin{array}{c}\Omega_1|v_{n+1}(t)-u_{n+1}(t)| \\ +\Omega_2|v_n(t)-u_n(t)|^2\end{array}\right] \\ &\quad +\alpha\beta\int_{t_0}^{t}\tau^{\beta-1}\left[\begin{array}{c}\Omega_1|v_{n+1}(\tau)-u_{n+1}(\tau)| \\ +\Omega_2|u_n(\tau)-u_n(\tau)|^2\end{array}\right]d\tau.\end{aligned} \tag{3.34}$$

By the induction formula, we have

$$\begin{aligned}|v_{n+1}(t)-u_{n+1}(t)| &\le (1-\alpha)\beta a^{\beta}\left[\begin{array}{c}\Omega_1|v_{n+1}(t)-u_{n+1}(t)| \\ +\Omega_2\frac{2^2\gamma^2}{2^{2^{n+1}}}\end{array}\right] +\alpha\beta\int_{t_0}^{t}\tau^{\beta-1}\left[\begin{array}{c}\Omega_1|v_{n+1}(\tau)-u_{n+1}(\tau)| \\ +\Omega_2\frac{2^2\gamma^2}{2^{2^{n+1}}}\end{array}\right]d\tau \\ &\le (1-\alpha)\beta a^{\beta}\Omega_1|v_{n+1}(t)-u_{n+1}(t)|+(1-\alpha)\beta a^{\beta}\Omega_2\frac{2^2\gamma^2}{2^{2^{n+1}}} \\ &\quad +\alpha\beta\Omega_1\int_{t_0}^{t}\tau^{\beta-1}|v_{n+1}(\tau)-u_{n+1}(\tau)|d\tau+\alpha\beta\Omega_2\int_{t_0}^{t}\tau^{\beta-1}\Omega_2\frac{2^2\gamma^2}{2^{2^{n+1}}}d\tau\end{aligned} \tag{3.35}$$

$$\begin{aligned}
&\leq \ (1-\alpha)\beta a^{\beta}\Omega_1 \left|v_{n+1}(t)-u_{n+1}(t)\right| + (1-\alpha)\beta a^{\beta}\Omega_2 \frac{2^2\gamma^2}{2^{2^{n+1}}} \\
&\quad +\alpha\beta\Omega_1 \int_{t_0}^{t} \tau^{\beta-1} \left|v_{n+1}(\tau)-u_{n+1}(\tau)\right| d\tau + \alpha\Omega_2 \frac{2^2\gamma^2}{2^{2^{n+1}}} a^{\beta}
\end{aligned}$$

$$\begin{aligned}
&\leq \ \frac{\left((1-\alpha)\beta+\alpha\right) a^{\beta}\Omega_2 \frac{2^2\gamma^2}{2^{2^{n+1}}}}{1+(\alpha-1)\beta a^{\beta}\Omega_1} \\
&\quad + \frac{\alpha\beta\Omega_1}{1+(\alpha-1)\beta a^{\beta}\Omega_1} \int_{t_0}^{t} \tau^{\beta-1} \left|v_{n+1}(\tau)-u_{n+1}(\tau)\right| d\tau.
\end{aligned}$$

With the help of the Gronwall inequality, we obtain

$$\left|v_{n+1}(t)-u_{n+1}(t)\right| \leq \frac{\left((1-\alpha)\beta+\alpha\right) a^{\beta}\Omega_2 \frac{2^2\gamma^2}{2^{2^{n+1}}}}{1+(\alpha-1)\beta a^{\beta}\Omega_1} \exp\left[\frac{\alpha\Omega_1 a^{\beta}}{1+(\alpha-1)\beta a^{\beta}\Omega_1}\right]. \tag{3.36}$$

To obtain the expected inequality, we get to choose

$$\gamma = \left(\frac{2\Omega_2\left((1-\alpha)\beta+\alpha\right) a^{\beta}}{1+(\alpha-1)\beta a^{\beta}\Omega_1} \exp\left[\frac{\alpha\Omega_1 a^{\beta}}{1+(\alpha-1)\beta a^{\beta}\Omega_1}\right]\right)^{-1}. \tag{3.37}$$

