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Research article

First-order differential subordinations associated with Carathéodory functions

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Abstract: In the present paper, we investigated some conditions to be in the class of Carathéodory functions by using the concept of the first-order differential subordinations. Moreover, various interesting special cases were considered in the geometric function theory as applications of main results presented here.

Keywords: differential subordination; Carathéodory function; univalent function; starlike function of order alpha; strongly starlike function
Mathematics Subject Classification: 30C45, 30C80

1. Introduction

Let $\mathcal{P}(\alpha)$ be the class of analytic functions p of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, with $\mathrm{Re}\,p(z) > \alpha$ for $z \in \mathbb{U}$. The class $\mathcal{P} \equiv \mathcal{P}(0)$ is known as the Carathéodory class or the class of functions with positive real part [2, 3], pioneered by Carathéodory. The theory of Carathéodory functions plays a very important role in the geometric function theory. For recent developments, the readers may refer to the works of Kim and Cho [5], Kwon and Sim [6], Nunokawa et al. [16], Sim et al. [18] and Wang [22].

Let $\mathcal{A}$ denote the class of all functions f analytic in $\mathbb{U}$ with the usual normalization $f(0) = f'(0)-1 = 0$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ in $\mathbb{U}$ such that $f(z) = g(w(z))$.

A function $f \in \mathcal{A}$ is said to be strongly starlike of order η $(0 < \eta \leq 1)$ if, and only if,

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\eta} \quad (z \in \mathbb{U}). \tag{1.1}$$

We note that the conditions (1.1) can be written by

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}[\eta]$ the subclass of $\mathcal{A}$ consisting of all strongly starlike functions of order η $(0 < \eta \leq 1)$. The class $\mathcal{S}[\eta]$ was introduced and studied by Brannan and Kirwan [1] and Stankiewicz [20, 21]. We also note that $\mathcal{S}[1] \equiv \mathcal{S}^*$ is the well-known class of all normalized starlike functions in $\mathbb{U}$. The class $\mathcal{S}[\eta]$ and the related classes have been extensively studied by Mocanu [14] and Nunokawa [15]. It is worth noticing that f belongs to $\mathcal{S}[\eta]$ if it satisfies

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\eta)} \quad (z \in \mathbb{U}),$$

where

$$\alpha(\eta) = \frac{2}{\pi} \arctan\left\{\tan \frac{\eta}{2}\pi + \frac{\beta}{(1-\eta)^{\frac{1-\eta}{2}}(1+\eta)^{\frac{1+\eta}{2}} \cos \frac{\eta}{2}\pi}\right\}.$$

Given $\alpha \in [0, 1)$, let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$, which consists of all starlike functions of order α, namely, $f \in \mathcal{A}$ belongs to $\mathcal{S}^*(\alpha)$ if, and only if, it satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (z \in \mathbb{U}).$$

The class $\mathcal{S}^*(\alpha)$ was introduced by Robertson [17]. Clearly, it holds that $\mathcal{S}^*(0) \equiv \mathcal{S}[1] \equiv \mathcal{S}^*$. A typical sufficient condition for starlike functions of order α is given by Wilken and Feng [23], which states that if $f \in \mathcal{A}$, then

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\beta)z}{1-z} \quad (z \in \mathbb{U})$$

implies $f \in \mathcal{S}^*(\alpha)$, where

$$\beta = \beta(\alpha) := \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})}, & \text{if } \alpha \neq 1/2, \\ \dfrac{1}{2\log 2}, & \text{if } \alpha = 1/2. \end{cases}$$

Given $\eta \in (0, 1]$, let $\mathcal{T}[\eta]$ be the class of $f \in \mathcal{A}$ such that

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\eta} \quad (z \in \mathbb{U}).$$

The class $\mathcal{T} \equiv \mathcal{T}[1]$ plays an important role in the theory of univalent functions, although all elements in $\mathcal{T}$ are functions that are not necessarily univalent. In [7], several sufficient conditions for functions in $\mathcal{T}[\eta]$ were introduced.

