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Research article

On rough generalized Marcinkiewicz integrals along surfaces of revolution on product spaces

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Abstract: In this paper, we prove the L^p boundedness of generalized Marcinkiewicz operators along surfaces of revolution on product spaces under very weak conditions on the the singular kernels. Our results generalize and improve many previously known results.

Keywords: singular kernels; Marcinkiewicz integrals; generalized Marcinkiewicz; product domains; surfaces of revolution
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1. Introduction

Throughout this paper, let $N \geq 2$ ($N = \kappa$ or τ) and $\mathbb{R}^N$ be the Euclidean space of dimension N. Let $\mathbb{S}^{N-1}$ be the unit sphere in $\mathbb{R}^N$ equipped with the normalized Lebesgue surface measure $d\sigma_{_N}(\cdot)$.

For $\eta_1 = a_1 + ib_1, \eta_2 = a_2 + ib_2$ $(a_1, b_1, a_2, b_2 \in \mathbb{R}$ *with* $a_1, a_2 > 0)$, let

$$\mathcal{K}_{\mho,h}(x,y) = \frac{h(|x|\,,|y|)\mho(x,y)}{|x|^{\kappa-\eta_1}|y|^{\tau-\eta_2}}$$

where h is a measurable function defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and $\mho$ is a measurable function defined on $\mathbb{R}^\kappa \times \mathbb{R}^\tau$, which is integrable over $\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1}$ with the following properties:

$$\mho(tx, sy) = \mho(x,y), \quad \forall\, t, s > 0 \tag{1.1}$$

and

$$\int_{\mathbb{S}^{\kappa-1}} \mho(x,.)d\sigma_\kappa(x) = \int_{\mathbb{S}^{\tau-1}} \mho(.,y)d\sigma_\tau(y) = 0. \tag{1.2}$$

For an appropriate mapping $\Theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, we consider the generalized parametric Marcinkiewicz integral operator $\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}$ along the surface of revolution $\Gamma_\Theta(u,v) = (u, v, \Theta(|u|, |v|))$ given by

$$\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}(f)(x,y,z) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|F_{t,s}(f)(x,y,z)\right|^{\varepsilon} \frac{dtds}{ts}\right)^{1/\varepsilon}, \tag{1.3}$$

where $f \in C_0^\infty(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})$, $\varepsilon > 1$ and

$$F_{t,s}(f)(x,y,z) = \frac{1}{t^{\eta_1} s^{\eta_2}} \int_{|v|\leq s} \int_{|u|\leq t} \mathcal{K}_{\mho,h}(u,v) f(x-u, y-v, z-\Theta(|u|, |v|))dudv.$$

We remark that the operator $\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}$ (in the two parameter setting) is a natural generalization of the Marcinkiewicz integral operator $\mathcal{G}^{\varphi,(\varepsilon)}_{\mho,h}$ along the surface of revolution $\Gamma_\varphi(u) = (u, \varphi(|u|))$ (in the one parameter setting), which is defined by

$$\mathcal{G}^{\varphi,(\varepsilon)}_{\mho,h}(f)(x, x_{\kappa+1}) = \left(\int_{\mathbb{R}_+} \left|\frac{1}{t^{\eta_1}} \int_{|u|\leq t} \frac{h(|u|)\mho(u)}{|u|^{\kappa-\eta_1}} f(x-u, x_{\kappa+1} - \varphi(|u|))du\right|^{\varepsilon} \frac{dt}{t}\right)^{1/\varepsilon}. \tag{1.4}$$

The study of the L^p boundedeness of the operator $\mathcal{G}^{\varphi,(2)}_{\mho,h}$ under various conditions on the functions $\mho, \varphi$, and h has received a large amount of attention by many authors. For a sample of past studies, we advise readers to refer to [1–10], among others.

The study of singular integrals on product spaces and the corresponding Marcinkiewicz integrals such as $\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}$, which may have singularities along subvarities, has attracted the attention of many authors in the past two decades. One of the principal motivations for the study of such operators is the requirement of several complex variables and large classes of "subelliptic" equations. For more background information, readers may refer to [10–12].