Then

$$\left|u_{n+1}(t)-v_{n+1}(t)\right| \leq \frac{2\gamma}{2^{2^{n+1}}}, \tag{3.38}$$

which completes the proof. The conclusion can be reached as previously shown that the following equation has a unique solution:

$$\begin{cases} {}^{FFE}_{t_0}D_t^{\alpha,\beta} y(t) = f(t, y(t)), & \text{if } t \in (t_0, t_0+a], \\ y(t_0) = y_0, & \text{if } t = t_0. \end{cases} \tag{3.39}$$

4. Chaplygin method for fractal-fractional nonlinear differential equations with the power law kernel

In this section, we extend Chaplygin's method to derive conditions for its applicability to the general nonlinear differential equation with the fractal-fractional differential operator with the power-law kernel. The equation under investigation is given by

$$\begin{cases} {}^{FFP}_{t_0}D_t^{\alpha,\beta} y(t) = f(t, y(t)), & \text{if } t \in (t_0, t_0+a], \\ y(t_0) = y_0, & \text{if } t = t_0. \end{cases} \tag{4.1}$$

We assume that all hypothesis of the Theorem 1 is satisfied, however, we first determine λ in this case

$$|y(t)| \ \leq \ \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} \left|f(\tau, y(\tau))\right| d\tau \tag{4.2}$$

$$
\begin{aligned}
&\leq \frac{M\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} \, d\tau \\
&\leq \frac{M\beta}{\Gamma(\alpha)} t^{\alpha+\beta-1} \left[B(\beta,\alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right] \\
&\leq \frac{M\beta}{\Gamma(\alpha)} a^{\alpha+\beta} \sup_{t\in[t_0,t_0+a]} \left[B(\beta,\alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right] < b.
\end{aligned}
$$

Thus, we have

$$
\lambda = \min\left\{ a, \left(\frac{b\Gamma(\alpha)}{\beta M \sup_{t\in[t_0,t_0+a]} \left[B(\beta,\alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right]} \right)^{\frac{1}{\alpha+\beta}} \right\}. \tag{4.3}
$$

Following the procedure presented earlier $t \in (t_0, t_0 + \lambda]$; we have

$$
u_0(t) < y(t) < v_0(t). \tag{4.4}
$$

Then, by hypothesis, we have

$$
\begin{aligned}
u_0(t) &< \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, u_0(\tau)) \, d\tau, \qquad (4.5) \\
&< \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, y(\tau)) \, d\tau \\
&< y(t).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
v_0(t) &> \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, v_0(\tau)) \, d\tau, \qquad (4.6) \\
&> \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, y(\tau)) \, d\tau \\
&> y(t).
\end{aligned}
$$

The linear differential equations

$$
\begin{aligned}
{}_{t_0}^{FFP}D_t^{\alpha,\beta} u_1(t) &= g_1(t_0, u_1(t); u_0, v_0), \; u_1(t_0) = y_0, \qquad (4.7) \\
{}_{t_0}^{FFP}D_t^{\alpha,\beta} v_1(t) &= g_2(t_0, v_1(t); u_0, v_0), \; v_1(t_0) = y_0
\end{aligned}
$$

are considered; they exist due to the definitions of g_1 and g_2 that are based on f $u_0(t)$ and $v_0(t)$. From the inequality (4.5), we have that

$$
u_0(t) < \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, u_0(\tau)) \, d\tau \tag{4.8}
$$

$$
\begin{aligned}
&< \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} g_1 (\tau, u_0 (\tau) ; u_0, v_0) \, d\tau \\
&= u_1 (t), \forall t \in (t_0, t_0 + \lambda].
\end{aligned}
$$

Therefore, $\forall t \in (t_0, t_0 + \lambda]$

$$
u_0 (t) < u_1 (t). \tag{4.9}
$$

On the other hand, we have

$$
\begin{aligned}
v_0 (t) &> \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f (\tau, v_0 (\tau)) \, d\tau \qquad (4.10) \\
&> \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} g_2 (\tau, u_0 (\tau) ; u_0, v_0) \, d\tau \\
&= v_1 (t), \forall t \in (t_0, t_0 + \lambda].
\end{aligned}
$$

Therefore, $\forall t \in (t_0, t_0 + \lambda]$

$$
v_0 (t) > v_1 (t). \tag{4.11}
$$

Following the routine presented earlier, we established that $\forall t \in (t_0, t_0 + \lambda]$

$$
u_0 (t) < u_1 (t) < y (t) < v_1 (t) < v_0 (t). \tag{4.12}
$$

The mapping used before yields

$$
(u_{n+1}, v_{n+1}) = \Lambda (u_n, v_n)
$$

with

$$
\begin{aligned}
u_n (t) &< \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f (\tau, u_n (\tau)) \, d\tau, u_n (t_0) = y_0, \qquad (4.13) \\
v_n (t) &> \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f (\tau, v_n (\tau)) \, d\tau, v_n (t_0) = y_0.
\end{aligned}
$$

