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Research article

The q-rung orthopair fuzzy-valued neutrosophic sets: Axiomatic properties, aggregation operators and applications

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Abstract: During the transitional phase spanning from the realm of fuzzy logic to the realm of neutrosophy, a multitude of hybrid models have emerged, each surpassing its predecessor in terms of superiority. Given the pervasive presence of indeterminacy in the world, a higher degree of precision is essential for effectively handling imprecision. Consequently, more sophisticated variants of neutrosophic sets (NSs) have been conceived. The key objective of this paper is to introduce yet another variant of NS, known as the q-rung orthopair fuzzy-valued neutrosophic set (q-ROFVNS). By leveraging the extended spatial range offered by q-ROFS, q-ROFVNS enables a more nuanced representation of indeterminacy and inconsistency. Our endeavor commences with the definitions of q-ROFVNS and q-ROFVN numbers (q-ROFVNNs). Then, we propose several types of score and accuracy functions to facilitate the comparison of q-ROFVNNs. Fundamental operations of q-ROFVNSs and some algebraic operational rules of q-ROFVNNs are also provided with their properties, substantiated by proofs and elucidated through illustrative examples. Drawing upon the operational rules of q-ROFVNNs, the q-ROFVN weighted average operator (q-ROFVNWAO) and q-ROFVN weighted geometric operator (q-ROFVNWGO) are proposed. Notably, we present the properties of these operators, including idempotency, boundedness and monotonicity. Furthermore, we emphasize the applicability and significance of the q-ROFVN operators, substantiating their utility through an algorithm and a numerical application. To further validate and evaluate the proposed model, we conduct a comparative analysis, examining its accuracy and performance in relation to existing

models.

Keywords: score function; optimization; fuzzy sets; aggregation operators; linear diophantine fuzzy sets; decision-making; T-spherical fuzzy sets
Mathematics Subject Classification: 03E72, 90B50, 68T35

1. Introduction

Multiple criteria decision-making (MCDM) is the most common method to help decision-makers opt for the most in-demand alternative from a given alternative set, where the MCDM requires ranking all results (alternatives) according to effective mathematical tools to pick the best one(s) in real-world systems. At the forefront of classical MCDM theory, fuzzy sets (FSs), conceptualized by the renowned pioneer Zadeh [1], have emerged as a transformative force. With their unique ability to effectively tackle the challenges posed by ambiguous and misleading information [2,3], fuzzy sets have established themselves as an invaluable tool for navigating the intricacies of complex decision-making processes [4–6]. However, there are situations where fuzzy sets alone may not provide an accurate representation of vague and incomplete information in MCDM problems. To address these challenging scenarios effectively, intuitionistic fuzzy sets (IFSs) [7] have been developed. The uncertain data in IFS is portrayed by the form of two membership functions, namely, the membership function (MF) $\hat{\zeta}$ and non-membership function (NMF) $\hat{\varpi}$, and both values fulfill the following condition: $\hat{\zeta} + \hat{\varpi} < 1$. Some further expansion and modernization of IFS are portrayed in [8–10]. Subsequently, Yager [11] proposed the Pythagorean fuzzy set (PyFS) to make up for the shortcomings of IFS when $\hat{\zeta} + \hat{\varpi} \geq 1$. In a simple way, the mechanism of action of PyFS is to square the values of each MF and NMF so that their sum is less than or equal to 1. This model attracted a large number of researchers and prompted them to make many contributions. For instance, in [12–15] authors conceptualized various measures like distance, similarity, divergence and fuzzy entropy in the PyF environment. Gao and Deng [16] presented PyFSs based on the negation of probability (NP) and applied the NPPyFS into technique for order preference by similarity to ideal solution (TOPSIS). Hussain et al. [17] studied the impact of Aczel-Alsina aggregation operators (AOs) on PyFSs when they developed novel types of PyFAOs by employing the Aczel-Alsina t-norm and Aczel-Alsina t-conorm. Ullah et al. [18] developed complex PyFSs and referred to their applications in pattern recognition. Subsequently, more and more contributions of PyFSs have been discussed by numerous scholars [19–21]. In some real-life situations, the handcuffs of the IFS and PyFS structures may be broken where the sum of both MF and NMF exceeds 1, i.e., $\hat{\zeta} + \hat{\varpi} > 1$ or the sum of the squares of both MF and NMF exceeds 1. For example, if we take the value (0.7, 0.9), we can simply note that $0.7 + 0.9 > 1$. To tackle this issue, Yager [22] again constructed an innovative notion called q-ROFS as an acclaimed generalization of both IFS and PyFS structures, with the following standard condition: The sum of the qth powers of MF and NMF is less than or equal to 1. Obviously, we can conclude that q-ROFS is more general than IFS and PyFS and offers a greater degree of flexibility and reliability. Therefore, q-ROFS has been successfully applied to treat obscure data where two or more grounds for doubt arise simultaneously. As a result, Liu and Wang [23] introduced the pioneering q-ROF weighted averaging operator (q-ROFWAO) and the q-ROF weighted geometric operator (q-ROFWGO) as effective tools for handling decision information. They [24] also proposed MADM based on Archimedean T-norm and T-conorm (ATT), Bonferroni mean (BM) operators of a q-ROF environment. Wang et al. [25] utilized ten similarity measures and ten weighted similarity measures between q-ROFSs to deal with MADM problems. Liu et al. [26] coined the cosine similarity measure and a Euclidean distance measure of q-ROFSs and studied their properties. Dhankhar et al. [27] initially defined the possibility degree-based measurement of q-ROFSs and clarified their theoretical structure. Deveci et al. [28] proposed a q-ROF OPA-RAFSI model to estimate three personal mobility alternative implementation options for autonomous cars in the metaverse. Lin et al. [29] initiated a general form of linguistic q-ROFSs and devised the operational laws, which are the linguistic q-ROF weighted averaging (LqROFWA) operator and the linguistic q-ROF weighted geometric (LqROFWG) operator. Li et al. [30] coined preference relations on q-ROFSs, and based on these preference relations, they built some algorithms for ranking and selecting the MCDM alternatives. Peng et al. [31] defined a new exponential operational law concerning q-ROFNs bases being positive real numbers and the exponents being q-ROFNs and applied it to derive the q-ROF weighted exponential aggregation operator (q-ROFWEAO). Deveci et al. [32] proposed a novel hybrid MCDM model named q-ROF full consistency method (q-ROF FUCOM) and q-ROF combined compromised solution (q-ROF CoCoSo), respectively, for the site picking of an offshore wind farm (OWF). Deveci et al. [33] developed a new approach to combinative distance-based assessment (CODAS)-based q-ROFSs, and this approach has been implemented to deal with the uncertain issues that occur in DM problems. Alnefaie et al. [34] formulated the q-ROFS for algebraic structures, and Habib et al. [35] investigated the formwork q-ROFS graph structures. The relationship between the q-ROFS and complex numbers (CN) was defined by Garg et al. [36] when they discussed several weighted averaging and geometric power aggregation operators for complex q-ROFSs (Cq-ROFSs).

On the other hand, Smarandache [37] came up with the idea of the NSs, which expand the MF and NMF of FS, IFS and PyFS in order to handle the MADM issues that have uncertain, incomplete and indeterminate decision information. The notion of NSs is summarized by terms namely, MF, NMF, and, in addition, the indeterminacy term (IMF) with the following condition: The sum of these terms is equal to or limited to three, so it can describe the real-life data more constitutionally and accurately. Due to these features that characterize the concept of NSs, NSs have been extensively studied by several researchers in the academic environment around the world, where it has invaded all mathematical sciences branches. For example, in neutrosophic statistics, it represents sample sizes and control chart design constants as neutrosophic numbers, while in neutrosophic algebra many algebraic concepts have appeared, such as the neutrosophic subgroup and group and the neutrosophic ring, whose operations and axioms are partially MF, partially IMF and partially NMF. In neutrosophic possibility, every neutrosophic term (MF, IMF, and NMF) has a possibility degree. However, from a precise scientific perspective, some weaknesses appear when applying NSs to common data analysis in many daily life scenarios. To overcome this issue, Ye [38] proposed the idea of simplified neutrosophic sets (SNSs), which are considered a sub-class form of NSs, and introduced some AOs, including a SN-weighted arithmetic average (SNWAA) operator and a SN-weighted geometric average operator (SNWGA). Mishra et al. [39] presented MCDM using the NS environment. Ali and Smarandache [40] expanded three NS memberships from a real to a complex environment. Building upon their work, Al-Quran et al. [41, 42] have further expanded NS by introducing Q-complex neutrosophic sets and fuzzy parameterized complex neutrosophic soft expert sets within the same environment. Expanding on these advancements, Al-Sharqi et al. [43] combined both NS and soft sets under the interval complex value. Karabasevi et al. [44] developed a novel extension of the TOPSIS method using NSs. Abdel-Basset et al. [45] suggested a neutrosophic MCDM (NMCDM) approach to assist patients and physicians in determining if a patient is suffering from heart failure. Jana and Pal [46] introduced a new aggregation operator of SVNSNs and utilized this operator to address medical diagnosis problems. Further, Ji et al. [47] studied the Frank normalized prioritized Bonferroni mean (FNPBM) under NS environment and defined some NS interaction FNPBM operators to solve MADM problems. Xu et al. [48] developed the neutrosophic TODIM method. Hu et al. [49] developed the local and global threshold criteria with the SVNS domain. Ye [50] extended the triangular NS to the trapezoidal NS, in which its main three characteristics (MF, NMF and IMF) are trapezoidal neutrosophic numbers rather than triangular neutrosophic numbers. Kaur and Garg [51] presented AOs based on generalized linguistic neutrosophic cubic weighted averages (GLNCWA) and generalized linguistic neutrosophic cubic weighted geometric using Archimedean norms. Jana et al. [52] further defined the score and accuracy functions on the interval trapezoidal neutrosophic set (ITNS), and then they defined the ITN-number weighted arithmetic averaging (ITNNWAA) operator and the ITN-number weighted geometric averaging (ITNNWGA) operator, along with their applications in real-life scenarios. Recently, a lot of emphasis has been placed on incorporating the features of NSs, IFSs and PyFs to increase accuracy and improve AOs to address data inaccuracies. Bhowmik and Pal [53] initiated the IFVNS and its operators with the condition that the sum of its MFs is less than or equal to two. Then, Unver et al. [54] redefined the IFVNS, when they defined the IF neutrosophic multi-sets (IFNMSs). They also presented some algebraic operations between IFVNSs in order to develop several AOs. Palanikumar et al. [55] discussed a new generalization of Pythagorean neutrosophic normal interval-valued weighted geometric (PNNIVWG) and obtained an algorithm that tackles the alternatives in MADM problems entrenched in these operators. Chellamani and Ajay [56] proposed several basic graphical ideas employing the Dombi operator within Pythagorean neutrosophic fuzzy graphs (PyNFG). Ajay and Chellamani [57] utilized soft parameters for the MCDM scenario under a PyFVNS environment. Palanikumar and Arulmozhi [58] developed a new approach to AOs using parameterized factors in the PyFVNS environment, and they proposed a score function based on aggregating of both TOPSIS and VIKOR techniques. Rajan and Krishnaswamy [59] developed clustering methods based on similarity measures between PyFVNSs. Siraj et al. [60] provoked the concept of a Pym-polar FNs (PmFNSs) for managing data that contains multi-polar facts. Lately, Bozyigit et al. [61] have redefined the PyFVNS, where each component of the NS encompasses a PyFVS under the condition: $\hat{\zeta}^2+\hat{\varpi}^2 \leq 1$. However, the scope of IFVNSs and PyFVNSs is restricted because their capability is limited to addressing decision making problems where the evaluation values are represented using IF and PyF values, and these values are insufficient to fully convey the actual decision-related information. In this article, we broaden the scope of the notions of IFVNSs and PyFVNSs by incorporating the q-ROF values to the construction of the SNS. It is important to highlight that as the rung q increases, the range of acceptable orthopairs expands, and a greater number of orthopairs meet the bounding constraint. Consequently, by utilizing q-ROF values, we are able to represent a broader spectrum of fuzzy information. Hence, this paper introduces several key contributions.

(1) The concept of the q-ROFVNS is introduced, which extends and incorporates the principles of previously published neutrosophic set-like literature.
(2) The q-ROFVNWAO and q-ROFVNWGO are investigated by leveraging the operational laws of

q-ROFVNNs.

(3) The ranking is accomplished using various types of SFs and AFs.

(4) An MADM problem is tackled by utilizing q-ROFVNNs and incorporating the q-ROFVNWAO and q-ROFVNWGO.

(5) Through comparative analysis, geometric interpretations are presented to demonstrate the benefits of the proposed approaches.

The subsequent sections of this manuscript are succinctly delineated as follows: Section 2 encompasses an exhaustive appraisal of the fundamental underpinnings surrounding the IFS, PyFS, q-ROFS, NS, SNS, IFVNS, and PyFVNS. Section 3, in its entirety, expounds upon the meticulous definition of the q-ROFVNS and delves into a comprehensive examination of its fundamental and algebraic operations. Furthermore, a multitude of diverse categories of SFs and AFs are meticulously introduced within this section. Section 4, on the other hand, engenders the conceptualization of the q-ROFVNWAO and q-ROFVNWGO, accompanied by an extensive discourse on their inherent properties. Notably, Section 5 unveils an exemplary MADM methodology that effectively harnesses the proposed operators. A compelling illustrative example is also presented here to vividly showcase the practical application of the proposed models. Section 6 culminates in a comprehensive comparative analysis meticulously elucidating the unequivocal superiority of the proposed methodologies. In the ultimate Section 7, conclusive remarks are expounded, encapsulating the key findings and outcomes.

2. Basic knowledge

A few elementary terms are recalled from previously published papers in this part. The symbols $\hat{\blacksquare}$ and $\hat{\Delta}$ will represent [0,1] and the universal set, respectively, throughout the paper.

Definition 2.1. [7] An IFS $\hat{\Xi}$ is defined on $\hat{\Delta}$ as

$$\hat{\Xi} = \{(\hat{\delta}, \langle \hat{\zeta}_{\hat{\Xi}}(\hat{\delta}), \hat{\varpi}_{\hat{\Xi}}(\hat{\delta})\rangle) : \hat{\delta} \in \hat{\Delta}\},$$

where $\hat{\zeta}_{\hat{\Xi}}$ and $\hat{\varpi}_{\hat{\Xi}} \in \hat{\blacksquare}$ are, respectively, the MF and NMF, such that $0 \leq \hat{\zeta}_{\hat{\Xi}}(\hat{\delta}) + \hat{\varpi}_{\hat{\Xi}}(\hat{\delta}) \leq 1, \forall \hat{\delta} \in \hat{\Delta}$.

Definition 2.2. [11] The PyFS $\hat{\mathfrak{A}}$ in $\hat{\Delta}$ is formalized as

$$\hat{\mathfrak{A}} = \{(\hat{\delta}, \langle \hat{\zeta}_{\hat{\mathfrak{A}}}(\hat{\delta}), \hat{\varpi}_{\hat{\mathfrak{A}}}(\hat{\delta})\rangle) : \hat{\delta} \in \hat{\Delta}\},$$

where $\hat{\zeta}_{\hat{\mathfrak{A}}} : \hat{\Delta} \longrightarrow \hat{\blacksquare}$ denotes the MF and $\hat{\varpi}_{\hat{\mathfrak{A}}} : \hat{\Delta} \longrightarrow \hat{\blacksquare}$ denotes the NMF with the condition that $0 \leq (\hat{\zeta}_{\hat{\mathfrak{A}}}(\hat{\delta}))^2 + (\hat{\varpi}_{\hat{\mathfrak{A}}}(\hat{\delta}))^2 \leq 1$.

Definition 2.3. [22] The q-ROFS $\hat{\Lambda}$ on $\hat{\Delta}$ is expressed as

$$\hat{\Lambda} = \{(\hat{\delta}, \hat{\zeta}_{\hat{\Lambda}}(\hat{\delta}), \hat{\varpi}_{\hat{\Lambda}}(\hat{\delta})) : \hat{\delta} \in \hat{\Delta}\},$$

where $\hat{\zeta}_{\hat{\Lambda}}(\hat{\delta})$ and $\hat{\varpi}_{\hat{\Lambda}}(\hat{\delta})$ lie in $\hat{\blacksquare}$ under the condition $0 \leq (\hat{\zeta}_{\hat{\Lambda}}(\hat{\delta}))^q + (\hat{\varpi}_{\hat{\Lambda}}(\hat{\delta}))^q \leq 1$ $(q \geq 1), \forall \hat{\delta} \in \hat{\Delta}$. The hesitancy part is given by: $\lambda_{\hat{\Lambda}}(\hat{\delta}) = \left((\hat{\zeta}_{\hat{\Lambda}}(\hat{\delta}))^q + (\hat{\varpi}_{\hat{\Lambda}}(\hat{\delta}))^q - (\hat{\zeta}_{\hat{\Lambda}}(\hat{\delta}))^q(\hat{\varpi}_{\hat{\Lambda}}(\hat{\delta}))^q\right)^{1/q}$.

Definition 2.4. [37] A NS $\hat{\mathbb{M}}$ in $\hat{\Delta}$ is a structure of the form

$$\hat{\mathbb{M}} = \{< \hat{\delta}; \mathbb{T}_{\hat{\mathbb{M}}}(\hat{\delta}), \mathbb{I}_{\hat{\mathbb{M}}}(\hat{\delta}), \mathbb{F}_{\hat{\mathbb{M}}}(\hat{\delta}) >: \hat{\delta} \in \hat{\Delta}\},$$

where the mappings $\mathbb{T}_{\hat{\mathbb{M}}}; \mathbb{I}_{\hat{\mathbb{M}}}; \mathbb{F}_{\hat{\mathbb{M}}} : \hat{\Delta} \rightarrow]^{-}0; 1^{+}[$ represent the MF, IMF and NMF functions, respectively, with $^{-}0 \leq \mathbb{T}_{\hat{\mathbb{M}}} + \mathbb{I}_{\hat{\mathbb{M}}} + \mathbb{F}_{\hat{\mathbb{M}}} \leq 3^{+}$.

Definition 2.5. [38] A SNS $\hat{\mathbb{N}}$ in $\hat{\Delta}$ with a generic element u in $\hat{\Delta}$ is characterized as.

$$\hat{\mathbb{N}} = \{< \hat{\delta}; \mathbb{T}_{\hat{\mathbb{N}}}(\hat{\delta}), \mathbb{I}_{\hat{\mathbb{N}}}(\hat{\delta}), \mathbb{F}_{\hat{\mathbb{N}}}(\hat{\delta}) >: \hat{\delta} \in \hat{\Delta}\},$$

where the mappings $\mathbb{T}_{\hat{\mathbb{N}}}$; $\mathbb{I}_{\hat{\mathbb{N}}}$; $\mathbb{F}_{\hat{\mathbb{N}}}$: $\hat{\Delta} \rightarrow \hat{\blacksquare}$ represent the MF, IMF and NMF functions, respectively, with $0 \leq \mathbb{T}_{\hat{\mathbb{N}}} + \mathbb{I}_{\hat{\mathbb{N}}} + \mathbb{F}_{\hat{\mathbb{N}}} \leq 3$.

3. The concept of q-rung orthopair fuzzy-valued neutrosophic set (q-ROFVNS)

This part exhibits the formal definitions of q-ROFVNS and q-ROFVNN, along with score functions (SF) of q-ROFVNN. Then, the basic and algebraic operations of q-ROFVNS are provided in the following parts.

