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Research article

An Erdélyi-Kober fractional coupled system: Existence of positive solutions

Mengjiao Zhao[1] **and Chen Yang**[2,*]

[1] School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, China
[2] Shanxi Vocational University of Engineering Science and Technology, Taiyuan, Shanxi 030619, China

* **Correspondence:** Email: yangchen0809@126.com.

Abstract: This paper studies an Erdélyi-Kober fractional coupled system where the variable is in an infinite interval, and the existence of positive solutions is considered. We first give proper conditions and then use the Guo-Krasnosel'skii fixed point theorem to discuss our problem in a special Banach space. The monotone iterative technique and the existence results of positive solutions for this system are established naturally. To show the plausibility of our main results, several concrete examples are given at the end.

Keywords: Erdélyi-Kober type fractional derivative; fractional coupled system; infinite interval; monotone iterative technique; positive solution
Mathematics Subject Classification: 26A33, 34A37, 34B15

1. Introduction

This paper considers a fractional coupled system on an infinite interval involving the Erdélyi-Kober derivative:

$$\begin{cases} D_\beta^{\gamma,\delta_1}u(x)+F(x,u(x),v(x))=0, x\in(0,+\infty),\\ D_\beta^{\gamma,\delta_2}v(x)+G(x,u(x),v(x))=0, x\in(0,+\infty),\\ \lim_{x\to 0} x^{\beta(2+\gamma)}I^{\delta_1+\gamma,2-\delta_1}u(x)=0, \lim_{x\to\infty} x^{\beta(1+\gamma)}I^{\delta_1+\gamma,2-\delta_1}u(x)=0,\\ \lim_{x\to 0} x^{\beta(2+\gamma)}I^{\delta_2+\gamma,2-\delta_2}v(x)=0, \lim_{x\to\infty} x^{\beta(1+\gamma)}I^{\delta_2+\gamma,2-\delta_2}v(x)=0,\end{cases} \tag{1.1}$$

where $\delta_1,\delta_2\in(1,2]$, $\gamma\in(-2,-1)$, and $\beta>0$. $D_\beta^{\gamma,\delta_1}$, $D_\beta^{\gamma,\delta_2}$ are Erdélyi-Kober fractional derivatives (EKFDs for short), and $I^{\delta_1+\gamma,2-\delta_1}, I^{\delta_2+\gamma,2-\delta_2}$ are the Erdélyi-Kober fractional integrals. F,G are continuous functions. We discuss the existence of positive solutions for (1.1).

During the past several decades, fractional equations have been studied widely; see [1–36] for instance. From the literature, we can see that there are many fractional derivatives used in differential equations. Among these various definitions, the widely used ones are the Riemann-Liouville and Caputo fractional derivatives, in many works. To generalize the Riemann-Liouville fractional derivative, Erdélyi-Kober defined a new fractional derivative, and we call it the Erdélyi-Kober fractional derivative. Moreover, the Erdélyi-Kober operator is very useful; we can refer to [6, 9, 14–17] and the references therein. The Erdélyi-Kober operator is a fractional integration operation which was given by Arthur Erdélyi and Hermann Kober in 1940 [23]. Some of these definitions and results were given in Samko et al. [3], Kiryakova [19], and McBride [20].

Nowadays, the theory of fractional operators in the Erdélyi-Kober frame has attracted much interest from researchers. The study of fractional systems is also very important, as these systems appear in various applications, especially in biological sciences. Recently, some problems of Erdélyi-Kober type fractional differential equations on infinite intervals received widespread attention from many scholars; see [8, 21, 22] for example.

Recently, in [8], the authors investigated the following equation:

$$\begin{cases}(D_{\theta}^{\vartheta,\sigma}u)(x)+F(u(x))=0, 0\leq x<\infty,\\ \lim_{t\to 0} x^{\theta(2-\sigma)}I^{\sigma+\vartheta,2-\sigma}u(x)=0,\\ \lim_{t\to +\infty} x^{\theta(2-\sigma)}I^{\sigma+\vartheta,2-\sigma}u(x)=0,\end{cases}$$

where $\sigma\in(1,2)$, $\vartheta\in(1,2)$, $\theta>0$, and F is a given continuous function, $D_{\theta}^{\vartheta,\sigma}$ denotes the EKFD, and $I^{\sigma+\vartheta,2-\sigma}$ denotes the Erdélyi-Kober fractional integral. The authors studied the existence and nonexistence of positive solutions for this problem by utilizing a fixed point result which uses the strongly positive-like operators and eigenvalue criteria.

In [9], the authors studied a fractional coupled system:

$$\begin{cases}{}^cD^{\varrho}u(\tau)=F(\tau,u(\tau),z(\tau),{}^cD^{\varsigma_1}z(\tau),I^{\xi}z(\tau)), \tau\in[0,T]:=K, 2<\varrho\leq 3, 1<\varsigma_1<2,\\ {}^cD^{\varsigma}z(\tau)=G(\tau,u(\tau),{}^cD^{\varrho_1}u(\tau),I^{\zeta}u(\tau),z(\tau)), \tau\in[0,T]:=K, 2<\varsigma\leq 3, 1<\varrho_1<2,\\ u(0)=\phi_1(z), u'(0)=\varepsilon_1 z'(k_1),\\ u(T)=\frac{\gamma\rho\vartheta^{-\rho(\varpi+\nu)}}{\Gamma(\varpi)}\int_0^{\vartheta}\frac{\sigma^{\rho\nu+\rho-1}z(\sigma)}{(\vartheta^{\rho}-\sigma^{\rho})^{1-\varpi}}d\sigma:=\gamma J_{\rho}^{\nu,\varpi}v(\vartheta),\\ z(0)=\phi_2(u), z'(0)=\varepsilon_2 z'(k_2),\\ z(T)=\frac{\delta v\varphi^{-v(\theta+\omega)}}{\Gamma(\theta)}\int_0^{\varphi}\frac{\sigma^{v\omega+v-1}u(\sigma)}{(\varphi^{v}-\sigma^{v})^{1-\theta}}d\sigma:=\delta J_{v}^{\omega,\theta}u(\varphi),\end{cases}$$

where ${}^cD^{\varrho},{}^cD^{\varsigma_1},{}^cD^{\varsigma},{}^cD^{\varrho_1}$ are the Liouville-Caputo fractional derivatives of order $2<\varrho,\varsigma\leq 3$, $1<\varsigma_1,\varrho_1<2$. I^{ξ},I^{ζ} are the Riemann-Liouville fractional integrals of order $1<\xi,\zeta<2$. $J_{\rho}^{\nu,\varpi},J_{v}^{\omega,\theta}$ are the Erdélyi-Kober fractional integrals of order $\varpi,\theta>0$, with $v,\omega>0$, ρ, $\vartheta\in(-\infty,+\infty)$. $F,G: K\times(-\infty,+\infty)^4\to(-\infty,+\infty)$ and $\phi_1,\phi_2: C(K,(-\infty,+\infty))\to(-\infty,+\infty)$ are continuous functions. $\gamma,\delta,\varepsilon_1,\varepsilon_2$ are positive real constants. The existence result was given by the Leray-Schauder alternative, and the uniqueness result was obtained due to Banach's fixed-point theorem. By the same methods, Arioua and Titraoui [18] studied system (1.1). Moreover, In [10], Arioua and Titraoui also investigated a new fractional problem involving the Erdélyi-Kober derivative. Inspired by the above articles, we use different methods to consider the fractional coupled system involving Erdélyi-Kober derivative (1.1). We employ the Guo-Krasnosel'skii fixed point theorem to discuss (1.1) in a special Banach space, and we also use the monotone iterative technique to study this system. Some existence results of positive solutions for system (1.1) are obtained, including the existence results of at least two positive solutions.

2. Preliminaries

Definition 2.1. (see [2]) Let $\alpha \in (-\infty, +\infty)$. C_α^n, $n \in N$, denotes a set of all functions $f(t), t > 0$, with $f(t) = t^p f_1(t)$ with $p > \alpha$ and $f_1 \in C^n[0, \infty)$.
Definition 2.2. (see [1, 2]) For a function $u \in C_\alpha$, the σ-order right-hand Erdélyi-Kober fractional integral is

$$(I_\beta^{\gamma,\sigma} u)(t) = \frac{\beta t^{-\beta(\gamma+\sigma)}}{\Gamma(\sigma)} \int_0^t \frac{s^{\beta(\gamma+1)-1} u(s)}{(t^\beta - s^\beta)^{1-\sigma}} ds, \sigma, \beta > 0, \gamma \in (-\infty, +\infty),$$

in which, Γ is the Euler gamma function.
Definition 2.3. (see [2]) Let $n-1 < \delta \leq n, n \in N$, and for $u \in C_\alpha$, the σ-order right-hand Erdélyi-Kober fractional derivative is

$$(D_\beta^{\gamma,\sigma} u)(t) = \prod_{j=1}^{n} (\gamma + j + \frac{t}{\beta}\frac{d}{dt})(I_\beta^{\gamma+\sigma, n-\sigma} u)(t),$$

where

$$\prod_{j=1}^{n} (\gamma + j + \frac{t}{\beta}\frac{d}{dt})(I_\beta^{\gamma+\sigma, n-\sigma} u) = (\gamma + 1 + \frac{t}{\beta}\frac{d}{dt}) \cdots (\gamma + n + \frac{t}{\beta}\frac{d}{dt})(I_\beta^{\gamma+\sigma, n-\sigma} u).$$

Lemma 2.1. (see [10]) Let $1 < \sigma \leq 2$, $-2 < \gamma < -1, \beta > 0$, and $h \in C_\alpha^2$, with $\int_0^\infty s^{\beta(\gamma+m)-1} h(\tau) d\tau < \infty$, $m = 1, 2$. The fractional problem

$$\begin{cases} D_\beta^{\gamma,\sigma} u(x) + h(x) = 0, x > 0, \\ \lim_{x\to 0} x^{\beta(2+\gamma)} I^{\sigma+\gamma, 2-\delta} u(x) = 0, \lim_{x\to\infty} x^{\beta(1+\gamma)} I^{\sigma+\gamma, 2-\sigma} u(x) = 0, \end{cases}$$

has a unique solution given by $u(x) = \int_0^\infty G_\sigma(x, s) s^{\beta(\gamma+1)-1} h(s) ds$, where

$$G_\sigma(x, s) = \begin{cases} \frac{\beta}{\Gamma(\sigma)} [x^{-\beta(\gamma+1)} - x^{-\beta(\delta+\gamma)} (x^\beta - s^\beta)^{\sigma-1}], 0 < s \leq x < \infty, \\ \frac{\beta}{\Gamma(\sigma)} x^{-\beta(\gamma+1)}, 0 < x \leq s < \infty. \end{cases} \tag{2.1}$$

