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Research article

First-order differential subordinations associated with Carathéodory functions

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Abstract: In the present paper, we investigated some conditions to be in the class of Carathéodory functions by using the concept of the first-order differential subordinations. Moreover, various interesting special cases were considered in the geometric function theory as applications of main results presented here.

Keywords: differential subordination; Carathéodory function; univalent function; starlike function of order alpha; strongly starlike function
Mathematics Subject Classification: 30C45, 30C80

1. Introduction

Let $\mathcal{P}(\alpha)$ be the class of analytic functions p of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, with $\mathrm{Re}\,p(z) > \alpha$ for $z \in \mathbb{U}$. The class $\mathcal{P} \equiv \mathcal{P}(0)$ is known as the Carathéodory class or the class of functions with positive real part [2, 3], pioneered by Carathéodory. The theory of Carathéodory functions plays a very important role in the geometric function theory. For recent developments, the readers may refer to the works of Kim and Cho [5], Kwon and Sim [6], Nunokawa et al. [16], Sim et al. [18] and Wang [22].

Let $\mathcal{A}$ denote the class of all functions f analytic in $\mathbb{U}$ with the usual normalization $f(0) = f'(0)-1 = 0$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ in $\mathbb{U}$ such that $f(z) = g(w(z))$.

A function $f \in \mathcal{A}$ is said to be strongly starlike of order η $(0 < \eta \leq 1)$ if, and only if,

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\eta} \quad (z \in \mathbb{U}). \tag{1.1}$$

We note that the conditions (1.1) can be written by

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}[\eta]$ the subclass of $\mathcal{A}$ consisting of all strongly starlike functions of order η $(0 < \eta \leq 1)$. The class $\mathcal{S}[\eta]$ was introduced and studied by Brannan and Kirwan [1] and Stankiewicz [20, 21]. We also note that $\mathcal{S}[1] \equiv \mathcal{S}^*$ is the well-known class of all normalized starlike functions in $\mathbb{U}$. The class $\mathcal{S}[\eta]$ and the related classes have been extensively studied by Mocanu [14] and Nunokawa [15]. It is worth noticing that f belongs to $\mathcal{S}[\eta]$ if it satisfies

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\eta)} \quad (z \in \mathbb{U}),$$

where

$$\alpha(\eta) = \frac{2}{\pi} \arctan\left\{\tan\frac{\eta}{2}\pi + \frac{\beta}{(1-\eta)^{\frac{1-\eta}{2}}(1+\eta)^{\frac{1+\eta}{2}}\cos\frac{\eta}{2}\pi}\right\}.$$

Given $\alpha \in [0,1)$, let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$, which consists of all starlike functions of order α, namely, $f \in \mathcal{A}$ belongs to $\mathcal{S}^*(\alpha)$ if, and only if, it satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (z \in \mathbb{U}).$$

The class $\mathcal{S}^*(\alpha)$ was introduced by Robertson [17]. Clearly, it holds that $\mathcal{S}^*(0) \equiv \mathcal{S}[1] \equiv \mathcal{S}^*$. A typical sufficient condition for starlike functions of order α is given by Wilken and Feng [23], which states that if $f \in \mathcal{A}$, then

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\beta)z}{1-z} \quad (z \in \mathbb{U})$$

implies $f \in \mathcal{S}^*(\alpha)$, where

$$\beta = \beta(\alpha) := \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})}, & \text{if } \alpha \neq 1/2, \\ \dfrac{1}{2\log 2}, & \text{if } \alpha = 1/2. \end{cases}$$

Given $\eta \in (0,1]$, let $\mathcal{T}[\eta]$ be the class of $f \in \mathcal{A}$ such that

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\eta} \quad (z \in \mathbb{U}).$$

The class $\mathcal{T} \equiv \mathcal{T}[1]$ plays an important role in the theory of univalent functions, although all elements in $\mathcal{T}$ are functions that are not necessarily univalent. In [7], several sufficient conditions for functions in $\mathcal{T}[\eta]$ were introduced.