Definition 3.1. A q-ROFVNS $\mathbb{S}$ over $\hat{\Delta}$ is signified by $\mathbb{S} = \{\langle \hat{\delta}, \mathbb{T}_{\mathbb{S}}, \mathbb{I}_{\mathbb{S}}, \mathbb{F}_{\mathbb{S}} \rangle : \hat{\delta} \in \hat{\Delta}\}$, where $\mathbb{T}_{\mathbb{S}}, \mathbb{I}_{\mathbb{S}}$ and $\mathbb{F}_{\mathbb{S}}$ represent the membership, indeterminacy membership and non-membership neutrosophic values. Each of them is a q-rung orthopair fuzzy value, where $\forall \hat{\delta} \in \hat{\Delta}, q \geq 1$, $\mathbb{T}_{\mathbb{S}} = (\hat{\zeta}_{\mathbb{S},\mathbb{T}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{T}}(\hat{\delta}))$ such that $\hat{\zeta}_{\mathbb{S},\mathbb{T}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{T}}(\hat{\delta}) \in \hat{\blacksquare}$, subject to the condition $(\hat{\zeta}_{\mathbb{S},\mathbb{T}}(\hat{\delta}))^q + (\hat{\varpi}_{\mathbb{S},\mathbb{T}}(\hat{\delta}))^q \leq 1$, $\mathbb{I}_{\mathbb{S}} = (\hat{\zeta}_{\mathbb{S},\mathbb{I}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{I}}(\hat{\delta}))$ such that $\hat{\zeta}_{\mathbb{S},\mathbb{I}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{I}}(\hat{\delta}) \in \hat{\blacksquare}$, subject to the condition $(\hat{\zeta}_{\mathbb{S},\mathbb{I}}(\hat{\delta}))^q + (\hat{\varpi}_{\mathbb{S},\mathbb{I}}(\hat{\delta}))^q \leq 1$, $\mathbb{F}_{\mathbb{S}} = (\hat{\zeta}_{\mathbb{S},\mathbb{F}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{F}}(\hat{\delta}))$ such that $\hat{\zeta}_{\mathbb{S},\mathbb{F}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{F}}(\hat{\delta}) \in \hat{\blacksquare}$, subject to the condition $(\hat{\zeta}_{\mathbb{S},\mathbb{F}}(\hat{\delta}))^q + (\hat{\varpi}_{\mathbb{S},\mathbb{F}}(\hat{\delta}))^q \leq 1$. By definition, $0 \leq \mathbb{T}_{\mathbb{S}} + \mathbb{I}_{\mathbb{S}} + \mathbb{F}_{\mathbb{S}} \leq 3$. A q-ROFVNS $\mathbb{S}$ over $\hat{\Delta}$ can be written as.

$$\mathbb{S} = \{\langle \hat{\delta}, (\hat{\zeta}_{\mathbb{S},\mathbb{T}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{T}}(\hat{\delta})), (\hat{\zeta}_{\mathbb{S},\mathbb{I}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{I}}(\hat{\delta})), (\hat{\zeta}_{\mathbb{S},\mathbb{F}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{F}}(\hat{\delta}))\rangle : \hat{\delta} \in \hat{\Delta}\}.$$

Definition 3.2. A collection of $\Gamma = \langle(\hat{\zeta}_{\mathbb{T}}, \hat{\varpi}_{\mathbb{T}}), (\hat{\zeta}_{\mathbb{I}}, \hat{\varpi}_{\mathbb{I}}), (\hat{\zeta}_{\mathbb{F}}, \hat{\varpi}_{\mathbb{F}})\rangle$ is called a q-ROFVN number (q-ROFVNN) with $(\hat{\zeta}_{\mathbb{T}})^q + (\hat{\varpi}_{\mathbb{T}})^q \leq 1$, $(\hat{\zeta}_{\mathbb{I}})^q + (\hat{\varpi}_{\mathbb{I}})^q \leq 1$ and $(\hat{\zeta}_{\mathbb{F}})^q + (\hat{\varpi}_{\mathbb{F}})^q \leq 1$, $(q \geq 1)$.

Example 3.3. Suppose $\hat{\Delta} = \{u_1, u_2, u_3\}$. Then,

$$\mathbb{S} = \left\{ \begin{array}{l} \langle u_1, (0.7, 0.9), (0.1, 0.8), (0.7, 0.6)\rangle, \\ \langle u_2, (0.2, 0.5), (0.3, 0.8), (0.8, 0.6)\rangle, \\ \langle u_3, (0.8, 0.9), (0.3, 0.8), (0.5, 0.6)\rangle \end{array} \right\}$$

is a q-ROFVNS $(q = 5)$.

Remark 3.4. Some particular cases are as follows

(1) When $q = 2$, a q-ROFVNN $\mathbb{S}$ becomes a PyFVNN.
(2) When $q = 1$, a q-ROFVNN $\mathbb{S}$ becomes an IFVNN.

Definition 3.5. Let $\mathbb{S} = \{\langle \hat{\delta}, (\hat{\zeta}_{\mathbb{S},\mathbb{T}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{T}}(\hat{\delta})), (\hat{\zeta}_{\mathbb{S},\mathbb{I}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{I}}(\hat{\delta})), (\hat{\zeta}_{\mathbb{S},\mathbb{F}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{F}}(\hat{\delta}))\rangle : \hat{\delta} \in \hat{\Delta}\}$ be a q-ROFVNS over $\hat{\Delta}$. $\mathbb{S}$ is said to be an absolute q-ROFVNS denoted by $\mathbb{S}_\Psi$ if $\hat{\zeta}_{\mathbb{S},\mathbb{T}}(\hat{\delta}) = \hat{\varpi}_{\mathbb{S},\mathbb{I}}(\hat{\delta}) = \hat{\varpi}_{\mathbb{S},\mathbb{F}}(\hat{\delta}) = 1$ and $\hat{\varpi}_{\mathbb{S},\mathbb{T}}(\hat{\delta}) = \hat{\zeta}_{\mathbb{S},\mathbb{I}}(\hat{\delta}) = \hat{\zeta}_{\mathbb{S},\mathbb{F}}(\hat{\delta}) = 0$, i.e., $\mathbb{S}_\Psi = \langle(1, 0), (0, 1), (0, 1)\rangle, \forall \hat{\delta} \in \hat{\Delta}$.

Definition 3.6. Let $\mathbb{S} = \{\langle \hat{\delta}, (\hat{\zeta}_{\mathbb{S},\mathbb{T}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{T}}(\hat{\delta})), (\hat{\zeta}_{\mathbb{S},\mathbb{I}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{I}}(\hat{\delta})), (\hat{\zeta}_{\mathbb{S},\mathbb{F}}(\hat{\delta}), \hat{\varpi}_{\mathbb{S},\mathbb{F}}(\hat{\delta}))\rangle : \hat{\delta} \in \hat{\Delta}\}$ be a q-ROFVNS over $\hat{\Delta}$. $\mathbb{S}$ is said to be a null q-ROFVNS denoted by $\mathbb{S}_\Phi$ if $\hat{\zeta}_{\mathbb{S},\mathbb{T}}(\hat{\delta}) = \hat{\varpi}_{\mathbb{S},\mathbb{I}}(\hat{\delta}) = \hat{\varpi}_{\mathbb{S},\mathbb{F}}(\hat{\delta}) = 0$ and $\hat{\varpi}_{\mathbb{S},\mathbb{T}}(\hat{\delta}) = \hat{\zeta}_{\mathbb{S},\mathbb{I}}(\hat{\delta}) = \hat{\zeta}_{\mathbb{S},\mathbb{F}}(\hat{\delta}) = 1$, i.e., $\mathbb{S}_\Phi = \langle(0, 1), (1, 0), (1, 0)\rangle, \forall \hat{\delta} \in \hat{\Delta}$.

3.1. Score functions of q-ROFVNNs

In this section, we define the SF, accuracy function (AF), quadratic SF (QSF) and QAF.

Definition 3.7. Let $\Gamma = \langle(\hat{\zeta}_{\mathbb{T}}, \hat{\varpi}_{\mathbb{T}}), (\hat{\zeta}_{\mathbb{I}}, \hat{\varpi}_{\mathbb{I}}), (\hat{\zeta}_{\mathbb{F}}, \hat{\varpi}_{\mathbb{F}})\rangle$ be q-ROFVNN. Then, the SF on Γ is signified by the mapping $\Pi : q - ROFVNN(\hat{\Delta}) \longrightarrow [-1, 1]$ and defined as

$$\Pi_\Gamma = \Pi(\Gamma) = \tfrac{1}{3}[[(\hat{\zeta}_{\mathbb{T}})^q - (\hat{\varpi}_{\mathbb{T}})^q] - [(\hat{\zeta}_{\mathbb{I}})^q - (\hat{\varpi}_{\mathbb{I}})^q] - [(\hat{\zeta}_{\mathbb{F}})^q - (\hat{\varpi}_{\mathbb{F}})^q]],\ q \geq 1. \tag{1}$$

$q - ROFVNN(\hat{\Delta})$ is the collection of q-ROFVNNs on $\hat{\Delta}$.

Definition 3.8. The AF $\urcorner$ is signified by the mapping $\urcorner : q - ROFVNN(\hat{\Delta}) \longrightarrow$ ☗ and defined as

$$2\urcorner_\Gamma = \urcorner(\Gamma) = \frac{1}{6}[[(\hat{\zeta}_{\mathbb{T}})^q + (\hat{\varpi}_{\mathbb{T}})^q] + [(\hat{\zeta}_{\mathbb{I}})^q + (\hat{\varpi}_{\mathbb{I}})^q] + [(\hat{\zeta}_{\mathbb{F}})^q + (\hat{\varpi}_{\mathbb{F}})^q]], q \geq 1.(2) \tag{2}$$

$q - ROFVNN(\hat{\Delta})$ is the collection of q-ROFVNNs on $\hat{\Delta}$.

Definition 3.9. Let Γ_1 and Γ_2 be two q-ROFVNNs.

(1) If $\Pi_{\Gamma_1} < \Pi_{\Gamma_2}$, then $\Gamma_1 < \Gamma_2$.
(2) If $\Pi_{\Gamma_1} > \Pi_{\Gamma_2}$, then $\Gamma_1 > \Gamma_2$.
(3) If $\Pi_{\Gamma_1} = \Pi_{\Gamma_2}$ and $\urcorner_{\Gamma_1} < \urcorner_{\Gamma_2}$, then $\Gamma_1 < \Gamma_2$.
(4) If $\Pi_{\Gamma_1} = \Pi_{\Gamma_2}$ and $\urcorner_{\Gamma_1} > \urcorner_{\Gamma_2}$, then $\Gamma_1 > \Gamma_2$.

Definition 3.10. Let $\Gamma = \langle(\hat{\zeta}_{\mathbb{T}}, \hat{\varpi}_{\mathbb{T}}), (\hat{\zeta}_{\mathbb{I}}, \hat{\varpi}_{\mathbb{I}}), (\hat{\zeta}_{\mathbb{F}}, \hat{\varpi}_{\mathbb{F}})\rangle$ be a q-ROFVNN. Then, the QSF on Γ is determined by the mapping $\Omega : q - ROFVNN(\hat{\Delta}) \longrightarrow [-1, 1]$ and defined as

$$Omega_\Gamma = \Omega(\Gamma) = \frac{1}{3}\Big[[(\hat{\zeta}_{\mathbb{T}})^{2q} - (\hat{\varpi}_{\mathbb{T}})^{2q}] - [(\hat{\zeta}_{\mathbb{I}})^{2q} - (\hat{\varpi}_{\mathbb{I}})^{2q}] - [(\hat{\zeta}_{\mathbb{F}})^{2q} - (\hat{\varpi}_{\mathbb{F}})^{2q}]\Big], q \geq 1. \tag{3}$$

$q - ROFVNN(\hat{\Delta})$ is the collection of q-ROFVNNs on $\hat{\Delta}$.

Definition 3.11. The QAF $\sqsupset$ is signified by the mapping $\sqsupset : q - ROFVNN(\hat{\Delta}) \longrightarrow$ ☗ and defined as

$$4\sqsupset_\Gamma = \sqsupset(\Gamma) = \frac{1}{6}\Big[[(\hat{\zeta}_{\mathbb{T}})^{2q} + (\hat{\varpi}_{\mathbb{T}})^{2q}] + [(\hat{\zeta}_{\mathbb{I}})^{2q} + (\hat{\varpi}_{\mathbb{I}})^{2q}] + [(\hat{\zeta}_{\mathbb{F}})^{2q} + (\hat{\varpi}_{\mathbb{F}})^{2q}]\Big], q \geq 1. \tag{4}$$

$q - ROFVNN(\hat{\Delta})$ is the collection of q-ROFVNNs on $\hat{\Delta}$.

Definition 3.12. Let Γ_1 and Γ_2 be two q-ROFVNNs.

(1) If $\Omega_{\Gamma_1} < \Omega_{\Gamma_2}$, then $\Gamma_1 < \Gamma_2$.
(2) If $\Omega_{\Gamma_1} > \Omega_{\Gamma_2}$, then $\Gamma_1 > \Gamma_2$.
(3) If $\Omega_{\Gamma_1} = \Omega_{\Gamma_2}$ and $\sqsupset_{\Gamma_1} < \sqsupset_{\Gamma_2}$, then $\Gamma_1 < \Gamma_2$.
(4) If $\Omega_{\Gamma_1} = \Omega_{\Gamma_2}$ and $\sqsupset_{\Gamma_1} > \sqsupset_{\Gamma_2}$, then $\Gamma_1 > \Gamma_2$.

3.2. Basic operations on q-ROFVNS

In order to present the basic operations on q-ROFVNS, we suppose that $\mathbb{H}$, $\mathbb{G}$ are two q-ROFVNSs over $\hat{\Delta}$, where $\mathbb{H} = \{\langle\hat{\delta}, \mathbb{T}_{\mathbb{H}}, \mathbb{I}_{\mathbb{H}}, \mathbb{F}_{\mathbb{H}}\rangle : \hat{\delta} \in \hat{\Delta}\} = \{\langle\hat{\delta}, (\hat{\zeta}_{\mathbb{H},\mathbb{T}}(\hat{\delta}), \hat{\varpi}_{\mathbb{H},\mathbb{T}}(\hat{\delta})), (\hat{\zeta}_{\mathbb{H},\mathbb{I}}(\hat{\delta}), \hat{\varpi}_{\mathbb{H},\mathbb{I}}(\hat{\delta})), (\hat{\zeta}_{\mathbb{H},\mathbb{F}}(\hat{\delta}), \hat{\varpi}_{\mathbb{H},\mathbb{F}}(\hat{\delta}))\rangle : \hat{\delta} \in \hat{\Delta}\}$, and $\mathbb{G} = \{\langle\hat{\delta}, \mathbb{T}_{\mathbb{G}}, \mathbb{I}_{\mathbb{G}}, \mathbb{F}_{\mathbb{G}}\rangle : \hat{\delta} \in \hat{\Delta}\}, = \{\langle\hat{\delta}, (\hat{\zeta}_{\mathbb{G},\mathbb{T}}(\hat{\delta}), \hat{\varpi}_{\mathbb{G},\mathbb{T}}(\hat{\delta})), (\hat{\zeta}_{\mathbb{G},\mathbb{I}}(\hat{\delta}), \hat{\varpi}_{\mathbb{G},\mathbb{I}}(\hat{\delta})), (\hat{\zeta}_{\mathbb{G},\mathbb{F}}(\hat{\delta}), \hat{\varpi}_{\mathbb{G},\mathbb{F}}(\hat{\delta}))\rangle : \hat{\delta} \in \hat{\Delta}\}$.

Definition 3.13. Let $\mathbb{H}$, $\mathbb{G}$ be two q-ROFVNSs over $\hat{\Delta}$. Then, $\mathbb{H}$ is a subset of $\mathbb{G}$, denoted by $\mathbb{H} \subseteq \mathbb{G}$ if and only if.
$\mathbb{T}_{\mathbb{H}} \subseteq_q \mathbb{T}_G$, i.e., $\hat{\zeta}_{\mathbb{H},\mathbb{T}}(\hat{\delta}) \leq \hat{\zeta}_{G,\mathbb{T}}(\hat{\delta})$ and $\hat{\varpi}_{\mathbb{H},\mathbb{T}}(\hat{\delta}) \geq \hat{\varpi}_{\mathbb{G},\mathbb{T}}(\hat{\delta})$,
$\mathbb{I}_{\mathbb{H}} \supseteq_q \mathbb{I}_G$, i.e., $\hat{\zeta}_{\mathbb{H},\mathbb{T}}(\hat{\delta}) \geq \hat{\zeta}_{G,\mathbb{I}}(\hat{\delta})$ and $\hat{\varpi}_{\mathbb{H},\mathbb{I}}(\hat{\delta}) \leq \hat{\varpi}_{\mathbb{G},\mathbb{I}}(\hat{\delta})$,
$\mathbb{F}_{\mathbb{H}} \supseteq_q \mathbb{F}_G$, i.e., $\hat{\zeta}_{\mathbb{H},\mathbb{F}}(\hat{\delta}) \geq \hat{\zeta}_{G,\mathbb{F}}(\hat{\delta})$ and $\hat{\varpi}_{\mathbb{H},\mathbb{F}}(\hat{\delta}) \leq \hat{\varpi}_{\mathbb{G},\mathbb{F}}(\hat{\delta})$. In this definition $\subseteq_q$ represents the q-rung orthopair fuzzy subset.

Definition 3.14. Let $\mathbb{H}$, $\mathbb{G}$ be two q-ROFVNSs over $\hat{\Delta}$. Then, $\mathbb{H}$ is equal to $\mathbb{G}$, denoted by $\mathbb{H} = \mathbb{G}$ if and only if.
$\mathbb{T}_{\mathbb{H}} = \mathbb{T}_G$, i.e., $\hat{\zeta}_{\mathbb{H},\mathbb{T}}(\hat{\delta}) = \hat{\zeta}_{G,\mathbb{T}}(\hat{\delta})$ and $\hat{\varpi}_{\mathbb{H},\mathbb{T}}(\hat{\delta}) = \hat{\varpi}_{\mathbb{G},\mathbb{T}}(\hat{\delta})$,
$\mathbb{I}_{\mathbb{H}} = \mathbb{I}_G$, i.e., $\hat{\zeta}_{\mathbb{H},\mathbb{T}}(\hat{\delta}) = \hat{\zeta}_{G,\mathbb{I}}(\hat{\delta})$ and $\hat{\varpi}_{\mathbb{H},\mathbb{I}}(\hat{\delta}) = \hat{\varpi}_{\mathbb{G},\mathbb{I}}(\hat{\delta})$,
$\mathbb{F}_{\mathbb{H}} = \mathbb{F}_G$, i.e., $\hat{\zeta}_{\mathbb{H},\mathbb{F}}(\hat{\delta}) = \hat{\zeta}_{G,\mathbb{F}}(\hat{\delta})$ and $\hat{\varpi}_{\mathbb{H},\mathbb{F}}(\hat{\delta}) = \hat{\varpi}_{\mathbb{G},\mathbb{F}}(\hat{\delta})$.

Definition 3.15. Let $\mathbb{H}$ be a q-ROFVNS over $\hat{\Delta}$. Then, the complement of $\mathbb{H}$ is denoted by $(\mathbb{H})^c$ and defined as
$(\mathbb{H})^c = \{\langle\hat{\delta}, \mathbb{F}_{\mathbb{H}}, (\mathbb{I}_{\mathbb{H}})^{c_q}, \mathbb{T}_{\mathbb{H}}\rangle : \hat{\delta} \in \hat{\Delta}\}$, where c_q is a q-ROF-complement, and $(\mathbb{I}_{\mathbb{H}})^{c_q} = (\hat{\varpi}_{\mathbb{H},\mathbb{I}}(\hat{\delta}), \hat{\zeta}_{\mathbb{H},\mathbb{I}}(\hat{\delta}))$.

