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Research article

New results about fuzzy γ-convex functions connected with the $\mathfrak{q}$-analogue multiplier-Noor integral operator

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Abstract: The features of analytical functions were mostly studied using a fuzzy subset and a $\mathfrak{q}$-difference operator in this study, as we investigate many fuzzy differential subordinations related to the $\mathfrak{q}$-analogue multiplier-Noor integral operator. By applying fuzzy subordination to univalent functions whose range is symmetric with respect to the real axis, we create a few new subclasses of analytical functions. We define numerous classes related to the family of linear $\mathfrak{q}$-operators and introduce them. Here, we focus on the inclusion results and other integral features.

Keywords: analytic functions; fuzzy $\mathfrak{q}$-starlike functions; fuzzy $\mathfrak{q}$-convex functions
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1. Introduction

Since the first paper investigate the concept of subordination in fuzzy set theory [1] appeared in 2011, the fuzzy set theory has been incorporated into studies of geometric function theory as it relates to complex analysis. We were inspired by the traditional subordination concepts first investigated by Miller and Mocanu [2, 3]. The subsequent publications that were published adopted concepts based on the previously accepted theory of differential subordination and continued the line of inquiry started by Oros and Oros [4–6]. They discussed fuzzy differential subordination. Geometric function theory scholars quickly adopted the concept, and all previous lines of inquiry into this field were modified to account for the novel fuzzy characteristics.

The concepts of fuzzy subordination and differential subordination were first studied by Oros and Oros [1,4]. Conducting investigations involving operators is a crucial aspect of the geometric function theory. New fuzzy subordination results were obtained using such investigations, which were published in 2013 [7], continued in the following years [8–11], and included the research of other scientists. For instance, see [12–18].

The complex plane's unit disc is written by **U**. $\mathcal{H}(\mathbf{U})$ represents the class of functions that are holomorphic in **U**. Let the class **A** of analytic functions $\mathfrak{f}(\zeta)$ in the open unit disk $\mathbf{U} = \{\zeta : |\zeta| < 1\}$ such that

$$\mathfrak{f}(\zeta) = \zeta + \sum_{\kappa=2}^{\infty} a_{\kappa}\zeta^{\kappa}, \ \ (\zeta \in \mathbf{U}). \tag{1.1}$$

We indicate the classes of univalent functions, starlike functions, and convex functions by S, S^*, and C, respectively.

For functions $\mathfrak{f}(\zeta) \in \mathbf{A}$, given by (1.1), and $\mathfrak{h}(\zeta) \in \mathbf{A}$ defined by

$$\mathfrak{h}(\zeta) = \zeta + \sum_{\kappa=2}^{\infty} b_{\kappa}\zeta^{\kappa}, \ \ (\zeta \in \mathbf{U}).$$

Hadamard product (or convolution) of $\mathfrak{f}(\zeta)$ and $\mathfrak{h}(\zeta)$ is given by

$$(\mathfrak{f} * \mathfrak{h})(\zeta) = \zeta + \sum_{\kappa=2}^{\infty} a_{\kappa}b_{\kappa}\zeta^{\kappa} = (\mathfrak{h} * \mathfrak{f})(\zeta), \ \ (\zeta \in \mathbf{U}).$$

For $\mathfrak{q} \in (0, 1)$, Jackson [19] given and investigated the $\mathfrak{q}$-difference operator, that is defined by:

$$\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta) = \frac{\mathfrak{f}(\zeta) - \mathfrak{f}(\mathfrak{q}\zeta)}{(1-\mathfrak{q})\zeta}; \quad \mathfrak{q} \neq 1, \ \zeta \neq 0.$$

We note that $\lim_{q\to 1^-} \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta) = \mathfrak{f}'(\zeta)$, where $\mathfrak{f}'(\zeta)$ is the function's ordinary derivative.

The use of $\mathfrak{q}$-difference equations in the setting of geometric function theory was pioneered by Jackson [19, 20], Carmichael [21], Mason [22], and Trjitzinsky [23]. Ismail et al. [24] introduced certain $\mathfrak{q}$-function theory-related characteristics for the first time. Additionally, various $\mathfrak{q}$-calculus applications related to generalized subclasses of analytic functions have been researched by numerous authors; see [25–39]. Several interesting applications of Jackson's $\mathfrak{q}$-difference operator $\mathfrak{D}_{\mathfrak{q}} : \mathbf{A} \to \mathbf{A}$ given by

$$\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta) := \begin{cases} \frac{\mathfrak{f}(\zeta)-\mathfrak{f}(\mathfrak{q}\zeta)}{(1-\mathfrak{q})\zeta} & (\zeta \neq 0; \ 0 < \mathfrak{q} < 1), \\ \\ \mathfrak{f}'(0) & (\zeta = 0). \end{cases} \tag{1.2}$$

It comes to light that, for $\kappa \in \mathbb{N}$ and $\zeta \in \mathbf{U}$

$$\mathfrak{D}_{\mathfrak{q}}\left\{\sum_{\kappa=1}^{\infty} a_{\kappa}\zeta^{\kappa}\right\} = \sum_{\kappa=1}^{\infty} [\kappa]_{\mathfrak{q}}\, a_{\kappa}\zeta^{\kappa-1}, \tag{1.3}$$

where

$$[\kappa]_{\mathfrak{q}} \ = \ \frac{1-\mathfrak{q}^{\kappa}}{1-\mathfrak{q}} = 1 + \sum_{n=1}^{\kappa-1} \mathfrak{q}^{n}, \qquad [0]_{\mathfrak{q}} = 0,$$

$$[\kappa]_{\mathfrak{q}}! \;=\; \begin{cases} [\kappa]_{\mathfrak{q}}\,[\kappa-1]_{\mathfrak{q}}\,.........\,[2]_{\mathfrak{q}}\,[1]_{\mathfrak{q}} & \kappa = 1,2,3,...,\\ 1 & \kappa = 0. \end{cases} \tag{1.4}$$