We have that

$$
u_n (t) < u_{n+1} (t) < y (t) < v_{n+1} (t) < v_n (t), \forall t \in (t_0, t_0 + \lambda]. \tag{4.14}
$$

Then, we write

$$
u_{n+1} (t) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} g_1 (\tau, u_{n+1} (\tau) ; u_n (\tau), v_n (\tau)) \, d\tau, \tag{4.15}
$$

and

$$
v_{n+1} (t) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} g_2 (\tau, v_{n+1} (\tau) ; u_n (\tau), v_n (\tau)) \, d\tau. \tag{4.16}
$$

These functions are also monotonic, equicontinuous and uniformly bounded. For the next part of the proof, we consider Ω_1 and Ω_2 as before; then, we evaluate the following:

$$\begin{aligned} v_0(t) - u_0(t) &> \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} \left[f(\tau, v_0(\tau)) - f(\tau, u_0(\tau)) \right] d\tau \\ &> \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f_y(\tau, \xi) \left[v_0(\tau) - u_0(\tau) \right] d\tau. \end{aligned} \tag{4.17}$$

We apply the mean value theorem and we have

$$u_0(t) < \xi < v_0(t), \forall t \in (t_0, t_0 + \lambda]. \tag{4.18}$$

Therefore,

$$\begin{aligned} v_0(t) - u_0(t) &> \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} \min_{l \in [t_0, \tau]} \left| f_y(l, \xi) \right| \left[v_0(\tau) - u_0(\tau) \right] d\tau \\ &> \frac{\beta}{\Gamma(\alpha)} M_f \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} \left[v_0(\tau) - u_0(\tau) \right] d\tau. \end{aligned} \tag{4.19}$$

By the Gronwall inequality, we have

$$v_0(t) - u_0(t) \leq \varepsilon \exp\left[\frac{\beta M_f t^{\alpha+\beta-1}}{\Gamma(\alpha)} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha \right) \right) \right]. \tag{4.20}$$

We assume that

$$v_0(t) - u_0(t) \leq \gamma. \tag{4.21}$$

We have the following for a fixed n,

$$|u_n(t) - v_n(t)| \leq \frac{2\gamma}{2^{2^n}}. \tag{4.22}$$

We apply

$$v_{n+1}(t) - u_{n+1}(t) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} \begin{bmatrix} \frac{f(\tau, u_n(\tau)) - f(\tau, v_n(\tau))}{u_n(\tau) - v_n(\tau)} (v_{n+1}(\tau) - u_n(\tau)) \\ -f_y(\tau, u_n(\tau)) (u_{n+1}(\tau) - u_n(\tau)) \end{bmatrix} d\tau. \tag{4.23}$$

Following the routine presented before, we get

$$|v_{n+1}(t) - u_{n+1}(t)| \leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} \begin{bmatrix} f_y(\tau, \xi) (v_{n+1}(\tau) - u_{n+1}(\tau)) \\ + \left[f_y(\tau, \xi) - f_y(\tau, u_n(\tau)) \right] (u_{n+1}(\tau) - u_n(\tau)) \end{bmatrix} d\tau \tag{4.24}$$

$$\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} \begin{bmatrix} f_y(\tau,\xi)(v_{n+1}(\tau) - u_{n+1}(\tau)) \\ +f_{yy}(\tau,\eta)|\xi - u_n|(u_{n+1}(\tau) - u_n(\tau)) \end{bmatrix} d\tau$$

$$\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} \begin{bmatrix} \Omega_1 |v_{n+1}(\tau) - u_{n+1}(\tau)| \\ +\Omega_2 |u_n(\tau) - u_n(\tau)|^2 \end{bmatrix} d\tau.$$