Definition 3.16. Let $\mathbb{H}$ and $\mathbb{G}$ be two q-ROFVNSs over $\hat{\Delta}$. The union of $\mathbb{H}$ and $\mathbb{G}$ is denoted by $(\mathbb{H} \cup \mathbb{G})$ and defined as:
$(\mathbb{H} \cup \mathbb{G}) = \{\langle\hat{\delta}, \mathbb{T}_{\mathbb{H}} \cup_q \mathbb{T}_{\mathbb{G}}, \mathbb{I}_{\mathbb{H}} \cap_q \mathbb{I}_{\mathbb{G}}, \mathbb{F}_{\mathbb{H}} \cap_q \mathbb{F}_{\mathbb{G}}\rangle : \hat{\delta} \in \hat{\Delta}\}$, where $\cup_q$ is the q-ROF-union, $\cap_q$ is the q-ROF-intersection, and
$\mathbb{T}_{\mathbb{H}} \cup_q \mathbb{T}_{\mathbb{G}} = \Big((\hat{\zeta}_{\mathbb{H},\mathbb{T}}(\hat{\delta}) \vee \hat{\zeta}_{\mathbb{G},\mathbb{T}}(\hat{\delta})), (\hat{\varpi}_{\mathbb{H},\mathbb{T}}(\hat{\delta}) \wedge \hat{\varpi}_{\mathbb{G},\mathbb{T}}(\hat{\delta}))\Big)$,
$\mathbb{I}_{\mathbb{H}} \cap_q \mathbb{I}_{\mathbb{G}} = \Big((\hat{\zeta}_{\mathbb{H},\mathbb{I}}(\hat{\delta}) \wedge \hat{\zeta}_{\mathbb{G},\mathbb{I}}(\hat{\delta})), (\hat{\varpi}_{\mathbb{H},\mathbb{I}}(\hat{\delta}) \vee \hat{\varpi}_{\mathbb{G},\mathbb{I}}(\hat{\delta}))\Big)$,
$\mathbb{F}_{\mathbb{H}} \cap_q \mathbb{F}_{\mathbb{G}} = \Big((\hat{\zeta}_{\mathbb{H},\mathbb{F}}(\hat{\delta}) \wedge \hat{\zeta}_{\mathbb{G},\mathbb{F}}(\hat{\delta})), (\hat{\varpi}_{\mathbb{H},\mathbb{F}}(\hat{\delta}) \vee \hat{\varpi}_{\mathbb{G},\mathbb{F}}(\hat{\delta}))\Big)$.
$\vee = \max, \quad \wedge = \min$.

Definition 3.17. Let $\mathbb{H}$ and $\mathbb{G}$ be two q-ROFVNSs over $\hat{\Delta}$. The intersection of $\mathbb{H}$ and $\mathbb{G}$ is denoted by $(\mathbb{H} \cap \mathbb{G})$ and defined as
$(\mathbb{H} \cap \mathbb{G}) = \{\langle\hat{\delta}, \mathbb{T}_{\mathbb{H}} \cap_q \mathbb{T}_{\mathbb{G}}, \mathbb{I}_{\mathbb{H}} \cup_q \mathbb{I}_{\mathbb{G}}, \mathbb{F}_{\mathbb{H}} \cup_q \mathbb{F}_{\mathbb{G}}\rangle : \hat{\delta} \in \hat{\Delta}\}$, where $\cup_q$ is the q-ROF-union, $\cap_q$ is the q-ROF-intersection, and
$\mathbb{T}_{\mathbb{H}} \cap_q \mathbb{T}_{\mathbb{G}} = \Big((\hat{\zeta}_{\mathbb{H},\mathbb{T}}(\hat{\delta}) \wedge \hat{\zeta}_{\mathbb{G},\mathbb{T}}(\hat{\delta})), (\hat{\varpi}_{\mathbb{H},\mathbb{T}}(\hat{\delta}) \vee \hat{\varpi}_{\mathbb{G},\mathbb{T}}(\hat{\delta}))\Big)$,
$\mathbb{I}_{\mathbb{H}} \cup_q \mathbb{I}_{\mathbb{G}} = \Big((\hat{\zeta}_{\mathbb{H},\mathbb{I}}(\hat{\delta}) \vee \hat{\zeta}_{\mathbb{G},\mathbb{I}}(\hat{\delta})), (\hat{\varpi}_{\mathbb{H},\mathbb{I}}(\hat{\delta}) \wedge \hat{\varpi}_{\mathbb{G},\mathbb{I}}(\hat{\delta}))\Big)$,
$\mathbb{F}_{\mathbb{H}} \cup_q \mathbb{F}_{\mathbb{G}} = \Big((\hat{\zeta}_{\mathbb{H},\mathbb{F}}(\hat{\delta}) \vee \hat{\zeta}_{\mathbb{G},\mathbb{F}}(\hat{\delta})), (\hat{\varpi}_{\mathbb{H},\mathbb{F}}(\hat{\delta}) \wedge \hat{\varpi}_{\mathbb{G},\mathbb{F}}(\hat{\delta}))\Big)$.
$\vee = \max, \quad \wedge = \min$.

Example 3.18. If $\hat{\Delta} = \{u_1, u_2\}$ such that

$$\mathbb{H} = \{\langle u_1, (0.7, 0.9), (0.1, 0.8), (0.7, 0.6)\rangle, \langle u_2, (0.2, 0.5), (0.3, 0.8), (0.8, 0.6)\rangle\}$$

and

$$\mathbb{G} = \{\langle u_1, (0.3, 0.9), (0.5, 0.7), (0.9, 0.6)\rangle, \langle u_2, (0.1, 0.7), (0.2, 0.8), (0.9, 0.7)\rangle\}$$

are two q-ROFVNSs, then

(1) $(\mathbb{H})^c = \{\langle u_1, (0.7, 0.6), (0.8, 0.1), (0.7, 0.9)\rangle, \langle u_2, (0.8, 0.6), (0.8, 0.3), (0.2, 0.5)\rangle\}$,
(2) $(\mathbb{H} \cup \mathbb{G}) = \{\langle u_1, (0.7, 0.9), (0.1, 0.8), (0.7, 0.6)\rangle, \langle u_2, (0.2, 0.5), (0.2, 0.8), (0.8, 0.7)\rangle\}$,
(3) $(\mathbb{H} \cap \mathbb{G}) = \{\langle u_1, (0.3, 0.9), (0.5, 0.7), (0.9, 0.6)\rangle, \langle u_2, (0.1, 0.7), (0.3, 0.8), (0.9, 0.6)\rangle\}$.

Proposition 3.19. *Let* $\mathbb{H} = \{\langle\hat{\delta}, \mathbb{T}_{\mathbb{H}}, \mathbb{I}_{\mathbb{H}}, \mathbb{F}_{\mathbb{H}}\rangle : \hat{\delta} \in \hat{\Delta}\}$, $\mathbb{G} = \{\langle\hat{\delta}, \mathbb{T}_{\mathbb{G}}, \mathbb{I}_{\mathbb{G}}, \mathbb{F}_{\mathbb{G}}\rangle : \hat{\delta} \in \hat{\Delta}\}$ *and* $\mathbb{K} = \{\langle\hat{\delta}, \mathbb{T}_{\mathbb{K}}, \mathbb{I}_{\mathbb{K}}, \mathbb{F}_{\mathbb{K}}\rangle : \hat{\delta} \in \hat{\Delta}\}$ *be three q-ROFVNSs. Then, the following properties hold:*

(1) $(\mathbb{H} \cup \mathbb{G}) \cup \mathbb{K} = \mathbb{H} \cup (\mathbb{G} \cup \mathbb{K}).$ (5)
(2) $(\mathbb{H} \cap \mathbb{G}) \cap \mathbb{K} = \mathbb{H} \cap (\mathbb{G} \cap \mathbb{K}).$ (6)
(3) $\mathbb{H} \cup (\mathbb{G} \cap \mathbb{K}) = (\mathbb{H} \cup \mathbb{G}) \cap (\mathbb{H} \cup \mathbb{K}).$ (7)
(4) $\mathbb{H} \cap (\mathbb{G} \cup \mathbb{K}) = (\mathbb{H} \cap \mathbb{G}) \cup (\mathbb{H} \cap \mathbb{K}).$ (8)
(5) $(\mathbb{H} \cup \mathbb{G})^c = (\mathbb{H})^c \cap (\mathbb{G})^c.$ (9)
(6) $(\mathbb{H} \cap \mathbb{G})^c = (\mathbb{H})^c \cup (\mathbb{G})^c.$ (10)

Proof. We will prove properties (5) and (6) as the proof of the remaining properties is trivial.

(5) For the left side, we have $(\mathbb{H} \cup \mathbb{G}) = \{\langle\hat{\delta}, \mathbb{T}_{\mathbb{H}} \cup_q \mathbb{T}_{\mathbb{G}}, \mathbb{I}_{\mathbb{H}} \cap_q \mathbb{I}_{\mathbb{G}}, \mathbb{F}_{\mathbb{H}} \cap_q \mathbb{F}_{\mathbb{G}}\rangle : \hat{\delta} \in \hat{\Delta}\}$. According to Definition 3.16, we have
$(\mathbb{H} \cup \mathbb{G})^c = \{\langle\hat{\delta}, \mathbb{F}_{\mathbb{H}} \cap_q \mathbb{F}_{\mathbb{G}}, (\mathbb{I}_{\mathbb{H}} \cap_q \mathbb{I}_{\mathbb{G}})^{c_q}, \mathbb{T}_{\mathbb{H}} \cup_q \mathbb{T}_{\mathbb{G}}\rangle : \hat{\delta} \in \hat{\Delta}\}$
$= \{\langle\hat{\delta}, \mathbb{F}_{\mathbb{H}} \cap_q \mathbb{F}_{\mathbb{G}}, (\mathbb{I}_{\mathbb{H}})^{c_q} \cup_q (\mathbb{I}_{\mathbb{G}})^{c_q}, \mathbb{T}_{\mathbb{H}} \cup_q \mathbb{T}_{\mathbb{G}}\rangle : \hat{\delta} \in \hat{\Delta}\}$,
$= (\mathbb{H})^c \cap (\mathbb{G})^c$

(6) For the left side, we have $(\mathbb{H} \cap \mathbb{G}) = \{\langle\hat{\delta}, \mathbb{T}_{\mathbb{H}} \cap_q \mathbb{T}_{\mathbb{G}}, \mathbb{I}_{\mathbb{H}} \cup_q \mathbb{I}_{\mathbb{G}}, \mathbb{F}_{\mathbb{H}} \cup_q \mathbb{F}_{\mathbb{G}}\rangle : \hat{\delta} \in \hat{\Delta}\}$. According to Definition 3.17, we have
$(\mathbb{H} \cap \mathbb{G})^c = \{\langle\hat{\delta}, \mathbb{F}_{\mathbb{H}} \cup_q \mathbb{F}_{\mathbb{G}}, (\mathbb{I}_{\mathbb{H}} \cup_q \mathbb{I}_{\mathbb{G}})^{c_q}, \mathbb{T}_{\mathbb{H}} \cap_q \mathbb{T}_{\mathbb{G}}\rangle : \hat{\delta} \in \hat{\Delta}\}$.
$= \{\langle\hat{\delta}, \mathbb{F}_{\mathbb{H}} \cup_q \mathbb{F}_{\mathbb{G}}, (\mathbb{I}_{\mathbb{H}})^{c_q} \cup_q (\mathbb{I}_{\mathbb{G}})^{c_q}, \mathbb{T}_{\mathbb{H}} \cap_q \mathbb{T}_{\mathbb{G}}\rangle : \hat{\delta} \in \hat{\Delta}\}$,
$= (\mathbb{H})^c \cup (\mathbb{G})^c$.

□

3.3. Algebraic operations for q-ROFVNNs

In this part, we present some algebraic operations for q-ROFVNNs.

Definition 3.20. Let
$\Gamma_1 = \langle({}^1\hat{\zeta}_{\mathbb{T}},\ {}^1\hat{\varpi}_{\mathbb{T}}), ({}^1\hat{\zeta}_{\mathbb{I}},\ {}^1\hat{\varpi}_{\mathbb{I}}), ({}^1\hat{\zeta}_{\mathbb{F}},\ {}^1\hat{\varpi}_{\mathbb{F}})\rangle$ and $\Gamma_2 = \langle({}^2\hat{\zeta}_{\mathbb{T}},\ {}^2\hat{\varpi}_{\mathbb{T}}), ({}^2\hat{\zeta}_{\mathbb{I}},\ {}^2\hat{\varpi}_{\mathbb{I}}), ({}^2\hat{\zeta}_{\mathbb{F}},\ {}^2\hat{\varpi}_{\mathbb{F}})\rangle$ be two q-ROFVNNs over $\hat{\Delta}$ and $\Theta > 0$. Then,

(1) $\Gamma_1 \oplus \Gamma_2 = \Big\langle((\,({}^1\hat{\zeta}_{\mathbb{T}})^q + ({}^2\hat{\zeta}_{\mathbb{T}})^q - ({}^1\hat{\zeta}_{\mathbb{T}})^q\,({}^2\hat{\zeta}_{\mathbb{T}})^q\,)^{\frac{1}{q}},\ {}^1\hat{\varpi}_{\mathbb{T}}\,{}^2\hat{\varpi}_{\mathbb{T}}), ({}^1\hat{\zeta}_{\mathbb{I}}\,{}^2\hat{\zeta}_{\mathbb{I}}, (\,({}^1\hat{\varpi}_{\mathbb{I}})^q + ({}^2\hat{\varpi}_{\mathbb{I}})^q - ({}^1\hat{\varpi}_{\mathbb{I}})^q\,({}^2\hat{\varpi}_{\mathbb{I}})^q\,)^{\frac{1}{q}}), ({}^1\hat{\zeta}_{\mathbb{F}}\,{}^2\hat{\zeta}_{\mathbb{F}}, (\,({}^1\hat{\varpi}_{\mathbb{F}})^q + ({}^2\hat{\varpi}_{\mathbb{F}})^q - ({}^1\hat{\varpi}_{\mathbb{F}})^q\,({}^2\hat{\varpi}_{\mathbb{F}})^q\,)^{\frac{1}{q}})\Big\rangle.$ (11)

(2) $\Gamma_1 \otimes \Gamma_2 = \Big\langle({}^1\hat{\zeta}_{\mathbb{T}}\,{}^2\hat{\zeta}_{\mathbb{T}}, (\,({}^1\hat{\varpi}_{\mathbb{T}})^q + ({}^2\hat{\varpi}_{\mathbb{T}})^q - ({}^1\hat{\varpi}_{\mathbb{T}})^q\,({}^2\hat{\varpi}_{\mathbb{T}})^q\,)^{\frac{1}{q}}), (\,({}^1\hat{\zeta}_{\mathbb{I}})^q + ({}^2\hat{\zeta}_{\mathbb{I}})^q - ({}^1\hat{\zeta}_{\mathbb{I}})^q\,({}^2\hat{\zeta}_{\mathbb{I}})^q\,)^{\frac{1}{q}},\ {}^1\hat{\varpi}_{\mathbb{I}}\,{}^2\hat{\varpi}_{\mathbb{I}}), (\,({}^1\hat{\zeta}_{\mathbb{F}})^q + ({}^2\hat{\zeta}_{\mathbb{F}})^q - ({}^1\hat{\zeta}_{\mathbb{F}})^q\,({}^2\hat{\zeta}_{\mathbb{F}})^q\,)^{\frac{1}{q}},\ {}^1\hat{\varpi}_{\mathbb{F}}\,{}^2\hat{\varpi}_{\mathbb{F}})\Big\rangle.$ (12)

(3) $\Theta\Gamma_1 = \Big\langle \Big((1-(1-({}^{1}\hat{\zeta}_{\mathbb{T}})^q)^{\Theta})^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{T}})^{\Theta}\Big), \Big(({}^{1}\hat{\zeta}_{\mathbb{I}})^{\Theta}, (1-(1-({}^{1}\hat{\varpi}_{\mathbb{I}})^q)^{\Theta})^{\frac{1}{q}}\Big), \Big(({}^{1}\hat{\zeta}_{\mathbb{F}})^{\Theta}, (1-(1-({}^{1}\hat{\varpi}_{\mathbb{F}})^q)^{\Theta})^{\frac{1}{q}}\Big)\Big\rangle.$ (13)

(4) $(\Gamma_1)^{\Theta} = \Big\langle\Big(({}^{1}\hat{\zeta}_{\mathbb{T}})^{\Theta}, (1-(1-({}^{1}\hat{\varpi}_{\mathbb{T}})^q)^{\Theta})^{\frac{1}{q}}\Big), \Big((1-(1-({}^{1}\hat{\zeta}_{\mathbb{I}})^q)^{\Theta})^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{I}})^{\Theta}\Big), \Big((1-(1-({}^{1}\hat{\zeta}_{\mathbb{F}})^q)^{\Theta})^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{F}})^{\Theta}\Big)\Big\rangle.$ (14)

Example 3.21. Suppose
$\Gamma_1 = \langle(0.7, 0.8), (0.6, 0.7), (0.4, 0.8)\rangle$ and $\Gamma_2 = \langle(0.6, 0.9), (0.4, 0.9), (0.7, 0.6)\rangle$ are two 3-ROFVNNs, and $\Theta = 4$. Then,

(1) $\Gamma_1 \oplus \Gamma_2 = \langle(0.79, 0.72), (0.24, 0.94), (0.28, 0.85)\rangle$,
(2) $\Gamma_1 \otimes \Gamma_2 = \langle(0.42, 0.95), (0.64, 0.63), (0.73, 0.48)\rangle$,
(3) $\Theta\Gamma_1 = \langle(0.93, 0.41), (0.13, 0.93), (0.03, 0.98)\rangle$,
(4) $(\Gamma_1)^{\Theta} = \langle(0.24, 0.98), (0.85, 0.24), (0.61, 0.41)\rangle$.

Proposition 3.22. *Let*
$\Gamma_1 = \langle({}^{1}\hat{\zeta}_{\mathbb{T}}, {}^{1}\hat{\varpi}_{\mathbb{T}}), ({}^{1}\hat{\zeta}_{\mathbb{I}}, {}^{1}\hat{\varpi}_{\mathbb{I}}), ({}^{1}\hat{\zeta}_{\mathbb{F}}, {}^{1}\hat{\varpi}_{\mathbb{F}})\rangle$, $\Gamma_2 = \langle({}^{2}\hat{\zeta}_{\mathbb{T}}, {}^{2}\hat{\varpi}_{\mathbb{T}}), ({}^{2}\hat{\zeta}_{\mathbb{I}}, {}^{2}\hat{\varpi}_{\mathbb{I}}), ({}^{2}\hat{\zeta}_{\mathbb{F}}, {}^{2}\hat{\varpi}_{\mathbb{F}})\rangle$ *and* $\Gamma_3 = \langle({}^{3}\hat{\zeta}_{\mathbb{T}}, {}^{3}\hat{\varpi}_{\mathbb{T}}), ({}^{3}\hat{\zeta}_{\mathbb{I}}, {}^{3}\hat{\varpi}_{\mathbb{I}}), ({}^{3}\hat{\zeta}_{\mathbb{F}}, {}^{3}\hat{\varpi}_{\mathbb{F}})\rangle$ *be three q-ROFVNNs over* $\hat{\Delta}$ *and* $\Theta > 0$. *Then, the following properties hold:*

(1) $(\Gamma_1 \oplus \Gamma_2) \oplus \Gamma_3 = \Gamma_1 \oplus (\Gamma_2 \oplus \Gamma_3).$ (15)

(2) $(\Gamma_1 \otimes \Gamma_2) \otimes \Gamma_3 = \Gamma_1 \otimes (\Gamma_2 \otimes \Gamma_3).$ (16)

(3) $\Theta(\Gamma_1 \oplus \Gamma_2) = \Theta\Gamma_1 \oplus \Theta\Gamma_2.$ (17)

(4) $(\Gamma_1 \otimes \Gamma_2)^{\Theta} = \Gamma_1^{\Theta} \otimes \Gamma_2^{\Theta}.$ (18)

Proof. We will prove properties (3) and (4) as the proof of the remaining properties is trivial.