If ψ is analytic in a domain $\mathbb{D} \subset \mathbb{C}^2$, h is univalent in $\mathbb{U}$ and p is analytic in $\mathbb{U}$ with $(p(z), zp'(z)) \in \mathbb{D}$ for $z \in \mathbb{U}$, then p is said to satisfy the first-order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z) \quad (z \in \mathbb{U}). \tag{1.2}$$

The univalent function q is said to be a dominant of the differential subordination (1.2) if $p \prec q$ for all p satisfying (1.2). If $\tilde{q}$ is a dominant of (1.2) and $\tilde{q} \prec q$ for all dominants of (1.2), then $\tilde{q}$ is said to be the best dominant of the differential subordination (1.2). The general theory of the first-order differential subordinations, with many interesting applications, especially in the theory of univalent functions, was developed by Miller and Mocanu [10] (also see [4, 8, 9, 11–13]).

In this paper, by applying the result obtained by Miller and Mocanu [10], we will investigate conditions to be in the class of Carathéodory functions. We will also find new sufficient conditions for $f \in \mathcal{A}$ to belong to the classes $\mathcal{S}[\eta]$, $\mathcal{S}^*(\alpha)$, and $\mathcal{T}[\eta]$ as some applications of the main results presented here. A differential subordination of the Briot-Bouquet type [12] (also see [13, Section 3]) will be considered for conditions for $f \in \mathcal{S}^*(\alpha)$ and $f \in \mathcal{T}[1]$, and an integral operator related to the differential subordination of this type will be discussed as our additional results. Moreover, more conditions for $f \in \mathcal{S}[\eta]$ and $f \in \mathcal{T}[\eta]$ will be introduced by using a nonlinear first-order differential subordination.

2. Main results

In proving our results, we shall need the following lemma due to Miller and Mocanu [10].

Lemma 1. *Let q be univalent in $\mathbb{U}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $q(\omega) \neq 0$ when $\omega \in q(\mathbb{U})$. Set $Q(z) = zq'(z)\varphi(q(z)), h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) *Q is starlike in $\mathbb{U}$,*
(ii) $\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in \mathbb{U})$.

If p is analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$, and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.1}$$

then $p \prec q$ and q is the best dominant of (2.1).

With the help of Lemma 1, we now derive the following Theorem 1.

Theorem 1. *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $\beta > 0$, $\beta + \gamma > 0$. If*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \neq -\frac{\gamma}{\beta} + ik \quad (z \in \mathbb{U}) \tag{2.2}$$

for all k ($|k| \geq \sqrt{2(\beta + \gamma) + 1}/\beta$), then

$$p(z) \prec \frac{1 + (1 + (2\gamma/\beta))\, z}{1 - z} \quad (z \in \mathbb{U}).$$

Proof. First, we note that $p(z) \neq -(\gamma/\beta)$ for $z \in \mathbb{U}$ under the condition (2.2). In fact, if $\beta p(z) + \gamma$ has a zero $z_0 \in \mathbb{U}$ of order n ($n \geq 1$) at a point $z_0 \in \mathbb{U}\backslash\{0\}$, then we may write

$$\beta p(z) + \gamma = (z - z_0)^n q(z) \quad (n \in \mathbb{N} := \{1,\ 2,\ 3, \cdots\}),$$

where p is analytic in $\mathbb{U}$ with $q(z_0) \neq 0$, then it follows that

$$\frac{\beta zp'(z)}{\beta p(z) + \gamma} = \frac{zq'(z)}{q(z)} + \frac{nz}{z - z_0}. \tag{2.3}$$

Therefore,

$$\lim_{z\to z_0}(z-z_0)\frac{\beta z p'(z)}{\beta p(z)+\gamma} = nz_0 \neq 0.$$

Letting z approach z_0 in the direction of $\arg z_0$, the righthand side of (2.3) takes infinite pure imaginary value. This contradicts the assumption (2.2).

Let $q(z) = (1+(1+2\gamma/\beta)z)/(1-z)$, $\theta(\omega) = \omega$, and $\varphi(\omega) = 1/(\beta\omega+\gamma)$ in Lemma 1, then θ and φ are analytic in $q(\mathbb{U})$ and $\varphi(\omega) \neq 0$ for $\varphi \in q(\mathbb{U})$. Setting

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{2z}{\beta(1-z^2)}$$

and

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= \frac{1}{\beta}\left\{(\beta+\gamma)\frac{1+z}{1-z} + \frac{2z}{1-z^2} - \gamma\right\}, \end{aligned}$$

the conditions (i) and (ii) of Lemma 1 can be verified. Therefore, Lemma 1 gives that if