Definition 1.1. *[19, 20] For* d, $\kappa \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathfrak{q}$*-shifted factorial defined by*

$$(\mathrm{d};\mathfrak{q})_0 = 1, \qquad (\mathrm{d};\mathfrak{q})_\kappa = \prod_{\ell=0}^{\kappa-1}(1-\mathrm{d}\mathfrak{q}^\ell),$$

and in terms of the basic (or q-) gamma function

$$(\mathfrak{q}^{\mathrm{d}};\mathfrak{q})_\kappa = \frac{(1-\mathfrak{q}^\kappa)\Gamma_{\mathfrak{q}}(\mathrm{d}+\kappa)}{\Gamma_{\mathfrak{q}}(\mathrm{d})}, \quad \kappa \in \mathbb{N}_0,$$

where the (or q-) gamma function is defined by

$$\Gamma_{\mathfrak{q}}(x) = \frac{(1-\mathfrak{q})^{1-x}(\mathfrak{q};\mathfrak{q})_\infty}{(\mathfrak{q}^x;\mathfrak{q})_\infty}, \quad (|\mathfrak{q}|<1,\; x \in \mathbb{N}_0),$$

where

$$(\mathrm{d};\mathfrak{q})_\infty = \prod_{\ell=0}^{\infty}(1-\mathrm{d}\mathfrak{q}^\ell), \quad |\mathfrak{q}|<1.$$

The $\mathfrak{q}$-difference operator is subject to the next fundamental laws.

$$\mathfrak{D}_{\mathfrak{q}}\left(c\mathfrak{f}(\zeta) \pm d\mathfrak{h}(\zeta)\right) = c\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta) \pm d\mathfrak{D}_{\mathfrak{q}}\mathfrak{h}(\zeta),$$

$$\mathfrak{D}_{\mathfrak{q}}\left(\mathfrak{f}(\zeta)\,\mathfrak{h}(\zeta)\right) = \mathfrak{f}(\mathfrak{q}\zeta)\,\mathfrak{D}_{\mathfrak{q}}\left(\mathfrak{h}(\zeta)\right) + \mathfrak{h}(\zeta)\mathfrak{D}_{\mathfrak{q}}\left(\mathfrak{f}(\zeta)\right),$$

$$\mathfrak{D}_{\mathfrak{q}}\left(\frac{\mathfrak{f}(\zeta)}{\mathfrak{h}(\zeta)}\right) = \frac{\mathfrak{D}_{\mathfrak{q}}\left(\mathfrak{f}(\zeta)\right)\mathfrak{h}(\zeta) - \mathfrak{f}(\zeta)\,\mathfrak{D}_{\mathfrak{q}}\left(\mathfrak{h}(\zeta)\right)}{\mathfrak{h}(\mathfrak{q}\zeta)\mathfrak{h}(\zeta)}, \quad \mathfrak{h}(\mathfrak{q}\zeta)\mathfrak{h}(\zeta) \neq 0,$$

$$\mathfrak{D}_{\mathfrak{q}}\left(\log\mathfrak{f}(\zeta)\right) = \frac{\ln\mathfrak{q}}{\mathfrak{q}-1}\,\frac{\mathfrak{D}_{\mathfrak{q}}\left(\mathfrak{f}(\zeta)\right)}{\mathfrak{f}(\zeta)},$$

where $\mathfrak{f}, \mathfrak{h} \in \mathbf{A}$, and c and d are real or complex constants.

Jackson in [20] investigated the $\mathfrak{q}$-integral of $\mathfrak{f}$ as:

$$\int_0^\zeta \mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t \;=\; \zeta(1-\mathfrak{q})\sum_{\kappa=0}^{\infty}\mathfrak{q}^\kappa\mathfrak{f}(\zeta\mathfrak{q}^\kappa)$$

and

$$\lim_{\mathfrak{q}\to 1-}\int_0^\zeta \mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t \;=\; \int_0^\zeta \mathfrak{f}(t)\mathfrak{d}t,$$

where $\int_0^\zeta \mathfrak{f}(t)\mathfrak{d}t$, is the usual integral.