Lemma 2.2. (see [10]) For $1 < \sigma \leq 2$, $-2 < \gamma < -1$, and $\beta > 0$, the function G_σ, defined in (2.1), has the following properties:
(i) $\frac{G_\sigma(x,s)}{1+x^{-\beta(1+\gamma)}} > 0$, for $x, s > 0$;
(ii) $\frac{G_\sigma(x,s)}{1+x^{-\beta(1+\gamma)}} \leq \frac{\beta}{\Gamma(\sigma)}$, for $x, s > 0$;
(iii) for $0 < \frac{\tau}{\lambda} \leq x \leq \tau$ and $s > \frac{\tau}{\lambda^2}$, where $\lambda > 1, \tau > 0$, we have

$$\frac{G_\sigma(x, s)}{1 + x^{-\beta(1+\gamma)}} \geq \frac{\beta(\sigma - 1)\tau^{-\beta(1+\gamma)}}{\Gamma(\sigma)\lambda^{-\beta(1-\gamma)}(1 + \tau^{-\beta(1+\gamma)})} = \frac{\beta p(\tau)}{\Gamma(\sigma)},$$

where $p(\tau) = \frac{(\sigma-1)\tau^{-\beta(1+\gamma)}}{\lambda^{\beta(1+\gamma)}(1+\tau^{-\beta(1+\gamma)})}$.
Lemma 2.3. (see [18]) Let $0 < \sigma_1, \sigma_2 \leq 1$ and $F, G \in C_\alpha^2$ with

$$\int_0^\infty s^{\beta(\gamma+m)-1} F(s, u(s), v(s)) ds < \infty, m = 1, 2,$$

$$\int_0^{\infty} s^{\beta(\gamma+m)-1} G(s, u(s), v(s))ds < \infty, m = 1, 2.$$

Then, (1.1) has a unique solution given by

$$u(x) = \int_0^{\infty} G_{\sigma_1}(x, s) s^{\beta(\gamma+1)-1} F(s, u(s), v(s))ds,$$

$$v(x) = \int_0^{\infty} G_{\sigma_2}(x, s) s^{\beta(\gamma+1)-1} G(s, u(s), v(s))ds,$$

where

$$G_{\sigma_1}(x, s) = \begin{cases} \frac{\beta}{\Gamma(\sigma_1)}[x^{-\beta(\gamma+1)} - x^{-\beta(\sigma_1+\gamma)}(x^{\beta} - s^{\beta})^{\sigma_1-1}], 0 < s \leq x < \infty, \\ \frac{\beta}{\Gamma(\sigma_1)} x^{-\beta(\gamma+1)}, 0 < x \leq s < \infty, \end{cases} \tag{2.2}$$

$$G_{\sigma_2}(x, s) = \begin{cases} \frac{\beta}{\Gamma(\sigma_2)}[x^{-\beta(\gamma+1)} - x^{-\beta(\sigma_2+\gamma)}(x^{\beta} - s^{\beta})^{\sigma_2-1}], 0 < s \leq x < \infty, \\ \frac{\beta}{\Gamma(\sigma_2)} x^{-\beta(\gamma+1)}, 0 < x \leq s < \infty. \end{cases} \tag{2.3}$$

The following result is our main tool.

Lemma 2.4. (Guo-Krasnosel'skii fixed point theorem; see [37]) P is a cone in a Banach space E, and D_1 and D_2 are bounded open sets in E with $\theta \in D_1, \overline{D_1} \subset D_2$. $A : P \cap (\overline{D_2} \setminus D_1) \to P$ is a completely continuous operator. Consider the following conditions (i), (ii):

(i) $\|Aw\| \leq \|w\|$ for $w \in P \cap \partial D_1$, $\|Aw\| \geq \|w\|$ for $w \in P \cap \partial D_2$;

(ii) $\|Aw\| \geq \|w\|$ for $w \in P \cap \partial D_1$, $\|Aw\| \leq \|w\|$ for $w \in P \cap \partial D_2$.

If one of the preceding conditions (i), (ii) holds, then A has at least one fixed point in $P \cap (\overline{D_2} \setminus D_1)$.

Next, we present some hypotheses that will play an important role in the subsequent discussion:

(H_1) $F, G : (0, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty) \to (0, +\infty)$ are continuous and nondecreasing with respect to the second, third variables on $(0, +\infty)$.

(H_2) For $(x, u, v) \in (0, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty)$,

$$F_1(x, u, v) = x^{\beta(1+\gamma)-1} F(x, (1 + x^{-\beta(1+\gamma)})u, (1 + x^{-\beta(1+\gamma)})v),$$

$$F_2(x, u, v) = x^{\beta(1+\gamma)-1} G(x, (1 + x^{-\beta(1+\gamma)})u, (1 + x^{-\beta(1+\gamma)})v),$$

such that

$$F_1(x, u, v) \leq \varphi_1(x)\omega_1(| u |) + \psi_1(x)\omega_2(| v |),$$

$$F_2(x, u, v) \leq \varphi_2(x)\widetilde{\omega_1}(| u |) + \psi_2(x)\widetilde{\omega_2}(| v |),$$

with $\omega_i, \widetilde{\omega_i} \in C((0, +\infty), (0, +\infty))$ nondecreasing and $\varphi_i, \psi_i \in L^1(0, +\infty)$, $i = 1, 2$.

(H_3) There are positive functions $q_i, \widetilde{q_i}, i = 1, 2$, with

$$q_i^* = \int_0^{\infty} (1 + x^{-\beta(1+\gamma)}) q_i(x)dx < \infty,$$

$$\widetilde{q_i^*} = \int_0^{\infty} (1 + x^{-\beta(1+\gamma)}) \widetilde{q_i}(x)dx < \infty,$$

such that

$$x^{\beta(\gamma+1)-1} \mid F(x, u, v) - F(x, \widetilde{u}, \widetilde{v}) \mid \leq q_1(x) \mid u - \widetilde{u} \mid + \widetilde{q_1}(x) \mid v - \widetilde{v} \mid,$$

$$x^{\beta(\gamma+1)-1} \mid G(x,u,v) - G(x,\widetilde{u},\widetilde{v}) \mid \leq q_2(x) \mid u - \widetilde{u} \mid + \widetilde{q}_2(t) \mid v - \widetilde{v} \mid,$$

for any $u, v, \widetilde{u}, \widetilde{v} \in (-\infty, +\infty)$ and $x \in (0, +\infty)$.

(H_4) $F, G : (0, +\infty) \times (0, +\infty) \times (0, +\infty) \to (0, +\infty)$ are continuous, such that

$$x^{\beta(1+\gamma)-1} F(x,u,v) = a_1(x) F_1(x,u,v),$$

$$x^{\beta(1+\gamma)-1} G(x,u,v) = a_2(x) G_1(x,u,v),$$

where $a_1, a_2 \in L^1((0,+\infty),(0,+\infty))$, $F_1, G_1 \in C((0,+\infty) \times (0,+\infty) \times (0,+\infty),(0,+\infty))$, $0 < \int_{\frac{\tau}{\lambda}}^{\tau} a_1(x)dx < \infty$, $0 < \int_{\frac{\tau}{\lambda}}^{\tau} a_2(x)dx < \infty$, with $\tau > 0$, $\lambda > 1$. Moreover, $x^{\beta(1+\gamma)-1}F(x,u,v)$, $x^{\beta(1+\gamma)-1}G(x,u,v) : [0,+\infty) \times (0,+\infty) \times (0,+\infty) \to [0,+\infty)$ also are continuous.

Remark 2.1. These conditions ensure the continuity and integrability of nonlinear terms in an infinite interval, which play a very important role in the proof of completely continuity for the relevant integral operators.

3. Main results

In this section, we use two Banach spaces defined by

$$X = \{u \in C((0,+\infty),(-\infty,+\infty)) \mid \lim_{x\to 0} \frac{u(x)}{1 + x^{-\beta(1+\gamma)}} \text{ and } \lim_{t\to+\infty} \frac{u(x)}{1 + x^{-\beta(1+\gamma)}} \text{ exist}\},$$

with the norm

$$\|u\|_X = \sup_{x>0} \mid \frac{u(x)}{1 + x^{-\beta(1+\gamma)}} \mid,$$

and

$$Y = \{v \in C((0,+\infty),(-\infty,+\infty)) \mid \lim_{x\to 0} \frac{v(x)}{1 + x^{-\beta(1+\gamma)}} \text{ and } \lim_{x\to+\infty} \frac{v(x)}{1 + x^{-\beta(1+\gamma)}} \text{ exist}\},$$

with the norm

$$\|v\|_Y = \sup_{x>0} \mid \frac{v(x)}{1 + x^{-\beta(1+\gamma)}} \mid .$$

So, $(X \times Y, \|(u,v)\|_{X\times Y})$ is a Banach space, with the norm $\|(u,v)\|_{X\times Y} = \|u\|_X + \|v\|_Y$.

Lemma 3.1. If F, G are continuous, then $(u,v) \in X \times Y$ is a solution of system (1.1)$\Leftrightarrow$ $(u,v) \in X \times Y$ is a solution of the following equations:

$$\begin{cases} u(x) = \int_0^{\infty} G_{\sigma_1}(x,s) s^{\beta(\gamma+1)-1} F(s,u(s),v(s)) ds, \\ v(x) = \int_0^{\infty} G_{\sigma_2}(x,s) s^{\beta(\gamma+1)-1} G(s,u(s),v(s)) ds. \end{cases}$$

For $(u,v) \in X \times Y$, we define an operator $A : X \times Y \to X \times Y$ as follows:

$$A(u,v)(x) = (A_1(u,v)(x), A_2(u,v)(x)),$$

where

$$A_1(u,v)(x) = \int_0^{\infty} G_{\sigma_1}(x,s) s^{\beta(\gamma+1)-1} F(s,u(s),v(s)) ds,$$

$$A_2(u,v)(x) = \int_0^{\infty} G_{\sigma_2}(x,s) s^{\beta(\gamma+1)-1} G(s,u(s),v(s)) ds,$$

with $G_{\sigma_i}(x,s), i=1,2$, given by (2.2) and (2.3).

Remark 3.1. Let $\sigma_1,\sigma_2,\beta,\gamma,\lambda,\tau\in\mathbb{R}$, such that $1<\sigma_1,\sigma_2\le 2, \beta>0, -2<\gamma<-1, \lambda>1, \tau>0$. If (H_2) and (H_4) hold, then for $(u,v)\in X\times Y$ with $u(x), v(x)>0$,

$$\int_0^{\infty} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds \le \eta\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds,$$

$$\int_0^{\infty} s^{\beta(\gamma+1)-1}G(s,u(s),v(s))ds \le \eta\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}G(s,u(s),v(s))ds,$$

where $\eta=\max\{\eta_1,\eta_2\}$ with $\eta_1=1+\frac{\iota}{\varrho_1(\lambda^2-1)}, \eta_2=1+\frac{\iota^*}{\varrho_2(\lambda^2-1)}>1, \varrho_1,\varrho_2,\iota,\iota^*>0$.