By the induction formula, we have

$$|v_{n+1}(t) - u_{n+1}(t)| \leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} \begin{bmatrix} \Omega_1 |v_{n+1}(\tau) - u_{n+1}(\tau)| \\ +\Omega_2 \frac{2^2\gamma^2}{2^{2^{n+1}}} \end{bmatrix} d\tau \tag{4.25}$$

$$\begin{aligned} &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} |v_{n+1}(\tau) - u_{n+1}(\tau)| d\tau + \frac{\beta}{\Gamma(\alpha)} \Omega_2 \frac{2^2\gamma^2}{2^{2^{n+1}}} t^{\alpha+\beta-1} \left(B(\beta,\alpha) - B\left(\frac{t_0}{t},\beta,\alpha\right)\right) \\ &\leq \frac{\beta\Omega_1}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} |v_{n+1}(\tau) - u_{n+1}(\tau)| d\tau + \frac{\beta\Omega_2}{\Gamma(\alpha)} \frac{2^2\gamma^2}{2^{2^{n+1}}} a^{\alpha+\beta} \sup_{t\in[t_0,t_0+\lambda]} \left(B(\beta,\alpha) - B\left(\frac{t_0}{t},\beta,\alpha\right)\right). \end{aligned}$$

By the Gronwall inequality, we get

$$\begin{aligned} |v_{n+1}(t) - u_{n+1}(t)| \leq\ & \frac{\beta\Omega_2}{\Gamma(\alpha)} \frac{2^2\gamma^2}{2^{2^{n+1}}} a^{\alpha+\beta} \sup_{t\in[t_0,t_0+\lambda]} \left(B(\beta,\alpha) - B\left(\frac{t_0}{t},\beta,\alpha\right)\right) \\ & \times \exp\left[\frac{\beta\Omega_1}{\Gamma(\alpha)} t^{\alpha+\beta-1} \left(B(\beta,\alpha) - B\left(\frac{t_0}{t},\beta,\alpha\right)\right)\right]. \end{aligned} \tag{4.26}$$

Then, we arrange the inequality (4.26) as

$$\begin{aligned} |v_{n+1}(t) - u_{n+1}(t)| \leq\ & \beta\Omega_2 \frac{2^2\gamma^2}{2^{2^{n+1}}} a^{\alpha+\beta} \sup_{t\in[t_0,t_0+\lambda]} \left(B(\beta,\alpha) - B\left(\frac{t_0}{t},\beta,\alpha\right)\right) \\ & \times \exp\left[\frac{\beta\Omega_1 a^{\alpha+\beta}}{\Gamma(\alpha)} \sup_{t\in[t_0,t_0+\lambda]} \left(B(\beta,\alpha) - B\left(\frac{t_0}{t},\beta,\alpha\right)\right)\right]. \end{aligned} \tag{4.27}$$

We choose

$$\gamma = \begin{pmatrix} \frac{2\beta\Omega_2 a^{\alpha+\beta}}{\Gamma(\alpha)} \sup_{t\in[t_0,t_0+\lambda]} \left(B(\beta,\alpha) - B\left(\frac{t_0}{t},\beta,\alpha\right)\right) \\ \times \exp\left[\frac{\beta\Omega_1 a^{\alpha+\beta}}{\Gamma(\alpha)} \sup_{t\in[t_0,t_0+\lambda]} \left(B(\beta,\alpha) - B\left(\frac{t_0}{t},\beta,\alpha\right)\right)\right] \end{pmatrix}^{-1} \tag{4.28}$$

such that

$$|u_{n+1}(t) - v_{n+1}(t)| \leq \frac{2\gamma}{2^{2^{n+1}}}, \tag{4.29}$$

which the completes the proof.

5. Chaplygin method for fractal-fractional nonlinear differential equations with the generalized Mittag-Leffler kernel

In this section, we shall present a detailed analysis of the extension of Chaplygin's method to derive conditions under which a general nonlinear ordinary differential equation has a unique solution. We assume that all hypothesis of the Theorem 1 is satisfied. The equation under investigation is given by

$$\begin{cases} {}_{t_0}^{FFM}D_t^{\alpha,\beta} y(t) = f(t, y(t)), & \text{if } t \in (t_0, t_0 + a] \\ y(t_0) = y_0, & \text{if } t = t_0. \end{cases} \tag{5.1}$$

Applying the corresponding integral yields

$$y(t) = (1-\alpha)\beta t^{\beta-1} f(t, y(t)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, y(\tau))\, d\tau. \tag{5.2}$$