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \quad (z \in \mathbb{U})$$

with

$$h(z) = \frac{1}{\beta}\left\{(\beta+\gamma)\frac{1+z}{1-z} + \frac{2z}{1-z^2} - \gamma\right\},$$

then

$$p(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Noting that

$$h(e^{i\theta}) = \frac{1}{\beta}\left\{(\beta+\gamma)\frac{1+e^{i\theta}}{1-e^{i\theta}} + \frac{2e^{i\theta}}{1-e^{i2\theta}} - \gamma\right\} \quad (0 < |\theta| < \pi),$$

we obtain

$$\mathrm{Re}\, h(e^{i\theta}) = -\frac{\gamma}{\beta}$$

and

$$\mathrm{Im}\, h(e^{i\theta}) = \frac{1}{\beta}\left\{(\beta+\gamma)\frac{\sin\theta}{1-\cos\theta} + \frac{1}{\sin\theta}\right\} \quad (0 < |\theta| < \pi).$$

Meanwhile, since the imaginary part of $h(e^{i\theta})$ is an odd function, we consider only the case $0 < \theta < \pi$. Putting $\tan(\theta/2) = t$ $(0 < \theta < \pi)$, we have

$$\begin{aligned} \mathrm{Im}\, h(e^{i\theta}) &= \frac{1}{\beta}\left\{(\beta+\gamma)\frac{\sin\theta}{1-\cos\theta} + \frac{1}{\sin\theta}\right\} \\ &= \frac{t^2+2(\beta+\gamma)+1}{2\beta t} \\ &= g(t). \end{aligned}$$

Here, the function $g(t)$ has a minimum value at $t_0 = \sqrt{2(\beta+\gamma)+1}$. Hence we have

$$|\mathrm{Im}\, h(e^{i\theta})| \geq |g(t_0)| = \frac{\sqrt{2(\beta+\gamma)+1}}{\beta}.$$

Applying Lemma 1 and the assumption (2.2), we conclude that

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \quad (z \in \mathbb{U}).$$

This completes the proof of Theorem 1. □

Taking $p(z) = zf'(z)/f(z)$, $\beta = 1$, and $\gamma = (1/\alpha) - 1$ $(0 < \alpha \leq 1)$ in Theorem 1, we have the following result.

Corollary 1. *Let $f \in \mathcal{A}$ and $0 < \alpha \leq 1$. If*

$$\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} \neq \alpha - 1 + ik \quad (z \in \mathbb{U})$$

for all k $(|k| \geq \sqrt{(2+\alpha)/\alpha}/\beta)$, then

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + (1 + 2(1-\alpha)/\alpha)z}{1-z} \quad (z \in \mathbb{U}).$$

Proof. Putting

$$p(z) = \frac{zf'(z)}{f(z)},$$

we have

$$\begin{aligned}
&\alpha z(zf'(z))' + (1-\alpha)zf'(z)\\
&= \alpha zf(z)p'(z) + \alpha zp(z)f'(z) + (1-\alpha)p(z)f(z)\\
&= \left(\alpha zp'(z) + p(z)(\alpha p(z) + 1 - \alpha)\right) f(z)
\end{aligned}$$

and

$$\alpha zf'(z) + (1-\alpha)f(z) = (\alpha p(z) + 1 - \alpha)f(z).$$

Hence,

$$\begin{aligned}
\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} &= \frac{\alpha zp'(z) + p(z)(\alpha p(z) + 1 - \alpha)}{\alpha p(z) + 1 - \alpha}\\
&= \frac{p(z) + zp'(z)}{p(z) + \left(\frac{1}{\alpha} - 1\right)}.
\end{aligned}$$

Therefore, applying Theorem 1, we have Corollary 1. □

Corollary 2. *Let $f \in \mathcal{A}$ and let*

$$F(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z t^{\frac{1}{\alpha}-2} f(t)dt \quad (0 < \alpha \leq 1).$$

If

$$\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} \neq \alpha - 1 + ik \quad (z \in \mathbb{U})$$

for all k ($|k| \geq \sqrt{(2+\alpha)/\alpha/\beta}$), then

$$\frac{\alpha z(zF'(z))' + (1-\alpha)zF'(z)}{\alpha zF'(z) + (1-\alpha)F(z)} \prec \frac{1 + (1 + 2(1-\alpha)/\alpha)z}{1-z} \quad (z \in \mathbb{U}).$$

