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Research article

The uniqueness of meromorphic function shared values with meromorphic solutions of a class of q-difference equations

Zhuo Wang[1,*] **and Weichuan Lin**[2]

[1] School of Mathematics, Renmin University of China, Beijing 100872, China
[2] Fujian Preschool Education College, Fuzhou 350007, China

* **Correspondence:** Email: zhuowangmaths@163.com.

Abstract: We first investigate the meromorphic solutions of a class of homogeneous second-order q-difference equations and the uniqueness problem for a meromorphic function with three shared values; then we discuss the uniqueness problem for the meromorphic solutions of a class of nonhomogeneous q-difference equations and a meromorphic function with four shared values.

Keywords: difference equations; meromorphic function; shared values; uniqueness
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1. Introduction

Throughout this paper, a meromorphic function will always mean meromorphic in the whole complex plane. In what follows, we assume that the reader is familiar with the fundamental concepts of Nevanlinna's value distribution theory [5, 8, 10].

Let f and g be meromorphic functions and a be a complex number. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share a counting multiplicity (CM). If $f - a$ and $g - a$ have the same zeros (ignoring multiplicity), then $f(z)$ and $g(z)$ share a IM. If $f(z)$ and $g(z)$ have the same poles (CM), then $f(z)$ and $g(z)$ share ∞ CM. In this paper, we suppose that $f(z)$ shares the value a partially with $g(z)$, and that $\overline{N}_{(m,n)}(r, a)$ denotes the reduced counting function of those zeros of $f(z) - a$ with multiplicity k, and of $g(z) - a$ with multiplicity l in $\{z : |z| < r\}$.

Definition 1. Let $f(z)$ be a meromorphic function in the complex plane. The order ρ and lower order μ of $f(z)$ are defined respectively by the order of $T(r, f)$, that is,

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}, \quad \mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}.$$

Definition 2. If the meromorphic function $a(z)(\not\equiv \infty)$ is satisfied, it follows that

$$T(r,a) = o(T(r,f)), r \to \infty, r \notin E,$$

where $E \subset [0,\infty)$ is a set of real numbers with finite measures, that is,

$$T(r,a) = S(r,f),$$

then, a is called a small function of $f(z)$.

Definition 3. Let $f(z)$ be meromorphic in the complex plane. If the order and the lower order of $f(z)$ are equal, then $f(z)$ is called a function with normal growth.

In recent years, with the research and development of the theory of difference equations, the value distribution and uniqueness of meromorphic solutions in complex domain difference equations (see [2, 3, 6]) has gradually became a hot research direction in the field of complex analysis. In 2017, Cui Ning and Chen Zongxuan [4] considered the problem that meromorphic solutions of a class of linear difference equations share values with arbitrary meromorphic function, and they obtained the following results.

Theorem 1.1. (see [4]) *Let $a_1(z)$, $a_0(z)$, and $F(z)$ be non-constant polynomials satisfying that $a_1(z) + a_0(z) \not\equiv 0$; $f(z)$ is the finite-order transcendental meromorphic solutions of the following difference equation:*

$$a_1(z)f(z+1) + a_0(z)f(z) = F(z).$$

If meromorphic functions $g(z)$ and $f(z)$ share 0, 1, ∞ CM, then one of the following situations must occur:

(i)$f(z) \equiv g(z)$;

(ii)$f(z) + g(z) \equiv f(z)g(z)$;

(iii) *There is a polynomial $\beta(z) = az + b_0$ and a constant a_0 satisfying that $e^{a_0} \neq e^{b_0}$; then, $f(z) = \dfrac{1-e^{\beta(z)}}{e^{\beta(z)}(e^{a_0-b_0}-1)}$ and $g(z) = \dfrac{1-e^{\beta(z)}}{1-e^{b_0-a_0}}$, where $a(\neq 0)$, b_0 are constants.*

Then, Yang Yin and Ye Yasheng [9] studied the problem for the solutions of q-difference equations with shared values for any meromorphic function.

Theorem 1.2. (see [9]) *Let $a_1(z)$, $a_0(z)$, and $F(z)$ be non-zero meromorphic functions whose order is less than 1; $f(z)$ is a finite transcendental meromorphic solution of the following difference equation:*

$$a_1(z)f(qz+p) + a_0(z)f(z) = F(z),$$

where p, q are constants, $n \in \mathbb{N}^+$, $q^n \neq \pm 1$, and $q \neq 0$. If $g(z)$ is any meromorphic function that shares 0, 1, ∞ CM with $f(z)$, then $f \equiv g$.

Based on the conclusion of the first-order q-difference equations of Theorems 1.1 and 1.2, we naturally consider the existence of the meromorphic solutions of the second-order q-difference equations with the meromorphic coefficients, as well as the uniqueness of the meromorphic solutions with any non-constant meromorphic function with shared values; it is a difficult and interesting problem. Therefore, we put forward the following question:

Question 1. *For the second order homogeneous q-difference equation:*

$$f(q^2z) + a_1(z)f(qz) + a_0(z)f(z) = 0, \tag{1.1}$$

with the meromorphic functions as coefficients. Are there uniqueness conclusions when meromorphic solutions share values with a non-constant meromorphic function?

The following result is obtained.

Theorem 1.3. *Let $a_1(z)$ and $a_0(z)$ be non-zero meromorphic functions whose order is less than 1; f is a transcendental meromorphic solution of finite order of (1.1), where $q \notin E$ and the set E satisfies that $E = \{q | A_4q^{4n} + A_3q^{3n} + A_2q^{2n} + A_1q^n + A_0 = 0,\ n \in \mathbf{N}^+,\ A_j(j = 0, 1, 2, 3, 4) \in \mathbb{Z}$ and $|A_4| \le 2,\ |A_3| \le 4,\ |A_2| \le 6,\ |A_1| \le 4,\ |A_0| \le 2\}$. If g is any non-constant meromorphic function that shares $0, 1, \infty$ CM with f, then $f \equiv g$.*

Furthermore, based on the above study of meromorphic solutions and meromorphic functions with shared values, we naturally have the following question:

Question 2. *Do meromorphic solutions and meromorphic functions of higher-order q-difference equations have the same uniqueness conclusions with shared values?*

In 1998, Bergweiler et al. (see [1]) studied the meromorphic solution existence of a class of non-homogeneous q-difference equations described by

$$\sum_{j=0}^{n} a_j(z)f(q^jz) = Q(z), \tag{1.2}$$

where $0 < |q| < 1$ is a complex number, $a_j(z)$, $j = 0, 1, \cdots, n$; $Q(z)$ denotes rational functions, and $a_0(z) \not\equiv 0$, $a_n(z) \equiv 1$.

For the study of Question 2, we obtain the following results.

Theorem 1.4. *Let f be a meromorphic solution of (1.2). If g is any non-constant meromorphic function that shares $0, 1, c\ (c \ne 0, 1)$ IM and ∞ CM with f, then $f \equiv g$.*

Remark 1.5. *The number of "3IM+1CM" shared values in Theorem 1.4 is accurate.*

Example 1.6. For the meromorphic solution of q-difference equation (1.2) to $f(z) = z$, for $g(z) = \dfrac{2z}{1+z^2}$, f and g share $1, -1$ IM and 0 CM, but $f(z) \not\equiv g(z)$.

2. Some lemmas

In order to prove our main results, we shall recall some lemmas as follows.

Lemma 2.1. (see [10], p. 65) *Let $h(z)$ be a non-constant entire function and $f(z) = e^{h(z)}$. Let ρ and μ be the order and the lower order of $f(z)$, respectively. We have the following:*

(i) *If $h(z)$ is a polynomial of degree p, then $\rho = \mu = p$.*

(ii) *If $h(z)$ is a transcendental entire function, then $\rho = \mu = \infty$.*

Lemma 2.2. (see [10], p. 75) *Suppose that $f_1(z)$, $f_2(z), \cdots, f_n(z)$ $(n \geq 2)$ are meromorphic functions and $g_1(z)$, $g_2(z), \cdots, g_n(z)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0.$

(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$.

(iii) *For* $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty, r \notin E).$$

Then $f_j(z) \equiv 0 (j = 1, 2, \cdots, n)$.