(3) Based on Definition 3.20 (items (1) and (3)), we have for the right side of the equation

$$\Theta(\Gamma_1 \oplus \Gamma_2) = \Theta\left(\left\langle\begin{array}{c} ((({}^{1}\hat{\zeta}_{\mathbb{T}})^q + ({}^{2}\hat{\zeta}_{\mathbb{T}})^q - ({}^{1}\hat{\zeta}_{\mathbb{T}})^q({}^{2}\hat{\zeta}_{\mathbb{T}})^q)^{\frac{1}{q}}, {}^{1}\hat{\varpi}_{\mathbb{T}}\,{}^{2}\hat{\varpi}_{\mathbb{T}}), \\ ({}^{1}\hat{\zeta}_{\mathbb{I}}\,{}^{2}\hat{\zeta}_{\mathbb{I}}, (({}^{1}\hat{\varpi}_{\mathbb{I}})^q + ({}^{2}\hat{\varpi}_{\mathbb{I}})^q - ({}^{1}\hat{\varpi}_{\mathbb{I}})^q({}^{2}\hat{\varpi}_{\mathbb{I}})^q)^{\frac{1}{q}}), \\ ({}^{1}\hat{\zeta}_{\mathbb{F}}\,{}^{2}\hat{\zeta}_{\mathbb{F}}, (({}^{1}\hat{\varpi}_{\mathbb{F}})^q + ({}^{2}\hat{\varpi}_{\mathbb{F}})^q - ({}^{1}\hat{\varpi}_{\mathbb{F}})^q({}^{2}\hat{\varpi}_{\mathbb{F}})^q)^{\frac{1}{q}}) \end{array}\right\rangle\right),$$

$$= \left\langle\begin{array}{c} \Big(\Big[1-[1-[(({}^{1}\hat{\zeta}_{\mathbb{T}})^q + ({}^{2}\hat{\zeta}_{\mathbb{T}})^q - ({}^{1}\hat{\zeta}_{\mathbb{T}})^q({}^{2}\hat{\zeta}_{\mathbb{T}})^q)^{\frac{1}{q}}]^q]^{\Theta}\Big]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{T}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{T}})^{\Theta}\Big), \\ \Big(({}^{1}\hat{\zeta}_{\mathbb{I}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{I}})^{\Theta}, \Big[1-[1-[(({}^{1}\hat{\varpi}_{\mathbb{I}})^q + ({}^{2}\hat{\varpi}_{\mathbb{I}})^q - ({}^{1}\hat{\varpi}_{\mathbb{I}})^q({}^{2}\hat{\varpi}_{\mathbb{I}})^q)^{\frac{1}{q}}]^q]^{\Theta}\Big]^{\frac{1}{q}}\Big), \\ \Big(({}^{1}\hat{\zeta}_{\mathbb{F}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{F}})^{\Theta}, \Big[1-[1-[(({}^{1}\hat{\varpi}_{\mathbb{F}})^q + ({}^{2}\hat{\varpi}_{\mathbb{F}})^q - ({}^{1}\hat{\varpi}_{\mathbb{F}})^q({}^{2}\hat{\varpi}_{\mathbb{F}})^q)^{\frac{1}{q}}]^q]^{\Theta}\Big]^{\frac{1}{q}}\Big) \end{array}\right\rangle,$$

$$= \left\langle\begin{array}{c} \Big(\Big[1-[1-[({}^{1}\hat{\zeta}_{\mathbb{T}})^q + ({}^{2}\hat{\zeta}_{\mathbb{T}})^q - ({}^{1}\hat{\zeta}_{\mathbb{T}})^q({}^{2}\hat{\zeta}_{\mathbb{T}})^q]]^{\Theta}\Big]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{T}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{T}})^{\Theta}\Big), \\ \Big(({}^{1}\hat{\zeta}_{\mathbb{I}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{I}})^{\Theta}, \Big[1-[1-[({}^{1}\hat{\varpi}_{\mathbb{I}})^q + ({}^{2}\hat{\varpi}_{\mathbb{I}})^q - ({}^{1}\hat{\varpi}_{\mathbb{I}})^q({}^{2}\hat{\varpi}_{\mathbb{I}})^q]]^{\Theta}\Big]^{\frac{1}{q}}\Big), \\ \Big(({}^{1}\hat{\zeta}_{\mathbb{F}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{F}})^{\Theta}, \Big[1-[1-[({}^{1}\hat{\varpi}_{\mathbb{F}})^q + ({}^{2}\hat{\varpi}_{\mathbb{F}})^q - ({}^{1}\hat{\varpi}_{\mathbb{F}})^q({}^{2}\hat{\varpi}_{\mathbb{F}})^q]]^{\Theta}\Big]^{\frac{1}{q}}\Big) \end{array}\right\rangle,$$

$$= \left\langle \begin{array}{c} \left([1-(1-({}^{1}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta}(1-({}^{2}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta}]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{T}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{T}})^{\Theta}\right), \\ \left(({}^{1}\hat{\zeta}_{\mathbb{I}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{I}})^{\Theta}, [1-(1-({}^{1}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta}(1-({}^{2}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta}]^{\frac{1}{q}}\right), \\ \left(({}^{1}\hat{\zeta}_{\mathbb{F}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{F}})^{\Theta}, [1-(1-({}^{1}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta}(1-({}^{2}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta}]^{\frac{1}{q}}\right) \end{array} \right\rangle.$$

For the right side of the equation, we have

$$\Theta\Gamma_1 = \left\langle \begin{array}{c} \left((1-(1-({}^{1}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{T}})^{\Theta}\right), \\ \left(({}^{1}\hat{\zeta}_{\mathbb{I}})^{\Theta}, (1-(1-({}^{1}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}\right), \\ \left(({}^{1}\hat{\zeta}_{\mathbb{F}})^{\Theta}, (1-(1-({}^{1}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}\right) \end{array} \right\rangle,$$

$$\Theta\Gamma_2 = \left\langle \begin{array}{c} \left((1-(1-({}^{2}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}}, ({}^{2}\hat{\varpi}_{\mathbb{T}})^{\Theta}\right), \\ \left(({}^{2}\hat{\zeta}_{\mathbb{I}})^{\Theta}, (1-(1-({}^{2}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}\right), \\ \left(({}^{2}\hat{\zeta}_{\mathbb{F}})^{\Theta}, (1-(1-({}^{2}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}\right) \end{array} \right\rangle,$$

$\Theta\Gamma_1 \oplus \Theta\Gamma_2$

$$= \left\langle \begin{array}{c} \left(\begin{array}{l} ([(1-(1-({}^{1}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} + [(1-(1-({}^{2}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} \\ -[(1-(1-({}^{1}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}}]^{q}[(1-(1-({}^{2}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}}]^{q})^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{T}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{T}})^{\Theta} \end{array} \right), \\ \left(\begin{array}{l} ({}^{1}\hat{\zeta}_{\mathbb{I}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{I}})^{\Theta},\ ([(1-(1-({}^{1}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} + [(1-(1-({}^{2}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} \\ -[(1-(1-({}^{1}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}]^{q}[(1-(1-({}^{2}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}]^{q})^{\frac{1}{q}} \end{array} \right), \\ \left(\begin{array}{l} ({}^{1}\hat{\zeta}_{\mathbb{F}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{F}})^{\Theta},\ ([(1-(1-({}^{1}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} + [(1-(1-({}^{2}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} \\ -[(1-(1-({}^{1}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}]^{q}[(1-(1-({}^{2}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}]^{q})^{\frac{1}{q}} \end{array} \right) \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{c} \left(\left(\begin{array}{l} 1-(1-({}^{1}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta} + 1-(1-({}^{2}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta} - \\ [1-(1-({}^{1}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta}][1-(1-({}^{2}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta}] \end{array} \right)^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{T}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{T}})^{\Theta} \right), \\ \left(({}^{1}\hat{\zeta}_{\mathbb{I}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{I}})^{\Theta}, \left(\begin{array}{l} 1-(1-({}^{1}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta} + 1-(1-({}^{2}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta} - \\ [1-(1-({}^{1}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta}][1-(1-({}^{2}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta}] \end{array} \right)^{\frac{1}{q}} \right), \\ \left(({}^{1}\hat{\zeta}_{\mathbb{F}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{F}})^{\Theta}, \left(\begin{array}{l} 1-(1-({}^{1}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta} + 1-(1-({}^{2}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta} - \\ [1-(1-({}^{1}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta}][1-(1-({}^{2}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta}] \end{array} \right)^{\frac{1}{q}} \right) \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{c} \left([1-(1-({}^{1}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta}(1-({}^{2}\hat{\zeta}_{\mathbb{T}})^{q})^{\Theta}]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{T}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{T}})^{\Theta}\right), \\ \left(({}^{1}\hat{\zeta}_{\mathbb{I}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{I}})^{\Theta}, [1-(1-({}^{1}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta}(1-({}^{2}\hat{\varpi}_{\mathbb{I}})^{q})^{\Theta}]^{\frac{1}{q}}\right), \\ \left(({}^{1}\hat{\zeta}_{\mathbb{F}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{F}})^{\Theta}, [1-(1-({}^{1}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta}(1-({}^{2}\hat{\varpi}_{\mathbb{F}})^{q})^{\Theta}]^{\frac{1}{q}}\right) \end{array} \right\rangle.$$

Thus, the right side of the equation equals the left side, which proves that $\Theta(\Gamma_1 \oplus \Gamma_2) = \Theta\Gamma_1 \oplus \Theta\Gamma_2$.

(4) Based on Definition 3.20 (item (2) and item (4)), we have for the right side of the equation $(\Gamma_1 \otimes \Gamma_2)^{\Theta} =$

$$\left(\left\langle \begin{array}{l} ({}^{1}\hat{\zeta}_{\mathbb{T}}\,{}^{2}\hat{\zeta}_{\mathbb{T}}, (({}^{1}\hat{\varpi}_{\mathbb{T}})^{q} + ({}^{2}\hat{\varpi}_{\mathbb{T}})^{q} - ({}^{1}\hat{\varpi}_{\mathbb{T}})^{q}({}^{2}\hat{\varpi}_{\mathbb{T}})^{q})^{\frac{1}{q}}), \\ (({}^{1}\hat{\zeta}_{\mathbb{I}})^{q} + ({}^{2}\hat{\zeta}_{\mathbb{I}})^{q} - ({}^{1}\hat{\zeta}_{\mathbb{I}})^{q}({}^{2}\hat{\zeta}_{\mathbb{I}})^{q})^{\frac{1}{q}}, \\ {}^{1}\hat{\varpi}_{\mathbb{I}}\,{}^{2}\hat{\varpi}_{\mathbb{I}}), (({}^{1}\hat{\zeta}_{\mathbb{F}})^{q} + ({}^{2}\hat{\zeta}_{\mathbb{F}})^{q} - ({}^{1}\hat{\zeta}_{\mathbb{F}})^{q}({}^{2}\hat{\zeta}_{\mathbb{F}})^{q})^{\frac{1}{q}},\ {}^{1}\hat{\varpi}_{\mathbb{F}}\,{}^{2}\hat{\varpi}_{\mathbb{F}}) \end{array} \right\rangle \right)^{\Theta},$$

$$= \left\langle \begin{array}{l} \left(({}^{1}\hat{\zeta}_{\mathbb{T}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{T}})^{\Theta}, \left[1 - [1 - [(({}^{1}\hat{\varpi}_{\mathbb{T}})^{q} + ({}^{2}\hat{\varpi}_{\mathbb{T}})^{q} - ({}^{1}\hat{\varpi}_{\mathbb{T}})^{q}({}^{2}\hat{\varpi}_{\mathbb{T}})^{q})^{\frac{1}{q}}]^{q}]^{\Theta}\right]^{\frac{1}{q}} \right), \\ \left(\left[1 - [1 - [(({}^{1}\hat{\zeta}_{\mathbb{I}})^{q} + ({}^{2}\hat{\zeta}_{\mathbb{I}})^{q} - ({}^{1}\hat{\zeta}_{\mathbb{I}})^{q}({}^{2}\hat{\zeta}_{\mathbb{I}})^{q})^{\frac{1}{q}}]^{q}]^{\Theta}\right]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{I}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{I}})^{\Theta} \right), \\ \left(\left[1 - [1 - [(({}^{1}\hat{\zeta}_{\mathbb{F}})^{q} + ({}^{2}\hat{\zeta}_{\mathbb{F}})^{q} - ({}^{1}\hat{\zeta}_{\mathbb{F}})^{q}({}^{2}\hat{\zeta}_{\mathbb{F}})^{q})^{\frac{1}{q}}]^{q}]^{\Theta}\right]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{F}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{F}})^{\Theta} \right) \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{l} \left(({}^{1}\hat{\zeta}_{\mathbb{T}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{T}})^{\Theta}, \left[1 - [1 - [({}^{1}\hat{\varpi}_{\mathbb{T}})^{q} + ({}^{2}\hat{\varpi}_{\mathbb{T}})^{q} - ({}^{1}\hat{\varpi}_{\mathbb{T}})^{q}({}^{2}\hat{\varpi}_{\mathbb{T}})^{q}]]^{\Theta}\right]^{\frac{1}{q}} \right), \\ \left(\left[1 - [1 - [({}^{1}\hat{\zeta}_{\mathbb{I}})^{q} + ({}^{2}\hat{\zeta}_{\mathbb{I}})^{q} - ({}^{1}\hat{\zeta}_{\mathbb{I}})^{q}({}^{2}\hat{\zeta}_{\mathbb{I}})^{q}]]^{\Theta}\right]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{I}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{I}})^{\Theta} \right), \\ \left(\left[1 - [1 - [({}^{1}\hat{\zeta}_{\mathbb{F}})^{q} + ({}^{2}\hat{\zeta}_{\mathbb{F}})^{q} - ({}^{1}\hat{\zeta}_{\mathbb{F}})^{q}({}^{2}\hat{\zeta}_{\mathbb{F}})^{q}]]^{\Theta}\right]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{F}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{F}})^{\Theta} \right) \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{l} \left(({}^{1}\hat{\zeta}_{\mathbb{T}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{T}})^{\Theta}, [1 - (1 - ({}^{1}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta}(1 - ({}^{2}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta}]^{\frac{1}{q}} \right), \\ \left([1 - (1 - ({}^{1}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta}(1 - ({}^{2}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta}]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{I}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{I}})^{\Theta} \right), \\ \left([1 - (1 - ({}^{1}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta}(1 - ({}^{2}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta}]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{F}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{F}})^{\Theta} \right), \end{array} \right\rangle.$$

For the right side of the equation, we have

$$(\Gamma_1)^{\Theta} = \left\langle \begin{array}{l} \left(({}^{1}\hat{\zeta}_{\mathbb{T}})^{\Theta}, (1 - (1 - ({}^{1}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}} \right), \\ \left((1 - (1 - ({}^{1}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{I}})^{\Theta} \right), \\ \left((1 - (1 - ({}^{1}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{F}})^{\Theta} \right) \end{array} \right\rangle,$$

$$(\Gamma_2)^{\Theta} = \left\langle \begin{array}{l} \left(({}^{2}\hat{\zeta}_{\mathbb{T}})^{\Theta}, (1 - (1 - ({}^{2}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}} \right), \\ \left((1 - (1 - ({}^{2}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}, ({}^{2}\hat{\varpi}_{\mathbb{I}})^{\Theta} \right), \\ \left((1 - (1 - ({}^{2}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}, ({}^{2}\hat{\varpi}_{\mathbb{F}})^{\Theta} \right) \end{array} \right\rangle,$$

$\Gamma_1^{\Theta} \otimes \Gamma_2^{\Theta} =$

$$\left\langle \begin{array}{l} \left(({}^{1}\hat{\zeta}_{\mathbb{T}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{T}})^{\Theta}, \left(\begin{array}{l} [(1 - (1 - ({}^{1}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} + \\ [(1 - (1 - ({}^{2}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} - \\ [(1 - (1 - ({}^{1}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}}]^{q}[(1 - (1 - ({}^{2}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} \end{array} \right)^{\frac{1}{q}} \right), \\ \left(\left(\begin{array}{l} [(1 - (1 - ({}^{1}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} + \\ [(1 - (1 - ({}^{2}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} - \\ [(1 - (1 - ({}^{1}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}]^{q}[(1 - (1 - ({}^{2}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} \end{array} \right)^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{I}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{I}})^{\Theta} \right), \\ \left(\left(\begin{array}{l} [(1 - (1 - ({}^{1}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} + \\ [(1 - (1 - ({}^{2}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} - \\ [(1 - (1 - ({}^{1}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}]^{q}[(1 - (1 - ({}^{2}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta})^{\frac{1}{q}}]^{q} \end{array} \right)^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{F}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{F}})^{\Theta} \right) \end{array} \right\rangle,$$

$$\left(\begin{array}{l}\left(({}^{1}\hat{\zeta}_{\mathbb{T}})^{\Theta}({}^{2}\hat{\zeta}_{\mathbb{T}})^{\Theta}, \left(\begin{array}{l} 1-(1-({}^{1}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta}+ \\ 1-(1-({}^{2}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta}- \\ [1-(1-({}^{1}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta}][1-(1-({}^{2}\hat{\varpi}_{\mathbb{T}})^{q})^{\Theta}]^{\frac{1}{q}} \end{array}\right)\right), \\ = \left\langle \left(\left(\begin{array}{l} 1-(1-({}^{1}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta}+ \\ 1-(1-({}^{2}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta}- \\ [1-(1-({}^{1}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta}][1-(1-({}^{2}\hat{\zeta}_{\mathbb{I}})^{q})^{\Theta}] \end{array}\right)^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{I}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{I}})^{\Theta} \right), \right\rangle, \\ \left(\left(\begin{array}{l} 1-(1-({}^{1}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta}+ \\ 1-(1-({}^{2}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta}- \\ [1-(1-({}^{1}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta}][1-(1-({}^{2}\hat{\zeta}_{\mathbb{F}})^{q})^{\Theta}] \end{array}\right)^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathbb{F}})^{\Theta}({}^{2}\hat{\varpi}_{\mathbb{F}})^{\Theta} \right)\end{array}\right.$$

$$= \left\langle \begin{array}{l} \left(({}^{1}\hat{\zeta}_{\mathcal{T}})^{\Theta}({}^{2}\hat{\zeta}_{\mathcal{T}})^{\Theta}, [1-(1-({}^{1}\hat{\varpi}_{\mathcal{T}})^{q})^{\Theta}(1-({}^{2}\hat{\varpi}_{\mathcal{T}})^{q})^{\Theta}]^{\frac{1}{q}} \right), \\ \left([1-(1-({}^{1}\hat{\zeta}_{\mathcal{I}})^{q})^{\Theta}(1-({}^{2}\hat{\zeta}_{\mathcal{I}})^{q})^{\Theta}]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathcal{I}})^{\Theta}({}^{2}\hat{\varpi}_{\mathcal{I}})^{\Theta} \right), \\ \left([1-(1-({}^{1}\hat{\zeta}_{\mathcal{F}})^{q})^{\Theta}(1-({}^{2}\hat{\zeta}_{\mathcal{F}})^{q})^{\Theta}]^{\frac{1}{q}}, ({}^{1}\hat{\varpi}_{\mathcal{F}})^{\Theta}({}^{2}\hat{\varpi}_{\mathcal{F}})^{\Theta} \right) \end{array} \right\rangle.$$

□

This proves that $(\Gamma_1 \otimes \Gamma_2)^{\Theta} = \Gamma_1^{\Theta} \otimes \Gamma_2^{\Theta}$.

4. q-ROFVN aggregation operators

Based on the algebraic operations of q-ROFVNNs, we go on with aggregation operators of q-ROFVNSs.

4.1. q-ROFVNWA operator

Here, we define the q-ROFVNWA operator and discuss its properties.

Definition 4.1. Let $\Gamma_{\varepsilon} = \{\langle({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{T}}), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{I}}), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})\rangle : \varepsilon = 1, ..., n\}$ be a set of q-ROFVNNs. The q-ROFVNWA operator is characterized by the transformation $q - ROFVNWA : q - ROFVNN(\hat{\Delta}) \longrightarrow q - ROFVNN(\hat{\Delta})$ and defined as.

$$q - ROFVNWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \eta_1\Gamma_1 \oplus \eta_2\Gamma_2 \oplus ...\eta_n\Gamma_n,$$

where $\eta_{\varepsilon} \in$ ☗ is the weight of Γ_{ε}, $\forall \varepsilon = 1, ..., n$ and $\sum_{\varepsilon=1}^{n} \eta_{\varepsilon} = 1$.

Theorem 4.2. *Let* $\Gamma_{\varepsilon} = \{\langle({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{T}}), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{I}}), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})\rangle : \varepsilon = 1, ..., n\}$ *be a set of q-ROFVNNs and* $\eta = (\eta_1, \eta_2, ...\eta_n)$ *be the weight vector of* Γ_{ε}*. Then,* $q - ROFVNWA(\Gamma_1, \Gamma_2, ..., \Gamma_n)$

$$= \left\langle \begin{array}{l} \left([1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^{q})^{\eta_{\varepsilon}}]^{\frac{1}{q}}, \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_{\varepsilon}}\right), \\ \left(\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}})^{\eta_{\varepsilon}}, [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^{q})^{\eta_{\varepsilon}}]^{\frac{1}{q}}\right), \\ \left(\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}})^{\eta_{\varepsilon}}, [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^{q})^{\eta_{\varepsilon}}]^{\frac{1}{q}}\right) \end{array} \right\rangle, q \geq 1. \tag{19}$$

Proof. This theorem can be proven using mathematical induction as follows.