Proof. Differentiating F with respect to z and multiplying by z, we have

$$\frac{\alpha z(zF'(z))' + (1-\alpha)zF'(z)}{\alpha zF'(z) + (1-\alpha)F(z)} = \frac{zf'(z)}{f(z)}.$$

Therefore, the result follows from Corollary 1. □

Letting $\beta = 1/\alpha$ ($\alpha > 0$), $\gamma = 0$, and $p(z) = zf'(z)/f(z)$ in Theorem 1, we have the following result.

Corollary 3. *Let $f \in \mathcal{A}$ and $\alpha > 0$. If*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \neq ik \quad (z \in \mathbb{U})$$

for all k ($|k| \geq \sqrt{\alpha(2+\alpha)}$), then f is starlike in $\mathbb{U}$.

Taking $\beta = 1$, $\gamma = 0$, and $p(z) = zf'(z)/f(z)$ in Theorem 1, we have the following result.

Corollary 4. *Let $f \in \mathcal{A}$. If*

$$1 + \frac{zf''(z)}{f'(z)} \neq ik \quad (z \in \mathbb{U})$$

for all k ($|k| \geq \sqrt{3}$), then f is a starlike in $\mathbb{U}$.

Example 1. *Consider a function $\tilde{f} : \mathbb{U} \to \mathbb{C}$ defined by*

$$\tilde{f}(z) = \frac{1}{\sqrt{3}-1}(e^{(\sqrt{3}-1)z} - 1).$$

Then we have

$$1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)} = 1 + (\sqrt{3}-1)z$$

and

$$\left|1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)}\right| < \sqrt{3}, \quad z \in \mathbb{U}.$$

Therefore, by Corollary 4, $\tilde{f}$ is starlike in $\mathbb{U}$ (see also the left side of Figure 1). In fact, we can check that $\mathrm{Re}\,\{z\tilde{f}'(z)/\tilde{f}(z)\} > 0$ *holds for all $z \in \mathbb{U}$, as shown in the right side of Figure 1.*

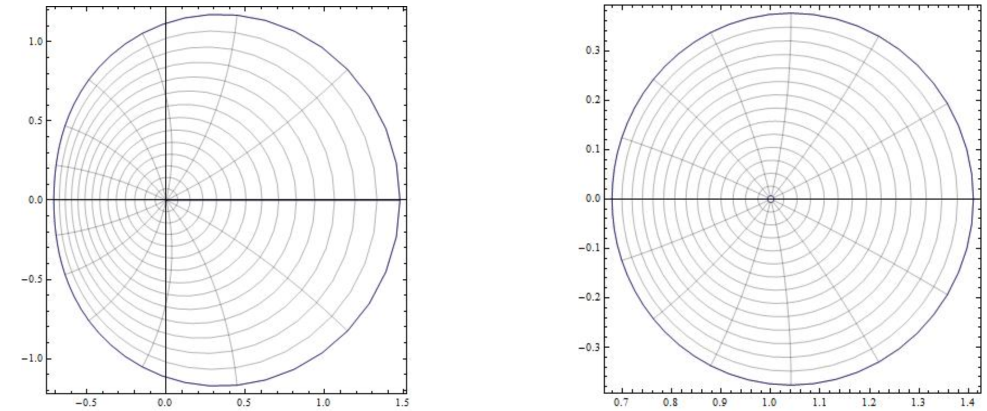

Figure 1. The images of $\tilde{f}(z)$ and $z\tilde{f}'(z)/\tilde{f}(z)$ in $\mathbb{U}$.

Letting $\beta = 1,\ \gamma = 0$ and $p(z) = f(z)/z$ in Theorem 1, we have the following result.

Corollary 5. *Let $f \in \mathcal{A}$. If*

$$\frac{f(z)}{z} + \frac{zf'(z)}{f(z)} \neq 1 + ik \quad (z \in \mathbb{U})$$

for all k with $|k| \geq \sqrt{3}$, then

$$\operatorname{Re} \frac{f(z)}{z} > 0 \quad (z \in \mathbb{U}).$$

Further, we derive the following corollary.