Lemma 2.3. (see [11], Theorem A) *Suppose that f_i $(i = 1, 2, \cdots, n, n+1, n+2, \cdots, n+m)$ represents meromorphic functions, where f_j $(j = 1, 2, \cdots, n)$ is not constant, $f_k \not\equiv 0$ $(k = n+1, n+2, \cdots, n+m)$, and*

$$\sum_{i=1}^{n+m} f_i \equiv 1.$$

If

$$\sum_{i=1}^{n+m} N(r, \frac{1}{f_i}) + (n+m-1)\sum_{i=1}^{n+m} \overline{N}(r, f_i) < (\lambda + o(1))T(r, f_j), \quad (r \in I, j = 1, 2, \cdots, n),$$

where $\lambda < 1$ and m is an arbitrary positive integer, then there is $t_i \in \{0, 1\}$ $(i = 1, 2, \cdots, m)$ such that

$$\sum_{i=1}^{m} t_i f_{n+i} \equiv 1.$$

Lemma 2.4. *Let*

$$f_1(z) = e^{l_1\alpha(q^2z)+l_2\alpha(qz)+l_3\alpha(z)+m_1\beta(q^2z)+m_2\beta(qz)+m_3\beta(z)},$$

$$f_2(z) = e^{u_1\alpha(q^2z)+u_2\alpha(qz)+u_3\alpha(z)+v_1\beta(q^2z)+v_2\beta(qz)+v_3\beta(z)},$$

where l_i, m_i, u_i, $v_i \in \{-1, 0, 1\}(i = 1, 2, 3)$, q is a non-zero constant, and $\alpha(z)$, $\beta(z)$ are polynomials with the degree $n(\geq 1)$. If $f_1(z)$, $f_2(z)$ are constants, then $q \in E$, where $E = \{q | A_4q^{4n} + A_3q^{3n} + A_2q^{2n} + A_1q^n + A_0 = 0,\ n \in \mathbf{N}^+,\ A_j(j = 0, 1, 2, 3, 4) \in \mathbb{Z}$ and $|A_4| \leq 2,\ |A_3| \leq 4,\ |A_2| \leq 6,\ |A_1| \leq 4,\ |A_0| \leq 2\}$.

Proof. If $f_1(z)$, $f_2(z)$ are constant functions, then the polynomials of $l_1\alpha(q^2z) + l_2\alpha(qz) + l_3\alpha(z) + m_1\beta(q^2z) + m_2\beta(qz) + m_3\beta(z)$, $u_1\alpha(q^2z) + u_2\alpha(qz) + u_3\alpha(z) + v_1\beta(q^2z) + v_2\beta(qz) + v_3\beta(z)$ have only constant terms; then,

$$a_n(l_1q^{2n} + l_2q^n + l_3) + b_n(m_1q^{2n} + m_2q^n + m_3) = 0,$$

$$a_n(u_1q^{2n} + u_2q^n + u_3) + b_n(v_1q^{2n} + v_2q^n + v_3) = 0,$$

where l_i, m_i, u_i, $v_i \in \{-1, 0, 1\}$.

As a result of $\alpha(z), \beta(z)$ being polynomials, that is, $a_n \neq 0$ and $b_n \neq 0$, it holds that

$$\frac{l_1q^{2n} + l_2q^n + l_3}{u_1q^{2n} + u_2q^n + u_3} = \frac{m_1q^{2n} + m_2q^n + m_3}{v_1q^{2n} + v_2q^n + v_3},$$

so we have

$$(l_1v_1 - m_1u_1)q^{4n} + (l_1v_2 + l_2v_1 - u_1m_2 - u_2m_1)q^{3n} + (l_1v_3 + l_2v_2 + l_3v_1 - u_1m_3 - u_2m_2 - u_3m_1)q^{2n} + (l_2v_3 + l_3v_2 - u_2m_3 - u_3m_2)q^n + (l_3v_3 - m_3u_3) = 0.$$

Namely

$$A_4q^{4n} + A_3q^{3n} + A_2q^{2n} + A_1q^n + A_0 = 0,$$

where A_j $(j = 0, 1, 2, 3, 4) \in \mathbb{Z}$ and $|A_4| \leq 2$, $|A_3| \leq 4$, $|A_2| \leq 6$, $|A_1| \leq 4$, $|A_0| \leq 2$.

This completes the proof of Lemma 2.4. □

Lemma 2.5. (see [10], p. 220) *Let f and g be non-constant meromorphic functions that share four distinct values a_j $(j = 1, 2, 3, 4)$ IM. If $f(z) \not\equiv g(z)$, then the following holds:*

(i) $T(r, f) = T(r, g) + O(\log r)$, $T(r, g) = T(r, f) + O(\log r)$;

(ii) $\sum_{j=1}^{4} \overline{N}(r, \frac{1}{f - a_j}) = 2T(r, f) + O(\log r)$;

(iii) $\overline{N}(r, \frac{1}{f - b}) = T(r, f) + O(\log r)$, $\overline{N}(r, \frac{1}{g - b}) = T(r, g) + O(\log r)$, *where* $b \neq a_j$ $(j = 1, 2, 3, 4)$;

(iv) $N_0(r, \frac{1}{f'}) = O(\log r)$, $N_0(r, \frac{1}{g'}) = O(\log r)$, *where $N_0(r, \frac{1}{f'})$ is the counting function of the zeros of f' but not the zeros of the $f - a_j$ $(j = 1, 2, 3, 4)$; the notation $N_0(r, \frac{1}{g'})$ can be similarly defined;*

(v) $\sum_{j=1}^{4} N^*(r, a_j) = O(\log r)$, *where $N^*(r, a_j)$ is the counting function of the multiple common zeros of $f - a_j$ and $g - a_j$, which counts multiplicities according to minor one.*

Lemma 2.6. (see [1], Theorems 1.1, 1.2) *Let f be a meromorphic solution of the following q-difference equation:*

$$\sum_{j=0}^{n} a_j(z)f(q^jz) = Q(z),$$

where $0 < |q| < 1$ is a complex number and $a_j(z)$, $j = 0, 1, \cdots, n$ and $Q(z)$ are rational functions with $a_0(z) \not\equiv 0$, $a_n(z) \equiv 1$. Then, the following holds:

(i) *All meromorphic solutions of the equation satisfy that* $T(r, f) = O(\log^2 r)$;

(ii) *All transcendental meromorphic solutions of the equation satisfy that* $\log^2 r = O(T(r, f))$.

Lemma 2.7. (see [10], p. 110) *Let f and g be non-constant rational functions. If f and g share distinct a_1, a_2, a_3 and a_4 IM, then $f \equiv g$.*

Lemma 2.8. (see [10], p. 30) *Suppose that $f(z)$ and $g(z)$ are two non-constant meromorphic functions in the complex plane with $\rho(f)$ and $\rho(g)$ as their orders, respectively. Then*

$$\rho(fg) \leq \max\{\rho(f), \rho(g)\},$$

and

$$\rho(f + g) \leq \max\{\rho(f), \rho(g)\}.$$

Lemma 2.9. (see [10], p. 28) *Let $f(z)$ be a non-constant meromorphic function in the complex plane and $R(f) = \frac{P(f)}{Q(f)}$, where*

$$P(f) = \sum_{k=0}^{p} a_k f^k \quad and \quad Q(f) = \sum_{j=0}^{q} a_j f^j$$

are two mutually prime polynomials in f. If the coefficients $\{a_k(z)\}$, $\{b_j(z)\}$ are small functions of f and $a_p(z) \not\equiv 0$, $b_q(z) \not\equiv 0$, then

$$T(r, R(f)) = \max\{p, q\}T(r, f) + S(r, f).$$

3. Proof of Theorem 1.3

The idea of proving this theorem is mainly derived from literature [4].

Since $f(z)$ and $g(z)$ share 0, 1, ∞ CM, by using the Nevanlinna second fundamental theorem, we have

$$\begin{aligned} T(r, g) &= \overline{N}(r, g) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{g-1}) + S(r, g) \\ &= \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-1}) + S(r, g) \\ &\leq 3T(r, f) + S(r, g). \end{aligned}$$

Similarly, we can get that $T(r, f) \leq 3T(r, g) + S(r, f)$.

By the definition of order, we arrive at

$$\rho(g) = \limsup_{r\to\infty} \frac{\log^+ T(r, g)}{\log r} \leq \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} = \rho(f).$$

Similarly, we have that $\rho(f) \leq \rho(g)$, that is $\rho(f) = \rho(g)$.

Since $f(z)$ is a finite-order meromorphic function, $g(z)$ is also a finite-order meromorphic function.

Suppose that

$$\frac{g(z)}{f(z)} = e^{\alpha(z)}, \qquad \frac{g(z)-1}{f(z)-1} = e^{\beta(z)}, \tag{3.1}$$

where $\alpha(z)$ and $\beta(z)$ are two polynomials.

If $e^{\alpha(z)} \equiv e^{\beta(z)}$, from (3.1), we have that $f(z) \equiv g(z)$.

Suppose, on the contrary, that $e^{\alpha(z)} \not\equiv e^{\beta(z)}$; then from (3.1) we have

$$f(z) = \frac{1 - e^{\beta(z)}}{e^{\alpha(z)} - e^{\beta(z)}}. \tag{3.2}$$

If $\alpha(z)$ and $\beta(z)$ are both constants, by (3.2), we have that $f(z)$ is a constant, which is a contradiction with $f(z)$ being a transcendental function.

Now, assume that at least one of the functions among $\alpha(z)$ and $\beta(z)$ is not a constant, and discuss it in three cases:

Case 1. Let $\alpha(z)$ be a constant and $\beta(z)$ be a non-constant polynomial. Sign $e^{\alpha} = c_1(\neq 0)$ is a constant; thus (3.2) can be rewritten as

$$f(z) = \frac{1 - e^{\beta(z)}}{c_1 - e^{\beta(z)}}. \tag{3.3}$$

If $c_1 = 1$, from (3.1), we have that $f(z) \equiv g(z)$.