(1) Take $n = 2$. Then, since

$$\eta_1\Gamma_1 = \left\langle \begin{array}{l} \Big((1-(1-\ ({}^1\hat{\zeta}_{\mathbb{T}})^q)^{\eta_1})^{\frac{1}{q}}, ({}^1\hat{\varpi}_{\mathbb{T}})^{\eta_1}\Big), \\ \Big(({}^1\hat{\zeta}_{\mathbb{I}})^{\eta_1}, (1-(1-\ ({}^1\hat{\varpi}_{\mathbb{I}})^q)^{\eta_1})^{\frac{1}{q}}\Big), \\ \Big(({}^1\hat{\zeta}_{\mathbb{F}})^{\eta_1}, (1-(1-\ ({}^1\hat{\varpi}_{\mathbb{F}})^q)^{\eta_1})^{\frac{1}{q}}\Big) \end{array} \right\rangle,$$

and

$$\eta_2\Gamma_2 = \left\langle \begin{array}{l} \Big((1-(1-\ ({}^2\hat{\zeta}_{\mathbb{T}})^q)^{\eta_2})^{\frac{1}{q}}, ({}^2\hat{\varpi}_{\mathbb{T}})^{\eta_2}\Big), \\ \Big(({}^2\hat{\zeta}_{\mathbb{I}})^{\eta_2}, (1-(1-\ ({}^2\hat{\varpi}_{\mathbb{I}})^q)^{\eta_2})^{\frac{1}{q}}\Big), \\ \Big(({}^2\hat{\zeta}_{\mathbb{F}})^{\eta_2}, (1-(1-\ ({}^2\hat{\varpi}_{\mathbb{F}})^q)^{\eta_2})^{\frac{1}{q}}\Big) \end{array} \right\rangle,$$

we have,
$\eta_1\Gamma_1 \oplus \eta_2\Gamma_2 = \Big\langle \Big((1-(1-\ ({}^1\hat{\zeta}_{\mathbb{T}})^q)^{\eta_1} + 1-(1-\ ({}^2\hat{\zeta}_{\mathbb{T}})^q)^{\eta_2} - [1-(1-\ ({}^1\hat{\zeta}_{\mathbb{T}})^q)^{\eta_1}][1-(1-\ ({}^2\hat{\zeta}_{\mathbb{T}})^q)^{\eta_2}])^{\frac{1}{q}}, ({}^1\hat{\varpi}_{\mathbb{T}})^{\eta_1}({}^2\hat{\varpi}_{\mathbb{T}})^{\eta_2}\Big), \Big(({}^1\hat{\zeta}_{\mathbb{I}})^{\eta_1}({}^2\hat{\zeta}_{\mathbb{I}})^{\eta_2},\ (1-(1-\ ({}^1\hat{\varpi}_{\mathbb{I}})^q)^{\eta_1} + 1-(1-\ ({}^2\hat{\varpi}_{\mathbb{I}})^q)^{\eta_2} - [1-(1-\ ({}^1\hat{\varpi}_{\mathbb{I}})^q)^{\eta_1}][1-(1-\ ({}^2\hat{\varpi}_{\mathbb{I}})^q)^{\eta_2}])^{\frac{1}{q}}\Big), \Big(({}^1\hat{\zeta}_{\mathbb{F}})^{\eta_1}({}^2\hat{\zeta}_{\mathbb{F}})^{\eta_2},\ (1-(1-\ ({}^1\hat{\varpi}_{\mathbb{F}})^q)^{\eta_1} + 1-(1-\ ({}^2\hat{\varpi}_{\mathbb{F}})^q)^{\eta_2} - [1-(1-\ ({}^1\hat{\varpi}_{\mathbb{F}})^q)^{\eta_1}][1-(1-\ ({}^2\hat{\varpi}_{\mathbb{F}})^q)^{\eta_2}])^{\frac{1}{q}}\Big)\Big\rangle,$
$= \Big\langle \Big([1-(1-({}^1\hat{\zeta}_{\mathbb{T}})^q)^{\eta_1}(1-({}^2\hat{\zeta}_{\mathbb{T}})^q)^{\eta_2}]^{\frac{1}{q}}, ({}^1\hat{\varpi}_{\mathbb{T}})^{\eta_1}({}^2\hat{\varpi}_{\mathbb{T}})^{\eta_2}\Big), \Big(({}^1\hat{\zeta}_{\mathbb{I}})^{\eta_1}({}^2\hat{\zeta}_{\mathbb{I}})^{\eta_2}, [1-(1-({}^1\hat{\varpi}_{\mathbb{I}})^q)^{\eta_1}(1-({}^2\hat{\varpi}_{\mathbb{I}})^q)^{\eta_2}]^{\frac{1}{q}}\Big), \Big(({}^1\hat{\zeta}_{\mathbb{F}})^{\eta_1}({}^2\hat{\zeta}_{\mathbb{F}})^{\eta_2}, [1-(1-({}^1\hat{\varpi}_{\mathbb{F}})^q)^{\eta_1}(1-({}^2\hat{\varpi}_{\mathbb{F}})^q)^{\eta_2}]^{\frac{1}{q}}\Big)\Big\rangle,$

$$= \left\langle \begin{array}{l} \Big([1-\prod_{\varepsilon=1}^{2}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}}, \prod_{\varepsilon=1}^{2}({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon}\Big), \\ \Big(\prod_{\varepsilon=1}^{2}({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}})^{\eta_\varepsilon}, [1-\prod_{\varepsilon=1}^{2}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}}\Big), \\ \Big(\prod_{\varepsilon=1}^{2}({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}})^{\eta_\varepsilon}, [1-\prod_{\varepsilon=1}^{2}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}}\Big) \end{array} \right\rangle.$$

This satisfies Eq (19).

(2) If Eq (19) is satisfied while $\varepsilon = n$, then $q-ROFVNWA(\Gamma_1, \Gamma_2, ..., \Gamma_n)$

$$= \left\langle \begin{array}{l} \Big([1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}}, \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon}\Big), \\ \Big(\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}})^{\eta_\varepsilon}, [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}}\Big), \\ \Big(\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}})^{\eta_\varepsilon}, [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}}\Big) \end{array} \right\rangle, q \geq 1.$$

Suppose $\varepsilon = n + 1$. Then, based on the algebraic operations of the q-ROFVNNs, we have $q-$

$$ROFVNWA(\Gamma_1,\Gamma_2,...,\Gamma_{n+1}) = q-ROFVNWA(\Gamma_1,\Gamma_2,...,\Gamma_n)\oplus\eta_{n+1}\Gamma_{n+1},$$

$$= \left\langle \begin{array}{l} \left([1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}}, \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon}\right), \\ \left(\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}})^{\eta_\varepsilon}, [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}}\right), \\ \left(\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}})^{\eta_\varepsilon}, [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}}\right) \end{array} \right\rangle \oplus \left\langle \begin{array}{l} \left(\begin{array}{l}(1-(1-({}^{n+1}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_{n+1}})^{\frac{1}{q}}, \\ ({}^{n+1}\hat{\varpi}_{\mathbb{T}})^{\eta_{n+1}}\end{array}\right), \\ \left(\begin{array}{l}({}^{n+1}\hat{\zeta}_{\mathbb{I}})^{\eta_{n+1}}, \\ (1-(1-({}^{n+1}\hat{\varpi}_{\mathbb{I}})^q)^{\eta_{n+1}})^{\frac{1}{q}}\end{array}\right), \\ \left(\begin{array}{l}({}^{n+1}\hat{\zeta}_{\mathbb{F}})^{\eta_{n+1}}, \\ (1-(1-({}^{n+1}\hat{\varpi}_{\mathbb{F}})^q)^{\eta_{n+1}})^{\frac{1}{q}}\end{array}\right), \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{l} \left(\left(\begin{array}{l} [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon}]+ \\ [1-(1-({}^{n+1}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_{n+1}}]- \\ [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon}]\,[1-(1-({}^{n+1}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_{n+1}}] \end{array}\right)^{\frac{1}{q}}, \\ (\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon})({}^{n+1}\hat{\varpi}_{\mathbb{T}})^{\eta_{n+1}} \end{array}\right), \\ \left(\begin{array}{l} (\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}})^{\eta_\varepsilon})({}^{n+1}\hat{\zeta}_{\mathbb{I}})^{\eta_{n+1}}, \\ \left(\begin{array}{l} [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^q)^{\eta_\varepsilon}]+ \\ [1-(1-({}^{n+1}\hat{\varpi}_{\mathbb{I}})^q)^{\eta_{n+1}}]- \\ [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^q)^{\eta_\varepsilon}]\,[1-(1-({}^{n+1}\hat{\varpi}_{\mathbb{I}})^q)^{\eta_{n+1}}] \end{array}\right)^{\frac{1}{q}} \end{array}\right), \\ \left(\begin{array}{l} (\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}})^{\eta_\varepsilon})({}^{n+1}\hat{\zeta}_{\mathbb{F}})^{\eta_{n+1}}, \\ \left(\begin{array}{l} [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^q)^{\eta_\varepsilon}]+ \\ [1-(1-({}^{n+1}\hat{\varpi}_{\mathbb{F}})^q)^{\eta_{n+1}}]- \\ [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^q)^{\eta_\varepsilon}]\,[1-(1-({}^{n+1}\hat{\varpi}_{\mathbb{F}})^q)^{\eta_{n+1}}] \end{array}\right)^{\frac{1}{q}} \end{array}\right) \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{l} \left([1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon}(1-({}^{n+1}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_{n+1}}]^{\frac{1}{q}}, \prod_{\varepsilon=1}^{n+1}({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon}\right), \\ \left(\prod_{\varepsilon=1}^{n+1}({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}})^{\eta_\varepsilon},\ [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^q)^{\eta_\varepsilon}(1-({}^{n+1}\hat{\varpi}_{\mathbb{I}})^q)^{\eta_{n+1}}]^{\frac{1}{q}}\right), \\ \left(\prod_{\varepsilon=1}^{n+1}({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}})^{\eta_\varepsilon},\ [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^q)^{\eta_\varepsilon}(1-({}^{n+1}\hat{\varpi}_{\mathbb{F}})^q)^{\eta_{n+1}}]^{\frac{1}{q}}\right) \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{c} \Big([1-\prod_{\varepsilon=1}^{n+1}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}}, \prod_{\varepsilon=1}^{n+1}({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon}\Big), \\ \Big(\prod_{\varepsilon=1}^{n+1}({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}})^{\eta_\varepsilon}, [1-\prod_{\varepsilon=1}^{n+1}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}}\Big), \\ \Big(\prod_{\varepsilon=1}^{n+1}({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}})^{\eta_\varepsilon}, [1-\prod_{\varepsilon=1}^{n+1}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}}\Big) \end{array} \right\rangle,$$

$\varepsilon = 1, 2, ..., n+1$. This proves that Eq (5) is satisfied for $\varepsilon = n+1$. According to (1) and (2), Eq (19) holds for any ε. This completes the proof.

□

Proposition 4.3. *Idempotent Property: Let* $\Gamma_\varepsilon = \{\langle({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{T}}), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{I}}), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})\rangle : \varepsilon = 1, ..., n\}$ *be a collection of q-ROFVNNs. If* $\Gamma_\varepsilon = \Gamma = \langle(\hat{\zeta}_{\mathbb{T}}, \hat{\varpi}_{\mathbb{T}}), (\hat{\zeta}_{\mathbb{I}}, \hat{\varpi}_{\mathbb{I}}), (\hat{\zeta}_{\mathbb{F}}, \hat{\varpi}_{\mathbb{F}})\rangle, \forall\varepsilon = 1, ..., n$, *then* $q-ROFVNWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \Gamma = \langle(\hat{\zeta}_{\mathbb{T}}, \hat{\varpi}_{\mathbb{T}}), (\hat{\zeta}_{\mathbb{I}}, \hat{\varpi}_{\mathbb{I}}), (\hat{\zeta}_{\mathbb{F}}, \hat{\varpi}_{\mathbb{F}})\rangle$.

Proof. $\Gamma_\varepsilon = \Gamma = \langle(\hat{\zeta}_{\mathbb{T}}, \hat{\varpi}_{\mathbb{T}}), (\hat{\zeta}_{\mathbb{I}}, \hat{\varpi}_{\mathbb{I}}), (\hat{\zeta}_{\mathbb{F}}, \hat{\varpi}_{\mathbb{F}})\rangle, \forall\varepsilon = 1, ..., n$. Then, based on Theorem 4.2,
$q-ROFVNWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) =$

$$\left\langle \begin{array}{c} \Big([1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}}, \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon}\Big), \\ \Big(\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}})^{\eta_\varepsilon}, [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}}\Big), \\ \Big(\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}})^{\eta_\varepsilon}, [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}}\Big) \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{c} \Big([1-(1-(\hat{\zeta}_{\mathbb{T}})^{q})^{\sum_{\varepsilon=1}^{n}\eta_\varepsilon}]^{\frac{1}{q}}, (\hat{\varpi}_{\mathbb{T}})^{\sum_{\varepsilon=1}^{n}\eta_\varepsilon}\Big), \\ \Big((\hat{\zeta}_{\mathbb{I}})^{\sum_{\varepsilon=1}^{n}\eta_\varepsilon}, [1-(1-(\hat{\varpi}_{\mathbb{I}})^{q})^{\sum_{\varepsilon=1}^{n}\eta_\varepsilon}]^{\frac{1}{q}}\Big), \\ \Big((\hat{\zeta}_{\mathbb{F}})^{\sum_{\varepsilon=1}^{n}\eta_\varepsilon}, [1-(1-(\hat{\varpi}_{\mathbb{F}})^{q})^{\sum_{\varepsilon=1}^{n}\eta_\varepsilon}]^{\frac{1}{q}}\Big) \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{c} \Big([1-(1-(\hat{\zeta}_{\mathbb{T}})^{q})]^{\frac{1}{q}}, (\hat{\varpi}_{\mathbb{T}})\Big), \\ \Big((\hat{\zeta}_{\mathbb{I}}), [1-(1-(\hat{\varpi}_{\mathbb{I}})^{q})]^{\frac{1}{q}}\Big), \\ \Big((\hat{\zeta}_{\mathbb{F}}), [1-(1-(\hat{\varpi}_{\mathbb{F}})^{q})]^{\frac{1}{q}}\Big) \end{array} \right\rangle,$$

$$= \left\langle (\hat{\zeta}_{\mathbb{T}}, \hat{\varpi}_{\mathbb{T}}), (\hat{\zeta}_{\mathbb{I}}, \hat{\varpi}_{\mathbb{I}}), (\hat{\zeta}_{\mathbb{F}}, \hat{\varpi}_{\mathbb{F}}) \right\rangle = \Gamma.$$

□

Proposition 4.4. *Boundedness Property: Let* $\Gamma_\varepsilon = \{\langle({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{T}}), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{I}}), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})\rangle : \varepsilon = 1, ..., n\}$ *be a collection of q-ROFVNNs. If* $\Gamma^- = \langle(\hat{\zeta}^-_{\mathbb{T}}, \hat{\varpi}^+_{\mathbb{T}}), (\hat{\zeta}^+_{\mathbb{I}}, \hat{\varpi}^-_{\mathbb{I}}), (\hat{\zeta}^+_{\mathbb{F}}, \hat{\varpi}^-_{\mathbb{F}})\rangle$ *and* $\Gamma^+ = \langle(\hat{\zeta}^+_{\mathbb{T}}, \hat{\varpi}^-_{\mathbb{T}}), (\hat{\zeta}^-_{\mathbb{I}}, \hat{\varpi}^+_{\mathbb{I}}), (\hat{\zeta}^-_{\mathbb{F}}, \hat{\varpi}^+_{\mathbb{F}})\rangle$, *where,* $\hat{\zeta}^-_{\mathbb{T}} = \min_{\varepsilon}\{{}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}\}, \hat{\zeta}^+_{\mathbb{T}} = \max_{\varepsilon}\{{}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}\}, \hat{\zeta}^-_{\mathbb{I}} = \min_{\varepsilon}\{{}^{\varepsilon}\hat{\zeta}_{\mathbb{I}}\}, \hat{\zeta}^+_{\mathbb{I}} = \max_{\varepsilon}\{{}^{\varepsilon}\hat{\zeta}_{\mathbb{I}}\}, \hat{\zeta}^-_{\mathbb{F}} = \min_{\varepsilon}\{{}^{\varepsilon}\hat{\zeta}_{\mathbb{F}}\}, \hat{\zeta}^+_{\mathbb{F}} = \max_{\varepsilon}\{{}^{\varepsilon}\hat{\zeta}_{\mathbb{F}}\}, \hat{\varpi}^-_{\mathbb{T}} = \min_{\varepsilon}\{{}^{\varepsilon}\hat{\varpi}_{\mathbb{T}}\}, \hat{\varpi}^+_{\mathbb{T}} = \max_{\varepsilon}\{{}^{\varepsilon}\hat{\varpi}_{\mathbb{T}}\}, \hat{\varpi}^-_{\mathbb{I}} = \min_{\varepsilon}\{{}^{\varepsilon}\hat{\varpi}_{\mathbb{I}}\}, \hat{\varpi}^+_{\mathbb{I}} = \max_{\varepsilon}\{{}^{\varepsilon}\hat{\varpi}_{\mathbb{I}}\}, \hat{\varpi}^-_{\mathbb{F}} = \min_{\varepsilon}\{{}^{\varepsilon}\hat{\varpi}_{\mathbb{F}}\}, \hat{\varpi}^+_{\mathbb{F}} = \max_{\varepsilon}\{{}^{\varepsilon}\hat{\varpi}_{\mathbb{F}}\}$, *then* $\Gamma^- \leq q-ROFVNWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) \leq \Gamma^+$.

Proof. Since $\hat{\zeta}_{\mathbb{T}}^{-} \leq {}^{\varepsilon}\hat{\zeta}_{\mathbb{T}} \leq \hat{\zeta}_{\mathbb{T}}^{+}$, for $q \geq 1$, we obtain

$(\hat{\zeta}_{\mathbb{T}}^{-})^q \leq ({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q \leq (\hat{\zeta}_{\mathbb{T}}^{+})^q \Rightarrow 1-(\hat{\zeta}_{\mathbb{T}}^{-})^q \geq 1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q \geq 1-(\hat{\zeta}_{\mathbb{T}}^{+})^q \Rightarrow (1-(\hat{\zeta}_{\mathbb{T}}^{-})^q)^{\eta_\varepsilon} \geq (1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon} \geq (1-(\hat{\zeta}_{\mathbb{T}}^{+})^q)^{\eta_\varepsilon} \Rightarrow \prod_{\varepsilon=1}^{n}(1-(\hat{\zeta}_{\mathbb{T}}^{-})^q)^{\eta_\varepsilon} \geq \prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon} \geq \prod_{\varepsilon=1}^{n}(1-(\hat{\zeta}_{\mathbb{T}}^{+})^q)^{\eta_\varepsilon} \Rightarrow 1-\prod_{\varepsilon=1}^{n}(1-(\hat{\zeta}_{\mathbb{T}}^{-})^q)^{\eta_\varepsilon} \leq 1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon} \leq 1-\prod_{\varepsilon=1}^{n}(1-(\hat{\zeta}_{\mathbb{T}}^{+})^q)^{\eta_\varepsilon} \Rightarrow [1-\prod_{\varepsilon=1}^{n}(1-(\hat{\zeta}_{\mathbb{T}}^{-})^q)^{\eta_\varepsilon}]^{\frac{1}{q}} \leq [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}} \leq [1-\prod_{\varepsilon=1}^{n}(1-(\hat{\zeta}_{\mathbb{T}}^{+})^q)^{\eta_\varepsilon}]^{\frac{1}{q}}$, since, $[1-\prod_{\varepsilon=1}^{n}(1-(\hat{\zeta}_{\mathbb{T}}^{-})^q)^{\eta_\varepsilon}]^{\frac{1}{q}} = \hat{\zeta}_{\mathbb{T}}^{-}$ and $[1-\prod_{\varepsilon=1}^{n}(1-(\hat{\zeta}_{\mathbb{T}}^{+})^q)^{\eta_\varepsilon}]^{\frac{1}{q}} = \hat{\zeta}_{\mathbb{T}}^{+}$.