Our main focus in this paper will be on the operator $\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}$. When $\Theta \equiv 0$, $h \equiv 1$, $\eta_1 = 1 = \eta_2$, and $\varepsilon = 2$, we denote $\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}$ by $\mathcal{M}_\mho$, which is essentially the classical Marcinkiewicz integral on product spaces. The study of L^p boundedness of the operator $\mathcal{M}_\mho$ has attracted the attention of many authors. For a sample of previous studies and more information about the applications as well as development of the integral operator $\mathcal{G}^{(2)}_{\Theta,\mho,h}$, we consult the readers to refer to [13–19] and the references therein. Let us now recall some pertinent results to our current study. In [13], the authors proved the L^p boundedness of $\mathcal{M}_\mho$ for all $p \in (1,\infty)$ under the assumption $\mho \in L(\log L)(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1})$. In addition, they pointed out the condition $\mho \in L(\log L)(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1})$ is optimal in the sense that the L^2 boundedness of $\mathcal{M}_\mho$ may that not hold if we replace this condition by any weaker condition $\mho \in L(\log L)^\alpha(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})$ with $\alpha \in (0,1)$. Also, in [14] the author showed that $\mathcal{M}_\mho$ is bounded on $L^p(\mathbb{R}^\kappa \times \mathbb{R}^\tau)$ for all $p \in (1,\infty)$, provided that $\mho \in B_q^{(0,0)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1})$ with $q > 1$. Moreover, they proved that the condition $\mho \in B_q^{(0,0)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1})$ is optimal in the sense that if we replace this condition by a weaker condition $\mho \in B_q^{(0,\alpha)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1})$ with $\alpha \in (-1,0)$, then the operator $\mathcal{M}_\mho$ may not be bounded on $L^2(\mathbb{R}^\kappa \times \mathbb{R}^\tau)$. Here, $B_q^{(0,\alpha)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1})$ is a special class of block spaces introduced in [20].

In [21], the authors proved the L^p boundedness of $\mathcal{G}^{(2)}_{0,\mho,h}$ for all $|1/p - 1/2| < \min\{1/2, 1/\ell'\}$ if $\mho \in L(\log L)(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1}) \cup B_q^{(0,0)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1})$ and $h \in \nabla_\ell(\mathbb{R}_+ \times \mathbb{R}_+)$ with $\ell > 1$, where $\nabla_\ell(\mathbb{R}_+ \times \mathbb{R}_+)$ (for $\ell > 1$) is the class of measurable functions h such that

$$\|h\|_{\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)} = \sup_{j,k\in\mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} |h(l,r)|^\ell \frac{dldr}{lr}\right)^{1/\ell} < \infty.$$

Very recently, under the assumptions $\mho \in L(\log L)(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1}) \cup B_{\kappa}^{(0,0)}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})$ and $h \in \nabla_{\ell}(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\ell > 1$, the authors of [22] established the L^p boundedness of $\mathcal{G}^{(2)}_{\Theta,\mho,h}$ for various classes of Θ.

On the other hand, the investigation of the boundedness of the generalized Marcinkiewicz integral operator $\mathcal{G}^{(\varepsilon)}_{0,\mho,h}$ and some of its extensions has attracted many authors. The readers may consult [23–27].

Recently, the authors of [28] proved that if either $\mho$ lies in $L(\log L)^{2/\varepsilon}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})$ or lies in $B_q^{(0,\frac{2}{\varepsilon}-1)}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})$, then the estimate

$$\left\|\mathcal{G}^{(\varepsilon)}_{0.\mho,1}(f)\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau)} \leq C_p \|f\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\tau)}$$

holds for all $p \in (1,\infty)$, where $\dot{F}_p^{\varepsilon,\vec{r}}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})$ is the homogeneous Triebel-Lizorkin space and its definition will be recalled in Section 2. This result was recently improved by the authors of [29]. Precisely, they established the L^p boundedness of $\mathcal{G}^{(\varepsilon)}_{0.\mho,h}$ provided that $h \in \nabla_{\ell}(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\ell > 1$ and $\mho$ belongs to either $L(\log L)^{2/\varepsilon}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})$ or to $B_{\kappa}^{(0,\frac{2}{\varepsilon}-1)}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})$.

In view of the results in [22] for the boundedness of Marcinkiewicz integral $\mathcal{G}^{(2)}_{\Theta,\mho,h}$ and of the results in [29] for the boundedness of the generalized Marcinkiewicz integral $\mathcal{G}^{(\varepsilon)}_{0,\mho,h}$, a question arises naturally is the following:

Question: Does the L^p boundedness of the operator $\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}$ hold under the conditions in [22] if $\varepsilon = 2$ is replaced by $\varepsilon > 1$?