If $c_1 \neq 1$, substituting (3.3) into (1.1), we have

$$\frac{1 - e^{\beta(q^2z)}}{c_1 - e^{\beta(q^2z)}} + a_1(z)\frac{1 - e^{\beta(qz)}}{c_1 - e^{\beta(qz)}} + a_0(z)\frac{1 - e^{\beta(z)}}{c_1 - e^{\beta(z)}} = 0,$$

therefore,

$$\begin{aligned}(1 - e^{\beta(q^2z)})(c_1 - e^{\beta(qz)})(c_1 - e^{\beta(z)}) + a_1(z)(1 - e^{\beta(qz)})(c_1 - e^{\beta(q^2z)})(c_1 - e^{\beta(z)})\\ +a_0(z)(1 - e^{\beta(z)})(c_1 - e^{\beta(q^2z)})(c_1 - e^{\beta(qz)}) = 0,\end{aligned}$$

hence,

$$\begin{aligned}-(1 + a_1(z) + a_0(z))e^{\beta(q^2z)+\beta(qz)+\beta(z)} + (c_1 + c_1a_1(z) + a_0(z))e^{\beta(q^2z)+\beta(qz)}\\ +(c_1 + a_1(z) + c_1a_0(z))e^{\beta(q^2z)+\beta(z)} + (1 + c_1a_1(z) + c_1a_0(z))e^{\beta(qz)+\beta(z)}\\ -({c_1}^2 + c_1a_1(z) + c_1a_0(z))e^{\beta(q^2z)} - (c_1 + {c_1}^2a_1(z) + c_1a_0(z))e^{\beta(qz)}\\ -(c_1 + c_1a_1(z) + {c_1}^2a_0(z))e^{\beta(z)} + {c_1}^2(1 + a_1(z) + a_0(z)) = 0.\end{aligned}$$

That is,

$$\begin{aligned}B_{18}(z)e^{\beta(q^2z)+\beta(qz)+\beta(z)} + B_{17}(z)e^{\beta(q^2z)+\beta(qz)} + B_{16}(z)e^{\beta(q^2z)+\beta(z)} + B_{15}(z)e^{\beta(qz)+\beta(z)}\\ +B_{14}(z)e^{\beta(q^2z)} + B_{13}(z)e^{\beta(qz)} + B_{12}(z)e^{\beta(z)} + B_{11}(z)e^{h_0(z)} \equiv 0,\end{aligned} \tag{3.4}$$

where $h_0(z) = 0$ and

$$\begin{cases} B_{18}(z) &= -1 - a_1(z) - a_0(z),\\ B_{17}(z) &= c_1 + c_1a_1(z) + a_0(z),\\ B_{16}(z) &= c_1 + a_1(z) + c_1a_0(z),\\ B_{15}(z) &= 1 + c_1a_1(z) + c_1a_0(z),\\ B_{14}(z) &= -{c_1}^2 - c_1a_1(z) - c_1a_0(z),\\ B_{13}(z) &= -c_1 - {c_1}^2a_1(z) - c_1a_0(z),\\ B_{12}(z) &= -c_1 - c_1a_1(z) - {c_1}^2a_0(z),\\ B_{11}(z) &= {c_1}^2 + {c_1}^2a_1(z) + {c_1}^2a_0(z).\end{cases}$$

Since we have $e^{\beta(z)}$ with normal growth, and by Lemma 2.1 we have that $\rho(e^{\beta(z)})$ is the degree of $\beta(z)$. Because $\beta(z)$ is a non-constant polynomial; hence, $\rho(e^{\beta(z)}) = \deg\beta(z) \geq 1$, which holds for other exponential function terms.

On the other hand, we have that $\rho(a_1(z)) < 1, \rho(a_0(z)) < 1$ and c_1 is a constant. So for $j = 1, 2, \cdots, 8$, we have

$$\begin{cases} T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(q^2z)+\beta(qz)+\beta(z)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(q^2z)+\beta(qz)-\beta(z)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(q^2z)+\beta(z)-\beta(qz)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(qz)+\beta(z)-\beta(q^2z)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(q^2z)-\beta(qz)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(q^2z)-\beta(z)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(qz)-\beta(z)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(q^2z)+\beta(qz)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(q^2z)+\beta(z)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(qz)+\beta(z)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(q^2z)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(qz)})\}, \\ T(r,B_{1j}(z)) &= o\{T(r,e^{\beta(z)})\}. \end{cases}$$

Thus, applying Lemma 2.2 to (3.4), we get that $B_{1j}(z)\equiv 0$ $(j=1,2,\cdots,8)$. So,

$$B_{18}(z)\equiv 0,\quad B_{15}(z)\equiv 0,$$

we have that $c_1=1$, which is a contradiction.

Case 2. $\beta(z)$ is a constant and $\alpha(z)$ is a non-constant polynomial. Let $e^{\beta}=c_2(\neq 0)$ be a constant; thus (3.2) can be rewritten as

$$f(z)=\frac{1-c_2}{e^{\alpha(z)}-c_2}. \tag{3.5}$$

If $c_2=1$, from (3.1), we have that $f(z)\equiv g(z)$.

If $c_2\neq 1$, substituting (3.5) into (1.1), we have

$$\frac{1}{e^{\alpha(q^2z)}-c_2}+a_1(z)\frac{1}{e^{\alpha(qz)}-c_2}+a_0(z)\frac{1}{e^{\alpha(z)}-c_2}=0,$$

therefore,

$$(e^{\alpha(qz)}-c_2)(e^{\alpha(z)}-c_2)+a_1(z)(e^{\alpha(q^2z)}-c_2)(e^{\alpha(z)}-c_2)+a_0(z)(e^{\alpha(q^2z)}-c_2)(e^{\alpha(qz)}-c_2)=0,$$

hence,

$$\begin{aligned} a_0(z)e^{\alpha(q^2z)+\alpha(qz)}+a_1(z)e^{\alpha(q^2z)+\alpha(z)}+e^{\alpha(qz)+\alpha(z)}-c_2(a_1(z)+a_0(z))e^{\alpha(q^2z)} \\ -c_2(a_0(z)+1)e^{\alpha(qz)}-c_2(1+a_1(z))e^{\alpha(z)}+{c_2}^2(1+a_1(z)+a_0(z))=0, \end{aligned}$$

that is,

$$\begin{aligned} B_{27}(z)e^{\alpha(q^2z)+\alpha(qz)}+B_{26}(z)e^{\alpha(q^2z)+\alpha(z)}+B_{25}(z)e^{\alpha(qz)+\alpha(z)}+B_{24}(z)e^{\alpha(q^2z)} \\ +B_{23}(z)e^{\alpha(qz)}+B_{22}(z)e^{\alpha(z)}+B_{21}(z)e^{h_0(z)}\equiv 0, \end{aligned} \tag{3.6}$$

where $h_0(z) = 0$ and

$$\begin{cases} B_{27}(z) &= a_0(z), \\ B_{26}(z) &= a_1(z), \\ B_{25}(z) &= 1, \\ B_{24}(z) &= -c_2(a_1(z) + a_0(z)), \\ B_{23}(z) &= -c_2(a_0(z) + 1), \\ B_{22}(z) &= -c_2(1 + a_1(z)), \\ B_{21}(z) &= {c_2}^2(1 + a_1(z) + a_0(z)). \end{cases}$$

Since we have $e^{\alpha(z)}$ with normal growth, and by Lemma 2.1, we have that $\rho(e^{\alpha(z)})$ is the degree of $\alpha(z)$. Because $\alpha(z)$ is a non-constant polynomial, $\rho(e^{\alpha(z)}) = \deg \alpha(z) \geq 1$, which holds for other exponential function terms.

On the other hand, we have that $\rho(a_1(z)) < 1$, $\rho(a_0(z)) < 1$, and c_2 is a constant. So, for $j = 1, 2, \cdots, 7$, we have

$$\begin{cases} T(r, B_{2j}(z)) &= o\{T(r, e^{\alpha(q^2z)+\alpha(qz)-\alpha(z)})\}, \\ T(r, B_{2j}(z)) &= o\{T(r, e^{\alpha(q^2z)+\alpha(z)-\alpha(qz)})\}, \\ T(r, B_{2j}(z)) &= o\{T(r, e^{\alpha(qz)+\alpha(z)-\alpha(q^2z)})\}, \\ T(r, B_{2j}(z)) &= o\{T(r, e^{\alpha(q^2z)-\alpha(qz)})\}, \\ T(r, B_{2j}(z)) &= o\{T(r, e^{\alpha(q^2z)-\alpha(z)})\}, \\ T(r, B_{2j}(z)) &= o\{T(r, e^{\alpha(qz)-\alpha(z)})\}, \\ T(r, B_{2j}(z)) &= o\{T(r, e^{\alpha(q^2z)})\}, \\ T(r, B_{2j}(z)) &= o\{T(r, e^{\alpha(qz)})\}, \\ T(r, B_{2j}(z)) &= o\{T(r, e^{\alpha(z)})\}. \end{cases}$$

Thus, applying Lemma 2.2 to (3.6), we get that $B_{2j}(z) \equiv 0$ ($j = 1, 2, \cdots, 7$). Clearly, for $B_{25}(z) = 1$, this is a contradiction.