Let us find λ. We wish to have that $t \in [t_0, t_0 + a]$ and $|y(t)| < b$; thus,

$$\begin{aligned} |y(t)| &\leq (1-\alpha)\beta t^{\beta-1} |f(t, y(t))| \\ &\quad + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} |f(\tau, y(\tau))|\, d\tau \\ &\leq (1-\alpha)\beta a^{\beta} M \\ &\quad + \frac{\alpha\beta M}{\Gamma(\alpha)} a^{\alpha+\beta} \sup_{t\in[t_0,t_0+a]} \left(B(\beta,\alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right)\right) < b \\ &\leq a^{\delta} \left\{ (1-\alpha)\beta M + \frac{\alpha\beta M}{\Gamma(\alpha)} \sup_{t\in[t_0,t_0+a]} \left(B(\beta,\alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right)\right)\right\}, \end{aligned} \tag{5.3}$$

where

$$a^{\delta} = \sup\left\{a^{\beta}, a^{\alpha+\beta}\right\} \tag{5.4}$$

implies that

$$a < \left(\frac{b\Gamma(\alpha)}{\beta M \left((1-\alpha) + \alpha \sup_{t\in[t_0,t_0+\lambda]} \left[B(\beta,\alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right)\right]\right)} \right)^{\frac{1}{\delta}}. \tag{5.5}$$

Therefore, we have

$$\lambda = \min\left\{ a, \left(\frac{b\Gamma(\alpha)}{\beta M \left((1-\alpha) + \alpha \sup_{t\in[t_0,t_0+\lambda]} \left[B(\beta,\alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right)\right]\right)} \right)^{\frac{1}{\delta}} \right\}. \tag{5.6}$$

From the hypothesis, we have that

$$u_0(t) < (1-\alpha)\beta t^{\beta-1} f(t, u_0(t)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, u_0(\tau))\, d\tau \tag{5.7}$$

$$
\begin{aligned}
&< (1-\alpha)\beta t^{\beta-1} f(t, y(t)) + \alpha\beta \int\limits_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, y(\tau))\, d\tau \\
&< y(t), \forall t \in (t_0, t_0+\lambda].
\end{aligned}
$$

Then, we get

$$u_0(t) < y(t), \forall t \in (t_0, t_0+\lambda]. \tag{5.8}$$

On the other hand, we have

$$
\begin{aligned}
v_0(t) &> (1-\alpha)\beta t^{\beta-1} f(t, v_0(t)) + \alpha\beta \int\limits_{t_0}^{t} \tau^{\beta-1} f(\tau, v_0(\tau))\, d\tau \qquad (5.9)\\
&> (1-\alpha)\beta t^{\beta-1} f(t, y(t)) + \alpha\beta \int\limits_{t_0}^{t} \tau^{\beta-1} f(\tau, y(\tau))\, d\tau \\
&> y(t), \forall t \in (t_0, t_0+\lambda].
\end{aligned}
$$

Thus, we get

$$v_0(t) > y(t), \forall t \in (t_0, t_0+\lambda]. \tag{5.10}$$

From the inequality (5.7), we have that

$$-u_0(t) > -(1-\alpha)\beta t^{\beta-1} f(t, u_0(t)) - \frac{\alpha\beta}{\Gamma(\alpha)} \int\limits_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, u_0(\tau))\, d\tau. \tag{5.11}$$

Therefore,

$$
\begin{aligned}
v_0(t) - u_0(t) &> (1-\alpha)\beta t^{\beta-1} \left[f(t, v_0(t)) - f(t, u_0(t)) \right] \\
&\quad + \frac{\alpha\beta}{\Gamma(\alpha)} \int\limits_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} \left[f(\tau, v_0(\tau)) - f(\tau, u_0(\tau)) \right] d\tau. \qquad (5.12)
\end{aligned}
$$

Using the mean value theorem, we can obtain

$$u_0(t) < \xi < v_0(t), \forall t \in (t_0, t_0+\lambda], \tag{5.13}$$

such that

$$f_y(t, \xi)(v_0(t) - u_0(t)) = f(t, v_0(t)) - f(t, u_0(t)). \tag{5.14}$$

Putting (5.14) into (5.12) yields

$$
\begin{aligned}
v_0(t) - u_0(t) &> (1-\alpha)\beta t^{\beta-1} f_y(t, \xi) \left[v_0(t) - u_0(t) \right] \\
&\quad + \frac{\alpha\beta}{\Gamma(\alpha)} \int\limits_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f_y(\tau, \xi) (v_0(\tau) - u_0(\tau))\, d\tau, \qquad (5.15)
\end{aligned}
$$

$$\begin{aligned}
&> (1-\alpha)\beta t_0^{\beta-1} \min_{t\in[t_0,t_0+\lambda]} \left\{f_y(t,\xi)\right\}[v_0(t)-u_0(t)] \\
&\quad +\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}\min_{l\in[t_0,\tau]}\left|f_y(l,\xi)\right|[v_0(\tau)-u_0(\tau)]\,d\tau \\
&> (1-\alpha)\beta\overline{M}_{f_y}[v_0(t)-u_0(t)]+\frac{\alpha\beta\overline{M}_{f_y}}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}[v_0(\tau)-u_0(\tau)]\,d\tau.
\end{aligned}$$