Proof. By (H_4), for $x\in[\frac{\tau}{\lambda^2},\tau]$, we know that there exist two constants $\varrho_1,\varrho_2>0$, such that

$$x^{\beta(\gamma+1)-1}F(s,u,v)\ge\varrho_1, x^{\beta(\gamma+1)-1}G(s,u,v)\ge\varrho_2, u,v\in(0,+\infty).$$

So, for $(u,v)\in X\times Y$ with $u(x),v(x)>0$,

$$\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds \ge \int_{\frac{\tau}{\lambda^2}}^{\tau} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds \ge \frac{\tau(\lambda^2-1)}{\lambda^2}\varrho_1,$$

$$\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}G(s,u(s),v(s))ds \ge \int_{\frac{\tau}{\lambda^2}}^{\tau} s^{\beta(\gamma+1)-1}G(s,u(s),v(s))ds \ge \frac{\tau(\lambda^2-1)}{\lambda^2}\varrho_2,$$

and hence,

$$\frac{\lambda^2}{\tau(\lambda^2-1)\varrho_1}\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds\ge 1,$$
$$\frac{\lambda^2}{\tau(\lambda^2-1)\varrho_2}\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}G(s,u(s),v(s))ds\ge 1.$$

By (H_4), we know that there exist two constants $\iota,\iota^*>0$, such that

$$x^{\beta(\gamma+1)-1}F(x,u(x),v(x))\le\iota, x^{\beta(\gamma+1)-1}G(x,u(x),v(x))\le\iota^*, for\ \ \forall x\in[0,\frac{\tau}{\lambda^2}].$$

Thus,

$$\int_0^{\frac{\tau}{\lambda^2}} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds\le\frac{\iota\tau}{\lambda^2},$$
$$\int_0^{\frac{\tau}{\lambda^2}} s^{\beta(\gamma+1)-1}G(s,u(s),v(s))ds\le\frac{\iota^*\tau}{\lambda^2}.$$

Therefore, we can obtain

$$\begin{aligned}\int_0^{\infty} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds &= \int_0^{\frac{\tau}{\lambda^2}} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds+\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds\\ &\le \frac{\iota\tau}{\lambda^2}+\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds\end{aligned}$$

$$
\begin{aligned}
&\leq (1+\frac{\iota}{\varrho_1(\lambda^2-1)})\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds\\
&= \eta_1\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_0^{\infty} s^{\beta(\gamma+1)-1}G(s,u(s),v(s))ds &\leq (1+\frac{\iota}{\varrho_2(\lambda^2-1)})\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}G(s,u(s),v(s))ds\\
&= \eta_2\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}G(s,u(s),v(s))ds.
\end{aligned}
$$

Take $\eta=\max\{\eta_1,\eta_2\}$, and thus

$$
\int_0^{\infty} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds \leq \eta\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds,
$$

$$
\int_0^{\infty} s^{\beta(\gamma+1)-1}G(s,u(s),v(s))ds \leq \eta\int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1}G(s,u(s),v(s))ds,
$$

hold. □

4. A positive solution

Define two cones

$$
K_1=\{u\in X \mid u(x)>0, x>0;\ \min_{x\in[\frac{\tau}{\lambda},\tau]}\frac{u(x)}{1+x^{-\beta(1+\gamma)}}\geq\frac{p(\tau)}{\eta}\|u\|_X\},
$$

$$
K_2=\{v\in Y \mid v(x)>0, x>0;\ \min_{x\in[\frac{\tau}{\lambda},\tau]}\frac{v(x)}{1+x^{-\beta(1+\gamma)}}\geq\frac{p(\tau)}{\eta}\|v\|_Y\}.
$$

Obviously, $K_1\times K_2=\{(u,v)\in X\times Y \mid u(x)>0, v(x)>0, \forall x>0;\ \min_{x\in[\frac{\tau}{\lambda},\tau]}\frac{u(x)}{1+x^{-\beta(1+\gamma)}}\geq\frac{p(\tau)}{\eta}\|u\|_X,\ \min_{x\in[\frac{\tau}{\lambda},\tau]}\frac{v(x)}{1+x^{-\beta(1+\gamma)}}\geq\frac{p(\tau)}{\eta}\|v\|_Y\}$ is also a cone. For convenience, we first list the following definitions:

$$
\begin{aligned}
F_0&=\lim_{(u,v)\to(0^+,0^+)}\sup_{x>0}\frac{F_1(t,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)}{u+v},\\
f_\infty&=\lim_{(u,v)\to(+\infty,+\infty)}\inf_{x>0}\frac{F_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)}{u+v},\\
f_0&=\lim_{(u,v)\to(0^+,0^+)}\inf_{x>0}\frac{F_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)}{u+v},\\
F_\infty&=\lim_{(u,v)\to(+\infty,+\infty)}\sup_{x>0}\frac{F_1(t,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)}{u+v},
\end{aligned}
$$

$$G_0^* = \lim_{(u,v)\to(0^+,0^+)} \sup_{x>0} \frac{G_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)}{u+v},$$

$$g_\infty^* = \lim_{(u,v)\to(+\infty,+\infty)} \inf_{x>0} \frac{G_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)}{u+v},$$

$$g_0^* = \lim_{(u,v)\to(0^+,0^+)} \inf_{x>0} \frac{G_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)}{u+v},$$

$$G_\infty^* = \lim_{(u,v)\to(+\infty,+\infty)} \sup_{x>0} \frac{G_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)}{u+v}.$$

Lemma 4.1. If assumptions (H_1) and (H_2) hold, then $A : K_1 \times K_2 \to K_1 \times K_2$ is completely continuous.
Proof. First, we show $A : K_1 \times K_2 \to K_1 \times K_2$. By (H_1) and (H_2), for $(u,v) \in K_1 \times K_2$,

$$\begin{aligned}
\|A_1(u,v)\|_X &= \sup_{t>0} \frac{|A_1(u,v)(x)|}{1+x^{-\beta(1+\gamma)}} \\
&= \sup_{x>0} \left| \int_0^\infty \frac{G_{\sigma_1}(x,s)}{1+x^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} F(s,u(s),v(s))ds \right| \\
&\le \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty | s^{\beta(\gamma+1)-1} F(s,u(s),v(s)) | ds \\
&= \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty | s^{\beta(\gamma+1)-1} F(s, \frac{(1+s^{-\beta(1+\gamma)})u(s)}{1+s^{-\beta(1+\gamma)}}, \frac{(1+s^{-\beta(1+\gamma)})v(s)}{1+s^{-\beta(1+\gamma)}}) | ds \\
&= \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty | F_1(s, \frac{u(s)}{1+s^{-\beta(1+\gamma)}}, \frac{v(s)}{1+s^{-\beta(1+\gamma)}}) | \\
&\le \frac{\beta}{\Gamma(\sigma_1)} [\omega_1(\|u\|_X) \int_0^\infty \varphi_1(s)ds + \omega_2(\|v\|_Y) \int_0^\infty \psi_1(s)ds] < +\infty.
\end{aligned}$$

Similarly,

$$\|A_2(u,v)\|_Y \le \frac{\beta}{\Gamma(\sigma_1)} [\widetilde{\omega_1}(\|u\|_X) \int_0^\infty \varphi_2(s)ds + \widetilde{\omega_2}(\|v\|_Y) \int_0^\infty \psi_2(s)ds] < +\infty.$$

By (H_1) and Lemma 2.2, for $(u,v) \in K_1 \times K_2$, we have $A_1(u,v)(x) > 0, A_2(u,v)(x) > 0, x > 0$. From Lemma 2.2 and Remark 3.1, for $x \in [\frac{\tau}{\lambda}, \tau]$, $\tau > 0$, and $\lambda > 1$,

$$\begin{aligned}
\frac{|A_1(u,v)(x)|}{1+x^{-\beta(1+\gamma)}} &= \int_0^\infty \frac{G_{\sigma_1}(x,s)}{1+x^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} F(s,u(s),v(s))ds \\
&= \int_0^{\frac{\tau}{\lambda^2}} \frac{G_{\sigma_1}(x,s)}{1+x^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} F(s,u(s),v(s))ds \\
&+ \int_{\frac{\tau}{\lambda^2}}^0 \frac{G_{\sigma_1}(x,s)}{1+x^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} F(s,u(s),v(s))ds \\
&\ge \int_{\frac{\tau}{\lambda^2}}^0 \frac{G_{\sigma_1}(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} F(s,u(s),v(s))ds \\
&\ge \frac{\beta p(\tau)}{\Gamma(\sigma_1)} \int_{\frac{\tau}{\lambda^2}}^0 s^{\beta(\gamma+1)-1} F(s,u(s),v(s))ds
\end{aligned}$$

$$\geq \frac{\beta p(\tau)}{\eta\Gamma(\sigma_1)}\int_0^\infty s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds$$
$$\geq \frac{p(\tau)}{\eta}\|A_1(u,v)\|_X.$$

So, $\frac{A_1(u,v)(x)}{1+x^{-\beta(1+\gamma)}} \geq \frac{p(\tau)}{\eta}\|A_1(u,v)\|_X$. Similarly, $\frac{A_2(u,v)(x)}{1+x^{-\beta(1+\gamma)}} \geq \frac{p(\tau)}{\eta}\|A_2(u,v)\|_Y$. Therefore,

$$\min_{x\in[\frac{\tau}{\lambda},\tau]}\frac{A_1(u,v)(x)}{1+x^{-\beta(1+\gamma)}} \geq \frac{p(\tau)}{\eta}\|A_1(u,v)\|_X,$$

$$\min_{x\in[\frac{\tau}{\lambda},\tau]}\frac{A_2(u,v)(x)}{1+x^{-\beta(1+\gamma)}} \geq \frac{p(\tau)}{\eta}\|A_2(u,v)\|_Y.$$

That is, $A : K_1 \times K_2 \to K_1 \times K_2$ is true.