Case 3. $\alpha(z)$ and $\beta(z)$ are non-constant polynomials. Substituting (3.2) into (1.2), we have

$$\frac{1 - e^{\beta(q^2z)}}{e^{\alpha(q^2z)} - e^{\beta(q^2z)}} + a_1(z)\frac{1 - e^{\beta(qz)}}{e^{\alpha(qz)} - e^{\beta(qz)}} + a_0(z)\frac{1 - e^{\beta(z)}}{e^{\alpha(z)} - e^{\beta(z)}} = 0, \tag{3.7}$$

therefore,

$$\begin{aligned} &e^{\alpha(qz)+\alpha(z)} - e^{\alpha(qz)+\beta(z)} - e^{\beta(qz)+\alpha(z)} + e^{\beta(qz)+\beta(z)} + a_1(z)e^{\alpha(q^2z)+\alpha(z)} - a_1(z)e^{\alpha(q^2z)+\beta(z)} \\ &-a_1(z)e^{\beta(q^2z)+\alpha(z)} + a_1(z)e^{\beta(q^2z)+\beta(z)} + a_0(z)e^{\alpha(q^2z)+\alpha(qz)} - a_0(z)e^{\alpha(q^2z)+\beta(qz)} \\ &-a_0(z)e^{\beta(q^2z)+\alpha(qz)} + a_0(z)e^{\beta(q^2z)+\beta(qz)} - e^{\beta(q^2z)+\alpha(qz)+\alpha(z)} + (1 + a_0(z))e^{\beta(q^2z)+\alpha(qz)+\beta(z)} \\ &+(1 + a_1(z))e^{\beta(q^2z)+\beta(qz)+\alpha(z)} - (1 + a_1(z) + a_0(z))e^{\beta(q^2z)+\beta(qz)+\beta(z)} - a_1(z)e^{\alpha(q^2z)+\beta(qz)+\alpha(z)} \\ &+(a_1(z) + a_0(z))e^{\alpha(q^2z)+\beta(qz)+\beta(z)} - a_0(z)e^{\alpha(q^2z)+\alpha(qz)+\beta(z)} = 0. \end{aligned} \tag{3.8}$$

Thus, by the degree relationship between $\alpha(z)$ and $\beta(z)$, there are three subcases:

Case 3.1. $\deg \alpha(z) > \deg \beta(z) \geq 1$; (3.8) can be rewritten as

$$\begin{aligned} &B_{37}(z)e^{\alpha(q^2z)+\alpha(qz)} + B_{36}(z)e^{\alpha(q^2z)+\alpha(z)} + B_{35}(z)e^{\alpha(qz)+\alpha(z)} + B_{34}(z)e^{\alpha(q^2z)} \\ &+B_{33}(z)e^{\alpha(qz)} + B_{32}(z)e^{\alpha(z)} + B_{31}(z)e^{h_0(z)} = 0, \end{aligned} \tag{3.9}$$

where $h_0(z) = 0$ and

$$\begin{cases} B_{37}(z) &= a_0(z)(1 - e^{\beta(z)}), \\ B_{36}(z) &= a_1(z)(1 - e^{\beta(qz)}), \\ B_{35}(z) &= 1 - e^{\beta(q^2z)}, \\ B_{34}(z) &= -a_1(z)e^{\beta(z)} - a_0(z)e^{\beta(qz)} + (a_1(z) + a_0(z))e^{\beta(qz)+\beta(z)}, \\ B_{33}(z) &= -e^{\beta(z)} - a_0(z)e^{\beta(q^2z)} + (1 + a_0(z))e^{\beta(q^2z)+\beta(z)}, \\ B_{32}(z) &= -e^{\beta(qz)} - a_1(z)e^{\beta(q^2z)} + (1 + a_1(z))e^{\beta(q^2z)+\beta(qz)}, \\ B_{31}(z) &= e^{\beta(qz)+\beta(z)} + a_1(z)e^{\beta(q^2z)+\beta(z)} + a_0(z)e^{\beta(q^2z)+\beta(qz)} \\ &\quad -(1 + a_1(z) + a_0(z))e^{\beta(q^2z)+\beta(qz)+\beta(z)}. \end{cases}$$

We have $e^{\alpha(z)}$, $e^{\beta(z)}$ with normal growth and $\deg\alpha(z) > \deg\beta(z) \geq 1$. Then, $\rho(a_1(z)) < 1$ and $\rho(a_0(z)) < 1$; hence, $T(r, e^{\beta(z)}) = o\{T(r, e^{\alpha(z)})\}$ and

$$\begin{cases} T(r, e^{\beta(q^2z)+\beta(qz)+\beta(z)}) = o\{T(r, e^{\alpha(z)})\}, \\ T(r, e^{\beta(q^2z)+\beta(qz)}) = o\{T(r, e^{\alpha(z)})\}, \\ T(r, e^{\beta(q^2z)+\beta(z)}) = o\{T(r, e^{\alpha(z)})\}, \\ T(r, e^{\beta(qz)+\beta(z)}) = o\{T(r, e^{\alpha(z)})\}, \\ T(r, e^{\beta(q^2z)}) = o\{T(r, e^{\alpha(z)})\}, \\ T(r, e^{\beta(qz)}) = o\{T(r, e^{\alpha(z)})\}. \end{cases}$$

Similarly, the above formulas have similar expressions for the other exponential terms of $B_{3j}(z)$ $(j = 1, 2, \cdots, 7)$ of (3.9); so, for $j = 1, 2, \cdots, 7$, we have

$$\begin{cases} T(r, B_{3j}(z)) &= o\{T(r, e^{\alpha(q^2z)+\alpha(qz)-\alpha(z)})\}, \\ T(r, B_{3j}(z)) &= o\{T(r, e^{\alpha(q^2z)+\alpha(z)-\alpha(qz)})\}, \\ T(r, B_{3j}(z)) &= o\{T(r, e^{\alpha(qz)+\alpha(z)-\alpha(q^2z)})\}, \\ T(r, B_{3j}(z)) &= o\{T(r, e^{\alpha(q^2z)-\alpha(qz)})\}, \\ T(r, B_{3j}(z)) &= o\{T(r, e^{\alpha(q^2z)-\alpha(z)})\}, \\ T(r, B_{3j}(z)) &= o\{T(r, e^{\alpha(qz)-\alpha(z)})\}, \\ T(r, B_{3j}(z)) &= o\{T(r, e^{\alpha(q^2z)})\}, \\ T(r, B_{3j}(z)) &= o\{T(r, e^{\alpha(qz)})\}, \\ T(r, B_{3j}(z)) &= o\{T(r, e^{\alpha(z)})\}. \end{cases}$$

Applying Lemma 2.2 to (3.9), we get that $B_{3j}(z) \equiv 0$ $(j = 1, 2, \cdots, 7)$. Therefore,

$$B_{35}(z) = 0,$$

we have that $e^{\beta(q^2z)} = 1$, where $\beta(q^2z) = 2k\pi i$, $k \in \mathbb{Z}$.

Since $\beta(z)$ is a polynomial, which is a contradiction.

Case 3.2. $\deg\beta(z) > \deg\alpha(z) \geq 1$; (3.8) can be rewritten as

$$\begin{aligned} B_{48}(z)e^{\beta(q^2z)+\beta(qz)+\beta(z)} + B_{47}(z)e^{\beta(q^2z)+\beta(qz)} + B_{46}(z)e^{\beta(q^2z)+\beta(z)} + B_{45}(z)e^{\beta(qz)+\beta(z)} \\ +B_{44}(z)e^{\beta(q^2z)} + B_{43}(z)e^{\beta(qz)} + B_{42}(z)e^{\beta(z)} + B_{41}(z)e^{h_0(z)} = 0, \end{aligned} \tag{3.10}$$

where $h_0(z) = 0$ and

$$\begin{cases} B_{48}(z) &= -1 - a_1(z) - a_0(z), \\ B_{47}(z) &= a_0(z) + (1 + a_1(z))e^{\alpha(z)}, \\ B_{46}(z) &= a_1(z) + (1 + a_0(z))e^{\alpha(qz)}, \\ B_{45}(z) &= 1 + (a_1(z) + a_0(z))e^{\alpha(q^2z)}, \\ B_{44}(z) &= -a_1(z)e^{\alpha(z)} - a_0(z)e^{\alpha(qz)} - e^{\alpha(qz)+\alpha(z)}, \\ B_{43}(z) &= -e^{\alpha(z)} - a_0(z)e^{\alpha(q^2z)} - a_1(z)e^{\alpha(q^2z)+\alpha(z)}, \\ B_{42}(z) &= -e^{\alpha(qz)} - a_1(z)e^{\alpha(q^2z)} - a_0(z)e^{\alpha(q^2z)+\alpha(qz)}, \\ B_{41}(z) &= e^{\alpha(qz)+\alpha(z)} + a_1(z)e^{\alpha(q^2z)+\alpha(z)} + a_0(z)e^{\alpha(q^2z)+\alpha(qz)}. \end{cases}$$