Under the condition that

$$1+(\alpha-1)\beta\overline{M}_{f_y}\geq 0, \tag{5.16}$$

then

$$\begin{aligned}
v_0(t)-u_0(t) &\geq \frac{\varepsilon}{1+(\alpha-1)\beta\overline{M}_{f_y}}+ \\
&\quad +\frac{\alpha\beta}{\left(1+(\alpha-1)\beta\overline{M}_{f_y}\right)\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}(v_0(\tau)-u_0(\tau))\,d\tau. \qquad (5.17)
\end{aligned}$$

$$\geq \frac{\varepsilon}{1+(\alpha-1)\beta\overline{M}_{f_y}}\exp\left[\frac{\alpha\beta}{\left(1+(\alpha-1)\beta\overline{M}_{f_y}\right)\Gamma(\alpha)}a^{\alpha+\beta}\left(B(\beta,\alpha)-B\left(\frac{t_0}{t},\beta,\alpha\right)\right)\right].$$

This shows that $\forall t\in(t_0,t_0+\lambda]$

$$u_0(t)<y(t)<v_0(t). \tag{5.18}$$

The functions $g_1(t,z;u_0,v_0)$ and $g_2(t,z;u_0,v_0)$ are defined as before. Notice that

$$g_1(t,u_1(t);u_0,v_0)=f(t,u_0(t))+f_{u_1}(t,u_0(t))(u_1(t)-u_0(t)),\forall t\in[t_0,t_0+\lambda], \tag{5.19}$$

where $f(t,u_0(t))$ exists and also by hypothesis $f_{u_1}(t,u_0(t))$ exists on R_0. Therefore, we can conclude that $g_1(t,u_1(t);u_0,v_0)$ exists on $[t_0,t_0+\lambda]$. The same holds for $g_2(t,v_1(t);u_0,v_0)$. We have by hypothesis that

$$\begin{aligned}
u_0(t) &< (1-\alpha)\beta t^{\beta-1}f(t,u_0(t))+\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}f(\tau,u_0(\tau))\,d\tau \qquad (5.20)\\
&< (1-\alpha)\beta t^{\beta-1}g_1(t,u_0(t);u_0,v_0)+\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}g_1(\tau,u_0(\tau);u_0,v_0)\,d\tau \\
&< (1-\alpha)\beta t^{\beta-1}g_1(t,u_1(t);u_0,v_0)+\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}g_1(\tau,u_1(\tau);u_0,v_0)\,d\tau \\
&< u_1(t),\forall t\in(t_0,t_0+\lambda].
\end{aligned}$$

Thus, we get

$$u_0(t) < u_1(t), \forall t \in (t_0, t_0 + \lambda].$$

On the other hand, we have that

$$\begin{aligned} u_0(t) &= (1-\alpha)\beta t^{\beta-1} g_1(t, u_1(t); u_0, v_0) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} g_1(\tau, u_1(\tau); u_0, v_0)\, d\tau \quad (5.21) \\ &< (1-\alpha)\beta t^{\beta-1} f(t, u_1(t)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, u_1(\tau))\, d\tau \\ &< (1-\alpha)\beta t^{\beta-1} f(t, y(t)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, y(\tau))\, d\tau \\ &< y(t), \forall t \in (t_0, t_0 + \lambda]. \end{aligned}$$