In [40] Aouf and Madian investigate the $\mathfrak{q}$-analogue Ĉătas operator $I^r_{\mathfrak{q}}(\lambda,\ell) : \mathbf{A} \to \mathbf{A}$ ($r \in \mathbb{N}_0, \ell, \lambda \geq 0, 0 < \mathfrak{q} < 1$) as follows:

$$I^r_{\mathfrak{q}}(\lambda,\ell)\mathfrak{f}(\zeta) = \zeta + \sum_{\kappa=2}^{\infty}\left(\frac{[1+\ell]_{\mathfrak{q}} + \lambda([\kappa+\ell]_{\mathfrak{q}} - [1+\ell]_{\mathfrak{q}})}{[1+\ell]_{\mathfrak{q}}}\right)^r a_\kappa\zeta^\kappa. \tag{1.5}$$

The operator $I^r_{\mathfrak{q}}(\lambda,\ell)\mathfrak{f}(\zeta)$ can be expressed as this way:

$$I^r_{\mathfrak{q}}(\lambda,\ell)\mathfrak{f}(\zeta) = \underbrace{\mathfrak{C}_q(\lambda,\ell) * ... * \mathfrak{C}_q(\lambda,\ell)}_{r-times} * \mathfrak{f}(\zeta), \tag{1.6}$$

where $\mathfrak{C}_q(\lambda,\ell)$ is the generating function given by (see El-Ashwah [41]),

$$\mathfrak{C}_q(\lambda,\ell) = \frac{z - \left(1 - \frac{\mathfrak{q}^\ell}{[1+\ell]_{\mathfrak{q}}}\lambda\right)\mathfrak{q}z^2}{(1-z)(1-\mathfrak{q}z)}.$$

Also in 2018, Arif et al. [42] investigated the $\mathfrak{q}$-analogue of Noor integral operator $\mathfrak{I}^\mu_{\mathfrak{q}}\mathfrak{f}(\zeta) : \mathbf{A} \to \mathbf{A}$ by

$$\mathfrak{I}^\mu_{\mathfrak{q}}\mathfrak{f}(\zeta) = \zeta + \sum_{\kappa=2}^{\infty} \frac{[\kappa,\mathfrak{q}]!}{[\mu+1,\mathfrak{q}]_{\kappa-1}} a_\kappa \zeta^\kappa,\ (\mu > -1, 0 < \mathfrak{q} < 1). \tag{1.7}$$

By virtue of (1.6) and (1.7), the operator $\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)$ essentially by the $\mathfrak{q}$-analogue of Noor integral operator and the $\mathfrak{q}$-analogue Catas operator, as follows:

$$\begin{aligned}\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta) &= \underbrace{\mathfrak{C}_{\mathfrak{q}}(\lambda,\ell) * ... * \mathfrak{C}_{\mathfrak{q}}(\lambda,\ell)}_{r-times} * \mathfrak{I}^\mu_{\mathfrak{q}}\mathfrak{f}(\zeta) \\ &= \zeta + \sum_{\kappa=2}^{\infty}\left(\frac{[1+\ell]_{\mathfrak{q}} + \lambda([\kappa+\ell]_{\mathfrak{q}} - [1+\ell]_{\mathfrak{q}})}{[1+\ell]_{\mathfrak{q}}}\right)^r \frac{[\kappa,\mathfrak{q}]!}{[\mu+1,\mathfrak{q}]_{\kappa-1}} a_\kappa\zeta^\kappa, \\ r &\in \mathbb{N}_0, \ell, \lambda \geq 0, \mu > -1, 0 < \mathfrak{q} < 1.\end{aligned} \tag{1.8}$$

We use (1.8) to deduce the following:

$$\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right) = \frac{[\ell+1]_{\mathfrak{q}}}{\lambda\mathfrak{q}^\ell}\mathcal{CN}^{r+1,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta) - \left(\frac{[\ell+1]_{\mathfrak{q}}}{\lambda\mathfrak{q}^\ell} - 1\right)\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta), \quad (\lambda > 0), \tag{1.9}$$

$$\mathfrak{q}^\mu\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu+1}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right) = [\mu+1,q]\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta) - [\mu,q]\mathcal{CN}^{r,\mu+1}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta), \quad (\mu > 0). \tag{1.10}$$

We note that :
(i) If $r = 0$ and $\mathfrak{q} \to 1^-$ the operator reduces to the familier Noor integral investigated in [43,44];
(ii) $\lim\limits_{q\to 1-}\mathcal{CN}^{r,0}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta) = \mathcal{CN}^r_{\lambda,\ell}\mathfrak{f}(\zeta)$ see [45];
(iii) $\mathcal{CN}^{r,0}_{\mathfrak{q},1,\ell}\mathfrak{f}(\zeta) = \mathcal{CN}^r_{\mathfrak{q},\ell}\mathfrak{f}(\zeta)$, $\lim\limits_{\mathfrak{q}\to 1-}\mathcal{CN}^r_{\mathfrak{q},\ell}\mathfrak{f}(\zeta) = \mathcal{CN}^r_\ell\mathfrak{f}(\zeta)$ see [46,47];
(iv) $\mathcal{CN}^{r,0}_{\mathfrak{q},\lambda,0}\mathfrak{f}(\zeta) = \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta)$, $\lim\limits_{\mathfrak{q}\to 1-}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta) = \mathcal{D}^r_\lambda\mathfrak{f}(\zeta)$ see [48,49];
(v) $\mathcal{CN}^{r,0}_{\mathfrak{q},1,0}\mathfrak{f}(\zeta) = \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta)$ (see [28]) $\lim\limits_{\mathfrak{q}\to 1-}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta) = \mathcal{D}^r\mathfrak{f}(\zeta)$ see Salagean [50].
We also observe that:
(i) $\mathcal{CN}^{r,\mu}_{\mathfrak{q},1,\ell}\mathfrak{f}(\zeta) = \mathcal{CN}^{r,\mu}_{\mathfrak{q},\ell}\mathfrak{f}(\zeta)$

$$\begin{aligned}\mathfrak{f}(\zeta) &\in \mathbf{A} : I^r_{\mathfrak{q},\mu}\mathfrak{f}(\zeta) = \zeta + \sum_{\kappa=2}^{\infty}\left(\frac{[\kappa+\ell]_{\mathfrak{q}}}{[1+\ell]_{\mathfrak{q}}}\right)^r \frac{[\kappa,q]!}{[\mu+1,q]_{\kappa-1}} a_\kappa\zeta^\kappa, \\ (r &\in \mathbb{N}_0, \ell > 0, \mu > -1, 0 < \mathfrak{q} < 1, \zeta \in \mathbf{U}).\end{aligned}$$