We have $e^{\alpha(z)}$, $e^{\beta(z)}$ with normal growth and $\deg\beta(z) > \deg\alpha(z) \geq 1$. Then, $\rho(a_1(z)) < 1$ and $\rho(a_0(z)) < 1$; thus, $T(r, e^{\alpha(z)}) = o\{T(r, e^{\beta(z)})\}$ and

$$\begin{cases} T(r, e^{\alpha(q^2z)+\alpha(qz)}) = o\{T(r, e^{\beta(z)})\}, \\ T(r, e^{\alpha(q^2z)+\alpha(z)}) = o\{T(r, e^{\beta(z)})\}, \\ T(r, e^{\alpha(qz)+\alpha(z)}) = o\{T(r, e^{\beta(z)})\}, \\ T(r, e^{\alpha(q^2z)}) = o\{T(r, e^{\beta(z)})\}, \\ T(r, e^{\alpha(qz)}) = o\{T(r, e^{\beta(z)})\}. \end{cases}$$

Similarly, for the other exponential terms of $B_{4j}(z)$ $(j = 1, 2, \cdots, 8)$, we have similar expressions; so, for $j = 1, 2, \cdots, 8$, we have

$$\begin{cases} T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(q^2z)+\beta(qz)+\beta(z)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(q^2z)+\beta(qz)-\beta(z)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(q^2z)+\beta(z)-\beta(qz)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(qz)+\beta(z)-\beta(q^2z)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(q^2z)-\beta(qz)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(q^2z)-\beta(z)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(qz)-\beta(z)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(q^2z)+\beta(qz)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(q^2z)+\beta(z)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(qz)+\beta(z)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(q^2z)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(qz)})\}, \\ T(r, B_{4j}(z)) &= o\{T(r, e^{\beta(z)})\}. \end{cases}$$

Applying Lemma 2.2 to (3.10) we get that $B_{4j}(z) \equiv 0$ $(j = 1, 2, \cdots, 8)$, we have

$$B_{45}(z) = 0, \quad B_{48}(z) = 0.$$

hence,

$$1 + (a_1(z) + a_0(z))e^{\alpha(q^2z)} = 0, \quad a_1(z) + a_0(z) = -1.$$

That is, $1 - e^{\alpha(q^2z)} = 0$, where $\alpha(z)$ is a non-constant polynomial, which is a contradiction.

Case 3.3. $\deg\beta(z) = \deg\alpha(z) = n \geq 1$. Set

$$\alpha(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0, \quad \beta(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0,$$

where $a_n(\neq 0), a_{n-1}, \cdots, a_0, b_n(\neq 0), b_{n-1}, \cdots, b_0$ are constants.

Let us consider the relationship between the highest coefficients of the polynomial number of each exponential function term in (3.8), and the items with the same coefficients are merged. Therefore, we discuss the following two situations according to whether a_n is equal to b_n:

Subcase 3.3.1. If $a_n = b_n$, (3.8) can be rewritten as

$$\begin{aligned} B_{54}(z)e^{\beta(q^2z)+\alpha(z)} + B_{53}(z)e^{\alpha(q^2z)+\beta(qz)} + B_{52}(z)e^{\beta(qz)+\alpha(z)} \\ +B_{51}(z)e^{\beta(q^2z)+\alpha(qz)+\beta(z)} = 0, \end{aligned} \tag{3.11}$$

where

$$\begin{cases} B_{54}(z) &= a_1(z)(e^{\alpha(q^2z)-\beta(q^2z)} - e^{\alpha(q^2z)+\beta(z)-\beta(q^2z)-\alpha(z)} - 1 + e^{\beta(z)-\alpha(z)}), \\ B_{53}(z) &= a_0(z)(e^{\alpha(qz)-\beta(qz)} - e^{\beta(q^2z)+\alpha(qz)-\alpha(q^2z)-\beta(qz)} - 1 + e^{\beta(q^2z)-\alpha(q^2z)}), \\ B_{52}(z) &= e^{\alpha(qz)-\beta(qz)} - e^{\alpha(qz)+\beta(z)-\beta(qz)-\alpha(z)} - 1 + e^{\beta(z)-\alpha(z)}, \\ B_{51}(z) &= -(1 + a_1(z) + a_0(z))e^{\beta(qz)-\alpha(qz)} + (a_1(z) + 1)e^{\beta(qz)+\alpha(z)-\alpha(qz)-\beta(z)} \\ &\quad +a_0(z) + 1 + (a_0(z) + a_1(z))e^{\alpha(q^2z)+\beta(qz)-\beta(q^2z)-\alpha(qz)} - e^{\alpha(z)-\beta(z)} \\ &\quad -a_0(z)e^{\alpha(q^2z)-\beta(q^2z)} - a_1(z)e^{\alpha(q^2z)+\beta(qz)+\alpha(z)-\beta(q^2z)-\alpha(qz)-\beta(z)}. \end{cases}$$

For $a_n = b_n$, we have

$$\alpha(z) - \beta(z) = (a_{n-1} - b_{n-1})z^{n-1} + (a_{n-2} - b_{n-2})z^{n-2} + \cdots + (a_0 - b_0),$$

hence $\deg(\alpha(z) - \beta(z)) \leq n - 1$.

Similarly, we can get the following formulas:

$$\begin{cases} \deg(\alpha(q^2z) + \beta(qz) + \alpha(z) - \beta(q^2z) - \alpha(qz) - \beta(z)) \leq n - 1, \\ \deg(\alpha(q^2z) + \beta(qz) - \beta(q^2z) - \alpha(qz)) \leq n - 1, \\ \deg(\beta(q^2z) + \alpha(qz) - \alpha(q^2z) - \beta(qz)) \leq n - 1, \\ \deg(\alpha(q^2z) + \beta(z) - \beta(q^2z) - \alpha(z)) \leq n - 1, \\ \deg(\alpha(qz) + \beta(z) - \beta(qz) - \alpha(z)) \leq n - 1, \\ \deg(\beta(qz) + \alpha(z) - \alpha(qz) - \beta(z)) \leq n - 1, \\ \deg(\alpha(q^2z) - \beta(q^2z)) \leq n - 1, \\ \deg(\beta(q^2z) - \alpha(q^2z)) \leq n - 1, \\ \deg(\alpha(qz) - \beta(qz)) \leq n - 1, \\ \deg(\beta(qz) - \alpha(qz)) \leq n - 1, \\ \deg(\beta(z) - \alpha(z)) \leq n - 1. \end{cases}$$

By Lemma 2.4, we have that $q \notin E$; then,

$$\begin{cases} \deg(\beta(q^2z) + \alpha(qz) + \beta(z) - \beta(qz) - \alpha(z)) = n, \\ \deg(\beta(q^2z) + \alpha(qz) + \beta(z) - \alpha(q^2z) - \beta(qz)) = n, \\ \deg(\beta(q^2z) + \alpha(qz) + \beta(z) - \beta(q^2z) - \alpha(z)) = n, \\ \deg(\beta(q^2z) + \alpha(z) - \alpha(q^2z) - \beta(qz)) = n, \\ \deg(\alpha(q^2z) + \beta(qz) - \beta(qz) - \alpha(z)) = n, \\ \deg(\beta(q^2z) + \alpha(z) - \beta(qz) - \alpha(z)) = n. \end{cases}$$

We have $e^{\alpha(z)}$, $e^{\beta(z)}$ with normal growth. Then, $\rho(a_1(z)) < 1$ and $\rho(a_0(z)) < 1$; so, for $B_{5j}(z)$ ($j = 1,2,3,4$), we have

$$\begin{cases} T(r, B_{5j}(z)) &= o\{T(r, e^{\beta(q^2z)+\alpha(qz)+\beta(z)-\beta(qz)-\alpha(z)})\}, \\ T(r, B_{5j}(z)) &= o\{T(r, e^{\beta(q^2z)+\alpha(qz)+\beta(z)-\alpha(q^2z)-\beta(qz)})\}, \\ T(r, B_{5j}(z)) &= o\{T(r, e^{\beta(q^2z)+\alpha(qz)+\beta(z)-\beta(q^2z)-\alpha(z)})\}, \\ T(r, B_{5j}(z)) &= o\{T(r, e^{\alpha(q^2z)+\beta(qz)-\beta(qz)-\alpha(z)})\}, \\ T(r, B_{5j}(z)) &= o\{T(r, e^{\beta(q^2z)+\alpha(z)-\beta(qz)-\alpha(z)})\}. \end{cases}$$

Thus, applying Lemma 2.2 to (3.11), we get that $B_{5j}(z) \equiv 0$ ($j = 1,2,3,4$). By

$$B_{52}(z) = 0.$$

that is,

$$e^{\alpha(qz)-\beta(qz)} - e^{\alpha(qz)+\beta(z)-\beta(qz)-\alpha(z)} - 1 + e^{\beta(z)-\alpha(z)} = 0,$$

it follows that

$$(1 - e^{\beta(z)-\alpha(z)})(e^{\alpha(qz)-\beta(qz)} - 1) = 0,$$

since $e^{\alpha(z)} \not\equiv e^{\beta(z)}$, $1 - e^{\beta(z)-\alpha(z)} \neq 0$, and $e^{\alpha(qz)-\beta(qz)} - 1 \neq 0$, thus, the above formula is obviously not equal to 0, which is a contradiction.