If ψ is analytic in a domain $\mathbb{D} \subset \mathbb{C}^2$, h is univalent in $\mathbb{U}$ and p is analytic in $\mathbb{U}$ with $(p(z), zp'(z)) \in \mathbb{D}$ for $z \in \mathbb{U}$, then p is said to satisfy the first-order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z) \quad (z \in \mathbb{U}). \tag{1.2}$$

The univalent function q is said to be a dominant of the differential subordination (1.2) if $p \prec q$ for all p satisfying (1.2). If $\tilde{q}$ is a dominant of (1.2) and $\tilde{q} \prec q$ for all dominants of (1.2), then $\tilde{q}$ is said to be the best dominant of the differential subordination (1.2). The general theory of the first-order differential subordinations, with many interesting applications, especially in the theory of univalent functions, was developed by Miller and Mocanu [10] (also see [4, 8, 9, 11–13]).

In this paper, by applying the result obtained by Miller and Mocanu [10], we will investigate conditions to be in the class of Carathéodory functions. We will also find new sufficient conditions for $f \in \mathcal{A}$ to belong to the classes $\mathcal{S}[\eta]$, $\mathcal{S}^*(\alpha)$, and $\mathcal{T}[\eta]$ as some applications of the main results presented here. A differential subordination of the Briot-Bouquet type [12] (also see [13, Section 3]) will be considered for conditions for $f \in \mathcal{S}^*(\alpha)$ and $f \in \mathcal{T}[1]$, and an integral operator related to the differential subordination of this type will be discussed as our additional results. Moreover, more conditions for $f \in \mathcal{S}[\eta]$ and $f \in \mathcal{T}[\eta]$ will be introduced by using a nonlinear first-order differential subordination.

2. Main results

In proving our results, we shall need the following lemma due to Miller and Mocanu [10].

Lemma 1. *Let q be univalent in $\mathbb{U}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $q(\omega) \neq 0$ when $\omega \in q(\mathbb{U})$. Set $Q(z) = zq'(z)\varphi(q(z)), h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) *Q is starlike in $\mathbb{U}$,*
(ii) $\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \mathrm{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in \mathbb{U})$.

If p is analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$, and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.1}$$

then $p \prec q$ and q is the best dominant of (2.1).

With the help of Lemma 1, we now derive the following Theorem 1.

Theorem 1. *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $\beta > 0$, $\beta + \gamma > 0$. If*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \neq -\frac{\gamma}{\beta} + ik \quad (z \in \mathbb{U}) \tag{2.2}$$

for all k ($|k| \geq \sqrt{2(\beta + \gamma) + 1}/\beta$), then

$$p(z) \prec \frac{1 + (1 + (2\gamma/\beta))\, z}{1 - z} \quad (z \in \mathbb{U}).$$

Proof. First, we note that $p(z) \neq -(\gamma/\beta)$ for $z \in \mathbb{U}$ under the condition (2.2). In fact, if $\beta p(z) + \gamma$ has a zero $z_0 \in \mathbb{U}$ of order n ($n \geq 1$) at a point $z_0 \in \mathbb{U}\backslash\{0\}$, then we may write

$$\beta p(z) + \gamma = (z - z_0)^n q(z) \quad (n \in \mathbb{N} := \{1,\ 2,\ 3, \cdots\}),$$

where p is analytic in $\mathbb{U}$ with $q(z_0) \neq 0$, then it follows that

$$\frac{\beta z p'(z)}{\beta p(z) + \gamma} = \frac{zq'(z)}{q(z)} + \frac{nz}{z - z_0}. \tag{2.3}$$

Therefore,

$$\lim_{z\to z_0}(z-z_0)\frac{\beta z p'(z)}{\beta p(z)+\gamma} = nz_0 \neq 0.$$

Letting z approach z_0 in the direction of $\arg z_0$, the righthand side of (2.3) takes infinite pure imaginary value. This contradicts the assumption (2.2).