Corollary 6. *Let $f \in \mathcal{A}$ and let*

$$F(z) = \left\{ \frac{\beta + \gamma}{z^\gamma} \int_0^z t^{\gamma - 1} f^\beta(t) dt \right\}^{\frac{1}{\beta}} \quad (\beta > 0,\ \beta + \gamma > 0).$$

If

$$\frac{zf'(z)}{f(z)} \neq -\frac{\gamma}{\beta} + ik \quad (z \in \mathbb{U})$$

for all k ($|k| \geq \sqrt{2(\beta + \gamma) + 1}/\beta$), then

$$\frac{zF'(z)}{F(z)} \prec \frac{1 + (1 + \frac{2\gamma}{\beta})z}{1 - z} \quad (z \in \mathbb{U}).$$

Proof. From the definition of F, we have

$$\frac{zF'(z)}{F(z)} + \frac{\gamma}{\beta} = \frac{\beta + \gamma}{\beta} \frac{f^\beta(z)}{F^\beta(z)}. \tag{2.4}$$

Let

$$p(z) = \frac{zF'(z)}{F(z)}.$$

Taking logarithmic derivatives in (2.4) and multiplying by z, we obtain, after some simple calculations,

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = \frac{zf'(z)}{f(z)}.$$

Therefore, applying Theorem 1, we have the result. □

Next, we prove the following theorem.

Theorem 2. *Let p be nonzero analytic in $\mathbb{U}$ with $p(0) = 1$ and $0 < \eta < 1$. If*

$$\left| \mathrm{Im} \left(1 - \frac{1}{p(z)} + \frac{zp'(z)}{p(z)^2} \right) \right| < C(\eta) \quad (z \in \mathbb{U}) \tag{2.5}$$

where

$$C(\eta) = {t_0}^{\eta} \sin \frac{\pi}{2}\eta + \frac{\eta}{2}({t_0}^{\eta-1} + {t_0}^{\eta+1}) \cos \frac{\pi}{2}\eta \tag{2.6}$$

and

$$t_0 = \frac{-\sin \frac{\pi}{2}\eta + \sqrt{1 - \eta^2 \cos^2 \frac{\pi}{2}\eta}}{(1+\eta)\cos \frac{\pi}{2}\eta},$$

then

$$|\arg\ p(z)| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

Proof. We choose $q(z) = \big((1+z)/(1-z)\big)^{\eta}$ $(0 < \eta < 1)$, $\theta(\omega) = 1 - 1/\omega$, and $\varphi(\omega) = 1/\omega^2$ in Lemma 1, then we see that θ and φ are analytic in $q(\mathbb{U})$ and $\varphi(\omega) \neq 0$ for $\omega \in q(\mathbb{U})$. Further,

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{2\eta z}{1-z^2}\left(\frac{1-z}{1+z}\right)^{\eta}$$

is starlike, and for the function

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= 1 - \left(\frac{1-z}{1+z}\right)^{\eta} + \frac{2\eta z}{1-z^2}\left(\frac{1-z}{1+z}\right)^{\eta}, \end{aligned}$$

we have

$$\mathrm{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \mathrm{Re} \left\{ 1 + \frac{zQ'(z)}{Q(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

Note that $h(0) = 0$ and

$$\begin{aligned} h(e^{i\theta}) &= 1 - \left(i \cot \frac{\theta}{2}\right)^{-\eta} + i\, \frac{\eta}{\sin\theta} \left(i \cot \frac{\theta}{2}\right)^{-\eta} \\ &= 1 - \left|\cot \frac{\theta}{2}\right|^{-\eta} (\cos \frac{\pi}{2}\eta - i \sin \frac{\pi}{2}\eta) + i\, \frac{\eta}{\sin\theta} \left|\cot \frac{\theta}{2}\right|^{-\eta} \left(\cos \frac{\pi}{2}\eta - i \sin(\pm \frac{\pi}{2}\eta)\right) \\ &= \left(1 - \left|\tan \frac{\theta}{2}\right|^{\eta} \cos \frac{\pi}{2}\eta + \frac{\eta}{\sin\theta} \left|\tan \frac{\theta}{2}\right|^{\eta} \sin(\pm \frac{\pi}{2}\eta)\right) \\ &\quad + i \left(\left|\tan \frac{\theta}{2}\right|^{\eta} \sin(\pm \frac{\pi}{2}\eta) + \frac{\eta}{\sin\theta} \left|\tan \frac{\theta}{2}\right|^{\eta} \cos \frac{\pi}{2}\eta \right), \end{aligned}$$