Second, it will give a simply prove that A is continuous. Let $D = \{(u,v)|(u,v) \in K_1\times K_2, \|(u,v)\|_{X\times Y} \leq K, K > 0\}$, a bounded subset in $K_1 \times K_2$. Let $(u_n, v_n) \in D$ be a sequence that converges to (u,v) in $\in K_1 \times K_2$. Then $\|(u_n,v_n)\|_{X\times Y} \leq K$. From Lemma 2.2,

$$\begin{aligned}\|A_1(u_n,v_n)-A_1(u,v)\|_X &= \sup_{x>0}\left|\frac{A_1(u_n,v_n)(x)-A_1(u,v)(x)}{1+x^{-\beta(1+\gamma)}}\right| \\ &\leq \frac{\beta}{\Gamma(\sigma_1)}\left|\int_0^\infty s^{\beta(\gamma+1)-1}F(s,u_n(s),v_n(s))ds-\int_0^\infty s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds\right| \\ &\leq \frac{\beta}{\Gamma(\sigma_1)}\int_0^\infty \left| s^{\beta(\gamma+1)-1}(F(s,u_n(s),v_n(s))-F(s,u(s),v(s)))\right| ds.\end{aligned}$$

By (H_2),

$$\begin{aligned}\left| s^{\beta(\gamma+1)-1}F(s,u_n(s),v_n(s))\right| &= \left| s^{\beta(\gamma+1)-1}F\left(s,\frac{(1+s^{-\beta(1+\gamma)})u_n(s)}{1+s^{-\beta(1+\gamma)}},\frac{(1+s^{-\beta(1+\gamma)})v_n(s)}{1+s^{-\beta(1+\gamma)}}\right)\right| \\ &= F_1\left(s,\frac{u_n(s)}{1+s^{-\beta(1+\gamma)}},\frac{v_n(s)}{1+s^{-\beta(1+\gamma)}}\right) \\ &\leq \varphi_1(s)\omega_1(\|u_n\|_X)+\psi_1(s)\omega_2(\|v_n\|_Y) \in L^1(0,\infty).\end{aligned}$$

By the continuity of $s^{\beta(\gamma+1)-1}F(s,u(s),v(s))$ and the Lebesgue dominated convergence theorem,

$$\int_0^\infty s^{\beta(\gamma+1)-1}F(s,u_n(s),v_n(s))ds \to \int_0^\infty s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds, n\to\infty.$$

Therefore, $\|A_1(u_n,v_n)-A_1(u,v)\|_X \to 0, n\to\infty$. Similarly, $\|A_2(u_n,v_n)-A_2(u,v)\|_Y \to 0, n\to\infty$.

So, $\|A(u_n,v_n)-A(u,v)\|_{X\times Y} \to 0, n\to\infty$. That is, A is continuous in D. In the end, we know that $A(D)$ is relatively compact on $(0,\infty)$ and is equi-convergent at ∞ by [18]. Therefore, $A : K_1 \times K_2 \to K_1 \times K_2$ is completely continuous. □

Theorem 4.1. Assume that (H_2) and (H_4) hold. If $F_0 = 0, G_0^* = 0, f_\infty = \infty, g_\infty^* = \infty$, then the system (1.1) has at least one positive solution.

Proof. We divide the proof into several steps.

Step 1. $A : K_1 \times K_2 \to K_1 \times K_2$ is completely continuous. This result easily follows from Lemma 4.1.

Step 2. We show that there exist $R_1 > 0$ and $D_1 = \{(u,v) \in X \times Y, \|(u,v)\|_{X\times Y} < R_1\}$ such that

$\|A(u,v)\|_{X\times Y} \leq \|(u,v)\|_{X\times Y},\ (u,v) \in (K_1 \times K_2) \cap \partial D_1.$
Because $F_0 = 0, G_0^* = 0$, we choose $R_1 > 0$, such that

$$F_1(x, (1 + x^{-\beta(1+\gamma)})u, (1 + x^{-\beta(1+\gamma)})v) \leq \epsilon_1(u + v),$$

$$G_1(x, (1 + x^{-\beta(1+\gamma)})u, (1 + x^{-\beta(1+\gamma)})v) \leq \epsilon_2(u + v),$$

for $0 < u + v \leq R_1, x > 0$, where $\epsilon_1, \epsilon_2 > 0$ satisfy

$$\epsilon_1 \leq \frac{1}{2}\frac{\Gamma(\sigma_1)}{\beta\int_0^\infty a_1(s)ds}, \epsilon_2 \leq \frac{1}{2}\frac{\Gamma(\sigma_2)}{\beta\int_0^\infty a_2(s)ds}.$$

So, for $(u,v) \in K_1 \times K_2$ and $\|(u,v)\|_{X\times Y} = R_1$, by Lemma 2.2,

$$\begin{aligned}
\frac{A_1(u,v)(x)}{1 + x^{-\beta(1+\gamma)}} &= \int_0^\infty \frac{G_{\sigma_1}(x,s)}{1 + x^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} F(s, u(s), v(s))ds \\
&\leq \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty s^{\beta(\gamma+1)-1} F(s, u(s), v(s))ds,
\end{aligned}$$

$$\begin{aligned}
\frac{A_2(u,v)(x)}{1 + x^{-\beta(1+\gamma)}} &= \int_0^\infty \frac{G_{\sigma_2}(x,s)}{1 + x^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} G(s, u(s), v(s))ds \\
&\leq \frac{\beta}{\Gamma(\sigma_2)} \int_0^\infty s^{\beta(\gamma+1)-1} G(s, u(s), v(s))ds.
\end{aligned}$$

By (H_4),

$$\begin{aligned}
\frac{A_1(u,v)(x)}{1 + x^{-\beta(1+\gamma)}} &\leq \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty a_1(s) F_1(s, u(s), v(s))ds \\
&= \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty a_1(s) F_1(s, (1 + s^{-\beta(1+\gamma)})\frac{u(s)}{1 + s^{-\beta(1+\gamma)}}, (1 + s^{-\beta(1+\gamma)})\frac{v(s)}{1 + s^{-\beta(1+\gamma)}})ds \\
&\leq \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty a_1(s)\epsilon_1 \frac{u(s) + v(s)}{1 + s^{-\beta(1+\gamma)}} ds \\
&\leq \frac{\beta}{\Gamma(\sigma_1)}\epsilon_1 \|(u,v)\|_{X\times Y} \int_0^\infty a_1(s)ds \\
&\leq \frac{1}{2}\|(u,v)\|_{X\times Y}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{A_2(u,v)(x)}{1 + x^{-\beta(1+\gamma)}} &\leq \frac{\beta}{\Gamma(\sigma_2)}\epsilon_2 \|(u,v)\|_{X\times Y} \int_0^\infty a_2(s)ds \\
&\leq \frac{1}{2}\|(u,v)\|_{X\times Y}.
\end{aligned}$$

Therefore,

$$\|A(u,v)\|_{X\times Y} \leq \|(u,v)\|_{X\times Y},\ for\ (u,v) \in K_1 \times K_2,\ and\ \|(u,v)\|_{X\times Y} = R_1.$$

Let $D_1 = \{(u,v) \in X \times Y, \|(u,v)\|_{X\times Y} < R_1\}$. Then,

$$\|A(u,v)\|_{X\times Y} \le \|(u,v)\|_{X\times Y},\ for\ (u,v) \in (K_1 \times K_2) \cap \partial D_1.$$

Step 3. We show that there exist $R_2 > 0$ and $D_2 = \{(u,v) \in X \times Y, \|(u,v)\|_{X\times Y} < R_2\}$ such that

$$\|A(u,v)\|_{X\times Y} \ge \|(u,v)\|_{X\times Y},\ for\ (u,v) \in (K_1 \times K_2) \cap \partial D_2.$$

Because $f_\infty = \infty, g^*_\infty = \infty$, there exists $R > 0$, such that

$$F_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v) \ge m_1(u+v),$$

$$G_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v) \ge m_2(u+v),$$

for $u + v \ge R, x > 0$, where $m_1, m_2 > 0$ satisfy

$$m_1 \ge \frac{1}{2}\frac{\eta_1\eta\Gamma(\sigma_1)}{\beta p^2(\tau)\int_\tau^{\frac{\tau}{\lambda}} a_1(s)ds}, m_2 \ge \frac{1}{2}\frac{\eta_2\eta\Gamma(\sigma_2)}{\beta p^2(\tau)\int_\tau^{\frac{\tau}{\lambda}} a_2(s)ds}, \eta = \max\{\eta_1,\eta_2\}.$$

Let $R_2 \ge \max\{R_1, \frac{\eta R}{p(\tau)}\}$, and $D_2 = \{(u,v) \in X \times Y, \|(u,v)\|_{X\times Y} < R_2\}$. Then, $D_1 \subset D_2$.

Thus, for $(u,v) \in K_1 \times K_2, \|(u,v)\|_{X\times Y} = R_2$, we have

$$\frac{u(x)}{1+x^{-\beta(1+\gamma)}} \ge \min_{x\in[\frac{\tau}{\lambda},\tau]} \frac{u(x)}{1+x^{-\beta(1+\gamma)}} \ge \frac{p(\tau)}{\eta_1}\|u\|_X,$$

$$\frac{v(x)}{1+x^{-\beta(1+\gamma)}} \ge \min_{x\in[\frac{\tau}{\lambda},\tau]} \frac{v(x)}{1+x^{-\beta(1+\gamma)}} \ge \frac{p(\tau)}{\eta_2}\|v\|_Y.$$

So,

$$\begin{aligned}\frac{u(x)+v(x)}{1+x^{-\beta(1+\gamma)}} &\ge \frac{p(\tau)}{\eta_1}\|u\|_X + \frac{p(\tau)}{\eta_2}\|v\|_Y \ge \frac{p(\tau)}{\eta}(\|u\|_X + \|v\|_Y)\\ &= \frac{p(\tau)}{\eta}\|(u,v)\|_{X\times Y} = \frac{p(\tau)}{\eta}R_2 \ge R.\end{aligned}$$

By (H_4), for $x \in [\frac{\tau}{\lambda},\tau]$, we can obtain

$$\begin{aligned}\frac{A_1(u,v)(x)}{1+x^{-\beta(1+\gamma)}} &\ge \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}\int_0^\infty s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds\\ &= \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}\int_0^\infty a_1(s)F_1(s,u(s),v(s))ds\\ &= \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}\int_0^\infty a_1(s)F_1(s,(1+s^{-\beta(1+\gamma)})\frac{u(s)}{1+s^{-\beta(1+\gamma)}},(1+s^{-\beta(1+\gamma)})\frac{v(s)}{1+s^{-\beta(1+\gamma)}})ds\\ &\ge \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}m_1\int_0^\infty a_1(s)\frac{u(s)+v(s)}{1+s^{-\beta(1+\gamma)}}ds\\ &\ge \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}m_1\int_0^\infty a_1(s)ds\frac{p(\tau)}{\eta_1}\|u\|_X + \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}m_1\int_0^\infty a_1(s)ds\frac{p(\tau)}{\eta_2}\|v\|_Y\end{aligned}$$

$$\begin{aligned}
&\geq \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}m_1\int_{\tau}^{\frac{\tau}{\lambda}}a_1(s)ds\frac{p(\tau)}{\eta_1}\|u\|_X+\frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}m_1\int_{\tau}^{\frac{\tau}{\lambda}}a_1(s)ds\frac{p(\tau)}{\eta_2}\|v\|_Y\\
&= \frac{\beta p^2(\tau)}{\eta_1\Gamma(\sigma_1)}m_1\int_{\tau}^{\frac{\tau}{\lambda}}a_1(s)ds(\frac{1}{\eta_1}\|u\|_X+\frac{1}{\eta_2}\|v\|_Y)\\
&\geq \frac{\beta p^2(\tau)}{\eta_1\Gamma(\sigma_1)}m_1\int_{\tau}^{\frac{\tau}{\lambda}}a_1(s)ds\frac{1}{\eta}\|(u,v)\|_{X\times Y}\\
&\geq \frac{1}{2}\|(u,v)\|_{X\times Y}.
\end{aligned}$$

Similarly, $\frac{A_2(u,v)(x)}{1+x^{-\beta(1+\gamma)}}\geq\frac{1}{2}\|(u,v)\|_{X\times Y}$. Therefore,

$$\|A(u,v)\|_{X\times Y}\geq\|(u,v)\|_{X\times Y},\ for\ (u,v)\in(K_1\times K_2)\cap\partial D_2.$$

Finally, by Lemma 2.4, A has a fixed point in $(K_1\times K_1)\cap\partial(\overline{D_2}\backslash D_1)$. So, (1.1) has at least one positive solution. □

Theorem 4.2. Assume that (H_2) and (H_4) hold. If $f_0=\infty, g_0^*=\infty, F_\infty=0, G_\infty^*=0$, then (1.1) has at least one positive solution.