The main purpose of this article is to answer the above question in the affirmative.

Let us present our main results. First, we present the conditions on Θ. Let **W** be the class of all functions $\Theta : \mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}$, which satisfies one of the following conditions (see [30]):

(a) $\Theta \in C^1(\mathbb{R}_+\times\mathbb{R}_+)$ such that for any fixed $l, r > 0$, we have $\varphi_l(.) = \Theta(l,.)$ and $\varphi_r(.) = \Theta(.,r)$ are in $C^2(\mathbb{R}_+)$, increasing and convex functions with $\varphi_l(0) = \varphi_r(0) = 0$.

(b) $\Theta(l,r) = \sum_{k=0}^{n}\sum_{j=0}^{m} C_{j,k} l^{\gamma_k} r^{\nu_j}$ $(\gamma_k, \nu_j > 0)$ is a generalized polynomial on $\mathbb{R}^2$.

(c) $\Theta(l,r) = \varphi_1(l) + \varphi_2(r)$, where $\varphi_k(\cdot)$ $(k = 1,2)$ is either a generalized polynomial or is in $C^2(\mathbb{R}_+)$, increasing and convex function with $\varphi_k(0) = 0$.

(d) $\Theta(l,r) = P(l)\varphi(r)$, where P is a generalized polynomial given by $P(l) = \sum_{k=0}^{n} C_k l^{\gamma_k}$ with $\gamma_k > 0$, and $\varphi \in C^2(\mathbb{R}_+)$, increasing and convex function with $\varphi(0) = 0$.

Model examples for functions Θ that are covered by the class **W** are $\Theta(l,r) = (e^{-1/l}+e^{-1/r})l^2r^2, (l,r > 0)$; $\Theta(l,r) = l^n r^m$ with $n,m > 0$; $\Theta(l,r) = P(l,r)$ is a polynomial; and $\Theta(l,r) = \varphi_1(r)\varphi_2(l)$, where each φ_j is in $C^2(\mathbb{R}_+)$ and a convex increasing function with $\varphi_j(0) = 0$.

In this article, our method of proof relies on obtaining some delicate estimates and following a similar argument as that employed in [28], which allows us to employ Yano's extrapolation argument so we can improve and extend the results in [13, 14, 21, 22, 28, 29]. In fact, we have the following results:

Theorem 1.1. *Let Θ belong to the class **W**, $h \in \nabla_{\ell}(\mathbb{R}_+\times\mathbb{R}_+)$ and $\mho \in L^q\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1}\right)$ with $\ell, q \in (1,2]$. Then there exists a positive constant $C_{p,\mho,h}$ such that the inequality*

$$\left\|\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}(f)\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \leq C_{p,\mho,h}\left(\frac{1}{(q-1)(\ell-1)}\right)^{2/\varepsilon} \|f\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \tag{1.5}$$

holds for all $p \in (\frac{\varepsilon\ell'}{\varepsilon+\ell'-1}, \frac{\varepsilon'\ell}{\varepsilon'-\ell})$ *if* $\varepsilon \leq \ell'$*, and it holds for all* $p \in (\ell', \infty)$ *if* $\varepsilon \geq \ell'$*, where* $C_{p,\mho,h} = C_p \|\mho\|_{L^q(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})} \|h\|_{\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)}$.

Theorem 1.2. *Suppose that* $\mho$ *lies in* $L^q\left(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1}\right)$ *for some* $q \in (1, 2]$ *and that* $h \in \nabla_\ell(\mathbb{R}_+ \times \mathbb{R}_+)$ *for some* $\ell \in (2, \infty)$*. If* Θ *belongs to the class* $\mathbf{W}$*, then the inequality*

$$\left\|\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}(f)\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \leq C_{p,\mho,h}\left(\frac{\ell}{q-1}\right)^{2/\varepsilon} \|f\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}$$

holds for all $p \in (1, \varepsilon)$ *if* $\varepsilon \leq \ell'$*, and it holds for all* $p \in (\ell', \infty)$ *if* $\varepsilon \geq \ell'$.