Subcase 3.3.2. If $a_n \neq b_n$, dividing (3.8) by $e^{\alpha(qz)+\beta(z)}$; we have

$$\begin{aligned} &e^{\alpha(z)-\beta(z)} - e^{\beta(qz)+\alpha(z)-\alpha(qz)-\beta(z)} + e^{\beta(qz)-\alpha(qz)} + a_1(z)e^{\alpha(q^2z)+\alpha(z)-\alpha(qz)-\beta(z)} \\ &-a_1(z)e^{\alpha(q^2z)-\alpha(qz)} - a_1(z)e^{\beta(q^2z)+\alpha(z)-\alpha(qz)-\beta(z)} + a_1(z)e^{\beta(q^2z)-\alpha(qz)} + a_0(z)e^{\alpha(q^2z)-\beta(z)} \\ &-a_0(z)e^{\alpha(q^2z)+\beta(qz)-\alpha(qz)-\beta(z)} - a_0(z)e^{\beta(q^2z)-\beta(z)} + a_0(z)e^{\beta(q^2z)+\beta(qz)-\alpha(qz)-\beta(z)} \\ &-e^{\beta(q^2z)+\alpha(z)-\beta(z)} + (1 + a_0(z))e^{\beta(q^2z)} + (1 + a_1(z))e^{\beta(q^2z)+\beta(qz)+\alpha(z)-\alpha(qz)-\beta(z)} \\ &+(-1 - a_1(z) - a_0(z))e^{\beta(q^2z)+\beta(qz)-\alpha(qz)} - a_1(z)e^{\alpha(q^2z)+\beta(qz)+\alpha(z)-\alpha(qz)-\beta(z)} \\ &+(a_1(z) + a_0(z))e^{\alpha(q^2z)+\beta(qz)-\alpha(qz)} - a_0(z)e^{\alpha(q^2z)} = 1. \end{aligned} \tag{3.12}$$

For the convenience of the following description, we use $f_j(z)$ and $g_j(z)$ ($j = 1, 2, \dots, 18$) to represent the coefficient functions and exponential functions, that is,

$$\sum_{j=1}^{18} f_j(z)e^{g_j(z)} = 1.$$

By Lemma 2.4, we deduce that $-e^{\beta(qz)+\alpha(z)-\alpha(qz)-\beta(z)}$, $e^{\alpha(z)-\beta(z)}$, $-a_1(z)e^{\alpha(q^2z)-\alpha(qz)}$, $-a_0(z)e^{\alpha(q^2z)}$, $-a_0(z)e^{\beta(q^2z)-\beta(z)}$, $(1 + a_0(z))e^{\beta(q^2z)}$ and $e^{\beta(qz)-\alpha(qz)}$, these are not constant functions in (3.12) for $q \notin E$; then applying Lemma 2.3 to (3.12), there exists $t_i \in \{0, 1\}$ ($i = 1, 2, \cdots, 11$) for the following equation:

$$\begin{aligned} &t_1a_1(z)e^{\alpha(q^2z)+\alpha(z)-\alpha(qz)-\beta(z)} - t_2a_1(z)e^{\beta(q^2z)+\alpha(z)-\alpha(qz)-\beta(z)} + t_3a_1(z)e^{\beta(q^2z)-\alpha(qz)} \\ &+t_4a_0(z)e^{\alpha(q^2z)-\beta(z)} - t_5a_0(z)e^{\alpha(q^2z)+\beta(qz)-\alpha(qz)-\beta(z)} + t_6a_0(z)e^{\beta(q^2z)+\beta(qz)-\alpha(qz)-\beta(z)} \\ &-t_7e^{\beta(q^2z)+\alpha(z)-\beta(z)} + t_8(1 + a_1(z))e^{\beta(q^2z)+\beta(qz)+\alpha(z)-\alpha(qz)-\beta(z)} \\ &-t_9(1 + a_1(z) + a_0(z))e^{\beta(q^2z)+\beta(qz)-\alpha(qz)} - t_{10}a_1(z)e^{\alpha(q^2z)+\beta(qz)+\alpha(z)-\alpha(qz)-\beta(z)} \\ &+t_{11}(a_1(z) + a_0(z))e^{\alpha(q^2z)+\beta(qz)-\alpha(qz)} \equiv 1, \end{aligned} \tag{3.13}$$

we assume that there are at least two values of t_i $(i = 1, 2, \cdots, 11)$ that are equal to 1; without loss of generality, set $t_1 = t_7 = 1$ and the rest are equal to 0; then,

$$a_1(z)e^{\alpha(q^2z)+\alpha(z)-\alpha(qz)-\beta(z)} - e^{\beta(q^2z)+\alpha(z)-\beta(z)} = 1, \tag{3.14}$$

from (3.13), we see that

$$g_4(z) = \alpha(q^2z) + \alpha(z) - \alpha(qz) - \beta(z), \quad g_{12}(z) = \beta(q^2z) + \alpha(z) - \beta(z).$$

If $e^{g_4(z)} = c_1$ is a constant, then, considering (3.14), we have

$$e^{g_{12}(z)} = a_1(z)e^{g_4(z)} - 1 = c_1a_1(z) - 1.$$

If $g_{12}(z)$ is a non-constant polynomial, we get contradiction from the following equation:

$$1 \leq \rho(e^{g_{12}(z)}) = \rho(a_1(z)) < 1.$$

If $g_{12}(z)$ is a constant, since $g_4(z)$, $g_{12}(z)$ are constants, it follows that

$$\begin{cases} a_n(q^{2n} - q^n + 1) - b_n = 0, \\ b_n(q^{2n} - 1) + a_n = 0, \end{cases}$$

we get that $q \in E$, which is a contradiction.

So, e^{g_4} and $e^{g_{12}}$ are not constants; considering (3.14) we have

$$a_1'(z)e^{g_4} + a_1g_4'e^{g_4} - g_{12}'e^{g_{12}} = 0,$$

if $a_1'(z) + a_1g_4' = 0$, we obtain that $\dfrac{a_1'(z)}{a_1(z)} = -g_4'(z)$, that is

$$a_1(z) = e^{-g_4(z)+C},$$

where C is an arbitrary constant.

Since $g_4(z)$ is a non-constant polynomial, we have that $\rho(a_1(z)) < 1$, which is a contradiction.

So, $a_1'(z) + a_1g_4' \neq 0$; then, there is

$$e^{g_4-g_{12}} = \frac{g_{12}'}{a_1' + a_1g_4'},$$

if $g_4 - g_{12}$ is not a constant, since $\rho(a_1) < 1$, we have

$$1 \leq \rho(e^{g_4-g_{12}}) = \rho(\frac{g_{12}'}{a_1' + a_1g_4'}) < 1,$$

which is a contradiction.

If $g_4 - g_{12} = c_2$ is a constant function, $g_4 = g_{12} + c_2$; then, also applying (3.14) we have

$$e^{g_{12}(z)} = \frac{1}{a_1(z)e^{c_2} - 1},$$

since g_{12} is a non-constant polynomial, by

$$1 \leq \rho(e^{g_{12}(z)}) = \rho(\frac{1}{a_1(z)e^{c_2} - 1}) < 1,$$

which is a contradiction.

Therefore, for (3.13), there is only one t_i ($i = 1, 2, \cdots, 11$) that is equal to 1, and the rest are all zeros. Without loss of generality, we assume that $t_1 = 1$; then,

$$a_1(z)e^{\alpha(q^2z)+\alpha(z)-\alpha(qz)-\beta(z)} = 1, \tag{3.15}$$

if $g_4(z) = \alpha(q^2z) + \alpha(z) - \alpha(qz) - \beta(z)$ is a non-constant polynomial, then

$$e^{\alpha(q^2z)+\alpha(z)-\alpha(qz)-\beta(z)} = \frac{1}{a_1(z)},$$

also consider the order of the following equation

$$1 \leq \rho(e^{g_4(z)}) = \rho(\frac{1}{a_1(z)}) < 1,$$

which is a contradiction.