Proof. We divide the proof into several steps.

Step 1. $A: K_1\times K_2\to K_1\times K_2$ is completely continuous. This result easily follows from Lemma 4.1.

Step 2. We show that there exist $r_1>0$ and $D_1=\{(u,v)\in X\times Y,\|(u,v)\|_{X\times Y}<r_1\}$ such that

$$\|A(u,v)\|_{X\times Y}\geq\|(u,v)\|_{X\times Y},\ for\ (u,v)\in(K_1\times K_2)\cap\partial D_1.$$

Because $f_0=\infty, g_0^*=\infty$, there exists $r_1>0$ such that

$$F_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)\geq M_1(u+v),$$

$$G_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)\geq M_2(u+v),$$

for $0<u+v\leq r_1, x>0$, where $M_1,M_2>0$, satisfy

$$M_1\geq\frac{1}{2}\frac{\eta_1\eta\Gamma(\sigma_1)}{\beta p^2(\tau)\int_{\tau}^{\frac{\tau}{\lambda}}a_1(s)ds}, M_2\geq\frac{1}{2}\frac{\eta_2\eta\Gamma(\sigma_2)}{\beta p^2(\tau)\int_{\tau}^{\frac{\tau}{\lambda}}a_2(s)ds}, \eta=\max\{\eta_1,\eta_2\}.$$

Let $D_1=\{(u,v)\in X\times Y,\|(u,v)\|_{X\times Y}<r_1\}$. So, for $(u,v)\in K_1\times K_2$ with $\|(u,v)\|_{X\times Y}=r_1$, and $x\in[\frac{\tau}{\lambda},\tau]$, then by (H_4),

$$\begin{aligned}
\frac{A_1(u,v)(x)}{1+x^{-\beta(1+\gamma)}} &\geq \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}\int_0^\infty s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds\\
&= \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}\int_0^\infty a_1(s)F_1(s,u(s),v(s))ds\\
&= \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}\int_0^\infty a_1(s)F_1(s,(1+s^{-\beta(1+\gamma)})\frac{u(s)}{1+s^{-\beta(1+\gamma)}},(1+s^{-\beta(1+\gamma)})\frac{v(s)}{1+s^{-\beta(1+\gamma)}})ds\\
&\geq \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}M_1\int_0^\infty a_1(s)\frac{u(s)+v(s)}{1+s^{-\beta(1+\gamma)}}ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}M_1\int_0^{\infty}a_1(s)ds\frac{p(\tau)}{\eta_1}\|u\|_X+\frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}M_1\int_0^{\infty}a_1(s)ds\frac{p(\tau)}{\eta_2}\|v\|_Y\\
&\geq \frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}M_1\int_{\tau}^{\frac{\tau}{\lambda}}a_1(s)ds\frac{p(\tau)}{\eta_1}\|u\|_X+\frac{\beta p(\tau)}{\eta_1\Gamma(\sigma_1)}M_1\int_{\tau}^{\frac{\tau}{\lambda}}a_1(s)ds\frac{p(\tau)}{\eta_2}\|v\|_Y\\
&= \frac{\beta p^2(\tau)}{\eta_1\Gamma(\sigma_1)}M_1\int_{\tau}^{\frac{\tau}{\lambda}}a_1(s)ds(\frac{1}{\eta_1}\|u\|_X+\frac{1}{\eta_2}\|v\|_Y)\\
&\geq \frac{\beta p^2(\tau)}{\eta_1\Gamma(\sigma_1)}M_1\int_{\tau}^{\frac{\tau}{\lambda}}a_1(s)ds\frac{1}{\eta}\|(u,v)\|_{X\times Y}\\
&\geq \frac{1}{2}\|(u,v)\|_{X\times Y}.
\end{aligned}
$$

Similarly, $\frac{A_2(u,v)(x)}{1+x^{-\beta(1+\gamma)}} \geq \frac{1}{2}\|(u,v)\|_{X\times Y}$. Thus,

$$\|A(u,v)\|_{X\times Y} \geq \|(u,v)\|_{X\times Y},\ for\ (u,v)\in(K_1\times K_2)\cap\partial D_1.$$

Step 3. We show that there exist $r_2>0$ and $D_2=\{(u,v)\in X\times Y,\|(u,v)\|_{X\times Y}<r_2\}$ such that

$$\|A(u,v)\|_{X\times Y}\leq\|(u,v)\|_{X\times Y} \text{ for } (u,v)\in(K_1\times K_2)\cap\partial D_2.$$

Because $F_\infty=0, G_\infty^*=0$, there exists $r>0$, such that

$$F_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)\leq\epsilon_1(u+v),$$

$$G_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)\leq\epsilon_2(u+v),$$

for $u+v>r, x>0$, where $\epsilon_1,\epsilon_2>0$ satisfy

$$\epsilon_1\leq\frac{1}{2}\frac{\Gamma(\sigma_1)}{\beta\int_0^\infty a_1(s)ds},\epsilon_2\leq\frac{1}{2}\frac{\Gamma(\sigma_2)}{\beta\int_0^\infty a_2(s)ds}.$$

Let $D_2=\{(u,v)\in X\times Y,\|(u,v)\|_{X\times Y}<r_2\}$, where $r_2>\max\{r_1,r\}$. Then $D_1\subset D_1$. We define two functions U_1,U_2 as follows:

$$U_1:(-\infty,+\infty)\to(-\infty,+\infty), U_1(a)=\sup_{0<u+v\leq a}\sup_{x>0}F_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v),$$

$$U_2:(-\infty,+\infty)\to(-\infty,+\infty), U_2(a)=\sup_{0<u+v\leq a}\sup_{x>0}G_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v).$$

For $(u,v)\in K_1\times K_2$ and $\|(u,v)\|_{X\times Y}=r_2$,

$$
\begin{aligned}
U_1(r_2) &= \sup_{0<u+v\leq r_2}\sup_{x>0}F_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)\\
&\leq \epsilon_1\sup_{0<u+v\leq r_2}(u+v)=\epsilon_1 r_2=\epsilon_1\|(u,v)\|_{X\times Y},
\end{aligned}
$$

$$U_2(r_2) = \sup_{0<u+v\leq r_2}\sup_{x>0}G_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)$$

$$\leq \epsilon_2 \sup_{0<u+v\leq r_2} (u+v) = \epsilon_2 r_2 = \epsilon_2 \|(u,v)\|_{X\times Y}.$$

By Lemma 2.2 and (H_4),

$$\begin{aligned}
\frac{A_1(u,v)(x)}{1+x^{-\beta(1+\gamma)}} &\leq \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty s^{\beta(\gamma+1)-1} F(s,u(s),v(s))ds \\
&= \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty a_1(s) F_1(s,u(s),v(s))ds \\
&= \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty a_1(s) F_1(s,(1+s^{-\beta(1+\gamma)})\frac{u(s)}{1+s^{-\beta(1+\gamma)}},(1+s^{-\beta(1+\gamma)})\frac{v(s)}{1+s^{-\beta(1+\gamma)}})ds \\
&\leq \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty a_1(s) \sup_{0<u+v\leq r_2} \sup_{x>0} F_1(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v)ds \\
&= \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty a_1(s) U_1(r_2)ds \\
&\leq \frac{\beta}{\Gamma(\sigma_1)} \int_0^\infty a_1(s)ds \epsilon_1 \|(u,v)\|_{X\times Y} \\
&\leq \frac{1}{2}\|(u,v)\|_{X\times Y}.
\end{aligned}$$

Similarly, $\frac{A_2(u,v)(x)}{1+x^{-\beta(1+\gamma)}} \leq \frac{1}{2}\|(u,v)\|_{X\times Y}$. Therefore, $\|A(u,v)\|_{X\times Y} \leq \|(u,v)\|_{X\times Y}$, for $(u,v) \in (K_1\times K_2)\cap \partial D_2$. Finally, by Lemma 2.4, A has a fixed point in $(K_1\times K_1)\cap \partial(\overline{D_2}\backslash D_1)$. So, the system (1.1) has at least one positive solution. □

5. Multiple positive solutions

In the section, we obtain the multiplicity of positive solution of (1.1) by using the monotone iterative technique.