where we take " + " for $0 < \theta < \pi$, and " - " for $-\pi < \theta < 0$. Since the imaginary part of $h(e^{i\theta})$ is an odd function of θ, we consider only the case $0 < \theta < \pi$. If we put $\tan(\theta/2) = t$ $(t > 0)$, then we have

$$\begin{aligned}\operatorname{Im} h(e^{i\theta}) &= t^{\eta} \sin \frac{\pi}{2}\eta + \frac{\eta}{2}(t^{\eta-1} + t^{\eta+1}) \cos \frac{\pi}{2}\eta \\ &\equiv g(t).\end{aligned}$$

It is easy to see that the function $g(t)$ has the minimum value at the point

$$t_0 = \frac{-\sin \frac{\pi}{2}\eta + \sqrt{1 - \eta^2 \cos^2 \frac{\pi}{2}\eta}}{(1+\eta) \cos \frac{\pi}{2}\eta}.$$

Therefore, we conclude that

$$|\operatorname{Im} h(e^{i\theta})| \geq {t_0}^{\eta} \sin \frac{\pi}{2}\eta + \frac{\eta}{2}({t_0}^{\eta-1} + {t_0}^{\eta+1}) \cos \frac{\pi}{2}\eta,$$

and so, by assumption (2.5),

$$1 - \frac{1}{p(z)} + \frac{zp'(z)}{p^2(z)} \prec h(z) \ (z \in \mathbb{U}).$$

Hence, from Lemma 1, we have $p(z) \prec q(z)$ $(z \in \mathbb{U})$, and this completes the proof of Theorem 2. □

From Theorem 2, we have the following result.

Corollary 7. *Let $f \in \mathcal{A}$ with $f(z)f'(z)/z \neq 0$ for $z \in \mathbb{U}$ and $0 < \eta < 1$. If*

$$\left| \operatorname{Im} \frac{f(z)f''(z)}{(f'(z))^2} \right| < C(\eta) \quad (z \in \mathbb{U}),$$

where $C(\eta)$ is given by (2.6)*, then*

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

Proof. Setting

$$p(z) = \frac{zf'(z)}{f(z)}$$

in Theorem 2, we see that p is regular in $\mathbb{U}$, $p(0) = 1$, and $p(z) \neq 0$ in $\mathbb{U}$. It can be derived that

$$\frac{f(z)f''(z)}{(f'(z))^2} = 1 - \frac{1}{p(z)} + \frac{zp'(z)}{(p(z))^2}.$$

Thus, from Theorem 2, we immediately have the result. □

Example 2. *Letting $\eta = 1/2$ in Corollary 7, we have $C(1/2) \doteqdot 0.72674$. Therefore, if*

$$\left| \operatorname{Im} \frac{f(z)f''(z)}{(f'(z))^2} \right| < C(1/2) \quad (z \in \mathbb{U}),$$

then

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{4} \quad (z \in \mathbb{U}).$$

Taking $p(z) = f(z)/z$ in Theorem 2, we have the following corollary.

Corollary 8. *Let $f \in \mathcal{A}$ with $f(z)/z \neq 0$ for $z \in \mathbb{U}$ and $0 < \eta < 1$. If*

$$\left|\operatorname{Im}\left(1 - \frac{2z}{f(z)} + \frac{z^2 f'(z)}{(f(z))^2}\right)\right| < C(\eta) \quad (z \in \mathbb{U}),$$

where $C(\eta)$ is given by (2.6)*, then*

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

Finally, by using a similar method of the proofs of Theorems 1 and 2, we have Theorem 3 below.