We have that $\forall t \in (t_0, t_0 + \lambda]$

$$u_0(t) < u_1(t) < y(t). \tag{5.22}$$

Similarly, we establish that $\forall t \in (t_0, t_0 + \lambda]$

$$y(t) < v_1(t) < v_0(t) \tag{5.23}$$

for $g_2(t_0, v_1(t); u_0, v_0)$. Therefore, also for this case, we have that, $\forall t \in (t_0, t_0 + \lambda]$

$$u_n(t) < u_1(t) < y(t) < v_1(t) < v_0(t). \tag{5.24}$$

Again the transformation for Λ, i.e.,

$$(u_{n+1}, v_{n+1}) = \Lambda(u_n, v_n),$$

helps us to obtain Chaplygin's sequences verifying

$$\begin{aligned} u_n(t) &< (1-\alpha)\beta t^{\beta-1} f(t, u_n(t)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, u_n(\tau))\, d\tau, u_n(t_0) = y_0, \quad (5.25) \\ v_n(t) &> (1-\alpha)\beta t^{\beta-1} f(t, v_n(t)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, v_n(\tau))\, d\tau, v_n(t_0) = y_0. \end{aligned}$$

We have that

$$u_n(t) < u_{n+1}(t) < y(t) < v_{n+1}(t) < v_n(t), \forall t \in (t_0, t_0 + \lambda]. \tag{5.26}$$

Then, we write

$$u_{n+1}(t) = (1-\alpha)\beta t^{\beta-1} g_1(t, u_{n+1}(t); u_n(t), v_n(t)) \tag{5.27}$$

$$+\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}g_1(\tau,u_{n+1}(\tau);u_n(\tau),v_n(\tau))\,d\tau,$$

and

$$\begin{aligned} v_{n+1}(t) &= (1-\alpha)\beta t^{\beta-1}g_2(t,v_{n+1}(t);u_n(t),v_n(t)) \\ &\quad +\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}g_2(\tau,v_{n+1}(\tau);u_n(\tau),v_n(\tau))\,d\tau. \end{aligned} \tag{5.28}$$

These functions are also monotonic, bounded uniformly and equicontinuous; therefore, they converge toward $y(t)$ as $n\to\infty$. For the second part of the Theorem 1, we have by hypothesis that

$$v_0(t)-u_0(t)\le\gamma. \tag{5.29}$$

Ω_1 and Ω_2 are the same as before. The assertion is correct for $n=0$. We assume for any fixed n that

$$|u_n(t)-v_n(t)|\le\frac{2\gamma}{2^{2^n}}. \tag{5.30}$$

Using the mean value theorem and following the routine presented earlier, we get

$$\begin{aligned} |v_{n+1}(t)-u_{n+1}(t)| &\le (1-\alpha)\beta t^{\beta-1}\begin{bmatrix}\Omega_1|v_{n+1}(t)-u_{n+1}(t)| \\ +\Omega_2\frac{2^2\gamma^2}{2^{2^{n+1}}}\end{bmatrix} \\ &\quad +\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}\begin{bmatrix}\Omega_1|v_{n+1}(\tau)-u_{n+1}(\tau)| \\ +\Omega_2\frac{2^2\gamma^2}{2^{2^{n+1}}}\end{bmatrix}d\tau \\ &\le (1-\alpha)\beta a^{\beta}\left[\Omega_1|v_{n+1}(t)-u_{n+1}(t)|+\Omega_2\frac{2^2\gamma^2}{2^{2^{n+1}}}\right] \\ &\quad +\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}\begin{bmatrix}\Omega_1|v_{n+1}(\tau)-u_{n+1}(\tau)| \\ +\Omega_2|u_n(\tau)-u_n(\tau)|^2\end{bmatrix}d\tau \\ &\le (1-\alpha)\beta a^{\beta}\Omega_1|v_{n+1}(t)-u_{n+1}(t)|+(1-\alpha)\beta a^{\beta}\Omega_2\frac{2^2\gamma^2}{2^{2^{n+1}}} \\ &\quad +\frac{\alpha\beta}{\Gamma(\alpha)}\Omega_1\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}|v_{n+1}(\tau)-u_{n+1}(\tau)|\,d\tau \\ &\quad +\frac{\alpha\beta}{\Gamma(\alpha)}\frac{2^2\gamma^2}{2^{2^{n+1}}}a^{\alpha+\beta}\Omega_1\sup_{t\in[t_0,t_0+a]}\left(B(\beta,\alpha)-B\left(\frac{t_0}{t},\beta,\alpha\right)\right). \end{aligned} \tag{5.31}$$