Let $q(z) = (1 + (1 + 2\gamma/\beta)z)/(1 - z)$, $\theta(\omega) = \omega$, and $\varphi(\omega) = 1/(\beta\omega + \gamma)$ in Lemma 1, then θ and φ are analytic in $q(\mathbb{U})$ and $\varphi(\omega) \neq 0$ for $\varphi \in q(\mathbb{U})$. Setting

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{2z}{\beta(1-z^2)}$$

and

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= \frac{1}{\beta}\left\{(\beta+\gamma)\frac{1+z}{1-z} + \frac{2z}{1-z^2} - \gamma\right\}, \end{aligned}$$

the conditions (i) and (ii) of Lemma 1 can be verified. Therefore, Lemma 1 gives that if

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \quad (z \in \mathbb{U})$$

with

$$h(z) = \frac{1}{\beta}\left\{(\beta+\gamma)\frac{1+z}{1-z} + \frac{2z}{1-z^2} - \gamma\right\},$$

then

$$p(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Noting that

$$h(e^{i\theta}) = \frac{1}{\beta}\left\{(\beta+\gamma)\frac{1+e^{i\theta}}{1-e^{i\theta}} + \frac{2e^{i\theta}}{1-e^{i2\theta}} - \gamma\right\} \quad (0 < |\theta| < \pi),$$

we obtain

$$\operatorname{Re} h(e^{i\theta}) = -\frac{\gamma}{\beta}$$

and

$$\operatorname{Im} h(e^{i\theta}) = \frac{1}{\beta}\left\{(\beta+\gamma)\frac{\sin\theta}{1-\cos\theta} + \frac{1}{\sin\theta}\right\} \quad (0 < |\theta| < \pi).$$

Meanwhile, since the imaginary part of $h(e^{i\theta})$ is an odd function, we consider only the case $0 < \theta < \pi$. Putting $\tan(\theta/2) = t$ $(0 < \theta < \pi)$, we have

$$\begin{aligned} \operatorname{Im} h(e^{i\theta}) &= \frac{1}{\beta}\left\{(\beta+\gamma)\frac{\sin\theta}{1-\cos\theta} + \frac{1}{\sin\theta}\right\} \\ &= \frac{t^2 + 2(\beta+\gamma)+1}{2\beta t} \\ &= g(t). \end{aligned}$$

Here, the function $g(t)$ has a minimum value at $t_0 = \sqrt{2(\beta+\gamma)+1}$. Hence we have

$$|\mathrm{Im}\, h(e^{i\theta})| \geq |g(t_0)| = \frac{\sqrt{2(\beta+\gamma)+1}}{\beta}.$$

Applying Lemma 1 and the assumption (2.2), we conclude that

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \quad (z \in \mathbb{U}).$$

This completes the proof of Theorem 1. □

Taking $p(z) = zf'(z)/f(z)$, $\beta = 1$, and $\gamma = (1/\alpha) - 1$ $(0 < \alpha \leq 1)$ in Theorem 1, we have the following result.

Corollary 1. *Let $f \in \mathcal{A}$ and $0 < \alpha \leq 1$. If*

$$\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} \neq \alpha - 1 + ik \quad (z \in \mathbb{U})$$

for all k ($|k| \geq \sqrt{(2+\alpha)/\alpha}/\beta$), then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1+2(1-\alpha)/\alpha)z}{1-z} \quad (z \in \mathbb{U}).$$

Proof. Putting

$$p(z) = \frac{zf'(z)}{f(z)},$$

we have

$$\begin{aligned}
&\alpha z(zf'(z))' + (1-\alpha)zf'(z)\\
&= \alpha zf(z)p'(z) + \alpha zp(z)f'(z) + (1-\alpha)p(z)f(z)\\
&= \left(\alpha zp'(z) + p(z)(\alpha p(z) + 1 - \alpha)\right) f(z)
\end{aligned}$$

and

$$\alpha zf'(z) + (1-\alpha)f(z) = (\alpha p(z) + 1 - \alpha)f(z).$$

Hence,

$$\begin{aligned}
\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} &= \frac{\alpha zp'(z) + p(z)(\alpha p(z) + 1 - \alpha)}{\alpha p(z) + 1 - \alpha}\\
&= \frac{p(z) + zp'(z)}{p(z) + \left(\frac{1}{\alpha} - 1\right)}.
\end{aligned}$$

Therefore, applying Theorem 1, we have Corollary 1. □

Corollary 2. *Let $f \in \mathcal{A}$ and let*

$$F(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z t^{\frac{1}{\alpha}-2} f(t)dt \quad (0 < \alpha \leq 1).$$

If

$$\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} \neq \alpha - 1 + ik \quad (z \in \mathbb{U})$$

for all k ($|k| \geq \sqrt{(2+\alpha)/\alpha/\beta}$), then

$$\frac{\alpha z(zF'(z))' + (1-\alpha)zF'(z)}{\alpha zF'(z) + (1-\alpha)F(z)} \prec \frac{1 + (1 + 2(1-\alpha)/\alpha)z}{1-z} \quad (z \in \mathbb{U}).$$