Theorem 3. *Let α, β, and η be real numbers satisfying $\alpha > 0$, $0 < \eta \leq 1$, and*

$$C(\alpha,\ \beta,\ \eta) > |1 - \beta|, \tag{2.7}$$

where

$$C(\alpha,\ \beta,\ \eta) = \begin{cases} \beta \sin \frac{\pi}{2}\eta + \alpha\eta \cos \frac{\pi}{2}\eta, & \text{if } \beta \cos \frac{\pi}{2}\eta > \alpha\eta \sin \frac{\pi}{2}\eta, \\ \sqrt{\beta^2 + \alpha^2\eta^2}, & \text{if } \beta \cos \frac{\pi}{2}\eta \leq \alpha\eta \sin \frac{\pi}{2}\eta. \end{cases} \tag{2.8}$$

Let p be analytic in $\mathbb{U}$ with $p(0) = 1$. If

$$\left|p(z) - \beta + \alpha\frac{zp'(z)}{p(z)}\right| < C(\alpha,\ \beta,\ \eta) \quad (z \in \mathbb{U}), \tag{2.9}$$

then

$$|\arg\ p(z)| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

Proof. We note that the inequality (2.9) is well-defined by (2.7). Applying the same method of the proof in Theorem 1, we can see that $p(z) \neq 0$ for $z \in \mathbb{U}$. Let $q(z) = \big((1 + z)/(1 - z)\big)^\eta$ $(0 < \eta \leq 1)$, $\theta(\omega) = \omega - \beta$, and $\varphi(\omega) = \alpha/\omega$ in Lemma 1, then

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{2\alpha\eta z}{1 - z^2}$$

and

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= \left(\frac{1 + z}{1 - z}\right)^\eta - \beta + \frac{2\alpha\eta z}{1 - z^2}. \end{aligned}$$

Also, the other conditions (i) and (ii) of Lemma 1 can be checked to be satisfied. Note that

$$h(e^{i\theta}) = \left(i \cot \frac{\theta}{2}\right)^\eta - \beta + i\,\frac{\alpha\eta}{\sin\theta} \quad (0 < |\theta| < \pi),$$

and

$$i \cot \frac{\theta}{2} = \begin{cases} e^{i\frac{\pi}{2}} \cot \frac{\theta}{2}, & \text{if } 0 < \theta < \pi, \\ -e^{-i\frac{\pi}{2}} \cot \frac{\theta}{2}, & \text{if } -\pi < \theta < 0. \end{cases}$$

Setting $t = \cot\ (\theta/2)$ $(0 < \theta < \pi)$ without loss of generality, we obtain

$$\begin{aligned}|h(e^{i\theta})|^2 &= \left(t^\eta \cos\frac{\pi}{2}\eta - \beta\right)^2 + \left(t^\eta \sin\frac{\pi}{2}\eta + \frac{\alpha\eta(1+t^2)}{2t}\right)^2 \\ &\geq t^{2\eta} + 2\left(\alpha\eta\sin\frac{\pi}{2}\eta - \beta\cos\frac{\pi}{2}\eta\right)t^\eta + \beta^2 + \alpha^2\eta^2 \\ &\equiv g(t), \quad t > 0.\end{aligned}$$

We first consider the case $\beta\cos(\pi\eta/2) > \alpha\eta\sin(\pi\eta/2)$, then the function $g(t)$ has the minimum value at

$$t_0 = \left(\beta\cos\frac{\pi}{2}\eta - \alpha\eta\sin\frac{\pi}{2}\eta\right)^{\frac{1}{\eta}}$$

so that

$$|h(e^{i\theta})|^2 \geq g(t_0) = \left(\beta\sin\frac{\pi}{2}\eta + \alpha\eta\cos\frac{\pi}{2}\eta\right)^2.$$

Hence we see that

$$|h(e^{i\theta})| \geq \beta\sin\frac{\pi}{2}\eta + \alpha\eta\cos\frac{\pi}{2}\eta = C(\alpha,\ \beta,\ \eta).$$

Therefore, by the assumption (2.9), we have

$$p(z) - \beta + \alpha\frac{zp'(z)}{p(z)} \prec h(z) \quad (z \in \mathbb{U}). \tag{2.10}$$

Next, we consider the case $\beta\cos(\pi\eta/2) \leq \alpha\eta\sin(\pi\eta/2)$, then the function g is increasing on $(0, \infty)$ and it follows that

$$|h(e^{i\theta})|^2 \geq g(0) = \beta^2 + \alpha^2\eta^2.$$

Hence, we get

$$|h(e^{i\theta})| \geq \sqrt{\beta^2 + \alpha^2\eta^2} = C(\alpha,\ \beta,\ \eta).$$

Therefore, by the assumption (2.9), we have (2.10) again. Finally, with the aid of Lemma 1, we obtain $p(z) \prec q(z)$ $(z \in \mathbb{U})$, that is, $|\arg\ p(z)| < \frac{\pi}{2}\eta$. □

Taking $\beta = \alpha$ in Theorem 3, we have the following result.