(ii) $\mathcal{CN}^{r,\mu}_{\mathfrak{q},1,1}\mathfrak{f}(\zeta) = \mathcal{CN}^{r,\mu}_{\mathfrak{q},}\mathfrak{f}(\zeta)$

$$\begin{aligned}\mathfrak{f}(\zeta) &\in \mathbf{A} : I^{r,\ell}_{\mathfrak{q},\mu}\mathfrak{f}(\zeta) = \zeta + \sum_{\kappa=2}^{\infty}\left([\kappa]_{\mathfrak{q}}\right)^{r}\frac{[\kappa,q]!}{[\mu+1,q]_{\kappa-1}}a_{\kappa}\zeta^{\kappa},\\ (r &\in \mathbb{N}_0, \mu > -1, 0 < \mathfrak{q} < 1, \zeta \in \mathbf{U}).\end{aligned}$$

(iii) $\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,0}\mathfrak{f}(\zeta) = \mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda}\mathfrak{f}(\zeta)$

$$\begin{aligned}\mathfrak{f}(\zeta) &\in \mathbf{A} : I^{r,\lambda}_{\mathfrak{q},\mu}\mathfrak{f}(\zeta) = \zeta + \sum_{\kappa=2}^{\infty}\left(1+\lambda([\kappa]_{\mathfrak{q}}-1)\right)^{r}\frac{[\kappa,q]!}{[\mu+1,q]_{\kappa-1}}a_{\kappa}\zeta^{\kappa},\\ (r &\in \mathbb{N}_0, \lambda > 0, \mu > -1, 0 < \mathfrak{q} < 1, \zeta \in \mathbf{U}).\end{aligned}$$

$\mathfrak{P}(\zeta) = \mathfrak{Q}(\mathfrak{w}(\zeta))$, where $\mathfrak{w}(\zeta)$ is the Schwartz function in $\mathbf{U}$, is the definition of the subordination of analytic functions $\mathfrak{P}$ and $\mathfrak{Q}$, represented as $\mathfrak{P} \prec \mathfrak{Q}$ (see [51]). Moreover, the writers in [2] presented and examined the concept of unequal subordination.

Here, we summarize some important basic concepts related to $\mathfrak{q}$-calculus and fuzzy differential subordination.

2. Definitions and preliminaries

Definition 2.1. *[52] Let $S \neq \phi$. A fuzzy subset of S is defined as F when it maps from S to* $[0, 1]$.

The next concept also applies to the fuzzy subset.

Definition 2.2. *[52] A Fuzzy subset of S is a pair $((\Im, F_{\Im})$, where $F_{\Im} : S \to [0,1]$ is known as the membership function of the fuzzy set $(\Im, F_{\Im})$ and $\Im = \{x \in \mathcal{S} : 0 < F_{\Im}(x) \leq 1\} = \sup(\Im, F_{\Im})$ is called the support of fuzzy set $(\Im, F_{\Im})$.*

Definition 2.3. *[52] Fuzzy subsets $\left(\Im_g, F_{\Im_g}\right)$ and $\left(\Im_y, F_{\Im_y}\right)$ of S are equal if and only if $\Im_g = \Im_y$, whereas $\left(\Im_g, F_{\Im_g}\right) \subseteq \left(\Im_y, F_{\Im_y}\right)$ if and only if $F_{\Im_g}(\eta) \leq F_{\Im_y}(\eta)$, $\eta \in S$.*

Definition 2.4. *[4] The fuzzy subordination of analytic functions $\mathfrak{f}$ and $\mathfrak{h}$ is given by $\mathfrak{f} \prec_F \mathfrak{h}$ (or $\mathfrak{f}(\zeta) \prec_F \mathfrak{h}(\zeta)$) if:*

$$\mathfrak{f}(\zeta_0) = \mathfrak{h}(\zeta_0) \text{ and } F(\mathfrak{f}(\zeta)) \leq F(\mathfrak{h}(\zeta)), \zeta \in \mathfrak{D},$$

since $\mathfrak{D} \subset \mathbb{C}$ and ζ_0 are a fixed point in $\mathfrak{D}$.

Remark 2.1. *From the next functions $\mathfrak{F}_i : \mathbb{C} \to [0,1]$, $(i = 1,2,3,4)$, could serve as an illustration*

$$\mathfrak{F}_1(\zeta) = \frac{|\zeta|}{1+|\zeta|}, \qquad \mathfrak{F}_2(\zeta) = \frac{1}{1+|\zeta|}, \qquad \mathfrak{F}_3(\zeta) = |\sin|\zeta||, \qquad \mathfrak{F}_4(\zeta) = |\cos|\zeta||.$$

Remark 2.2. *When $\mathfrak{D} = \mathbf{U}$ in Definition 2.3, the concepts of fuzzy subordination and classical subordination coincide.*

The lemma next is required to verify our studies.