Therefore, we deduce that $g_4(z)$ is a constant; thus, we have

$$a_n(q^{2n} - q^n + 1) - b_n = 0. \tag{3.16}$$

On the other hand, together with (3.12) and (3.15), we have

$$\begin{aligned}
&e^{\alpha(z)-\beta(z)} - e^{\beta(qz)+\alpha(z)-\alpha(qz)-\beta(z)} + e^{\beta(qz)-\alpha(qz)} - a_1(z)e^{\alpha(q^2z)-\alpha(qz)} \\
&-a_1(z)e^{\beta(q^2z)+\alpha(z)-\alpha(qz)-\beta(z)} + a_1(z)e^{\beta(q^2z)-\alpha(qz)} + a_0(z)e^{\alpha(q^2z)-\beta(z)} \\
&-a_0(z)e^{\alpha(q^2z)+\beta(qz)-\alpha(qz)-\beta(z)} - a_0(z)e^{\beta(q^2z)-\beta(z)} + a_0(z)e^{\beta(q^2z)+\beta(qz)-\alpha(qz)-\beta(z)} \\
&-e^{\beta(q^2z)+\alpha(z)-\beta(z)} + (1 + a_0(z))e^{\beta(q^2z)} + (1 + a_1(z))e^{\beta(q^2z)+\beta(qz)+\alpha(z)-\alpha(qz)-\beta(z)} \\
&+(-1 - a_1(z) - a_0(z))e^{\beta(q^2z)+\beta(qz)-\alpha(qz)} - a_1(z)e^{\alpha(q^2z)+\beta(qz)+\alpha(z)-\alpha(qz)-\beta(z)} \\
&+(a_1(z) + a_0(z))e^{\alpha(q^2z)+\beta(qz)-\alpha(qz)} - a_0(z)e^{\alpha(q^2z)} = 0,
\end{aligned} \tag{3.17}$$

that is,

$$\sum_{j=1, j\neq 4}^{18} f_j e^{g_j} = 0. \tag{3.18}$$

We assert that g_k, $g_k - g_j$ ($1 \leq k < j \leq 18, k, j \neq 4$) in (3.18) are not constants. If $g_j(z)$ ($1 \leq j \leq 18, j \neq 4$) is a constant, we take $j = 6$; then $\beta(q^2z) + \alpha(z) - \alpha(qz) - \beta(z)$ is a constant; taking into considering with (3.16), we have the following:

$$\begin{cases} a_n(q^{2n} - q^n + 1) - b_n = 0, \\ a_n(-q^n + 1) + b_n(q^{2n} - 1) = 0, \end{cases}$$

we get that $q \in E$, which is a contradiction.

Similarly, if $g_k - g_j$ $(1 \leq k < j \leq 18, k, j \neq 4)$ is a constant, we can take $k = 1$ and $j = 6$; then, $\alpha(qz) - \beta(q^2z)$ is a constant; incorporating (3.16), we have the following:

$$\begin{cases} a_n(q^{2n} - q^n + 1) - b_n = 0, \\ a_nq^n - b_nq^{2n} = 0, \end{cases}$$

we have that $q \in E$, which is contradictory.

Above all, for (3.18), we have g_k, $g_k - g_j$ $(1 \leq k < j \leq 18, k, j \neq 4)$ are polynomials. Given that $\rho(f_j) < 1$ $(1 \leq k < j \leq 18)$, we can deduce that when $1 \leq k < j \leq 18, k, j \neq 4$, we have

$$T(r, f_j) = o\{T(r, e^{g_k - g_j})\}, (r \to \infty, r \notin E),$$

where E is a set of finite logarithmic measures.

So, the formula (3.18) satisfies all conditions of Lemma 2.2; then, we have that $f_j(z) \equiv 0$ $(j = 1, 2, \cdots, 18)$. It follows that $f_1(z) = 1$, $f_2(z) = -1$, $f_3(z) = 1$ and $f_{12}(z) = -1$ are non-zero constants, which is a contradiction.

Hence, $f(z) \equiv g(z)$. This completes the proof of Theorem 1.3.

4. Proof of Theorem 1.4

The idea of proving this theorem is mainly derived from literature [7].

Since $f(z)$ and $g(z)$ are non-constant meromorphic functions, assume that $f(z) \not\equiv g(z)$; we have

$$H(z) := \frac{f'(z)g(z)(g(z) - 1)(g(z) - c)}{g'(z)f(z)(f(z) - 1)(f(z) - c)}. \tag{4.1}$$

By (4.1), we deduce the following.

Assertion 1: $N(r, H) = O(\log r)$ and $N(r, \frac{1}{H}) = O(\log r)$.

The possible poles of $H(z)$ are derived from the zeros of $g'(z)$, $f(z)$, $f(z) - 1$, $f(z) - c$ and the poles of $f(z)$ and $g(z)$.

We first prove that the poles of $f(z)$ and $g(z)$ are not zeros or poles of $H(z)$. Since $f(z)$ and $g(z)$ share ∞ CM, suppose that z_0 represents $k(\geq 1)$ multiplicities of $f(z)$ and $g(z)$; then,

$$f(z) = \frac{a_{-k}}{(z - z_0)^k}(1 + O(z - z_0)),$$

$$g(z) = \frac{b_{-k}}{(z - z_0)^k}(1 + O(z - z_0)),$$

where a_{-k}, b_{-k} are non-zero constants.

So, we have

$$\frac{f'(z)}{g'(z)} = \frac{a_{-k}}{b_{-k}} + O(z - z_0),$$

then,

$$\frac{f'(z_0)}{g'(z_0)} = \frac{a_{-k}}{b_{-k}}. \tag{4.2}$$

Similarly,

$$\frac{g(z_0)}{f(z_0)} = \frac{b_{-k}}{a_{-k}}, \quad \frac{g(z_0)-1}{f(z_0)-1} = \frac{b_{-k}}{a_{-k}}, \quad \frac{g(z_0)-c}{f(z_0)-c} = \frac{b_{-k}}{a_{-k}}, \tag{4.3}$$

substituting (4.2) and (4.3) into (4.1), we have

$$H(z_0) = \left(\frac{b_{-k}}{a_{-k}}\right)^2.$$

Since a_{-k}, b_{-k} is not 0, $z = z_0$ is not the zero or pole of $H(z)$; that is, the poles of $f(z)$ and $g(z)$ are not zeros or poles of $H(z)$.

Next, we prove that the same zeros of the zeros, 1−value points, c−value points of $f(z)$ and the zeros, 1−value points, c−value points of $g(z)$ are not the zeros or poles of $H(z)$. Assume that z_1 is the public zero of $f(z) - a$ and $g(z) - a$, and that the multiplicities of $f(z)$ and $g(z)$ are $s(> 0)$, $t(> 0)$ respectively, where $a \in \{0, 1, c\}$. For the two sides of the above formula with the following Laurent expansion in the neighborhood of z_1, assume that

$$f(z) = (z - z_1)^s f_1(z), \quad f_1(z_1) \neq 0,$$

$$g(z) = (z - z_1)^t g_1(z), \quad g_1(z_1) \neq 0.$$

Hence,

$$\frac{f'(z)}{f(z)} = \frac{s(z-z_1)^{s-1} f_1 + (z-z_1)^s f_1{}'}{(z-z_1)^s f_1} = \frac{s f_1 + (z-z_1) f_1{}'}{(z-z_1) f_1}, \tag{4.4}$$

$$\frac{g(z)}{g'(z)} = \frac{(z-z_1)^t g_1}{t(z-z_1)^{t-1} g_1 + (z-z_1)^t g_1{}'} = \frac{(z-z_1) g_1}{t g_1 + (z-z_1) g_1{}'}, \tag{4.5}$$

substituting (4.4) and (4.5) into (4.1), we have

$$H(z_1) = \frac{s f_1(z_1) g_1(z_1)}{t f_1(z_1) g_1(z_1)}.$$

Since $f_1(z_1)$ and $g_1(z_1)$ are not zero, then

$$H(z_1) = \frac{s}{t}, \tag{4.6}$$

then, the public zeros of $f(z)$ and $g(z)$ are not zeros or poles of $H(z)$.

Similarly, the 1−value points of $f(z)$ and the 1−value points of $g(z)$ are not zeros or poles of $H(z)$. Moreover, the c−value points of $f(z)$ and the c−value points of $g(z)$ are not zeros or poles of $H(z)$. So the poles of $H(z)$ can only come from the zero points of $g'(z)$ but not the zero points of $g(z)$, $g(z) - 1$ and $g(z) - c$; by Lemma 2.5(iv), we have

$$N(r, H) \leq N_0(r, \frac{1}{g'}) = O(\log r),$$

where $N_0(r, \frac{1}{g'})$ denotes the zero points of g' but not the counting function of the zero points of $g - a_i$ $(i = 1, 2, 3)$.

Above all, we can draw a conclusion that assertion 1 is established. Therefore

$$\frac{H(z)}{R(z)} = e^{\alpha(z)}, \tag{4.7}$$

where $R(z)$ is a rational function and $\alpha(z)$ is a polynomial.

By Lemma 2.6 for the meromorphic solution f of (1.2), we have

$$k_1 \log^2 r \leq T(r, f) \leq k_2 \log^2 r, \tag{4.8}$$

where k_1, k_2 are non-zero constants.

Since

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} \leq \limsup_{r\to\infty} \frac{\log^+ k_2 \log^2 r}{\log r} = 0,$$
$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} \leq \liminf_{r\to\infty} \frac{\log^+ k_1 \log^2 r}{\log r} = 0,$$

we have that $\mu(f) = \rho(f) = 0$.

By Lemma 2.5(i),

$$T(r, f) = T(r, g) + O(\log r), \quad T(r, g) = T(r, f) + O(\log r),$$

hence,

$$\rho(f) = \rho(g) = \mu(f) = \mu(g) = 0,$$

thus, for $H(z)$, we have that $\rho(H(z)) = 0$. Therefore, $\alpha(z)$ is a constant. Otherwise,

$$\rho(H(z)) = \rho(R(z)e^{\alpha(z)}) > 0,$$

which is a contradiction. So, $H(z)$ is a rational function.