Corollary 9. *Let* α *and* η *be real numbers such that* $\alpha > 0$, $0 < \eta \leq 1$, *and*

$$\sin\frac{\pi}{2}\eta + \eta\cos\frac{\pi}{2}\eta > \frac{1-\alpha}{\alpha}.$$

Let $x^* = 0.638\ldots$ *be the unique root of the equation* $x = \cot(\pi x/2)$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\alpha\frac{zf''(z)}{f'(z)} + (1-\alpha)\frac{zf'(z)}{f'(z)}\right| < C(\alpha,\ \eta) \quad (z \in \mathbb{U}),$$

where

$$C(\alpha,\ \eta) = \begin{cases}\alpha(\sin\frac{\pi}{2}\eta + \eta\cos\frac{\pi}{2}\eta), & \textit{if}\ \ 0 < \eta < x^*, \\ \alpha\sqrt{1+\eta^2}, & \textit{if}\ \ x^* \leq \eta \leq 1,\end{cases}$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

Example 3. *Choosing* $\alpha = 1$ *and* $\eta = 1/2$ *in Corollary 9, we have* $C(1,\ 1/2) = 3\sqrt{2}/4$. *Therefore, we obtain that if*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3\sqrt{2}}{4} \quad (z \in \mathbb{U}),$$

then

$$\left|\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4} \quad (z \in \mathbb{U}).$$

Making $p(z) = f(z)/z$ in Theorem 3, we have the following result.

Corollary 10. *Let* α, β, *and* η *be real numbers satisfying* (2.7). *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\frac{f(z)}{z} - (\beta + 1) + \alpha\frac{zf'(z)}{f(z)}\right| < C(\alpha,\ \beta,\ \eta) \quad (z \in \mathbb{U}),$$

where $C(\alpha,\ \beta,\ \eta)$ *is given by* (2.8)*, then*

$$\left|\arg\ \frac{f(z)}{z}\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

We remark that, for the case $\eta = 1$ in Theorem 3, we have $C(\alpha,\ \beta,\ 1) = \sqrt{\alpha^2 + \beta^2}$. We end this paper with showing that this quantity can be improved as follows:

Corollary 11. *Let* α *and* β *be real numbers such that* $\alpha > 0$ *and* $\sqrt{\alpha(\alpha + 2) + \beta^2} > |1 - \beta|$. *Let* p *be analytic in* $\mathbb{U}$ *with* $p(0) = 1$. *If*

$$\left|p(z) - \beta + \alpha\frac{zp'(z)}{p(z)}\right| < \sqrt{\alpha(\alpha + 2) + \beta^2} \quad (z \in \mathbb{U}),$$

then Re $p(z) > 0$ *for all* $z \in \mathbb{U}$.

Proof. By defining the same functions q, θ, φ, Q, and h with $\eta = 1$, as in the proof of Theorem 3, we will reach the following equality:

$$\left|h(e^{i\theta})\right|^2 = \beta^2 + \left(t + \frac{\alpha(1 + t^2)}{2t}\right)^2, \tag{2.11}$$

where $t = \cot(\theta/2)$ with $0 < \theta < \pi$. Furthermore, since $t > 0$, we get

$$t + \frac{\alpha(1 + t^2)}{2t} = \frac{1}{2}\left[\alpha \cdot t^{-1} + (\alpha + 2)t\right] \geq \sqrt{\alpha(\alpha + 2)}. \tag{2.12}$$

Hence, combining (2.11) and (2.12) leads us to get

$$\left|h(e^{i\theta})\right| \geq \sqrt{\alpha(\alpha + 2) + \beta^2} \quad (0 < \theta < \pi).$$

Thus, it follows from the same proof of Theorem 3 that $|\arg p(z)| < \pi/2$ $(z \in \mathbb{U})$, or Re $p(z) > 0$ $(z \in \mathbb{U})$. □

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

Prof. Dr. Nak Eun Cho is the Guest Editor of special issue "Geometric Function Theory and Special Functions" for AIMS Mathematics. Prof. Dr. Nak Eun Cho was not involved in the editorial review and the decision to publish this article.

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