Theorem 5.1. If (H_1) and (H_2) hold, then (1.1) has two positive solutions (u^*,v^*) and (w^*,z^*) satisfying $0 \leq \|(u^*,v^*)\|_{X\times Y} \leq \Upsilon$ and $0 \leq \|(w^*,z^*)\|_{X\times Y} \leq \Upsilon$, where Υ is a positive preset constant. Moreover, $\lim_{n\to\infty}(u_n,v_n) = (u^*,v^*)$ and $\lim_{n\to\infty}(w_n,z_n) = (w^*,z^*)$, where (u_n,v_n) and (w_n,z_n) are given by

$$(u_n(x),v_n(x)) = (A_1(u_{n-1},v_{n-1})(x),A_2(u_{n-1},v_{n-1})(x)), n=1,2,\ldots, \tag{5.1}$$

with

$$(u_0(x),v_0(x)) = (\Upsilon_1[1+x^{-\beta(\gamma+1)}],\Upsilon_2[1+x^{-\beta(\gamma+1)}]),\Upsilon_1,\Upsilon_2>0,\Upsilon_1+\Upsilon_2\leq\Upsilon,$$

and

$$(w_n(x),z_n(x)) = (A_1(w_{n-1},z_{n-1})(x),A_2(w_{n-1},z_{n-1})(x)), n=1,2,\ldots, \tag{5.2}$$

with $(w_0(x),z_0(x)) = (0,0)$. In addition,

$$\begin{aligned}
&(w_0(x),z_0(x)) \leq (w_1(x),z_1(x)) \leq \cdots \leq (w_n(x),z_n(x)) \leq \cdots \leq (w^*,z^*) \leq (u^*,v^*) \\
&\leq \cdots \leq (u_n(x),v_n(x)) \leq \cdots \leq (u_1(x),v_1(x)) \leq (u_0(x),v_0(x)).
\end{aligned} \tag{5.3}$$

Proof. First, from Lemma 4.1, $A(K_1\times K_2)\subset K_1\times K_2$ for $(u,v)\in K_1\times K_2$. Let

$$\Upsilon_1 = \frac{\beta}{\Gamma(\sigma_1)}[\omega_1(\Upsilon)\int_0^\infty \varphi_1(s)ds + \omega_2(\Upsilon)\int_0^\infty \psi_1(s)ds] < \infty,$$

$$\Upsilon_2 = \frac{\beta}{\Gamma(\sigma_2)}[\widetilde{\omega_1}(\Upsilon)\int_0^\infty \varphi_2(s)ds + \widetilde{\omega_2}(\Upsilon)\int_0^\infty \psi_2(s)ds] < \infty,$$

and $\Upsilon \geq \Upsilon_1 + \Upsilon_2$ with $D_\Upsilon = \{(u, v) \in K_1 \times K_2 : \|(u, v)\|_{X\times Y} \leq \Upsilon\}$. For any $(u, v) \in D_\Upsilon$, from (H_2) and Lemma 2.2,

$$\begin{aligned}
\|A_1(u,v)\|_X &= \sup_{x>0}\frac{|A_1(u,v)(x)|}{1+x^{-\beta(1+\gamma)}}\\
&= \sup_{x>0} \mid \int_0^\infty \frac{G_{\sigma_1}(x,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds \mid\\
&\leq \frac{\beta}{\Gamma(\sigma_1)}\int_0^\infty \mid s^{\beta(\gamma+1)-1}F(s,u(s),v(s))ds \mid\\
&\leq \frac{\beta}{\Gamma(\sigma_1)}[\omega_1(\frac{\mid u(s)\mid}{1+s^{-\beta(1+\gamma)}})\int_0^\infty \varphi_1(s)ds + \omega_2(\frac{\mid v(s)\mid}{1+s^{-\beta(1+\gamma)}})\int_0^\infty \psi_1(s)ds]\\
&\leq \frac{\beta}{\Gamma(\sigma_1)}[\omega_1(\|u\|_X)\int_0^\infty \varphi_1(s)ds + \omega_2(\|v\|_Y)\int_0^\infty \psi_1(s)ds]\\
&\leq \frac{\beta}{\Gamma(\sigma_1)}[\omega_1(\Upsilon)\int_0^\infty \varphi_1(s)ds + \omega_2(\Upsilon)\int_0^\infty \psi_1(s)ds] = \Upsilon_1.
\end{aligned}$$

Similarly, $\|A_2(u, v)\|_Y \leq \Upsilon_2$ for $(u, v) \in D_\Upsilon$. Thus,

$$\|A(u,v)\|_{X\times Y} = \|A_1(u,v)\|_X + \|A_2(u,v)\|_Y \leq \Upsilon_1 + \Upsilon_2 \leq \Upsilon.$$

That is, $A(D_\Upsilon) \subset D_\Upsilon$. We construct two sequences as follows:

$$(u_n, v_n) = A(u_{n-1}, v_{n-1}), (w_n, z_n) = A(w_{n-1}, z_{n-1}), \quad n = 1, 2, 3, \ldots.$$

Obviously, $(u_0(x), v_0(x)), (w_0(x), z_0(x)) \in D_\Upsilon$. Because $A(D_\Upsilon) \subset D_\Upsilon$, $(u_n, v_n), (w_n, z_n) \in D_\Upsilon, n = 1, 2, \ldots$. We need to show that there exist (u^*, v^*) and (w^*, z^*) satisfying $\lim_{n\to\infty}(u_n, v_n) = (u^*, v^*)$ and $\lim_{n\to\infty}(w_n, z_n) = (w^*, z^*)$ which are two monotone sequences for approximating positive solutions of the system (1.1).

For $x \in (0, +\infty), (u_n, v_n) \in D_\Upsilon$, from Lemma 2.2 and (5.1),

$$\begin{aligned}
u_1(x) &= A_1(u_0,v_0)(x) = \int_0^\infty G_{\sigma_1}(x,s)s^{\beta(\gamma+1)-1}F(s,u_0(s),v_0(s))ds\\
&\leq \frac{\beta}{\Gamma(\sigma_1)}\int_0^\infty (1+t^{-\beta(1+\gamma)})s^{\beta(\gamma+1)-1}F(s,u_0(s),v_0(s))ds\\
&\leq \frac{\beta}{\Gamma(\sigma_1)}(1+x^{-\beta(1+\gamma)})[\omega_1(\frac{\mid u_0(s)\mid}{1+s^{-\beta(1+\gamma)}})\int_0^\infty \varphi_1(s)ds + \omega_2(\frac{\mid v_0(s)\mid}{1+s^{-\beta(1+\gamma)}})\int_0^\infty \psi_1(s)ds]\\
&\leq \frac{\beta}{\Gamma(\sigma_1)}(1+x^{-\beta(1+\gamma)})[\omega_1(\|u_0\|_X)\int_0^\infty \varphi_1(s)ds + \omega_2(\|v_0\|_Y)\int_0^\infty \psi_1(s)ds]\\
&\leq \frac{\beta}{\Gamma(\sigma_1)}(1+x^{-\beta(1+\gamma)})[\omega_1(\Upsilon)\int_0^\infty \varphi_1(s)ds + \omega_2(\Upsilon)\int_0^\infty \psi_1(s)ds]\\
&= (1+x^{-\beta(1+\gamma)})\Upsilon_1 = u_0(x)
\end{aligned}$$

and

$$\begin{aligned}
v_1(x)=A_2(u_0,v_0)(x) &= \int_0^\infty G_{\sigma_2}(x,s)s^{\beta(\gamma+1)-1}G(s,u_0(s),v_0(s))ds\\
&\le \frac{\beta}{\Gamma(\sigma_2)}\int_0^\infty (1+x^{-\beta(1+\gamma)})s^{\beta(\gamma+1)-1}G(s,u_0(s),v_0(s))ds\\
&\le \frac{\beta}{\Gamma(\sigma_2)}(1+t^{-\beta(1+\gamma)})[\widetilde{\omega_1}(\frac{\mid u_0(s)\mid}{1+s^{-\beta(1+\gamma)}})\int_0^\infty \varphi_2(s)ds+\widetilde{\omega_2}(\frac{\mid v_0(s)\mid}{1+s^{-\beta(1+\gamma)}})\int_0^\infty \psi_2(s)ds]\\
&\le \frac{\beta}{\Gamma(\sigma_2)}(1+t^{-\beta(1+\gamma)})[\widetilde{\omega_1}(\|u_0\|_X)\int_0^\infty \varphi_2(s)ds+\widetilde{\omega_2}(\|v_0\|_Y)\int_0^\infty \psi_2(s)ds]\\
&\le \frac{\beta}{\Gamma(\sigma_2)}(1+x^{-\beta(1+\gamma)})[\widetilde{\omega_1}(\Upsilon)\int_0^\infty \varphi_2(s)ds+\widetilde{\omega_2}(\Upsilon)\int_0^\infty \psi_2(s)ds]\\
&= (1+x^{-\beta(1+\gamma)})\Upsilon_2=v_0(x),
\end{aligned}$$

that is,

$$(u_1(x),v_1(x))=(A_1(u_0,v_0)(x),A_2(u_0,v_0)(x))\le((1+x^{-\beta(1+\gamma)})\Upsilon_1,(1+x^{-\beta(1+\gamma)})\Upsilon_2)=(u_0(x),v_0(x)).$$

So, by the condition (H_1),

$$(u_2(x),v_2(x))=(A_1(u_1,v_1)(x),A_2(u_1,v_1)(x))\le(A_1(u_0,v_0)(x),A_2(u_0,v_0)(x))=(u_1(x),v_1(x)).$$

For $x\in(0,+\infty)$, the sequences $\{(u_n,v_n)\}_{n=0}^\infty$ satisfy $(u_{n+1}(x),v_{n+1}(x))\le(u_n(x),v_n(x))$. By the iterative sequences $(u_{n+1},v_{n+1})=A(u_n,v_n)$ and the complete continuity of the operator A, $(u_n,v_n)\to(u^*,v^*)$, and $A(u^*,v^*)=(u^*,v^*)$.

Similarly, for the sequences $\{(w_n,z_n)\}_{n=0}^\infty$, we have

$$\begin{aligned}
(w_1(x),z_1(x)) &= (A_1(w_0,z_0)(x),A_2(w_0,z_0)(x))\\
&= (\int_0^\infty G_{\sigma_1}(x,s)s^{\beta(\gamma+1)-1}F(s,w_0(s),z_0(s))ds,\int_0^\infty G_{\sigma_2}(x,s)s^{\beta(\gamma+1)-1}G(s,w_0(s),z_0(s))ds)\\
&\ge (0,0)=(w_0(x),z_0(x)).
\end{aligned}$$

Then, by the condition (H_1),

$$(w_2(x),z_2(x))=(A_1(w_1,z_1)(x),A_2(w_1,z_1)(x))\ge(A_1(w_0,z_0)(x),A_2(w_0,z_0)(x))=(w_1(x),z_1(x)).$$

Analogously, for $x\in(0,+\infty)$, we have $(w_{n+1}(x),z_{n+1}(x))\ge(w_n(x),z_n(x))$. By the iterative sequences $(w_{n+1},z_{n+1})=A(w_n,z_n)$ and the complete continuity of the operator A, $(w_n,z_n)\to(w^*,z^*)$, and $A(w^*,z^*)=(w^*,z^*)$.