Then, $\hat{\zeta}_{\mathbb{T}}^{-} \leq [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}} \leq \hat{\zeta}_{\mathbb{T}}^{+}$. Similarly, since $\hat{\varpi}_{\mathbb{I}}^{-} \leq {}^{\varepsilon}\hat{\varpi}_{\mathbb{I}} \leq \hat{\varpi}_{\mathbb{I}}^{+}$, and $\hat{\varpi}_{\mathbb{F}}^{-} \leq {}^{\varepsilon}\hat{\varpi}_{\mathbb{F}} \leq \hat{\varpi}_{\mathbb{F}}^{+}$, we obtain, $\hat{\varpi}_{\mathbb{I}}^{-} \leq [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}} \leq \hat{\varpi}_{\mathbb{I}}^{+}$ and $\hat{\varpi}_{\mathbb{F}}^{-} \leq [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}} \leq \hat{\varpi}_{\mathbb{F}}^{+}$.

Now, since $\hat{\varpi}_{\mathbb{T}}^{-} \leq {}^{\varepsilon}\hat{\varpi}_{\mathbb{T}} \leq \hat{\varpi}_{\mathbb{T}}^{+} \Rightarrow (\hat{\varpi}_{\mathbb{T}}^{-})^{\eta_\varepsilon} \leq ({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon} \leq (\hat{\varpi}_{\mathbb{T}}^{+})^{\eta_\varepsilon} \Rightarrow \prod_{\varepsilon=1}^{n}(\hat{\varpi}_{\mathbb{T}}^{-})^{\eta_\varepsilon} \leq \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon} \leq \prod_{\varepsilon=1}^{n}(\hat{\varpi}_{\mathbb{T}}^{+})^{\eta_\varepsilon}$, and $\prod_{\varepsilon=1}^{n}(\hat{\varpi}_{\mathbb{T}}^{-})^{\eta_\varepsilon} = \hat{\varpi}_{\mathbb{T}}^{-}$ and $\prod_{\varepsilon=1}^{n}(\hat{\varpi}_{\mathbb{T}}^{+})^{\eta_\varepsilon} = \hat{\varpi}_{\mathbb{T}}^{+}$. Then, $\hat{\varpi}_{\mathbb{T}}^{-} \leq \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon} \leq \hat{\varpi}_{\mathbb{T}}^{+}$.

In the same manner, as $\hat{\zeta}_{\mathbb{I}}^{-} \leq {}^{\varepsilon}\hat{\zeta}_{\mathbb{I}} \leq \hat{\zeta}_{\mathbb{I}}^{+}$ and $\hat{\zeta}_{\mathbb{F}}^{-} \leq {}^{\varepsilon}\hat{\zeta}_{\mathbb{F}} \leq \hat{\zeta}_{\mathbb{F}}^{+}$, we obtain $\hat{\zeta}_{\mathbb{I}}^{-} \leq \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}})^{\eta_\varepsilon} \leq \hat{\zeta}_{\mathbb{I}}^{+}$, and $\hat{\zeta}_{\mathbb{F}}^{-} \leq \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}})^{\eta_\varepsilon} \leq \hat{\zeta}_{\mathbb{F}}^{+}$.

Now, let $q-ROFVNWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \Gamma = \langle(\hat{\zeta}_{\mathbb{T}}, \hat{\varpi}_{\mathbb{T}}), (\hat{\zeta}_{\mathbb{I}}, \hat{\varpi}_{\mathbb{I}}), (\hat{\zeta}_{\mathbb{F}}, \hat{\varpi}_{\mathbb{F}})\rangle$. Then,

$\Pi(\Gamma) = \frac{1}{3}\Big[[(\hat{\zeta}_{\mathbb{T}})^q - (\hat{\varpi}_{\mathbb{T}})^q] - [(\hat{\zeta}_{\mathbb{I}})^q - (\hat{\varpi}_{\mathbb{I}})^q] - [(\hat{\zeta}_{\mathbb{F}})^q - (\hat{\varpi}_{\mathbb{F}})^q]\Big] \geq \frac{1}{3}\Big[[(\hat{\zeta}_{\mathbb{T}}^{-})^q - (\hat{\varpi}_{\mathbb{T}}^{+})^q] - [(\hat{\zeta}_{\mathbb{I}}^{+})^q - (\hat{\varpi}_{\mathbb{I}}^{-})^q] - [(\hat{\zeta}_{\mathbb{F}}^{+})^q - (\hat{\varpi}_{\mathbb{F}}^{-})^q]\Big] = \Pi(\Gamma^{-})$, and

$\Pi(\Gamma) = \frac{1}{3}\Big[[(\hat{\zeta}_{\mathbb{T}})^q - (\hat{\varpi}_{\mathbb{T}})^q] - [(\hat{\zeta}_{\mathbb{I}})^q - (\hat{\varpi}_{\mathbb{I}})^q] - [(\hat{\zeta}_{\mathbb{F}})^q - (\hat{\varpi}_{\mathbb{F}})^q]\Big] \leq \frac{1}{3}\Big[[(\hat{\zeta}_{\mathbb{T}}^{+})^q - (\hat{\varpi}_{\mathbb{T}}^{-})^q] - [(\hat{\zeta}_{\mathbb{I}}^{-})^q - (\hat{\varpi}_{\mathbb{I}}^{+})^q] - [(\hat{\zeta}_{\mathbb{F}}^{-})^q - (\hat{\varpi}_{\mathbb{F}}^{+})^q]\Big] = \Pi(\Gamma^{+})$.

This implies $\Gamma^{-} \leq q-ROFVNWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) \leq \Gamma^{+}$. □

Proposition 4.5. *Monotonicity Property: Let* $\Gamma_\varepsilon = \{\langle({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{T}}), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{I}}), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})\rangle : \varepsilon = 1, ..., n\}$ *and* $\Gamma_\varepsilon^* = \{\langle({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}^*, {}^{\varepsilon}\hat{\varpi}_{\mathbb{T}}^*), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}}^*, {}^{\varepsilon}\hat{\varpi}_{\mathbb{I}}^*), ({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}}^*, {}^{\varepsilon}\hat{\varpi}_{\mathbb{F}}^*)\rangle : \varepsilon = 1, ..., n\}$ *be two collections of q-ROFVNNs. If* ${}^{\varepsilon}\hat{\zeta}_{\mathbb{T}} \leq^{\varepsilon} \hat{\zeta}_{\mathbb{T}}^*$, ${}^{\varepsilon}\hat{\varpi}_{\mathbb{T}} \geq^{\varepsilon} \hat{\varpi}_{\mathbb{T}}^*$, ${}^{\varepsilon}\hat{\zeta}_{\mathbb{I}} \geq^{\varepsilon} \hat{\zeta}_{\mathbb{I}}^*$, ${}^{\varepsilon}\hat{\varpi}_{\mathbb{I}} \leq^{\varepsilon} \hat{\varpi}_{\mathbb{I}}^*$, ${}^{\varepsilon}\hat{\zeta}_{\mathbb{F}} \geq^{\varepsilon} \hat{\zeta}_{\mathbb{F}}^*$ *and* ${}^{\varepsilon}\hat{\varpi}_{\mathbb{F}} \leq^{\varepsilon} \hat{\varpi}_{\mathbb{F}}^*$, $\forall\varepsilon = 1, 2, ..., n$, *then,* $q-ROFVNWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) \leq q-ROFVNWA(\Gamma_1^*, \Gamma_2^*, ..., \Gamma_n^*)$.

Proof. Since ${}^{\varepsilon}\hat{\zeta}_{\mathbb{T}} \leq^{\varepsilon} \hat{\zeta}_{\mathbb{T}}^*$, for $q \geq 1$, we obtain

$({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q \leq ({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}^*)^q \Rightarrow 1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q \geq 1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}^*)^q \Rightarrow (1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon} \geq (1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}^*)^q)^{\eta_\varepsilon} \Rightarrow \prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon} \geq \prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}^*)^q)^{\eta_\varepsilon} \Rightarrow 1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon} \leq 1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}^*)^q)^{\eta_\varepsilon} \Rightarrow [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^q)^{\eta_\varepsilon}]^{\frac{1}{q}} \leq [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}^*)^q)^{\eta_\varepsilon}]^{\frac{1}{q}}$. Similarly, since ${}^{\varepsilon}\hat{\varpi}_{\mathbb{I}} \leq^{\varepsilon} \hat{\varpi}_{\mathbb{I}}^*$, and ${}^{\varepsilon}\hat{\varpi}_{\mathbb{F}} \leq^{\varepsilon} \hat{\varpi}_{\mathbb{F}}^*$, we obtain

$[1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}} \leq [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}^{*}_{\mathbb{I}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}}$, and $[1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}} \leq [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}^{*}_{\mathbb{F}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}}$.

Now, ${}^{\varepsilon}\hat{\varpi}_{\mathbb{T}} \geq {}^{\varepsilon}\hat{\varpi}^{*}_{\mathbb{T}} \Rightarrow ({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon} \geq ({}^{\varepsilon}\hat{\varpi}^{*}_{\mathbb{T}})^{\eta_\varepsilon} \Rightarrow \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{\eta_\varepsilon} \geq \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\varpi}^{*}_{\mathbb{T}})^{\eta_\varepsilon}$.

In the same manner, as ${}^{\varepsilon}\hat{\zeta}_{\mathbb{I}} \geq {}^{\varepsilon}\hat{\zeta}^{*}_{\mathbb{I}}$ and ${}^{\varepsilon}\hat{\zeta}_{\mathbb{F}} \geq {}^{\varepsilon}\hat{\zeta}^{*}_{\mathbb{F}}$, we obtain $\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}})^{\eta_\varepsilon} \geq \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}^{*}_{\mathbb{I}})^{\eta_\varepsilon}$ and $\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}})^{\eta_\varepsilon} \geq \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}^{*}_{\mathbb{F}})^{\eta_\varepsilon}$.

Now, let $q-ROFVNWA(\Gamma_1,\Gamma_2,...,\Gamma_n) = \Gamma = \langle(\hat{\zeta}_{\mathbb{T}}, \hat{\varpi}_{\mathbb{T}}),(\hat{\zeta}_{\mathbb{I}}, \hat{\varpi}_{\mathbb{I}}),(\hat{\zeta}_{\mathbb{F}}, \hat{\varpi}_{\mathbb{F}})\rangle$ and
$q-ROFVNWA(\Gamma^{*}_1,\Gamma^{*}_2,...,\Gamma^{*}_n) = \Gamma^{*} = \langle(\hat{\zeta}^{*}_{\mathbb{T}}, \hat{\varpi}^{*}_{\mathbb{T}}),(\hat{\zeta}^{*}_{\mathbb{I}}, \hat{\varpi}^{*}_{\mathbb{I}}),(\hat{\zeta}^{*}_{\mathbb{F}}, \hat{\varpi}^{*}_{\mathbb{F}})\rangle$. Then,
$\Pi(\Gamma) = \frac{1}{3}\Big[[(\hat{\zeta}_{\mathbb{T}})^{q}-(\hat{\varpi}_{\mathbb{T}})^{q}]-[(\hat{\zeta}_{\mathbb{I}})^{q}-(\hat{\varpi}_{\mathbb{I}})^{q}]-[(\hat{\zeta}_{\mathbb{F}})^{q}-(\hat{\varpi}_{\mathbb{F}})^{q}]\Big] \leq \frac{1}{3}\Big[[(\hat{\zeta}^{*}_{\mathbb{T}})^{q}-(\hat{\varpi}^{*}_{\mathbb{T}})^{q}]-[(\hat{\zeta}^{*}_{\mathbb{I}})^{q}-(\hat{\varpi}^{*}_{\mathbb{I}})^{q}]-[(\hat{\zeta}^{*}_{\mathbb{F}})^{q}-(\hat{\varpi}^{*}_{\mathbb{F}})^{q}]\Big] = \Pi(\Gamma^{*})$.

This implies $q-ROFVNWA(\Gamma_1,\Gamma_2,...,\Gamma_n) \leq q-ROFVNWA(\Gamma^{*}_1,\Gamma^{*}_2,...,\Gamma^{*}_n)$. □

4.2. q-ROFVNWG operator

In this part the q-ROFVNWG operator and its properties are presented.

Definition 4.6. Let $\Gamma_\varepsilon = \{\langle({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{T}}),({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{I}}),({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})\rangle : \varepsilon = 1,...,n\}$ be a set of q-ROFVNNs. The q-ROFVNWG operator is characterized by the transformation $q-ROFVNWG : q-ROFVNN(\hat{\Delta}) \longrightarrow q-ROFVNN(\hat{\Delta})$ and defined as

$$q-ROFVNWG(\Gamma_1,\Gamma_2,...,\Gamma_n) = \Gamma_1^{\eta_1} \otimes \Gamma_2^{\eta_2} \otimes ...\Gamma_n^{\eta_n},$$

where $\eta_\varepsilon \in \hat{\blacksquare}$ is the weight of Γ_ε, $\forall\varepsilon = 1,...,n$, and $\sum_{\varepsilon=1}^{n}\eta_\varepsilon = 1$.

Theorem 4.7. *Let $\Gamma_\varepsilon = \{\langle({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{T}}),({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{I}}),({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}}, {}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})\rangle : \varepsilon = 1,...,n\}$ be a set of q-ROFVNNs and $\eta = (\eta_1,\eta_2,...\eta_n)$ be the weight vector of Γ_ε. Then,*

$$q-ROFVNWG(\Gamma_1,\Gamma_2,...,\Gamma_n) = \left\langle \begin{array}{c} \Big(\prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\zeta}_{\mathbb{T}})^{\eta_\varepsilon}, [1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\varpi}_{\mathbb{T}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}}\Big), \\ \Big([1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{I}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}}, \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\varpi}_{\mathbb{I}})^{\eta_\varepsilon}\Big), \\ \Big([1-\prod_{\varepsilon=1}^{n}(1-({}^{\varepsilon}\hat{\zeta}_{\mathbb{F}})^{q})^{\eta_\varepsilon}]^{\frac{1}{q}}, \prod_{\varepsilon=1}^{n}({}^{\varepsilon}\hat{\varpi}_{\mathbb{F}})^{\eta_\varepsilon}\Big) \end{array} \right\rangle, q \geq 1. \tag{20}$$

Proof. Proof of this theorem is similar to the proof of Theorem 4.2. □

The q-ROFVNWG operator has the following properties, which are stated without proof, as the proof is similar to that of the q-ROFVNWA operator.

Proposition 4.8. *Idempotent Property: Let* $\Gamma_\varepsilon = \{\langle({}^\varepsilon\hat{\zeta}_{\mathbb{T}},\ {}^\varepsilon\hat{\varpi}_{\mathbb{T}}), ({}^\varepsilon\hat{\zeta}_{\mathbb{I}},\ {}^\varepsilon\hat{\varpi}_{\mathbb{I}}), ({}^\varepsilon\hat{\zeta}_{\mathbb{F}},\ {}^\varepsilon\hat{\varpi}_{\mathbb{F}})\rangle : \varepsilon = 1, ..., n\}$ *be a collection of q-ROFVNNs. If* $\Gamma_\varepsilon = \Gamma = \langle(\hat{\zeta}_{\mathbb{T}},\ \hat{\varpi}_{\mathbb{T}}), (\hat{\zeta}_{\mathbb{I}},\ \hat{\varpi}_{\mathbb{I}}), (\hat{\zeta}_{\mathbb{F}},\ \hat{\varpi}_{\mathbb{F}})\rangle, \forall\varepsilon = 1, ..., n$, *then,* $q-ROFVNWG(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \Gamma = \langle(\hat{\zeta}_{\mathbb{T}},\ \hat{\varpi}_{\mathbb{T}}), (\hat{\zeta}_{\mathbb{I}},\ \hat{\varpi}_{\mathbb{I}}), (\hat{\zeta}_{\mathbb{F}},\ \hat{\varpi}_{\mathbb{F}})\rangle$.

Proposition 4.9. *Boundedness Property: Let* $\Gamma_\varepsilon = \{\langle({}^\varepsilon\hat{\zeta}_{\mathbb{T}},\ {}^\varepsilon\hat{\varpi}_{\mathbb{T}}), ({}^\varepsilon\hat{\zeta}_{\mathbb{I}},\ {}^\varepsilon\hat{\varpi}_{\mathbb{I}}), ({}^\varepsilon\hat{\zeta}_{\mathbb{F}},\ {}^\varepsilon\hat{\varpi}_{\mathbb{F}})\rangle : \varepsilon = 1, ..., n\}$ *be a collection of q-ROFVNNs. If* $\Gamma^- = \langle(\hat{\zeta}^-_{\mathbb{T}},\ \hat{\varpi}^+_{\mathbb{T}}), (\hat{\zeta}^+_{\mathbb{I}},\ \hat{\varpi}^-_{\mathbb{I}}), (\hat{\zeta}^+_{\mathbb{F}},\ \hat{\varpi}^-_{\mathbb{F}})\rangle$ *and* $\Gamma^+ = \langle(\hat{\zeta}^+_{\mathbb{T}},\ \hat{\varpi}^-_{\mathbb{T}}), (\hat{\zeta}^-_{\mathbb{I}}, \hat{\varpi}^+_{\mathbb{I}}), (\hat{\zeta}^-_{\mathbb{F}},\ \hat{\varpi}^+_{\mathbb{F}})\rangle$, *where* $\hat{\zeta}^-_{\mathbb{T}} = \min_\varepsilon\{{}^\varepsilon\hat{\zeta}_{\mathbb{T}}\}, \hat{\zeta}^+_{\mathbb{T}} = \max_\varepsilon\{{}^\varepsilon\hat{\zeta}_{\mathbb{T}}\}, \hat{\zeta}^-_{\mathbb{I}} = \min_\varepsilon\{{}^\varepsilon\hat{\zeta}_{\mathbb{I}}\}, \hat{\zeta}^+_{\mathbb{I}} = \max_\varepsilon\{{}^\varepsilon\hat{\zeta}_{\mathbb{I}}\}, \hat{\zeta}^-_{\mathbb{F}} = \min_\varepsilon\{{}^\varepsilon\hat{\zeta}_{\mathbb{F}}\}, \hat{\zeta}^+_{\mathbb{F}} = \max_\varepsilon\{{}^\varepsilon\hat{\zeta}_{\mathbb{F}}\}, \hat{\varpi}^-_{\mathbb{T}} = \min_\varepsilon\{{}^\varepsilon\hat{\varpi}_{\mathbb{T}}\}, \hat{\varpi}^+_{\mathbb{T}} = \max_\varepsilon\{{}^\varepsilon\hat{\varpi}_{\mathbb{T}}\}, \hat{\varpi}^-_{\mathbb{I}} = \min_\varepsilon\{{}^\varepsilon\hat{\varpi}_{\mathbb{I}}\}, \hat{\varpi}^+_{\mathbb{I}} = \max_\varepsilon\{{}^\varepsilon\hat{\varpi}_{\mathbb{I}}\}, \hat{\varpi}^-_{\mathbb{F}} = \min_\varepsilon\{{}^\varepsilon\hat{\varpi}_{\mathbb{F}}\}, \hat{\varpi}^+_{\mathbb{F}} = \max_\varepsilon\{{}^\varepsilon\hat{\varpi}_{\mathbb{F}}\}$, *then,* $\Gamma^- \leq q-ROFVNWG(\Gamma_1, \Gamma_2, ..., \Gamma_n) \leq \Gamma^+$.