Proof. Differentiating F with respect to z and multiplying by z, we have

$$\frac{\alpha z(zF'(z))' + (1-\alpha)zF'(z)}{\alpha zF'(z) + (1-\alpha)F(z)} = \frac{zf'(z)}{f(z)}.$$

Therefore, the result follows from Corollary 1. □

Letting $\beta = 1/\alpha$ $(\alpha > 0)$, $\gamma = 0$, and $p(z) = zf'(z)/f(z)$ in Theorem 1, we have the following result.

Corollary 3. *Let $f \in \mathcal{A}$ and $\alpha > 0$. If*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \neq ik \quad (z \in \mathbb{U})$$

for all k ($|k| \geq \sqrt{\alpha(2+\alpha)}$), then f is starlike in $\mathbb{U}$.

Taking $\beta = 1$, $\gamma = 0$, and $p(z) = zf'(z)/f(z)$ in Theorem 1, we have the following result.

Corollary 4. *Let $f \in \mathcal{A}$. If*

$$1 + \frac{zf''(z)}{f'(z)} \neq ik \quad (z \in \mathbb{U})$$

for all k ($|k| \geq \sqrt{3}$), then f is a starlike in $\mathbb{U}$.

Example 1. *Consider a function $\tilde{f} : \mathbb{U} \to \mathbb{C}$ defined by*

$$\tilde{f}(z) = \frac{1}{\sqrt{3}-1}(e^{(\sqrt{3}-1)z} - 1).$$

Then we have

$$1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)} = 1 + (\sqrt{3}-1)z$$

and

$$\left|1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)}\right| < \sqrt{3}, \quad z \in \mathbb{U}.$$

Therefore, by Corollary 4, $\tilde{f}$ is starlike in $\mathbb{U}$ (see also the left side of Figure 1). In fact, we can check that $\mathrm{Re}\,\{z\tilde{f}'(z)/\tilde{f}(z)\} > 0$ *holds for all $z \in \mathbb{U}$, as shown in the right side of Figure 1.*

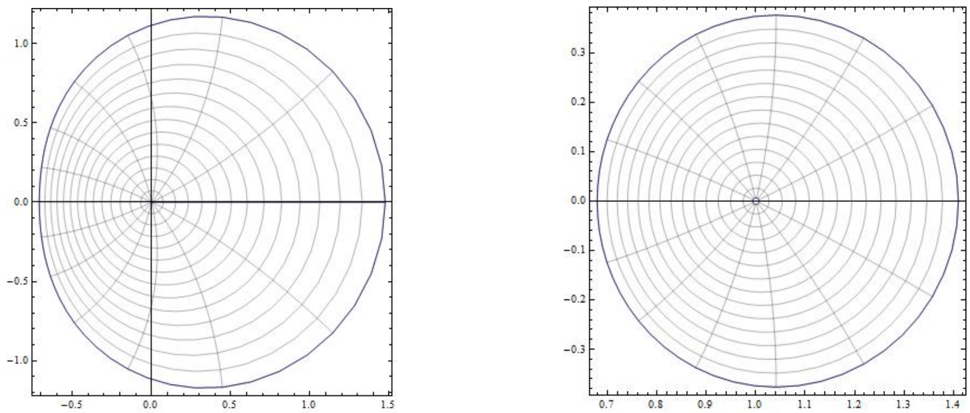

Figure 1. The images of $\tilde{f}(z)$ and $z\tilde{f}'(z)/\tilde{f}(z)$ in $\mathbb{U}$.

Letting $\beta = 1,\ \gamma = 0$ and $p(z) = f(z)/z$ in Theorem 1, we have the following result.

Corollary 5. *Let $f \in \mathcal{A}$. If*

$$\frac{f(z)}{z} + \frac{zf'(z)}{f(z)} \neq 1 + ik \quad (z \in \mathbb{U})$$

for all k with $|k| \geq \sqrt{3}$, then

$$\mathrm{Re}\,\frac{f(z)}{z} > 0 \quad (z \in \mathbb{U}).$$

Further, we derive the following corollary.