Finally, we prove that (u^*,v^*) and (w^*,z^*) are the minimal and maximal positive solutions of (1.1). Assume that $(\varsigma(x),\mu(x))$ is any positive solution of (1.1). Then, $A(\varsigma(x),\mu(x))=(\varsigma(x),\mu(x))$, and

$$(w_0(x),z_0(x))=(0,0)\le(\varsigma(x),\mu(x))\le((1+x^{-\beta(1+\gamma)})\Upsilon_1,(1+x^{-\beta(1+\gamma)})\Upsilon_2)=(u_0(x),v_0(x)).$$

Therefore,

$$(w_1(x),z_1(x))=(A_1(w_0,z_0)(x),A_2(w_0,z_0)(x))\le(\varsigma(x),\mu(x))\le(A_1(u_0,v_0)(x),A_2(u_0,v_0)(x))=(u_1(x),v_1(x)).$$

That is, $(w_1(x),z_1(x))\le(\varsigma(x),\mu(x))\le(u_n(x),v_n(x))$. So, (5.3) holds. By (H_1), $(0,0)$ is not a solution of (1.1). From (5.1), (w^*,z^*) and (u^*,v^*) are two extreme positive solutions of (1.1), which can be constructed via limitS of two monotone iterative sequences in (5.1) and (5.2). □

6. Examples

Example 6.1. We consider the following system:

$$\begin{cases} D_1^{-\frac{3}{2},\frac{5}{3}}u(x) + x^{\frac{3}{2}}(\frac{u}{1+x^{\frac{1}{2}}})^2 e^{-x} + x^{\frac{3}{2}}(\frac{v}{1+x^{\frac{1}{2}}})^2 e^{-x} = 0, t \in (0,+\infty), \\ D_1^{-\frac{3}{2},\frac{3}{2}}v(x) + x^{\frac{5}{2}}e^{-2x^2}(\frac{u}{1+x^{\frac{1}{2}}})^2 \ln(1+(\frac{u}{1+x^{\frac{1}{2}}})^2) + x^{\frac{5}{2}}e^{-2x^2}(\frac{u}{1+x^{\frac{1}{2}}})^2 \ln(1+(\frac{u}{1+x^{\frac{1}{2}}})^2), x \in (0,+\infty), \\ \lim_{x\to 0} x^{\frac{1}{2}} I^{\frac{1}{6},\frac{1}{3}}u(x) = 0, \lim_{x\to\infty} x^{-\frac{1}{2}} I^{\frac{1}{6},\frac{1}{3}}u(x) = 0, \\ \lim_{x\to 0} x^{\frac{1}{2}} I^{0,\frac{1}{2}}v(x) = 0, \lim_{x\to\infty} x^{-\frac{1}{2}} I^{0,\frac{1}{2}}v(x) = 0, \end{cases} \tag{6.1}$$

where $\sigma_1 = \frac{5}{3}, \sigma_2 = \frac{3}{2}, \gamma = -\frac{3}{2}, \beta = 1$,

$$F(x,u,v) = x^{\frac{3}{2}}e^{-x}[(\frac{u}{1+x^{\frac{1}{2}}})^2 + (\frac{v}{1+x^{\frac{1}{2}}})^2],$$

$$G(x,u,v) = x^{\frac{5}{2}}e^{-2x^2}[(\frac{u}{1+x^{\frac{1}{2}}})^2 \ln(1+(\frac{u}{1+x^{\frac{1}{2}}})^2) + (\frac{u}{1+x^{\frac{1}{2}}})^2 \ln(1+(\frac{u}{1+x^{\frac{1}{2}}})^2)].$$

First, for $F_1(x,u,v) = x^{\beta(1+\gamma)-1}F(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v) = e^{-x}(u^2,v^2)$, we choose $\omega_1(u) = u^2 \in C((0,+\infty),(0,+\infty))$, $\omega_2(v) = v^2 \in C((0,+\infty),(0,+\infty))$, and $\varphi_1(x) = \psi_1(x) = e^{-x} \in L^1(0,+\infty)$. Then,

$$| F_1(x,u,v) | \leq \varphi_1(x)\omega_1(| u |) + \psi_1(t)\omega_2(| v |), \quad (0,+\infty)\times(-\infty,+\infty)\times(-\infty,+\infty).$$

Similarly, for $F_2(x,u,v) = x^{\beta(1+\gamma)-1}G(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v) = xe^{-2x^2}[u^2\ln(u^2+1)+v^2\ln(v^2+1)]$, we choose $\widetilde{\omega_1}(u) = u^2\ln(u^2+1) \in C((0,+\infty),(0,+\infty))$, $\widetilde{\omega_2}(v) = v^2\ln(v^2+1) \in C((0,+\infty),(0,+\infty))$, and $\varphi_2(x) = \psi_2(x) = xe^{-2x^2} \in L^1(0,+\infty)$. Then,

$$| F_2(x,u,v) | \leq \varphi_2(x)\widetilde{\omega_1}(| u |) + \psi_2(x)\widetilde{\omega_2}(| v |), \quad (0,+\infty)\times(-\infty,+\infty)\times(-\infty,+\infty).$$

So, the condition (H_2) holds. Obviously, $F,G : (0,+\infty)\times(0,+\infty)\times(0,+\infty) \to (0,+\infty)$ are continuous.

$$x^{-\frac{3}{2}}F(x,u,v) = e^{-x}[(\frac{u}{1+x^{\frac{1}{2}}})^2 + (\frac{v}{1+x^{\frac{1}{2}}})^2] = a_1(x)F_1(x,u,v),$$

$$x^{-\frac{3}{2}}G(x,u,v) = xe^{-2x^2}[(\frac{u}{1+x^{\frac{1}{2}}})^2 \ln(1+(\frac{u}{1+x^{\frac{1}{2}}})^2) + (\frac{u}{1+x^{\frac{1}{2}}})^2 \ln(1+(\frac{u}{1+x^{\frac{1}{2}}})^2)] = a_2(x)G_1(x,u,v),$$

where $a_1(x) = e^{-x}, a_2(x) = xe^{-2x^2}$, $F_1(x,u,v) = (\frac{u}{1+x^{\frac{1}{2}}})^2 + (\frac{v}{1+x^{\frac{1}{2}}})^2$, $G_1(t,u,v) = (\frac{u}{1+x^{\frac{1}{2}}})^2\ln(1+(\frac{u}{1+x^{\frac{1}{2}}})^2) + (\frac{u}{1+x^{\frac{1}{2}}})^2\ln(1+(\frac{u}{1+x^{\frac{1}{2}}})^2)$. So, $x^{-\frac{3}{2}}f(x,u,v), x^{-\frac{3}{2}}G(x,u,v) : [0,+\infty)\times(0,+\infty)\times(0,+\infty) \to [0,+\infty)$ are continuous. Hence, the condition (H_4) holds. Finally,

$$F_0 = \lim_{(u,v)\to(0^+,0^+)} \frac{u^2+v^2}{u+v} = 0, G_0^* = \lim_{(u,v)\to(0^+,0^+)} \frac{u^2\ln(u^2+1)+v^2\ln(v^2+1)}{u+v} = 0,$$

$$f_\infty = \lim_{(u,v)\to(+\infty,+\infty)} \frac{u^2+v^2}{u+v} = \infty, g_\infty^* = \lim_{(u,v)\to(+\infty,+\infty)} \frac{u^2\ln(u^2+1)+v^2\ln(v^2+1)}{u+v} = \infty.$$

Therefore, from Theorem 4.1, (6.1) has at least one positive solution $(u(x), v(x))$. Further,

$$\begin{cases} u(x) = \frac{3}{2\Gamma(\frac{2}{3})}[x^{\frac{1}{2}}\int_0^\infty s^{-\frac{3}{2}}F(s,u(s),v(s))ds - x^{-\frac{8}{3}}\int_x^\infty (x-s)^{\frac{2}{3}}s^{-\frac{3}{2}}F(s,u(s),v(s))ds], \\ v(x) = \frac{2}{\sqrt{\pi}}[x^{\frac{1}{2}}\int_0^\infty s^{-\frac{3}{2}}G(s,u(s),v(s))ds - \int_x^\infty (x-s)^{\frac{1}{2}}s^{-\frac{3}{2}}G(s,u(s),v(s))ds]. \end{cases}$$

Example 6.2. We consider the following system:

$$\begin{cases} D_1^{-\frac{3}{2},\frac{3}{2}}u(x) + x^{\frac{5}{2}}e^{-2x^2+1}[\arctan(\frac{u}{1+x^{\frac{1}{2}}})^2 + \frac{1}{\pi}] + x^{\frac{5}{2}}e^{-2x^2+1}[\arctan(\frac{u}{1+x^{\frac{1}{2}}})^2 + \pi] = 0, x \in (0,+\infty),\\ D_1^{-\frac{3}{2},\frac{7}{6}}v(x) + x^{\frac{3}{2}}e^{-x}[\arctan(\ln((\frac{u}{1+x^{\frac{1}{2}}})^2 + 1)) + \frac{3}{2}\pi] + x^{\frac{3}{2}}e^{-x}[\arctan(\ln((\frac{v}{1+x^{\frac{1}{2}}})^2 + 1)) + 1], x \in (0,+\infty),\\ \lim_{x\to 0} x^{\frac{1}{2}}I^{0\frac{1}{2}}u(x) = 0, \lim_{x\to\infty} x^{-\frac{1}{2}}I^{0,\frac{1}{2}}u(x) = 0,\\ \lim_{x\to 0} x^{\frac{1}{2}}I^{-\frac{1}{3},\frac{5}{6}}v(x) = 0, \lim_{x\to\infty} x^{-\frac{1}{2}}I^{-\frac{1}{3},\frac{5}{6}}v(x) = 0, \end{cases} \tag{6.2}$$

where $\sigma_1 = \frac{3}{2}, \sigma_2 = \frac{7}{6}, \gamma = -\frac{3}{2}, \beta = 1$,

$$F(x,u,v) = x^{\frac{5}{2}}e^{-2x^2+1}[\arctan(\frac{u}{1+x^{\frac{1}{2}}})^2 + \frac{1}{\pi}] + x^{\frac{5}{2}}e^{-2x^2+1}[\arctan(\frac{u}{1+x^{\frac{1}{2}}})^2 + \frac{1}{\pi}],$$

$$G(x,u,v) = x^{\frac{3}{2}}e^{-x}[\arctan(\ln((\frac{u}{1+x^{\frac{1}{2}}})^2 + 1)) + \frac{3}{2}\pi] + x^{\frac{3}{2}}e^{-x}[\arctan(\ln((\frac{v}{1+x^{\frac{1}{2}}})^2 + 1)) + 1].$$

First, for

$$F_1(x,u,v) = x^{\beta(1+\gamma)-1}F(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v) = xe^{-2x^2+1}[\arctan u^2 + \frac{1}{\pi} + \arctan v^2 + \pi],$$

we choose $\omega_1(u) = \arctan u^2 + \frac{1}{\pi} \in C((0,+\infty),(0,+\infty)), \omega_2(v) = \arctan v^2 + \pi \in C((0,+\infty),(0,+\infty))$, and $\varphi_1(x) = \psi_1(x) = xe^{-2x^2+1} \in L^1(0,+\infty)$. Then,

$$| F_1(x,u,v) |\leq \varphi_1(x)\omega_1(| u |) + \psi_1(x)\omega_2(| v |),\ (0,+\infty)\times(-\infty,+\infty)\times(-\infty,+\infty).$$