Lemma 2.1. *[18] Let $\beta, \alpha \in \mathbb{C}$ with $\beta \neq 0$, and $\hbar(\zeta) \in T$ for*

$$\Re\{\beta\hbar(\zeta) + \alpha\} > 0. \tag{2.1}$$

If $\omega(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + ...$ is analytic in $\mathbf{U}$, then

$$\omega(\zeta) + \frac{\zeta\mathfrak{D}_{\mathfrak{q}}\omega(\zeta)}{\beta\omega(\zeta) + \alpha} \prec_F \hbar(\zeta) \text{ implies } \omega(\zeta) \prec_F \hbar(\zeta),$$

where $F : \mathbb{C} \to [0, 1]$.

In this paper, we investigate many fuzzy differential subordinations related with the $\mathfrak{q}$-analogue multiplier-Noor integral operator. By applying fuzzy subordination to univalent functions whose range is symmetric with respect to the real axis, we obtain a few new subclasses of analytical functions and define numerous classes related to the family of linear $\mathfrak{q}$-operators. Our purpose in this article is to obtain inclusion results and other integral features for functions belonging to the class $F\mathcal{M}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma; \varphi)$.

3. Main results

Suppose that T be the class of analytic functions $\varphi(\zeta)$ which are univalent convex functions in $\mathbf{U}$ with $\varphi(0) = 1$ and $\Re(\varphi(\zeta)) > 0$ in $\mathbf{U}$ and where $\varphi(\mathbf{U})$ is symmetric with respect to the real axis. Now, for $\varphi(\zeta) \in T$ and $q \in (0, 1)$ with $F : \mathbb{C} \to [0, 1]$, $0 \neq \eta \in \mathbb{C}$, $r \geq 0$ and $\ell, \lambda \geq 0, \mu > -1, 0 < \mathfrak{q} < 1$, we define the following.

Definition 3.1. *Let $\mathfrak{f} \in \mathbf{A}$, $\varphi \in T$, $0 \leq \gamma \leq 1$ and $0 < \mathfrak{q} < 1$. Then, $\mathfrak{f} \in F\mathcal{M}_q(\gamma; \varphi)$ if and only if*

$$(1 - \gamma)\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta)}{\mathfrak{f}(\zeta)} + \gamma\frac{\mathfrak{D}_{\mathfrak{q}}(\zeta\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta))}{\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta)} \prec_F \varphi(\zeta).$$

Additionally, let us say

$$F\mathcal{M}_q(0; \varphi) = FST_q(\varphi), \qquad F\mathcal{M}_q(1; \varphi) = FC_q(\varphi).$$

A function $\mathfrak{f} \in \mathbf{A}$ is in $FST_q(\varphi)$ and $FC_q(\varphi)$ iff

$$\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta)}{\mathfrak{f}(\zeta)} \prec_F \varphi(\zeta) \text{ and } \frac{\mathfrak{D}_{\mathfrak{q}}(\zeta\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta))}{\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta)} \prec_F \varphi(\zeta),$$

respectively.

Special cases:
(i) If $q \to 1^-$, then $F\mathcal{M}_q(\gamma; \varphi) \equiv F\mathcal{M}_\gamma(\varphi)$ investigated in [16].
(ii) If $q \to 1^-$ and $\gamma = 0$, then $F\mathcal{M}_q(\gamma; \varphi) \equiv FS^*(\varphi)$ studied by [16].
(iii) If $q \to 1^-$ and $\gamma = 1$, then $F\mathcal{M}_q(\gamma; \varphi) \equiv FC(\varphi)$ studied by [16].
At this point we define a few new classes using the $\mathfrak{q}$-analogue operator supplied by (1.8):

Definition 3.2. *Let $\mathfrak{f} \in \mathbf{A}$, $\varphi \in T$, $0 \leq \gamma \leq 1$, $\ell, \lambda \geq 0, \mu > -1, 0 < \mathfrak{q} < 1$ and $r \in \mathbb{N}_0$. Then,*

$$\mathfrak{f} \in F\mathcal{M}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma; \varphi) \quad \text{if and only if } \; C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta) \in F\mathcal{M}_{\mathfrak{q}}(\gamma; \varphi).$$

Furthermore,

$$\mathfrak{f} \in \mathcal{FST}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi) \;\; \textit{iff} \;\; \mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta) \in \mathcal{FST}_{\mathfrak{q}}(\varphi)$$

and

$$\mathfrak{f} \in \mathcal{FC}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi) \;\; \textit{iff} \;\; \mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta) \in \mathcal{FC}_{\mathfrak{q}}(\varphi).$$

We note that

$$\mathfrak{f} \in \mathcal{FC}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi) \;\; \textit{iff} \;\; \zeta\left(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}\right) \in \mathcal{FST}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi). \tag{3.1}$$

Special cases:

If $r = 0 = \mu$, then $\mathcal{FM}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma;\varphi) \equiv \mathcal{FM}_{\mathfrak{q}}(\gamma;\varphi)$, $\mathcal{FST}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi) \equiv \mathcal{FST}_{\mathfrak{q}}(\varphi)$ and $\mathcal{FC}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi) \equiv \mathcal{FC}_{\mathfrak{q}}(\varphi)$.

Theorem 3.1. *Let* $0 \le \gamma \le 1$, $\varphi \in T$, $\ell \ge 0$, $\mu, \lambda > -1, 0 < \mathfrak{q} < 1$ *and* $r \in \mathbb{N}_0$, *Then,*

(i) $\mathcal{FM}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma;\varphi) \subset \mathcal{FST}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)$ *for* $0 \le \gamma \le 1$.