Since $f(z)$, $g(z)$ share $0, 1, c(\neq 0, 1)$ IM and ∞ CM and $f(z) \not\equiv g(z)$, by Lemma 2.7 for the transcendental functions $f(z)$, we have

$$T(r, H(z)) = S(r, f).$$

Thus, given (4.1), we have

$$\frac{f'(z)}{f(z)(f(z)-1)(f(z)-c)} = \frac{H(z)g'(z)}{g(z)(g(z)-1)(g(z)-c)}. \tag{4.9}$$

For the convenience of the proof, suppose that $f(z)$ shares the value a partially with $g(z)$, and that $\overline{N}_{(m,n)}(r, a)$ denotes the reduced counting function of those zeros of $f(z) - a$ with multiplicity m, and of $g(z) - a$ with multiplicity n in $\{z : |z| < r\}$. We make the following assumption:

Assertion 2. For any positive integer pair (m, n), we have

$$\overline{N}_{(m,n)}(r, a) = S(r, f), \quad a \in \{0, 1, c\}.$$

Suppose, on the contrary, that $\overline{N}_{(m,n)}(r, a) \neq S(r, f)$, $a \in \{0, 1, c\}$. Next, we consider the following two cases.

Case 1. Suppose that $m = n$.

Let $z_2 \in \{z : |z| < r\}$ be the public zeros of $f(z) - a$ and $g(z) - a$ for m multiplicities; applying this in consideration with (4.6), we have that $H(z_2) = 1$.

If $H(z) \not\equiv 1$, combining (4.7) with the first fundamental Nevanlinna theorem, we have

$$\overline{N}_{(m,m)}(r, a) \leq N\left(r, \frac{1}{H - 1}\right) \leq T(r, H) = S(r, f),$$

which is a contradiction.

If $H(z) \equiv 1$, by (4.9), we have

$$\frac{f'(z)}{f(z)(f(z) - 1)(f(z) - c)} = \frac{g'(z)}{g(z)(g(z) - 1)(g(z) - c)}, \tag{4.10}$$

for all $z \in \mathbb{C}$.

Since $f(z)$ and $g(z)$ share 0, 1, $c(\neq 0, 1)$ IM, taking into consideration (4.10), we arrive at $f(z)$ and $g(z)$ sharing 0, 1, c, ∞ CM. Then, there is

$$\frac{f(z)}{g(z)} = e^{\alpha(z)}, \quad \frac{f(z) - 1}{g(z) - 1} = e^{\beta(z)},$$

given that $\rho(f) = \rho(g) = 0$, we have that $\alpha(z)$ and $\beta(z)$ are constants. Thus, $f(z)$ is a constant, which is a contradiction.

Case 2. Suppose that $m \neq n$.

Let z_3 be the public zeros of $f(z) - a$ and $g(z) - a$ for m and n multiplicities respectively, where $a \in \{0, 1, c\}$. Also, by (4.6), we have that $H(z_3) = \frac{m}{n}$.

If $H(z_3) \not\equiv \frac{m}{n}$, combining (4.7) with the first fundamental Nevanlinna theorem, we have

$$\overline{N}_{(m,n)}(r, a) \leq N\left(r, \frac{1}{H - \frac{m}{n}}\right) \leq T(r, H) = S(r, f),$$

which is a contradiction.

If $H(z_3) \equiv \frac{m}{n}$, then

$$\frac{nf'(z)}{f(z)(f(z) - 1)(f(z) - c)} = \frac{mg'(z)}{g(z)(g(z) - 1)(g(z) - c)}, \tag{4.11}$$

for all $z \in \mathbb{C}$.

Let z_3 be the public zeros of $f(z) - a$ and $g(z) - a$ and the corresponding multiplicities be p and q, where $a \in \{0, 1, c\}$. Combining (4.6) with (4.11) on both sides with Laurent expansion of z_3, then, for all $a \in \{0, 1, c\}$ we have

$$nq = mp.$$

Set

$$L(z) = \frac{nf'(z)}{f(z)(f(z) - 1)} - \frac{mg'(z)}{g(z)(g(z) - 1)}, \tag{4.12}$$

from the above we get that $L(z)$ is analytic in z_3. Thus,

$$\begin{aligned} L(z) &= -nf'(z)(\frac{1}{f(z)} - \frac{1}{f(z)-1}) + mg'(z)(\frac{1}{g(z)} - \frac{1}{g(z)-1}) \\ &= -n(\frac{f'(z)}{f(z)} - \frac{f'(z)}{f(z)-1}) + m(\frac{g'(z)}{g(z)} - \frac{g'(z)}{g(z)-1}). \end{aligned} \tag{4.13}$$

Given that $m \neq n$ and $m(r, \frac{f'}{f}) = O(\log r)$, we deduce that

$$m(r, L(z)) = O(\log r),$$

that is, $L(z)$ is a polynomial.

Then for, (4.13), we have

$$\begin{aligned} \int L(z)dz &= -n\int f'(z)(\frac{1}{f(z)} - \frac{1}{f(z)-1})dz + m\int g'(z)(\frac{1}{g(z)} - \frac{1}{g(z)-1})dz \\ &= -n(\int \frac{1}{f(z)}df - \int \frac{1}{f(z)-1}df) + m(\int \frac{1}{g(z)}dg - \int \frac{1}{g(z)-1}dg) \\ &= -n(\ln f(z) - \ln(f(z)-1)) + m(\ln g(z) - \ln(g(z)-1)) \\ &= -n\ln\frac{f(z)}{f(z)-1} + m\ln\frac{g(z)}{g(z)-1} \\ &= \ln(\frac{f(z)-1}{f(z)})^n + \ln(\frac{g(z)}{g(z)-1})^m, \end{aligned}$$

therefore,

$$\left(\frac{g(z)-1}{g(z)}\right)^m e^{\int L(z)dz} = \left(\frac{f(z)-1}{f(z)}\right)^n,$$

since $L(z)$ is a polynomial, by Lemma 2.9, we get

$$mT(r, g) = nT(r, f) + S(r, f),$$

then, by $m \neq n$ and Lemma 2.5(i), we have

$$T(r, f) = S(r, f),$$

which is a contradiction.

In summary, for any positive integer pair (m, n), we have

$$\overline{N}_{(m,n)}(r, a) = S(r, f), \quad a \in \{0, 1, c\},$$

this completes the proof of Assertion 2.

Above all, we give the rest of the proof of the theorem.

$$\begin{aligned} \overline{N}\left(r, \tfrac{1}{f}\right) &= \textstyle\sum_{m\geq 1}\sum_{n\geq 1}\overline{N}_{(m,n)}(r, 0) \\ &= \textstyle\sum_{1\leq m\leq 9}\sum_{1\leq n\leq 9}\overline{N}_{(m,n)}(r, 0) + \sum_{m\geq 10}\sum_{1\leq n\leq 9}\overline{N}_{(m,n)}(r, 0) \\ &\quad + \textstyle\sum_{1\leq m\leq 9}\sum_{n\geq 10}\overline{N}_{(m,n)}(r, 0) + \sum_{m\geq 10}\sum_{n\geq 10}\overline{N}_{(m,n)}(r, 0), \end{aligned} \tag{4.14}$$

the first item on the right sides of (4.14) is equivalent to $S(r, f)$; the second item, the third item and the fourth item can be estimated by applying the upper bound as $\frac{1}{10}N(r, \frac{1}{f})$, $\frac{1}{10}N(r, \frac{1}{g})$ and $\frac{1}{10}N(r, \frac{1}{f})$ (or $\frac{1}{10}N(r, \frac{1}{g})$). Combining Lemma 2.5(i) with (4.14), we have

$$\overline{N}\left(r, \frac{1}{f}\right) \leq \frac{1}{10}T(r, f) + \frac{1}{10}T(r, g) + \frac{1}{10}T(r, f) + S(r, f) \leq \frac{3}{10}T(r, f) + S(r, f).$$

Similarly,

$$\overline{N}\left(r, \frac{1}{f-1}\right) \leq \frac{3}{10}T(r, f) + S(r, f),$$
$$\overline{N}\left(r, \frac{1}{f-c}\right) \leq \frac{3}{10}T(r, f) + S(r, f).$$

Combining this with the above estimates and Lemma 2.5(ii), we have

$$\begin{aligned} 2T(r, f) &= \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}\left(r, \frac{1}{fc}\right) + S(r, f) \\ &\leq \overline{N}(r, f) + \frac{9}{10}T(r, f) + S(r, f) \leq \frac{19}{10}T(r, f) + S(r, f), \end{aligned}$$

we get $T(r, f) = S(r, f)$, which is a contradiction.

This completes the proof of Theorem 1.4.

5. Conclusions

In this paper, we have investigated the meromorphic solutions of a class of homogeneous second-order q-difference equations and the uniqueness problem of a meromorphic function with three shared values. We have also discussed the uniqueness problem of the meromorphic solutions of a class of nonhomogeneous q-difference equations and a meromorphic function with four shared values.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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