Proposition 4.10. *Monotonicity Property: Let* $\Gamma_\varepsilon = \{\langle({}^\varepsilon\hat{\zeta}_{\mathbb{T}},\ {}^\varepsilon\hat{\varpi}_{\mathbb{T}}), ({}^\varepsilon\hat{\zeta}_{\mathbb{I}},\ {}^\varepsilon\hat{\varpi}_{\mathbb{I}}), ({}^\varepsilon\hat{\zeta}_{\mathbb{F}},\ {}^\varepsilon\hat{\varpi}_{\mathbb{F}})\rangle : \varepsilon = 1, ..., n\}$ *and* $\Gamma^*_\varepsilon = \{\langle({}^\varepsilon\hat{\zeta}^*_{\mathbb{T}},\ {}^\varepsilon\hat{\varpi}^*_{\mathbb{T}}), ({}^\varepsilon\hat{\zeta}^*_{\mathbb{I}},\ {}^\varepsilon\hat{\varpi}^*_{\mathbb{I}}), ({}^\varepsilon\hat{\zeta}^*_{\mathbb{F}},\ {}^\varepsilon\hat{\varpi}^*_{\mathbb{F}})\rangle : \varepsilon = 1, ..., n\}$ *be two collections of q-ROFVNNs. If* ${}^\varepsilon\hat{\zeta}_{\mathbb{T}} \leq^\varepsilon \hat{\zeta}^*_{\mathbb{T}},\ {}^\varepsilon\hat{\varpi}_{\mathbb{T}} \geq^\varepsilon \hat{\varpi}^*_{\mathbb{T}},\ {}^\varepsilon\hat{\zeta}_{\mathbb{I}} \geq^\varepsilon \hat{\zeta}^*_{\mathbb{I}},\ {}^\varepsilon\hat{\varpi}_{\mathbb{I}} \leq^\varepsilon \hat{\varpi}^*_{\mathbb{I}},\ {}^\varepsilon\hat{\zeta}_{\mathbb{F}} \geq^\varepsilon \hat{\zeta}^*_{\mathbb{F}}$ *and* ${}^\varepsilon\hat{\varpi}_{\mathbb{F}} \leq^\varepsilon \hat{\varpi}^*_{\mathbb{F}}, \forall\varepsilon = 1, 2, ..., n$, *then,* $q-ROFVNWG(\Gamma_1, \Gamma_2, ..., \Gamma_n) \leq q-ROFVNWG(\Gamma^*_1, \Gamma^*_2, ..., \Gamma^*_n)$.

5. Applicability of the q-ROFVN operators in decision making

This section emphasizes the applicability and materiality of the q-ROFVN operators when making a decision. To verify this, we pose a MCDM problem, where the evaluation outcome is presented in terms of q-ROFVNNs. We utilize the q-ROFVNWA and q-ROFVNWG operators to solve the MCDM problem. For this purpose, we assume that the alternatives $\Re_{i=1,2,...,n}$ can be deduced from the DMs with the attributes $\mathfrak{S}_{j=1,2,...,m}$ that have the weights $\eta_{j=1,2,...,m}$ with the condition $\eta_j \in$ ☗ and $\sum_{j=1}^{m}\eta_j = 1, \forall j = 1, 2, ..., m$. Experts are invited to evaluate q-ROFVN data of each attribute for the selection of the optimal candidate. In this setting, to choose the optimal candidate, we propose the following algorithm. (see Algorithm 1)

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Algorithm 1:

(1) The evaluated attributes for each alternative are delivered in the form of q-ROFVNNs as a matrix called the decision matrix.

(2) The obtained decision matrix, which has two types of attributes, is normalized to keep consistency of the attributes. For this, we use the following equation:

$$\Gamma_\varepsilon = \begin{cases} \langle({}^\varepsilon\hat{\zeta}_{\mathbb{T}}, {}^\varepsilon\hat{\varpi}_{\mathbb{T}}), ({}^\varepsilon\hat{\zeta}_{\mathbb{I}}, {}^\varepsilon\hat{\varpi}_{\mathbb{I}}), ({}^\varepsilon\hat{\zeta}_{\mathbb{F}}, {}^\varepsilon\hat{\varpi}_{\mathbb{F}})\rangle & \text{for benefit attributes} \\ \langle({}^\varepsilon\hat{\zeta}_{\mathbb{F}}, {}^\varepsilon\hat{\varpi}_{\mathbb{F}}), ({}^\varepsilon\hat{\varpi}_{\mathbb{I}}, {}^\varepsilon\hat{\zeta}_{\mathbb{I}}), ({}^\varepsilon\hat{\zeta}_{\mathbb{T}}, {}^\varepsilon\hat{\varpi}_{\mathbb{T}})\rangle & \text{for cost attributes} \end{cases} \tag{21}$$

(3) Using q-ROFVNWA or q-ROFVNWG operators, the multiple attribute values of each candidate amount to a single value symbolized as $\mathfrak{L}_{i=1,2,...,n}$.

(4) SF of each candidate is computed using Definition 3.7.

(5) The candidate which has the highest score value is considered the optimal candidate.

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Figure 1 exemplifies the intricate procedure of the groundbreaking and innovative method put forth in this research.

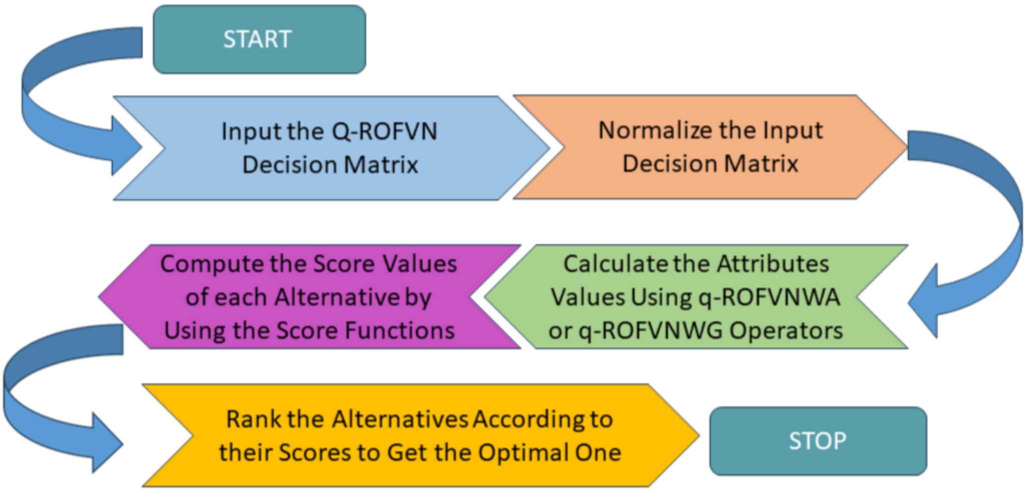


Figure 1. Illustration of the procedural workflow for the proposed method.

5.1. Numerical application

In this part, we employ the above algorithm to solve the following decision making problem:

Bridges in cities are vital for enhancing connectivity, easing traffic congestion, supporting economic growth, improving accessibility, and adding aesthetic value to urban landscapes. However, when it comes to building bridges, finding the most suitable construction contractor can be challenging due to their claims of offering affordable and dependable packages. In this proposed model, our focus is on selecting a contractor specifically for building bridges in urban areas. We consider various objectives, such as $\mathfrak{S}_1$: Bridge design and engineering expertise, $\mathfrak{S}_2$: Adherence to quality standards, $\mathfrak{S}_3$: Technical competence, $\mathfrak{S}_4$: Cost competitiveness, and $\mathfrak{S}_5$: Safety measures. For instance, let us consider a scenario where a construction owner has four contractors $\mathfrak{R}_{i=1,2,3,4}$ available to complete a project. The construction owner aims to evaluate these contractors based on the aforementioned attributes, assigning different levels of importance or weights to each attribute. Suppose the weights for the attributes are 0.2 for bridge design and engineering expertise, 0.1 for adherence to quality standards, 0.1 for technical competence, 0.3 for cost competitiveness, and 0.3 for safety measures. In this case study the qualities of the alternatives $\mathfrak{R}_i$ with respect to attributes $\mathfrak{S}_j$ are expressed by q-ROFVNNs with $q = 3$. Subsequently, we will use the proposed algorithm to choose the best contractor for the bridge construction project as discussed below.

Step 1. The decision makers evaluated each alternative concerning its attributes using q-ROFVN values and assembled the below decision matrix as in Table 1.

Table 1. The original decision matrix.

	$\Re_1$	$\Re_2$
$\mathfrak{S}_1$	⟨(0.9 , 0.3) , (0.6 , 0.9) ,(0.5 , 0.7) ⟩	⟨(0.9 , 0.6),(0.5 , 0.7),(0.6 , 0.8) ⟩
$\mathfrak{S}_2$	⟨(0.6 , 0.1) , (0.5 , 0.9) ,(0.8 , 0.7) ⟩	⟨(0.8 , 0.5),(0.4 , 0.8),(0.5 , 0.7) ⟩
$\mathfrak{S}_3$	⟨(0.7 , 0.3) , (0.3 , 0.7) ,(0.1 , 0.4) ⟩	⟨(0.6 , 0.8),(0.3 ,0.9),(0.7 , 0.8) ⟩
$\mathfrak{S}_4$	⟨(0.5 , 0.3) , (0.2 , 0.5) ,(0.8 , 0.7) ⟩	⟨(0.6 , 0.7),(0.8 , 0.2),(0.7 , 0.4) ⟩
$\mathfrak{S}_5$	⟨(0.5 , 0.3) , (0.8 , 0.7) ,(0.6 , 0.7) ⟩	⟨(0.8 , 0.7), (0.6 , 0,8),(0.6, 0.9) ⟩
	$\Re_3$	$\Re_4$
$\mathfrak{S}_1$	⟨(0.5 , 0.1), (0.8 , 0.5),(0.5 , 0.4) ⟩	⟨(0.6 , 0.2),(0.9 , 0.6), (0.5 , 0.5) ⟩
$\mathfrak{S}_2$	⟨(0.8 , 0.6), (0.6 , 0.8), (0.4 , 0.7) ⟩	⟨(0.8 ,0.7),(0.5 ,0.3), (0.2 , 0.7) ⟩
$\mathfrak{S}_3$	⟨(0.7 , 0.2), (0.3 , 0.5),(0.7 , 0.1) ⟩	⟨(0.5 , 0.6),(0.7 , 0.9),(0.1 , 0.2) ⟩
$\mathfrak{S}_4$	⟨(0.6 , 0.5), (0.8 , 0.7),(0.2 , 0.4) ⟩	⟨(0.7 , 0.8),(0.2 , 0.9), (0.5, 0.7) ⟩
$\mathfrak{S}_5$	⟨(0.5 , 0.1), (0.7 ,0.8) ,(0.5 , 0.6) ⟩	⟨(0.8 , 0.2), (0.7 , 0.8), (0.9, 0.1) ⟩

Step 2. We normalize the original decision matrix by taking the complement of the cost attribute, which is $\mathfrak{S}_4$ in our case study. Table 2 portrays the normalized decision matrix.

Table 2. The normalized decision matrix.

	$\Re_1$	$\Re_2$
$\mathfrak{S}_1$	⟨(0.9 , 0.3) , (0.6 , 0.9) ,(0.5 , 0.7) ⟩	⟨(0.9 , 0.6),(0.5 , 0.7),(0.6 , 0.8) ⟩
$\mathfrak{S}_2$	⟨(0.6 , 0.1) , (0.5 , 0.9) ,(0.8 , 0.7) ⟩	⟨(0.8 , 0.5),(0.4 , 0.8),(0.5 , 0.7) ⟩
$\mathfrak{S}_3$	⟨(0.7 , 0.3) , (0.3 , 0.7) ,(0.1 , 0.4) ⟩	⟨(0.6 , 0.8),(0.3 ,0.9),(0.7 , 0.8) ⟩
$\mathfrak{S}_4$	⟨(0.8 , 0.7) , (0.5 , 0.2) ,(0.5 , 0.3) ⟩	⟨(0.7 , 0.4),(0.2 , 0.8),(0.6 , 0.7) ⟩
$\mathfrak{S}_5$	⟨(0.5 , 0.3) , (0.8 , 0.7) ,(0.6 , 0.7) ⟩	⟨(0.8 , 0.7), (0.6 , 0,8),(0.6, 0.9) ⟩
	$\Re_3$	$\Re_4$
$\mathfrak{S}_1$	⟨(0.5 , 0.1), (0.8 , 0.5),(0.5 , 0.4) ⟩	⟨(0.6 , 0.2),(0.9 , 0.6), (0.5 , 0.5) ⟩
$\mathfrak{S}_2$	⟨(0.8 , 0.6), (0.6 , 0.8), (0.4 , 0.7) ⟩	⟨(0.8 ,0.7),(0.5 ,0.3), (0.2 , 0.7) ⟩
$\mathfrak{S}_3$	⟨(0.7 , 0.2), (0.3 , 0.5),(0.7 , 0.1) ⟩	⟨(0.5 , 0.6),(0.7 , 0.9),(0.1 , 0.2) ⟩
$\mathfrak{S}_4$	⟨(0.2 , 0.4), (0.7 , 0.8),(0.6 , 0.5) ⟩	⟨(0.5 , 0.7),(0.9 , 0.2), (0.7, 0.8) ⟩
$\mathfrak{S}_5$	⟨(0.5 , 0.1), (0.7 ,0.8) ,(0.5 , 0.6) ⟩	⟨(0.8 , 0.2), (0.7 , 0.8), (0.9, 0.1) ⟩

Step 3. In this step we use Eq (19) to find the q-ROFVNWA operator for each alternative. The resulting values are given below.
$\mathfrak{L}_1 = \langle(0.7615, 0.3466), (0.5674, 0.7548), (0.4712, 0.6164)\rangle$,
$\mathfrak{L}_2 = \langle(0.7951, 0.5623), (0.3728, 0.7998), (0.5983, 0.814)\rangle$,
$\mathfrak{L}_3 = \langle(0.5509, 0.1943), (0.6504, 0.748), (0.5341, 0.5371)\rangle$,
and $\mathfrak{L}_4 = \langle(0.6856, 0.3684), (0.7675, 0.6906), (0.5125, 0.6283)\rangle$.

Step 4. The score value of each alternative is calculated. We obtained $\Pi(\mathfrak{L}_1) = 0.259, \Pi(\mathfrak{L}_2) = 0.37, \Pi(\mathfrak{L}_3) = 0.1019$ and $\Pi(\mathfrak{L}_4) = 0.0877$.

Step 5. From Step 4, the ranking results are $\Re_2 \geq \Re_1 \geq \Re_3 \geq \Re_4$.

The q-ROFVNWG operator can be used in Step 3. The results are given below.
$\mathfrak{L}_1 = \langle(0.682, 0.5103), (0.6494, 0.5183), (0.5756, 0.5133)\rangle$,
$\mathfrak{L}_2 = \langle(0.7646, 0.6248), (0.4725, 0.7881), (0.6048, 0.7857)\rangle$,

$\mathfrak{L}_3 = \langle(0.4117, 0.3543), (0.7011, 0.6948), (0.5558, 0.4447)\rangle$, and
$\mathfrak{L}_4 = \langle(0.6258, 0.5628), (0.8271, 0.4571), (0.7491, 0.3352)\rangle$.
Then, Step 4 is applied to find the score value of each alternative. We obtain $\Pi(\mathfrak{L}_1) = -0.0019, \Pi(\mathfrak{L}_2) = 0.2836, \Pi(\mathfrak{L}_3) = -0.0226$ and $\Pi(\mathfrak{L}_4) = -0.2621$.
According to Step 4, the ranking results are $\mathfrak{R}_2 \geq \mathfrak{R}_1 \geq \mathfrak{R}_3 \geq \mathfrak{R}_4$. We can see that the ranking results are the same for both proposed operators.

6. Discussion and comparative analysis

This section encompasses three parts. In part 1, we discuss the stability of the proposed operators while taking different q values. Part 2 presents the performances of the proposed q-ROFVNWA and q-ROFVNWG operators by using different types of SFs. Comparison of the proposed method with existing methods is provided in part 3.

6.1. Accuracy analysis

In this part, we examine the stability and reliability of derived approaches by changing the values of q . Here, we check q-ROFVNWA and q-ROFVNWG operators along with SF and see the ranking results. Tables 3 and 4 exhibit the ranking positions of the alternatives according to parameter q.

Table 3. Ranking of the q-ROFVNWA operator with the values of the parameter q.

q	SF	Order of alternatives
$q = 3$	$\Pi(\mathfrak{L}_1) = 0.259, \Pi(\mathfrak{L}_2) = 0.37, \Pi(\mathfrak{L}_3) = 0.1019, \Pi(\mathfrak{L}_4) = 0.0877$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_4$
$q = 5$	$\Pi(\mathfrak{L}_1) = 0.1974, \Pi(\mathfrak{L}_2) = 0.2972, \Pi(\mathfrak{L}_3) = 0.0713, \Pi(\mathfrak{L}_4) = 0.066$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_4$
$q = 7$	$\Pi(\mathfrak{L}_1) = 0.1413, \Pi(\mathfrak{L}_2) = 0.2185, \Pi(\mathfrak{L}_3) = 0.0491, \Pi(\mathfrak{L}_4) = 0.0467$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_4$
$q = 10$	$\Pi(\mathfrak{L}_1) = 0.0867, \Pi(\mathfrak{L}_2) = 0.134, \Pi(\mathfrak{L}_3) = 0.0269, \Pi(\mathfrak{L}_4) = 0.0285$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_4 > \mathfrak{R}_3$
$q = 13$	$\Pi(\mathfrak{L}_1) = 0.0556, \Pi(\mathfrak{L}_2) = 0.0836, \Pi(\mathfrak{L}_3) = 0.0143, \Pi(\mathfrak{L}_4) = 0.0179$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_4 > \mathfrak{R}_3$
$q = 15$	$\Pi(\mathfrak{L}_1) = 0.0423, \Pi(\mathfrak{L}_2) = 0.0619, \Pi(\mathfrak{L}_3) = 0.0093, \Pi(\mathfrak{L}_4) = 0.0133$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_4 > \mathfrak{R}_3$
$q = 20$	$\Pi(\mathfrak{L}_1) = 0.0227, \Pi(\mathfrak{L}_2) = 0.031, \Pi(\mathfrak{L}_3) = 0.0031, \Pi(\mathfrak{L}_4) = 0.0065$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_4 > \mathfrak{R}_3$
$q = 40$	$\Pi(\mathfrak{L}_1) = 0.0025, \quad \Pi(\mathfrak{L}_2) = 0.003, \Pi(\mathfrak{L}_3) = 0, \Pi(\mathfrak{L}_4) = 0.0005$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_4 > \mathfrak{R}_3$

Tables 3 and 4 provide a comprehensive overview of the ranking results obtained at different values of q. Our analysis encompasses a wide range of q values between 3 and 40, revealing a consistent and robust optimal solution throughout this entire range. This remarkable stability serves as a testament to the reliability and effectiveness of the proposed method.