(ii) $\mathcal{FM}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma;\varphi) \subset \mathcal{FST}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)$ *for* $\gamma \ge 1$.

(iii) $\mathcal{FM}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma_2;\varphi) \subset \mathcal{FM}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma_1;\varphi)$ *for* $0 \le \gamma_1 < \gamma_2 < 1$.

Proof. (i) Let $\mathfrak{f} \in \mathcal{FM}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma;\varphi)$. We set

$$\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)} = \omega(\zeta), \tag{3.2}$$

for analytic $\omega(\zeta)$ in $\mathbf{U}$ with $\omega(0) = 1$. The $\mathfrak{q}$-logarithmic differentiation of (3.2) yields:

$$\frac{\mathfrak{D}_{\mathfrak{q}}\left(\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)\right)}{\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)} - \frac{\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)} = \frac{\mathfrak{D}_{\mathfrak{q}}\omega(\zeta)}{\omega(\zeta)}.$$

Equivalently,

$$\frac{\mathfrak{D}_{\mathfrak{q}}\left(\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)\right)}{\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)} = \omega(\zeta) + \frac{\zeta\mathfrak{D}_{\mathfrak{q}}\omega(\zeta)}{\omega(\zeta)}.$$

Since $\mathfrak{f} \in \mathcal{FM}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma;\varphi)$, we obtain:

$$\begin{aligned}(1-\gamma)\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)} + \gamma\frac{\mathfrak{D}_{\mathfrak{q}}\left(\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)\right)}{\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)} &= \omega(\zeta) + \gamma\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\omega(\zeta)}{\omega(\zeta)} \\ &\prec_F \varphi(\zeta).\end{aligned} \tag{3.3}$$

We use Lemma 2.1 to obtain $\omega(\zeta) \prec_F \varphi(\zeta)$. Consequently, $\mathfrak{f} \in \mathcal{FST}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)$.

(ii) Suppose that $\mathfrak{f} \in \mathcal{FM}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma;\varphi)$. Then,

$$(1-\gamma)\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)} + \gamma\frac{\mathfrak{D}_{\mathfrak{q}}\left(\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)\right)}{\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)} = \omega_1(\zeta) \prec_F \varphi(\zeta).$$

Now,

$$\gamma\frac{\mathfrak{D}_{\mathfrak{q}}\left(\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)\right)}{\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)} = (1-\gamma)\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)} + \gamma\frac{\mathfrak{D}_{\mathfrak{q}}\left(\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)\right)}{\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}$$

$$+(\gamma-1)\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)}$$
$$=\ (\gamma-1)\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)}+\omega_1(\zeta).$$

This implies

$$\frac{\mathfrak{D}_{\mathfrak{q}}\left(\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)\right)}{\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)} = \left(1-\frac{1}{\gamma}\right)\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)}+\frac{1}{\gamma}\omega_1(\zeta)$$
$$= \left(1-\frac{1}{\gamma}\right)\omega_2(\zeta)+\frac{1}{\gamma}\omega_1(\zeta).$$

Since $\omega_1,\omega_2\prec_F\varphi(\zeta)$, we can write $\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)}\prec_F\varphi(\zeta)$. This completes the proof of (ii).
(*iii*) For $\gamma_1=0$, the result of section (*i*) is correct.
Thus, we assume that $\mathfrak{f}\in F\mathcal{M}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma_2;\varphi)$. Then,

$$(1-\gamma_2)\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)}+\gamma_2\frac{\mathfrak{D}_{\mathfrak{q}}\left(\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)\right)}{\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}=\omega_1(\zeta)\prec_F\varphi(\zeta). \tag{3.4}$$

Now, we can easily write

$$J_q(\gamma_1,\mathfrak{f}(\zeta))=\frac{\gamma_1}{\gamma_2}\omega_1(\zeta)+\left(1-\frac{\gamma_1}{\gamma_2}\right)\omega_2(\zeta), \tag{3.5}$$

with

$$J_q(\gamma_1,\mathfrak{f}(\zeta))=(1-\gamma_1)\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)}+\gamma_1\frac{\mathfrak{D}_{\mathfrak{q}}\left(\zeta\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)\right)}{\mathfrak{D}_{\mathfrak{q}}\left(\mathcal{CN}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)},$$

where we have used (3.4) and $\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\zeta)}{\mathfrak{f}(\zeta)}=\omega_2(\zeta)\prec_F\varphi(\zeta)$ (3.5) implies our required result. □

Remark 3.1. *If* $\gamma_2=1$ *and letting* $\mathfrak{f}\in F\mathcal{M}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(1;\varphi)=F\mathcal{C}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)$. *Then, by Theorem 3.1* (*iii*), *we have:*

$$\mathfrak{f}\in F\mathcal{M}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma_1;\varphi),\ for\ \ 0\le\gamma_1<1.$$

We utilize Theorem 3.1 (*i*), *to get* $\mathfrak{f}\in F\mathcal{ST}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)$. *Consequently,* $F\mathcal{C}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)\subset F\mathcal{ST}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)$.