Corollary 6. *Let $f \in \mathcal{A}$ and let*

$$F(z) = \left\{\frac{\beta+\gamma}{z^{\gamma}}\int_0^z t^{\gamma-1} f^{\beta}(t)dt\right\}^{\frac{1}{\beta}} \quad (\beta > 0,\ \beta+\gamma > 0).$$

If

$$\frac{zf'(z)}{f(z)} \neq -\frac{\gamma}{\beta} + ik \quad (z \in \mathbb{U})$$

for all k ($|k| \geq \sqrt{2(\beta+\gamma)+1}/\beta$), then

$$\frac{zF'(z)}{F(z)} \prec \frac{1 + (1 + \frac{2\gamma}{\beta})z}{1 - z} \quad (z \in \mathbb{U}).$$

Proof. From the definition of F, we have

$$\frac{zF'(z)}{F(z)} + \frac{\gamma}{\beta} = \frac{\beta+\gamma}{\beta}\frac{f^{\beta}(z)}{F^{\beta}(z)}. \tag{2.4}$$

Let

$$p(z) = \frac{zF'(z)}{F(z)}.$$

Taking logarithmic derivatives in (2.4) and multiplying by z, we obtain, after some simple calculations,

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = \frac{zf'(z)}{f(z)}.$$

Therefore, applying Theorem 1, we have the result. □

Next, we prove the following theorem.

Theorem 2. *Let p be nonzero analytic in $\mathbb{U}$ with $p(0) = 1$ and $0 < \eta < 1$. If*

$$\left| \mathrm{Im} \left(1 - \frac{1}{p(z)} + \frac{zp'(z)}{p(z)^2} \right) \right| < C(\eta) \quad (z \in \mathbb{U}) \tag{2.5}$$

where

$$C(\eta) = {t_0}^{\eta} \sin \frac{\pi}{2}\eta + \frac{\eta}{2}({t_0}^{\eta-1} + {t_0}^{\eta+1}) \cos \frac{\pi}{2}\eta \tag{2.6}$$

and

$$t_0 = \frac{-\sin \frac{\pi}{2}\eta + \sqrt{1 - \eta^2 \cos^2 \frac{\pi}{2}\eta}}{(1+\eta)\cos \frac{\pi}{2}\eta},$$

then

$$|\arg\ p(z)| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

Proof. We choose $q(z) = \big((1+z)/(1-z)\big)^{\eta}$ $(0 < \eta < 1)$, $\theta(\omega) = 1 - 1/\omega$, and $\varphi(\omega) = 1/\omega^2$ in Lemma 1, then we see that θ and φ are analytic in $q(\mathbb{U})$ and $\varphi(\omega) \neq 0$ for $\omega \in q(\mathbb{U})$. Further,

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{2\eta z}{1 - z^2} \left(\frac{1-z}{1+z} \right)^{\eta}$$

is starlike, and for the function

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= 1 - \left(\frac{1-z}{1+z} \right)^{\eta} + \frac{2\eta z}{1-z^2} \left(\frac{1-z}{1+z} \right)^{\eta}, \end{aligned}$$

we have

$$\mathrm{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \mathrm{Re} \left\{ 1 + \frac{zQ'(z)}{Q(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

Note that $h(0) = 0$ and

$$\begin{aligned} h(e^{i\theta}) &= 1 - \left(i \cot \frac{\theta}{2} \right)^{-\eta} + i\, \frac{\eta}{\sin\theta} \left(i \cot \frac{\theta}{2} \right)^{-\eta} \\ &= 1 - \left| \cot \frac{\theta}{2} \right|^{-\eta} (\cos \frac{\pi}{2}\eta - i \sin \frac{\pi}{2}\eta) + i\, \frac{\eta}{\sin\theta} \left| \cot \frac{\theta}{2} \right|^{-\eta} \left(\cos \frac{\pi}{2}\eta - i \sin(\pm \frac{\pi}{2}\eta) \right) \\ &= \left(1 - \left| \tan \frac{\theta}{2} \right|^{\eta} \cos \frac{\pi}{2}\eta + \frac{\eta}{\sin\theta} \left| \tan \frac{\theta}{2} \right|^{\eta} \sin(\pm \frac{\pi}{2}\eta) \right) \\ &\quad + i \left(\left| \tan \frac{\theta}{2} \right|^{\eta} \sin(\pm \frac{\pi}{2}\eta) + \frac{\eta}{\sin\theta} \left| \tan \frac{\theta}{2} \right|^{\eta} \cos \frac{\pi}{2}\eta \right), \end{aligned}$$