Table 4. Ranking of the q-ROFVNWG operator with the values of the parameter q.

q	SF	Order of alternatives
$q = 3$	$\Pi(\mathfrak{L}_1) = -0.0019, \Pi(\mathfrak{L}_2) = 0.2836, \Pi(\mathfrak{L}_3) = -0.0226, \Pi(\mathfrak{L}_4) = -0.2621$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_4$
$q = 5$	$\Pi(\mathfrak{L}_1) = -0.0163, \Pi(\mathfrak{L}_2) = 0.2133, \Pi(\mathfrak{L}_3) = -0.019, \Pi(\mathfrak{L}_4) = -0.2155$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_4$
$q = 7$	$\Pi(\mathfrak{L}_1) = -0.0164, \Pi(\mathfrak{L}_2) = 0.1433, \Pi(\mathfrak{L}_3) = -0.0124, \Pi(\mathfrak{L}_4) = -0.1655$	$\mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_4$
$q = 10$	$\Pi(\mathfrak{L}_1) = -0.0109, \Pi(\mathfrak{L}_2) = 0.0732, \Pi(\mathfrak{L}_3) = -0.0063, \Pi(\mathfrak{L}_4) = -0.111$	$\mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_4$
$q = 13$	$\Pi(\mathfrak{L}_1) = -0.0063, \Pi(\mathfrak{L}_2) = 0.036, \Pi(\mathfrak{L}_3) = -0.0033, \Pi(\mathfrak{L}_4) = -0.0761$	$\mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_4$
$q = 15$	$\Pi(\mathfrak{L}_1) = -0.0042, \Pi(\mathfrak{L}_2) = 0.0222, \Pi(\mathfrak{L}_3) = -0.0022, \Pi(\mathfrak{L}_4) = -0.06$	$\mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_4$
$q = 20$	$\Pi(\mathfrak{L}_1) = -0.0015, \Pi(\mathfrak{L}_2) = 0.0066, \Pi(\mathfrak{L}_3) = -0.0007, \Pi(\mathfrak{L}_4) = -0.034$	$\mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_4$
$q = 40$	$\Pi(\mathfrak{L}_1) = 0, \quad \Pi(\mathfrak{L}_2) = 0, \quad \Pi(\mathfrak{L}_3) = 0, \quad \Pi(\mathfrak{L}_4) = -0.0039$	$\mathfrak{R}_2 = \mathfrak{R}_1 = \mathfrak{R}_3 > \mathfrak{R}_4$

Upon closer examination of Table 3, it is evident that $\mathfrak{R}2$ emerges as the dominant alternative, closely followed by $\mathfrak{R}1$. Meanwhile, $\mathfrak{R}3$ lags behind these alternatives and switches roles with $\mathfrak{R}4$. Notably, a clear inverse relationship is observed between the values of q and the corresponding scores. As q increases, the scores decrease, eventually converging towards zero. In Table 4, we observe that the optimal solution remains consistent with $\mathfrak{R}2$ when using the q-ROFVNWG operator. However, it is worth noting that as the values of q increase, the score of $\mathfrak{R}2$ decreases, while the scores of the other alternatives increase. Ultimately, all alternative scores converge to zero. It is to be noted that if we obtain the same score value of two alternatives, we refer to the accuracy value, Definition 3.8. Figures 2 and 3 show the tendency of the ranking of the alternatives produced by the q-ROFVNWA and q-ROFVNWG operators as discussed in Tables 3 and 4.

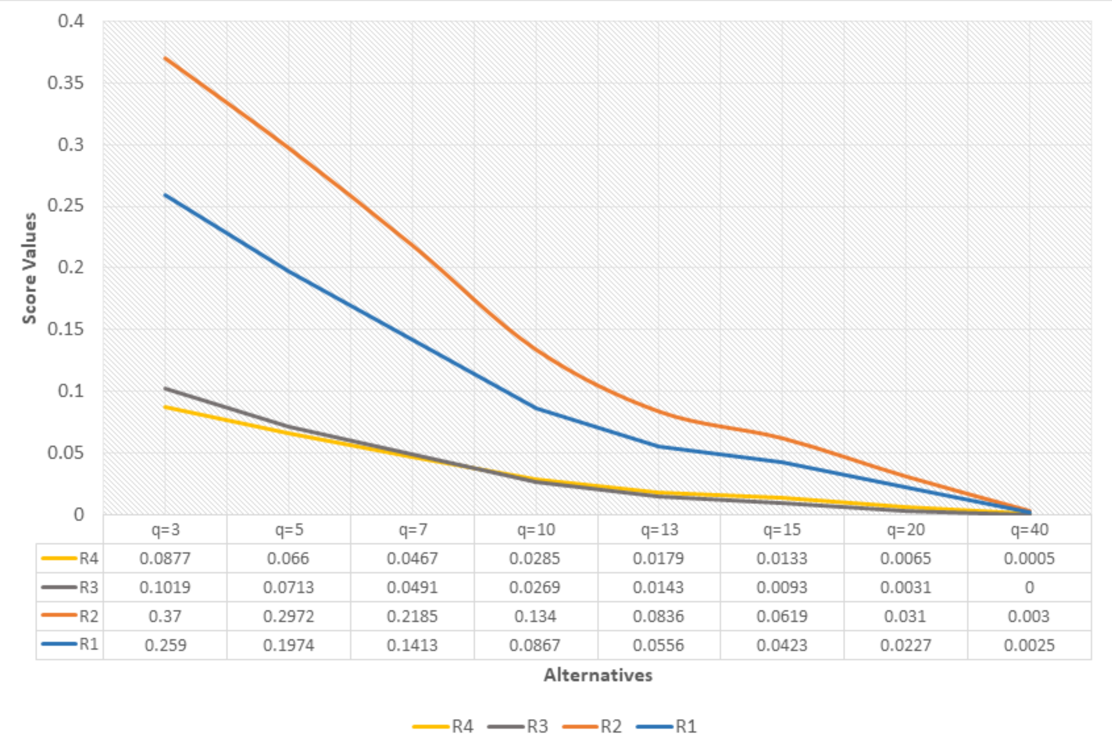


Figure 2. Accuracy of the proposed q-ROFVNWA operator with different q values.

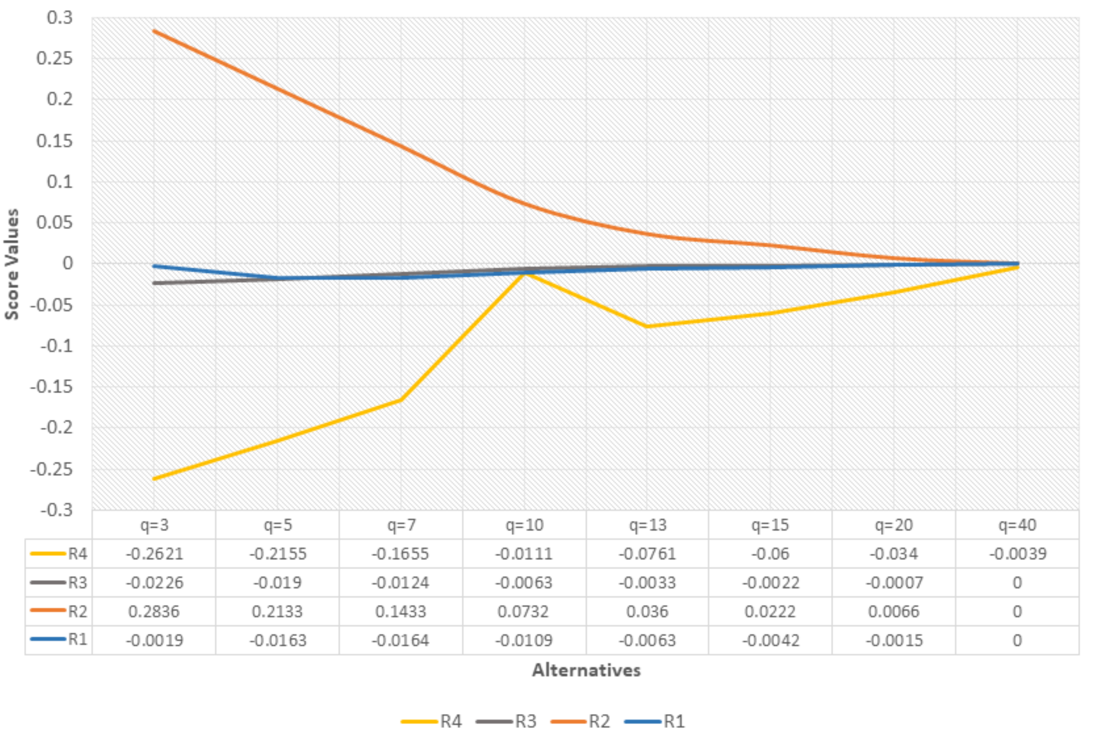


Figure 3. Accuracy of the proposed q-ROFVNWG operator with different q values.

6.2. *Performances of the proposed operators and SFs*

In Section 5.1, we utilized the q-ROFVNWA and q-ROFVNWG operators in conjunction with SF to address the decision making problem. In this part, we employ the same operators with QSF to solve the same decision-making problem mentioned earlier. By employing QSF with both the q-ROFVNWA and q-ROFVNWG operators, we obtained the subsequent outcomes.

For the q-ROFVNWA operator, we get $\Omega(\mathfrak{L}_1) = 0.1296, \Omega(\mathfrak{L}_2) = 0.2417, \Omega(\mathfrak{L}_3) = 0.0427$ and $\Omega(\mathfrak{L}_4) = 0.0163$. Thus, the ranking results are $\mathfrak{R}_2 \geq \mathfrak{R}_1 \geq \mathfrak{R}_3 \geq \mathfrak{R}_4$.

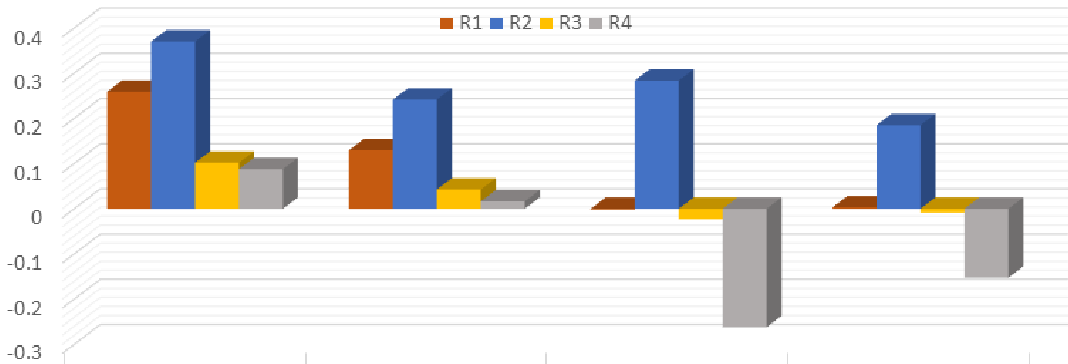
For the q-ROFVNWG operator, we get $\Omega(\mathfrak{L}_1) = 0.0031, \Omega(\mathfrak{L}_2) = 0.185, \Omega(\mathfrak{L}_3) = -0.0084$ and $\Omega(\mathfrak{L}_4) = -0.1527$. Thus, the ranking results are $\mathfrak{R}_2 \geq \mathfrak{R}_1 \geq \mathfrak{R}_3 \geq \mathfrak{R}_4$.

Table 5 summarizes the obtained results employing the proposed operators and SFs.

Table 5 clearly shows that the ranking results are exactly the same for all the proposed models, which reveals the consistency and accuracy of measures. Figure 4 illustrates the performance of the proposed operators and SFs, as presented in Table 5.

Table 5. Performances of the proposed operators and SFs.

Proposed Operators	Score Values	Order of alternatives
q-ROFVNWA (SF)	$\Pi(\mathfrak{L}_1) = 0.259, \Pi(\mathfrak{L}_2) = 0.37, \Pi(\mathfrak{L}_3) = 0.1019, \Pi(\mathfrak{L}_4) = 0.0877$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_4$
q-ROFVNWA (QSF)	$\Omega(\mathfrak{L}_1) = 0.1296, \Omega(\mathfrak{L}_2) = 0.2417, \Omega(\mathfrak{L}_3) = 0.0427, \Omega(\mathfrak{L}_4) = 0.0163$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_4$
q-ROFVNWG (SF)	$\Pi(\mathfrak{L}_1) = -0.0019, \Pi(\mathfrak{L}_2) = 0.2836, \Pi(\mathfrak{L}_3) = -0.0226, \Pi(\mathfrak{L}_4) = -0.2621$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_4$
q-ROFVNWG (QSF)	$\Omega(\mathfrak{L}_1) = 0.0031, \Omega(\mathfrak{L}_2) = 0.185, \Omega(\mathfrak{L}_3) = -0.0084, \Omega(\mathfrak{L}_4) = -0.1527$	$\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_3 > \mathfrak{R}_4$



	q-ROFVNWA (SF)	q-ROFVNWA (QSF)	q-ROFVNWG (SF)	q-ROFVNWG (QSF)
R1	0.259	0.1296	-0.0019	0.0031
R2	0.37	0.2417	0.2836	0.185
R3	0.1019	0.0427	-0.0226	-0.0084
R4	0.0877	0.0163	-0.2621	-0.1527

Figure 4. A visual depiction of the information presented in Table 5.

6.3. Comparison of the proposed method with existing methods

In this section, we will compare the proposed method with other commonly used approaches and discuss their strengths and weaknesses to determine the effectiveness of the presented method.

Besides the q-ROFVN model, there are also some other models proposed in the literature to address the MCDM problems. Among them, we present for their relevance in this comparison, the IFS [7], PyFS [11], q-ROFS [22], SNS [38], IFVNS [53] and PyFVNS [61]. In what follows, let us give some comparison analysis over these variant models. In order to conduct the comparison, we try to apply the above mentioned models to the same data presented in Section 5.1. In this comparison υ, ξ, γ refer to MF, IMF and NMF degrees, respectively.

First, IFS is characterized by υ and γ, where $\upsilon + \gamma \leq 1$. The AOs of this model are proposed under this condition. In circumstances where $\upsilon + \gamma > 1$, these AOs fail to give the requested outcomes in such situation. Moreover, this model is not prepared to handle the indeterminate situations. Thus, it can not be applied to solve the DM problem in Section 5.1.

Second, PyFS came to enlarge the space of IFS, although in a limited fashion, with the condition $\upsilon^2 + \gamma^2 \leq 1$. For example, if we pick the value (6, 9) from Table 2, according to this condition $6^2 + 9^2 = 0.36 + 0.81 = 1.17 > 1$. Thus, such model can not handle the DM problem at hand.

Third, q-ROFS replaces the conditions of IFS and PyFS with the constraint $\upsilon^q + \gamma^q \leq 1,\ q \geq 1$. Obviously, the q-ROFS can express efficiently such kinds of data, i.e., $6^3 + 9^3 = 0.216 + 0.729 = 0.945 < 1$ ($q = 3$). However, q-ROFS ignores indeterminacy circumstances, which makes it unable to be a descriptor of the data given in Table 2.

Fourth, SNS has three components, υ, ξ and γ, for MF, IMF and NMF degrees, respectively. Each component is represented by a single value, while the q-ROFVNS is constructed by considering q-ROF values instead of single values for the MF, IMF and NMF degrees. It can be seen that the SVNS cannot model the data presented in Table 2, as its memberships are unable to express two dimensional data. However, the structure of q-ROFVNS provides the ability to describe these data, as its memberships are two-dimensional.

Fifth, IFVNS serves as a generalized form of SNS, incorporating three membership functions, namely, υ, ξ and γ, each encompassing an IF value subject to the condition $\upsilon + \gamma \leq 1$. While IFVNS shares the same construction as q-ROFVNS, it imposes different conditions. However, the limitations of IFVNS become apparent when confronted with certain data, as exemplified in Table 2. Notably, data such as $< (0.8\ ,\ 0.7)\ ,\ (0.5\ ,\ 0.2)\ , (0.5\ ,\ 0.3) >$ cannot be effectively represented by IFVNS, as the condition $\upsilon + \gamma \leq 1$ is violated in the case of (0.8, 0.7). Conversely, q-ROFVNS exhibits exceptional flexibility, effortlessly accommodating such data within its adaptable conditions.

Finally, PyFVNS emerged as a pivotal expansion of the IFVNS domain with the condition $(\upsilon)^2 + (\gamma)^2 \leq 1$ for each of its MFs. This revised condition significantly broadens the scope of data that can be effectively accommodated, surpassing the limitations of IFVNS. For instance, when considering the data set $< (0.7, 0.4), (0.2, 0.8), (0.6, 0.7) >$ from Table 2, it becomes evident that IFVNS fails to capture the essence of such data. Conversely, PyFVNS effortlessly represents this type of data. However, in our case study, there exist some data that cannot be adequately described by PyFVNS, such as the data set $< (0.8, 0.7), (0.6, 0.8), (0.6, 0.9) >$. This necessity led to the exploration of a new model, namely, q-ROFVNS, specifically designed to handle such data. In q-ROFVNS, each membership function consists of q-ROF value with the condition $(\upsilon)^q + (\gamma)^q \leq 1$, where $q \geq 1$. Notably, q-ROFVNS proves to be more comprehensive, encompassing IFS and PyFS as special cases (when q = 1 and q = 2, respectively). To effectively represent the value $< (0.8, 0.7), (0.6, 0.8), (0.6, 0.9) >$ using q-ROFVNS, the parameter q is increased to q = 3. It is important to note that as the rung q increases, the acceptable orthopair space expands, allowing for a greater number of orthopairs to satisfy the bounding constraint. Consequently, q-ROFVNS enables the expression of a wider range of fuzzy information. In essence, the flexibility of q-ROFVNS lies in dynamically adjusting the value of parameter q to determine the range of information expression. This flexibility renders q-ROFVNS more suitable for effectively describing uncertain information. Table 6 compares current models based on suitable criteria, including the existence of three membership degrees, representation as 2D information in each degree, existence of constraints on 2D information in each degree, degree of flexibility of the constraints, and ranking values.

Table 6. Comparative analysis of current models based on relevant criteria.

Method	Existence of three membership degrees	Representation as 2D information in each degree	Existence of constraints on 2D information in each degree	Degree of flexibility of the constraints	Ranking values
IFS [7]	x	x	Non-applicable	Non-applicable	Non-computable
PyFS [11]	x	x	Non-applicable	Non-applicable	Non-computable
q-ROFS [22]	x	x	Non-applicable	Non-applicable	Non-computable
SNS [38]	✓	x	Non-applicable	Non-applicable	Non-computable
IFVNS [53]	✓	✓	✓	Low	Non-computable
PyFVNS [61]	✓	✓	✓	Mid	Non-computable
The proposed method	✓	✓	✓	High	Algorithmic

7. Conclusions

This manuscript provides a thorough and comprehensive analysis of the proposed theory of q-ROFVNS, showcasing its remarkable capacity to encompass and generalize prevailing methodologies. q-ROFVNS stands as a profound advancement, enabling a more precise representation of indeterminate information and facilitating the simulation of intricate decision-making scenarios through the strategic incorporation of the novel q-ROFS model in the construction of SNS. The manuscript expounds upon the formal definition of q-ROFVNS, accompanied by the development of operational laws and a comprehensive delineation of diverse aggregation operators within the q-ROFVN environment. Rigorous verification of the properties inherent to these operators is undertaken. Moreover, an innovative MADM methodology is meticulously devised, hinging on the proposed operators and the adept utilization of SFs. A numerical application is conducted to rank various construction contractors, utilizing q-ROFVNNs to assess their performance across different features. Subsequently, q-ROFVNWA and q-ROFVNWG operators are applied to aggregate attribute values, while SF is employed to derive ranking results. A rigorous examination of the robustness and dependability of the proposed q-ROFVNWA and q-ROFVNWG operators and SF methodologies is conducted by systematically varying the values of q. Remarkably, we observed that the optimal solution remained unaltered throughout these variations, thereby unequivocally affirming the unwavering stability and resilience of the proposed techniques. We also examined the proposed QSF. To evaluate its efficacy, we tackled the same numerical application that was previously addressed using the q-ROFVNWA and q-ROFVNWG operators and SF methodologies. We discovered that the ranking outcomes obtained from all the proposed models were wholly identical. This remarkable consistency unequivocally attests to the unwavering accuracy and precision exhibited by these measures. Furthermore, this manuscript facilitated a comparative examination of the proposed models in relation to the existing models, employing a detailed and insightful discussion to elucidate and interpret the findings. In this research, we strive to tackle more complex decision making problems. However, it is important to acknowledge the limitations of our proposed work. We have solely considered the evaluation information provided by q-ROFVNS, whereas in practical decision problems, decision makers have the ability to employ hybrid evaluation methods that incorporate features of bipolar fuzzy hypersoft sets [62], T-spherical fuzzy sets [63], hesitant q-rung orthopair fuzzy sets [64], and interval-valued neutrosophic sets [65]. These hybrid methods can effectively capture the vagueness and uncertainties present in complex data. Furthermore, our study has focused exclusively on two aggregation operators, namely, the q-ROFVNWA and q-ROFVNWG operators. To broaden the scope of our research, future investigations should explore other generalizations of q-ROFVNS such as q-rung orthopair bipolar neutrosophic sets, T-spherical fuzzy valued neutrosophic sets, hesitant q-rung orthopair neutrosophic sets and interval-valued q-ROFVNS. Additionally, the proposed operators could be extended to incorporate Heronian mean, Yager's ordered weighted averaging, Hamacher product, Einstein product, Choquet average and Dombi's aggregation operators.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

Authors declare no conflicts of interest.

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