Similarly, for

$$F_2(x,u,v) = x^{\beta(1+\gamma)-1}g(x,(1+x^{-\beta(1+\gamma)})u,(1+x^{-\beta(1+\gamma)})v) = e^{-x}[\arctan(\ln(u^2+1))+\frac{3}{2}\pi+\arctan(\ln(v^2+1))+1],$$

we choose $\widetilde{\omega_1}(u) = \arctan(\ln(u^2+1)) + \frac{3}{2}\pi \in C((0,+\infty),(0,+\infty)), \widetilde{\omega_2}(v) = \arctan(\ln(v^2+1)) + 1 \in C((0,+\infty),(0,+\infty))$, and $\varphi_2(x) = \psi_2(x) = e^{-x} \in L^1(0,+\infty)$. Then,

$$| F_2(x,u,v) |\leq \varphi_2(x)\widetilde{\omega_1}(| u |) + \psi_2(x)\widetilde{\omega_2}(| v |),\ \ (0,+\infty)\times(-\infty,+\infty)\times(-\infty,+\infty).$$

That is, (H_2) holds. Second, $F, G : (0,+\infty)\times(0,+\infty)\times(0,+\infty) \to (0,+\infty)$ are continuous. And

$$x^{-\frac{3}{2}}F(x,u,v) = xe^{-2x^2+1}[\arctan(\frac{u}{1+x^{\frac{1}{2}}})^2 + \frac{1}{\pi} + \arctan(\frac{v}{1+x^{\frac{1}{2}}})^2 + \pi] = a_1(x)F_1(x,u,v),$$

$$x^{-\frac{3}{2}}G(x,u,v) = e^{-x}[\arctan(\ln((\frac{u}{1+x^{\frac{1}{2}}})^2 + 1)) + \frac{3}{2}\pi + \arctan(\ln((\frac{v}{1+x^{\frac{1}{2}}})^2 + 1)) + 1] = a_2(x)G_1(x,u,v),$$

where $a_1(x) = xe^{-2x^2+1}, a_2(x) = e^{-x}$, $F_1(x,u,v) = \arctan(\frac{u}{1+x^{\frac{1}{2}}})^2 + \frac{1}{\pi} + \arctan(\frac{v}{1+x^{\frac{1}{2}}})^2 + \pi$, $G_1(x,u,v) = \arctan(\ln((\frac{u}{1+x^{\frac{1}{2}}})^2 + 1)) + \frac{3}{2}\pi + \arctan(\ln((\frac{v}{1+x^{\frac{1}{2}}})^2 + 1)) + 1$. So, $x^{-\frac{3}{2}}F(x,u,v), x^{-\frac{3}{2}}G(x,u,v) : [0,+\infty)\times(0,+\infty)\times(0,+\infty) \to [0,+\infty)$ are continuous. That is, (H_4) holds. In addition,

$$f_0 = \lim_{(u,v)\to(0^+,0^+)} \frac{\arctan u^2 + \frac{1}{\pi} + \arctan v^2 + \pi}{u+v} = \infty,$$

$$g_0^* = \lim_{(u,v)\to(0^+,0^+)} \frac{\arctan(\ln(u^2+1)) + \frac{3}{2}\pi + \arctan(\ln(v^2+1)) + 1}{u+v} = \infty,$$

$$F_\infty = \lim_{(u,v)\to(+\infty,+\infty)} \frac{\arctan u^2 + \frac{1}{\pi} + \arctan v^2 + \pi}{u+v} = 0,$$

$$G_\infty^* = \lim_{(u,v)\to(+\infty,+\infty)} \frac{\arctan(\ln(u^2+1)) + \frac{3}{2}\pi + \arctan(\ln(v^2+1)) + 1}{u+v} = 0.$$

Therefore, from Theorem 4.2, (6.2) has at least one positive solution $(u(x), v(x))$. Further,

$$\begin{cases} u(x) = \frac{2}{\sqrt{\pi}}[x^{\frac{1}{2}} \int_0^\infty s^{-\frac{3}{2}} F(s,u(s),v(s))ds - \int_x^\infty (x-s)^{\frac{1}{2}} s^{-\frac{3}{2}} F(s,u(s),v(s))ds], \\ v(x) = \frac{6}{\Gamma(\frac{1}{6})}[x^{\frac{1}{2}} \int_0^\infty s^{-\frac{3}{2}} G(s,u(s),v(s))ds - x^{\frac{1}{3}} \int_x^\infty (x-s)^{\frac{1}{6}} s^{-\frac{3}{2}} G(s,u(s),v(s))ds]. \end{cases}$$

Example 6.3. We consider the following system:

$$\begin{cases} D_1^{-\frac{3}{2},\frac{5}{3}} u(x) + x^{\frac{3}{2}} \frac{e^{-x}}{3} \mid \frac{u}{1+x^{\frac{1}{2}}} \mid + x^{\frac{5}{2}} \ln(\mid \frac{v}{1+x^{\frac{1}{2}}} \mid + 1) \frac{e^{-2x^2+1}}{10} = 0, x \in (0,+\infty), \\ D_1^{-\frac{3}{2},\frac{3}{2}} v(x) + x^{\frac{5}{2}} e^{-2x^2+1} \arctan(\mid \frac{u}{1+x^{\frac{1}{2}}} \mid + \frac{1}{\sqrt{\pi}}) + x^{\frac{5}{2}} \frac{e^{-2x^2+1}}{5} \mid \frac{v}{1+x^{\frac{1}{2}}} \mid = 0, x \in (0,+\infty), \\ \lim_{x\to 0} x^{\frac{1}{2}} I^{\frac{1}{6},\frac{1}{3}} u(x) = 0, \lim_{x\to\infty} x^{-\frac{1}{2}} I^{\frac{1}{6},\frac{1}{3}} u(x) = 0, \\ \lim_{x\to 0} x^{\frac{1}{2}} I^{0,\frac{1}{2}} v(x) = 0, \lim_{x\to\infty} x^{-\frac{1}{2}} I^{0,\frac{1}{2}} v(x) = 0, \end{cases} \tag{6.3}$$

where $\sigma_1 = \frac{5}{3}, \sigma_2 = \frac{3}{2}, \gamma = -\frac{3}{2}, \beta = 1$,

$$F(x,u,v) = x^{\frac{3}{2}} \frac{e^{-x}}{3} \mid \frac{u}{1+x^{\frac{1}{2}}} \mid + x^{\frac{5}{2}} \ln(\mid \frac{v}{1+x^{\frac{1}{2}}} \mid + 1) \frac{e^{-2x^2+1}}{10},$$

$$G(x,u,v) = x^{\frac{5}{2}} e^{-2x^2+1} \arctan(\mid \frac{u}{1+x^{\frac{1}{2}}} \mid + \frac{1}{\sqrt{\pi}}) + x^{\frac{5}{2}} \frac{e^{-2x^2+1}}{5} \mid \frac{v}{1+x^{\frac{1}{2}}} \mid .$$

Obviously, $F, G : (0,+\infty) \times (-\infty,+\infty) \times (-\infty,+\infty) \to (0,+\infty)$ are continuous and nondecreasing with respect to the second and the third variables on $(0,+\infty)$. That is, (H_1) holds. Next,

$$F_1(x,u,v) = x^{\beta(1+\gamma)-1} F(x, (1+x^{-\beta(1+\gamma)})u, (1+x^{-\beta(1+\gamma)})v) = \frac{e^{-x}}{3} \mid u \mid + x \frac{e^{-2x^2+1}}{10} \ln(\mid v \mid + 1).$$

We choose $\omega_1(u) = \mid u \mid \in C((0,+\infty),(0,+\infty))$, $\omega_2(v) = \ln(\mid v \mid + 1) \in C((0,+\infty),(0,+\infty))$, and $\varphi_1(x) = \frac{e^{-x}}{3}, \psi_1(x) = \frac{xe^{-2x^2+1}}{10} \in L^1(0,+\infty)$. Then,

$$\mid F_1(x,u,v) \mid \le \varphi_1(x)\omega_1(\mid u \mid) + \psi_1(x)\omega_2(\mid v \mid), \quad (0,+\infty) \times (-\infty,+\infty) \times (-\infty,+\infty).$$

Similarly, for

$$F_2(x,u,v) = x^{\beta(1+\gamma)-1} G(x, (1+x^{-\beta(1+\gamma)})u, (1+x^{-\beta(1+\gamma)})v) = xe^{-2x^2+1} \arctan(\mid u \mid + \frac{1}{\sqrt{\pi}}) + x \frac{e^{-2x^2+1}}{5} \mid v \mid,$$

we choose $\widetilde{\omega_1}(u) = \arctan(\mid u \mid + \frac{1}{\sqrt{\pi}}) \in C((0,+\infty),(0,+\infty))$, $\widetilde{\omega_2}(v) = \mid v \mid \in C((0,+\infty),(0,+\infty))$, and $\varphi_2(x) = xe^{-2x^2+1}, \psi_2(x) = x\frac{e^{-2x^2+1}}{5} \in L^1(0,+\infty)$. Then,

$$\mid F_2(x,u,v) \mid \le \varphi_2(x)\widetilde{\omega_1}(\mid u \mid) + \psi_2(x)\widetilde{\omega_2}(\mid v \mid), \quad (0,+\infty) \times (-\infty,+\infty) \times (-\infty,+\infty).$$

That is, (H_2) holds. Therefore, from Theorem 5.1, (6.3) has two positive solutions (u^*, v^*) and (w^*, z^*) with $(0,0) \leq (u^*(x), v^*(x)), (w^*(x), z^*(x)) \leq ((1+x^{\frac{1}{2}})\Upsilon_1, (1+x^{\frac{1}{2}})\Upsilon_2)$, where $\Upsilon_1 + \Upsilon_2 \leq \Upsilon$, and Υ satisfies

$$\frac{95.58}{191.86}\Upsilon - 0.69\arctan(\Upsilon + 0.56) \geq \frac{1}{36}.$$

7. Conclusions

This paper studies the Erdélyi-Kober fractional coupled system (1.1), where the variable is in an infinite interval. We give some proper conditions and set a special Banach space. We obtain the existence of at least one positive solution for (1.1) by using the Guo-Krasnosel'skii fixed point theorem, and we get the existence of at least two positive solutions for (1.1) by using the monotone iterative technique. Our methods and results are different from ones in [18]. Moreover, we give three examples to show the plausibility of our main results. For future work, we intend to use other fixed point theorems to solve some Erdélyi-Kober fractional differential equations.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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