where we take " + " for $0 < \theta < \pi$, and " - " for $-\pi < \theta < 0$. Since the imaginary part of $h(e^{i\theta})$ is an odd function of θ, we consider only the case $0 < \theta < \pi$. If we put $\tan(\theta/2) = t$ $(t > 0)$, then we have

$$\begin{aligned}\operatorname{Im} h(e^{i\theta}) &= t^{\eta} \sin\frac{\pi}{2}\eta + \frac{\eta}{2}(t^{\eta-1} + t^{\eta+1})\cos\frac{\pi}{2}\eta \\ &\equiv g(t).\end{aligned}$$

It is easy to see that the function $g(t)$ has the minimum value at the point

$$t_0 = \frac{-\sin\frac{\pi}{2}\eta + \sqrt{1 - \eta^2\cos^2\frac{\pi}{2}\eta}}{(1+\eta)\cos\frac{\pi}{2}\eta}.$$

Therefore, we conclude that

$$|\operatorname{Im} h(e^{i\theta})| \geq {t_0}^{\eta} \sin\frac{\pi}{2}\eta + \frac{\eta}{2}({t_0}^{\eta-1} + {t_0}^{\eta+1})\cos\frac{\pi}{2}\eta,$$

and so, by assumption (2.5),

$$1 - \frac{1}{p(z)} + \frac{zp'(z)}{p^2(z)} \prec h(z) \ (z \in \mathbb{U}).$$

Hence, from Lemma 1, we have $p(z) \prec q(z)$ $(z \in \mathbb{U})$, and this completes the proof of Theorem 2. □

From Theorem 2, we have the following result.

Corollary 7. *Let* $f \in \mathcal{A}$ *with* $f(z)f'(z)/z \neq 0$ *for* $z \in \mathbb{U}$ *and* $0 < \eta < 1$. *If*

$$\left|\operatorname{Im} \frac{f(z)f''(z)}{(f'(z))^2}\right| < C(\eta) \quad (z \in \mathbb{U}),$$

where $C(\eta)$ *is given by* (2.6), *then*

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

Proof. Setting

$$p(z) = \frac{zf'(z)}{f(z)}$$

in Theorem 2, we see that p is regular in $\mathbb{U}$, $p(0) = 1$, and $p(z) \neq 0$ in $\mathbb{U}$. It can be derived that

$$\frac{f(z)f''(z)}{(f'(z))^2} = 1 - \frac{1}{p(z)} + \frac{zp'(z)}{(p(z))^2}.$$

Thus, from Theorem 2, we immediately have the result. □

Example 2. *Letting* $\eta = 1/2$ *in Corollary 7, we have* $C(1/2) \doteqdot 0.72674$. *Therefore, if*

$$\left|\operatorname{Im} \frac{f(z)f''(z)}{(f'(z))^2}\right| < C(1/2) \quad (z \in \mathbb{U}),$$

then

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4} \quad (z \in \mathbb{U}).$$

Taking $p(z) = f(z)/z$ in Theorem 2, we have the following corollary.

Corollary 8. *Let $f \in \mathcal{A}$ with $f(z)/z \neq 0$ for $z \in \mathbb{U}$ and $0 < \eta < 1$. If*

$$\left|\operatorname{Im}\left(1 - \frac{2z}{f(z)} + \frac{z^2 f'(z)}{(f(z))^2}\right)\right| < C(\eta) \quad (z \in \mathbb{U}),$$

where $C(\eta)$ is given by (2.6)*, then*

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

Finally, by using a similar method of the proofs of Theorems 1 and 2, we have Theorem 3 below.