Theorem 3.2. *Let* $\varphi\in T$, $0\le\gamma\le1$, $\ell\ge0$, $\mu,\lambda>0, 0<\mathfrak{q}<1$ *and* $r\in\mathbb{N}_0$ *with* $[\ell+1]_{\mathfrak{q}}>\lambda\mathfrak{q}^{\ell}$ *Then,*

$$F\mathcal{ST}^{r+1,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)\subset F\mathcal{ST}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)\subset F\mathcal{ST}^{r,\mu+1}_{\mathfrak{q},\lambda,\ell}(\varphi).$$

Proof. Let $\mathfrak{f}\in F\mathcal{ST}^{r+1,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)$. Then,

$$\frac{\zeta\mathfrak{D}_{\mathfrak{q}}(\mathcal{CN}^{r+1,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta))}{\mathcal{CN}^{r+1,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)}\prec_F\varphi(\zeta).$$

Now, let

$$\frac{\zeta\mathfrak{D}_{\mathfrak{q}}(C\mathcal{N}_{\mathfrak{q},\lambda,\ell}^{r,\mu}\mathfrak{f}(\zeta))}{C\mathcal{N}_{\mathfrak{q},\lambda,\ell}^{r,\mu}\mathfrak{f}(\zeta)}=\chi(\zeta), \tag{3.6}$$

for analytic $\chi(\zeta)$ in $\mathbf{U}$ with $\chi(0)=1$. Using (1.9) and (3.6), we obtain

$$\frac{[\ell+1]_{\mathfrak{q}}}{\lambda\mathfrak{q}^{\ell}}\frac{C\mathcal{N}_{\mathfrak{q},\lambda,\ell}^{r+1,\mu}\mathfrak{f}(\zeta)}{C\mathcal{N}_{\mathfrak{q},\lambda,\ell}^{r,\mu}\mathfrak{f}(\zeta)}=\chi(\zeta)+\left(\frac{[\ell+1]_{\mathfrak{q}}}{\lambda\mathfrak{q}^{\ell}}-1\right).$$

The q-logarithmic differentiation yields:

$$\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(C\mathcal{N}_{\mathfrak{q},\lambda,\ell}^{r+1,\mu}\mathfrak{f}(\zeta)\right)}{C\mathcal{N}_{\mathfrak{q},\lambda,\ell}^{r+1,\mu}\mathfrak{f}(\zeta)}=\omega(\zeta)+\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\chi(\zeta)}{\chi(\zeta)+\left(\frac{[\ell+1]_{\mathfrak{q}}}{\lambda\mathfrak{q}^{\ell}}-1\right)}. \tag{3.7}$$

Since $\mathfrak{f}\in FST_{\mathfrak{q},\lambda,\ell}^{r+1,\mu}(\varphi)$, (3.7) implies

$$\omega(\zeta)+\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\chi(\zeta)}{\chi(\zeta)+\left(\frac{[\ell+1]_{\mathfrak{q}}}{\lambda\mathfrak{q}^{\ell}}-1\right)}\prec_F\varphi(\zeta).$$

We assume that $\Re\left\{\varphi(\zeta)+\left(\frac{[\ell+1]_{\mathfrak{q}}}{\lambda\mathfrak{q}^{\ell}}-1\right)\right\}>0$ and we use Lemma 2.1 to get

$$\chi(\zeta)\prec_F\varphi(\zeta).$$

Consequently, $\mathfrak{f}\in FST_{\mathfrak{q},\lambda,\ell}^{r,\mu}(\varphi)$.

To prove second part, we follow a similar technique to that used in first part. □

Theorem 3.3. *Let $\varphi\in T$, $0\le\gamma\le1$, $\ell\ge0$, $\mu,\lambda>0, 0<\mathfrak{q}<1$ and $r\in\mathbb{N}_0$ with $[\ell+1]_{\mathfrak{q}}>\lambda\mathfrak{q}^{\ell}$ Then,*

$$FC_{\mathfrak{q},\lambda,\ell}^{r+1,\mu}(\varphi)\subset FC_{\mathfrak{q},\lambda,\ell}^{r,\mu}(\varphi)\subset FC_{\mathfrak{q},\lambda,\ell}^{r,\mu+1}(\varphi).$$

Proof. Let $\mathfrak{f}\in FC_{\mathfrak{q},\lambda,\ell}^{r+1,\mu}(\varphi)$. Then, by (3.1),

$$\zeta\left(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}\right)\in FST_{\mathfrak{q},\lambda,\ell}^{r+1,\mu}(\varphi).$$

We utilize first part of Theorem 3.2 to get:

$$\zeta\left(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}\right)\in FST_{\mathfrak{q},\lambda,\ell}^{r,\mu}(\varphi).$$

Once more, by utilizing association (3.1), we get

$$\mathfrak{f}\in FC_{\mathfrak{q},\lambda,\ell}^{r,\mu}(\varphi).$$

Similarly, one can prove second part by applying second part of Theorem 3.2 through using the relation (3.1). □

Remark 3.2. *We can expand the inclusions according to using Theorems 3.1–3.3.*

$$F\mathcal{M}_{\mathfrak{q},\lambda,\ell}^{r+1,\mu}(\gamma;\varphi)\subset FST_{\mathfrak{q},\lambda,\ell}^{r+1,\mu}(\varphi)\subset FST_{\mathfrak{q},\lambda,\ell}^{r,\mu}(\varphi)\subset\ldots\subset FST_{\mathfrak{q},\lambda,\ell}^{\mu}(\varphi).$$

$$FC_{\mathfrak{q},\lambda,\ell}^{r+1,\mu}(\varphi)\subset FC_{\mathfrak{q},\lambda,\ell}^{r,\mu}(\varphi)\subset\ldots\subset FC_{\mathfrak{q},\lambda,\ell}^{\mu}(\varphi).$$