Rearranging (5.31), we get

$$|v_{n+1}(t)-u_{n+1}(t)| \le (1-\alpha)\beta a^{\beta}\Omega_1|v_{n+1}(t)-u_{n+1}(t)| \tag{5.32}$$

$$+\frac{\alpha\beta}{\Gamma(\alpha)}\Omega_1\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}|v_{n+1}(\tau)-u_{n+1}(\tau)|\,d\tau$$
$$+\frac{2^2\gamma^2}{2^{2^{n+1}}}\beta\left\{(1-\alpha)a^{\beta}\Omega_2+\frac{\alpha a^{\alpha+\beta}}{\Gamma(\alpha)}\Omega_1\sum_{t\in[t_0,t_0+a]}\left(B(\beta,\alpha)-B\left(\frac{t_0}{t},\beta,\alpha\right)\right)\right\}$$

$$\leq \frac{\beta\frac{2^2\gamma^2}{2^{2^{n+1}}}}{1+(\alpha-1)\beta a^{\beta}\Omega_1}\left\{(1-\alpha)a^{\beta}\Omega_2+\frac{\alpha a^{\alpha+\beta}}{\Gamma(\alpha)}\Omega_1\sup_{t\in[t_0,t_0+a]}\left(B(\beta,\alpha)-B\left(\frac{t_0}{t},\beta,\alpha\right)\right)\right\}$$
$$+\frac{\alpha\beta\Omega_1}{\Gamma(\alpha)(1+(\alpha-1)\beta a^{\beta}\Omega_1)}\int_{t_0}^{t}\tau^{\beta-1}|v_{n+1}(\tau)-u_{n+1}(\tau)|\,d\tau.$$

By the Gronwall inequality, we get

$$|v_{n+1}(t)-u_{n+1}(t)| \leq \frac{\beta\frac{2^2\gamma^2}{2^{2^{n+1}}}}{1+(\alpha-1)\beta a^{\beta}\Omega_1}\left\{\begin{array}{c}(1-\alpha)a^{\beta}\Omega_2+\frac{\alpha a^{\alpha+\beta}}{\Gamma(\alpha)}\Omega_1\\ \times\sup_{t\in[t_0,t_0+a]}\left(B(\beta,\alpha)-B\left(\frac{t_0}{t},\beta,\alpha\right)\right)\end{array}\right\}$$
$$\times\exp\left[\frac{\alpha\beta\Omega_1 a^{\alpha+\beta}}{\Gamma(\alpha)(1+(\alpha-1)\beta a^{\beta}\Omega_1)}\sup_{t\in[t_0,t_0+a]}\left(B(\beta,\alpha)-B\left(\frac{t_0}{t},\beta,\alpha\right)\right)\right]. \tag{5.33}$$

We then choose

$$\gamma=\left(\begin{array}{c}\frac{2\beta}{1+(\alpha-1)\beta a^{\beta}\Omega_1}\left\{\begin{array}{c}(1-\alpha)a^{\beta}\Omega_2+\frac{\alpha a^{\alpha+\beta}}{\Gamma(\alpha)}\Omega_1\\ \times\sup_{t\in[t_0,t_0+a]}\left(B(\beta,\alpha)-B\left(\frac{t_0}{t},\beta,\alpha\right)\right)\end{array}\right\}\\ \times\exp\left[\frac{\alpha\beta\Omega_1 a^{\alpha+\beta}}{\Gamma(\alpha)\left(1+(\alpha-1)\beta a^{\beta}\Omega_1\right)}\sup_{t\in[t_0,t_0+a]}\left(B(\beta,\alpha)-B\left(\frac{t_0}{t},\beta,\alpha\right)\right)\right]\end{array}\right)^{-1}, \tag{5.34}$$

such that

$$|u_{n+1}(t)-v_{n+1}(t)|\leq\frac{2\gamma}{2^{2^n}}. \tag{5.35}$$

Therefore, the assertion is correct for $n\geq 0$; thus, we have

$$\begin{aligned}|y(t)-u_n(t)| &\leq \frac{2\gamma}{2^{2^n}},\\ |y(t)-v_n(t)| &\leq \frac{2\gamma}{2^{2^n}}.\end{aligned} \tag{5.36}$$

This completes the proof.

6. Conclusions

This study has incorporated Chaplygin's method for fractal-fractional nonlinear ordinary differential equations, which is an existence and uniqueness method that involves creating lower and uppersequences that converge toward the unique solution of a nonlinear differential equation. The case with the Dirac-delta, exponential, power law, and generalized Mittag-Leffler kernels. There were four categories that were taken into consideration.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Abdon Atangana is an editorial board member for AIMS Mathematics and was not involved in the editorial review or the decision to publish this article.

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