Theorem 3. *Let α, β, and η be real numbers satisfying $\alpha > 0$, $0 < \eta \leq 1$, and*

$$C(\alpha,\ \beta,\ \eta) > |1 - \beta|, \tag{2.7}$$

where

$$C(\alpha,\ \beta,\ \eta) = \begin{cases} \beta \sin \frac{\pi}{2}\eta + \alpha\eta \cos \frac{\pi}{2}\eta, & \text{if } \beta \cos \frac{\pi}{2}\eta > \alpha\eta \sin \frac{\pi}{2}\eta, \\ \sqrt{\beta^2 + \alpha^2\eta^2}, & \text{if } \beta \cos \frac{\pi}{2}\eta \leq \alpha\eta \sin \frac{\pi}{2}\eta. \end{cases} \tag{2.8}$$

Let p be analytic in $\mathbb{U}$ with $p(0) = 1$. If

$$\left|p(z) - \beta + \alpha \frac{zp'(z)}{p(z)}\right| < C(\alpha,\ \beta,\ \eta) \quad (z \in \mathbb{U}), \tag{2.9}$$

then

$$|\arg\ p(z)| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

Proof. We note that the inequality (2.9) is well-defined by (2.7). Applying the same method of the proof in Theorem 1, we can see that $p(z) \neq 0$ for $z \in \mathbb{U}$. Let $q(z) = \big((1+z)/(1-z)\big)^\eta$ $(0 < \eta \leq 1)$, $\theta(\omega) = \omega - \beta$, and $\varphi(\omega) = \alpha/\omega$ in Lemma 1, then

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{2\alpha\eta z}{1 - z^2}$$

and

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= \left(\frac{1+z}{1-z}\right)^\eta - \beta + \frac{2\alpha\eta z}{1 - z^2}. \end{aligned}$$

Also, the other conditions (i) and (ii) of Lemma 1 can be checked to be satisfied. Note that

$$h(e^{i\theta}) = \left(i \cot \frac{\theta}{2}\right)^\eta - \beta + i\,\frac{\alpha\eta}{\sin\theta} \quad (0 < |\theta| < \pi),$$

and

$$i \cot \frac{\theta}{2} = \begin{cases} e^{i\frac{\pi}{2}} \cot \frac{\theta}{2}, & \text{if } 0 < \theta < \pi, \\ -e^{-i\frac{\pi}{2}} \cot \frac{\theta}{2}, & \text{if } -\pi < \theta < 0. \end{cases}$$

Setting $t = \cot\ (\theta/2)$ $(0 < \theta < \pi)$ without loss of generality, we obtain

$$\begin{aligned}|h(e^{i\theta})|^2 &= \left(t^\eta \cos\frac{\pi}{2}\eta - \beta\right)^2 + \left(t^\eta \sin\frac{\pi}{2}\eta + \frac{\alpha\eta(1+t^2)}{2t}\right)^2 \\ &\geq t^{2\eta} + 2\left(\alpha\eta\sin\frac{\pi}{2}\eta - \beta\cos\frac{\pi}{2}\eta\right)t^\eta + \beta^2 + \alpha^2\eta^2 \\ &\equiv g(t), \quad t > 0.\end{aligned}$$

We first consider the case $\beta\cos(\pi\eta/2) > \alpha\eta\sin(\pi\eta/2)$, then the function $g(t)$ has the minimum value at

$$t_0 = \left(\beta\cos\frac{\pi}{2}\eta - \alpha\eta\sin\frac{\pi}{2}\eta\right)^{\frac{1}{\eta}}$$

so that

$$|h(e^{i\theta})|^2 \geq g(t_0) = \left(\beta\sin\frac{\pi}{2}\eta + \alpha\eta\cos\frac{\pi}{2}\eta\right)^2.$$

Hence we see that

$$|h(e^{i\theta})| \geq \beta\sin\frac{\pi}{2}\eta + \alpha\eta\cos\frac{\pi}{2}\eta = C(\alpha,\ \beta,\ \eta).$$

Therefore, by the assumption (2.9), we have

$$p(z) - \beta + \alpha\frac{zp'(z)}{p(z)} \prec h(z) \quad (z \in \mathbb{U}). \tag{2.10}$$

Next, we consider the case $\beta\cos(\pi\eta/2) \leq \alpha\eta\sin(\pi\eta/2)$, then the function g is increasing on $(0, \infty)$ and it follows that

$$|h(e^{i\theta})|^2 \geq g(0) = \beta^2 + \alpha^2\eta^2.$$

Hence, we get

$$|h(e^{i\theta})| \geq \sqrt{\beta^2 + \alpha^2\eta^2} = C(\alpha,\ \beta,\ \eta).$$

Therefore, by the assumption (2.9), we have (2.10) again. Finally, with the aid of Lemma 1, we obtain $p(z) \prec q(z)$ $(z \in \mathbb{U})$, that is, $|\arg\ p(z)| < \frac{\pi}{2}\eta$. □

Taking $\beta = \alpha$ in Theorem 3, we have the following result.