Theorem 3.4. *Let* $\mathfrak{f} \in F\mathcal{M}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma;\varphi)$*. Then,*

$$F_{\upsilon,\mathfrak{q}}(\zeta) = \frac{[\upsilon+1]_{\mathfrak{q}}}{\zeta^{\upsilon}} \int_0^{\zeta} t^{\upsilon-1}\mathfrak{f}(t) d_{\mathfrak{q}} t \tag{3.8}$$

is in $FST^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)$*.*

Proof. Let $\mathfrak{f} \in F\mathcal{M}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma;\varphi)$. Let

$$\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(F_{\upsilon,\mathfrak{q}}(\zeta))\right)}{C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(F_{\upsilon,\mathfrak{q}}(\zeta))} = \varpi(\zeta), \tag{3.9}$$

for analytic $\varpi(\zeta)$ in $\mathbf{U}$ with $\varpi(0) = 1$.

Simple calculations (3.9) imply that

$$\frac{\mathfrak{D}_{\mathfrak{q}}\left(\zeta^{\upsilon}C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(F_{\upsilon,\mathfrak{q}}(\zeta))\right)}{[\upsilon+1]_{\mathfrak{q}}} = \zeta^{\upsilon-1}C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta).$$

This implies

$$\zeta\mathfrak{D}_{\mathfrak{q}}\left(C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}F_{\upsilon,\mathfrak{q}}(\zeta)\right) = \left(1+\frac{[\upsilon]_{\mathfrak{q}}}{\mathfrak{q}^{\upsilon}}\right)C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta) - \frac{[\upsilon]_{\mathfrak{q}}}{\mathfrak{q}^{\upsilon}}C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}F_{\upsilon,\mathfrak{q}}(\zeta). \tag{3.10}$$

From (3.9), (3.10) and (1.8), we obtain

$$\varpi(\zeta) = \left(1+\frac{[\upsilon]_{\mathfrak{q}}}{\mathfrak{q}^{\upsilon}}\right)\frac{\zeta\left(C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}\mathfrak{f}(\zeta)\right)}{C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(F_{\upsilon,\mathfrak{q}}(\zeta))} - \frac{[\upsilon]_{\mathfrak{q}}}{\mathfrak{q}^{\upsilon}},$$

we take $\mathfrak{q}$-logarithmic differentiation:

$$\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\mathfrak{f}(\zeta))\right)}{C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\mathfrak{f}(\zeta))} = \varpi(\zeta) + \frac{\zeta\mathfrak{D}_{\mathfrak{q}}\varpi(\zeta)}{\varpi(\zeta)+L_{\mathfrak{q}}}, \quad \left(\text{for } L_{\mathfrak{q}} = \frac{[\upsilon]_{\mathfrak{q}}}{\mathfrak{q}^{\upsilon}}\right). \tag{3.11}$$

Since $\mathfrak{f} \in F\mathcal{M}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\gamma;\varphi) \subset FST^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)$, (3.11) implies

$$\varpi(\zeta) + \frac{\zeta\mathfrak{D}_{\mathfrak{q}}\varpi(\zeta)}{\varpi(\zeta)+L_{\mathfrak{q}}} \prec_F \varphi(\zeta).$$

We now use Lemma 2.1 to conclude $\varpi(\zeta) \prec_F \varphi(\zeta)$. Consequently,

$$\frac{\zeta\mathfrak{D}_{\mathfrak{q}}\left(C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(F_{\upsilon,\mathfrak{q}}(\zeta))\right)}{C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(F_{\upsilon,\mathfrak{q}}(\zeta))} \prec_F \varphi(\zeta).$$

Hence, $C\mathcal{N}^{r,\mu}_{\mathfrak{q},\lambda,\ell}(F_{\upsilon,\mathfrak{q}}(\zeta)) \in FST^{r,\mu}_{\mathfrak{q},\lambda,\ell}(\varphi)$. □

4. Conclusions

The primary outcomes of our work are the extensions of a number of traditional findings regarding $\mathfrak{q}$-theory and fuzzy subordination. In this paper, we looked at the $\mathfrak{q}$-theory principles related to a fuzzy differential subordination. Using two well-known $\mathfrak{q}$-calculus, we first introduced the $\mathfrak{q}$-linear operator and subsequently generated numerous analytic function subclasses using this operator. We looked into a few inclusion results and integral properties for the newly specified classes. The operator may also be examined using the dual theory of fuzzy differential superordination, potentially yielding theorems of the sandwhich-type. This would link the current findings to a standard result in the geometric function theory. Since this operator has been defined with specific parameter values, it could be interesting to experiment with different values to obtain some possibly interesting operators. Given the widespread usage of hypergeometric functions in statistics, engineering, and physics, it stands to reason that other fields may find use for the operators they involve. Since fuzzy differential subordination is a relatively young theory, its potential uses in other scientific fields or in real life are unknown. Future research projects with a longer time frame should look into those topics.

Consequently, using the new operators linked to the $\mathfrak{q}$-analogue and other appropriate operators in an analogous manner, it may be possible to generalize the findings reported in this work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Authors' contributions

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Conflict of interest

The authors declare that they have no conflicts of interest.

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