Corollary 9. *Let α and η be real numbers such that $\alpha > 0$, $0 < \eta \leq 1$, and*

$$\sin\frac{\pi}{2}\eta + \eta\cos\frac{\pi}{2}\eta > \frac{1-\alpha}{\alpha}.$$

Let $x^* = 0.638\ldots$ be the unique root of the equation $x = \cot(\pi x/2)$. If $f \in \mathcal{A}$ satisfies

$$\left|\alpha\frac{zf''(z)}{f'(z)} + (1-\alpha)\frac{zf'(z)}{f'(z)}\right| < C(\alpha,\ \eta) \quad (z \in \mathbb{U}),$$

where

$$C(\alpha,\ \eta) = \begin{cases}\alpha(\sin\frac{\pi}{2}\eta + \eta\cos\frac{\pi}{2}\eta), & \text{if } 0 < \eta < x^*, \\ \alpha\sqrt{1+\eta^2}, & \text{if } x^* \leq \eta \leq 1,\end{cases}$$

then

$$\left|\arg\ \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

Example 3. *Choosing* $\alpha = 1$ *and* $\eta = 1/2$ *in Corollary 9, we have* $C(1,\ 1/2) = 3\sqrt{2}/4$. *Therefore, we obtain that if*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3\sqrt{2}}{4} \quad (z \in \mathbb{U}),$$

then

$$\left|\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4} \quad (z \in \mathbb{U}).$$

Making $p(z) = f(z)/z$ in Theorem 3, we have the following result.

Corollary 10. *Let* α*,* β*, and* η *be real numbers satisfying* (2.7)*. If* $f \in \mathcal{A}$ *satisfies*

$$\left|\frac{f(z)}{z} - (\beta + 1) + \alpha\frac{zf'(z)}{f(z)}\right| < C(\alpha,\ \beta,\ \eta) \quad (z \in \mathbb{U}),$$

where $C(\alpha,\ \beta,\ \eta)$ *is given by* (2.8)*, then*

$$\left|\arg\ \frac{f(z)}{z}\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

We remark that, for the case $\eta = 1$ in Theorem 3, we have $C(\alpha,\ \beta,\ 1) = \sqrt{\alpha^2 + \beta^2}$. We end this paper with showing that this quantity can be improved as follows:

Corollary 11. *Let* α *and* β *be real numbers such that* $\alpha > 0$ *and* $\sqrt{\alpha(\alpha+2)+\beta^2} > |1-\beta|$*. Let* p *be analytic in* $\mathbb{U}$ *with* $p(0) = 1$*. If*

$$\left|p(z) - \beta + \alpha\frac{zp'(z)}{p(z)}\right| < \sqrt{\alpha(\alpha+2)+\beta^2} \quad (z \in \mathbb{U}),$$

then Re $p(z) > 0$ *for all* $z \in \mathbb{U}$*.*

Proof. By defining the same functions q, θ, φ, Q, and h with $\eta = 1$, as in the proof of Theorem 3, we will reach the following equality:

$$\left|h(e^{i\theta})\right|^2 = \beta^2 + \left(t + \frac{\alpha(1+t^2)}{2t}\right)^2, \tag{2.11}$$

where $t = \cot(\theta/2)$ with $0 < \theta < \pi$. Furthermore, since $t > 0$, we get

$$t + \frac{\alpha(1+t^2)}{2t} = \frac{1}{2}\left[\alpha \cdot t^{-1} + (\alpha+2)t\right] \geq \sqrt{\alpha(\alpha+2)}. \tag{2.12}$$

Hence, combining (2.11) and (2.12) leads us to get

$$\left|h(e^{i\theta})\right| \geq \sqrt{\alpha(\alpha+2)+\beta^2} \quad (0 < \theta < \pi).$$

Thus, it follows from the same proof of Theorem 3 that $|\arg p(z)| < \pi/2$ $(z \in \mathbb{U})$, or Re $p(z) > 0$ $(z \in \mathbb{U})$. ◻

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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