By the estimates in Theorems 1.1 and 1.2 and by employing the extrapolation argument of Yano (see [31, 32]) we obtain the following results:

Theorem 1.3. *Suppose that* $h \in \nabla_\ell(\mathbb{R}_+ \times \mathbb{R}_+)$ *for some* $\ell \in (1, 2]$ *and* $\Theta \in \mathbf{W}$.
(*i*) *If* $\mho \in L(\log L)^{2/\varepsilon}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1})$*, then we have*

$$\left\|\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}(f)\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \leq C_p \|h\|_{\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)}\left(1 + \|\mho\|_{L(logL)^{2/\varepsilon}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}\right) \|f\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}$$

for $p \in (\ell', \infty)$ *if* $\varepsilon \geq \ell'$ *and for* $p \in (\frac{\varepsilon\ell'}{\varepsilon+\ell'-1}, \frac{\varepsilon'\ell}{\varepsilon'-\ell})$ *if* $\varepsilon \leq \ell'$*;*
(*ii*) *If* $\mho \in B_q^{(0,\frac{2}{\varepsilon}-1)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1})$ *for some* $q > 1$*, then the inequality*

$$\left\|\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}(f)\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \leq C_p \|h\|_{\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)}\left(1 + \|\mho\|_{B_q^{(0,\frac{2}{\varepsilon}-1)}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}\right) \|f\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}$$

holds for $p \in (\ell', \infty)$ *if* $\varepsilon \geq \ell'$ *and it holds for* $p \in (\frac{\varepsilon\ell'}{\varepsilon+\ell'-1}, \frac{\varepsilon'\ell}{\varepsilon'-\ell})$ *if* $\varepsilon \leq \ell'$.

Theorem 1.4. *Let* $\Theta \in \mathbf{W}$*,* $h \in \nabla_\ell(\mathbb{R}_+ \times \mathbb{R}_+)$ *with* $\ell \in (2, \infty)$ *and* $\mho \in L(\log L)^{2/\varepsilon}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1}) \cup B_q^{(0,\frac{2}{\varepsilon}-1)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1})$ *with* $q > 1$*. Then the generalized Marcinkiewicz operator* $\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}$ *is bounded on* $L^p(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})$ *for* $p \in (1, \varepsilon)$ *if* $\varepsilon \leq \ell'$*, and for* $p \in (\ell', \infty)$ *if* $\varepsilon \geq \ell'$.

Remarks:

(*i*) We notice that in the special case $\varepsilon = 2$, Theorems 1.3 and 1.4 recover the results obtained in [22]. Thus, our results improve the main results in [22].

(*ii*) We notice that Theorem 2.7 in [28] is obtained directly from Theorem 1.4 if we take $\Theta \equiv 0$ and $h \equiv 1$.

(*iii*) For the special case $\Theta \equiv 0$, Theorems 1.3 and 1.4 give the main results in [29]. Thus, our results generalize the results in [29].

(*iv*) For the special case $\Theta \equiv 0$, $\varepsilon = 2$, and $1 < \ell \leq 2$, Theorem 1.3 gives that $\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}$ is bounded for $p \in (\frac{\varepsilon\ell'}{\varepsilon+\ell'-1}, \frac{\varepsilon'\ell}{\varepsilon'-\ell})$, which essentially improves the results in [21] in which the authors showed that $\mathcal{G}^{(2)}_{\Theta,\mho,h}$ is bounded for $p \in (\frac{2\ell'}{\ell'-2}, \frac{2\ell}{2-\ell})$. Therefore, the range of p in Theorem 1.3 is better than the range of p obtained in [21].

(*v*) In Theorem 1.4, the conditions on $\mho$ are the weakest conditions in their respective classes for the case $\Theta \equiv 0$, $h \equiv 1$, and $\varepsilon = 2$ (see [13, 14]).

(*vi*) For the case $\varepsilon = \ell'$ with $2 < \ell < \infty$, Theorem 1.4 implies the boundedness of $\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}$ for all $p \in (1, \infty)$, which is the full range.

From now on, the constant C denotes a positive number that may vary at each occurrence but it is independent of the essential variables. Also, ℓ' denotes the exponent conjugate of ℓ, that is, $1/\ell' + 1/\ell = 1$.

2. Some definitions and lemmas

Let us start recalling the definition of the homogeneous Triebel-Lizorkin space $\dot{F}_p^{\varepsilon,\overrightarrow{r}}(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})$. Assume that $p, \varepsilon \in (1, \infty)$ and $\overrightarrow{r} = (\gamma, \nu) \in \mathbb{R} \times \mathbb{R}$. Then the homogeneous Triebel-Lizorkin space $\dot{F}_p^{\varepsilon,\overrightarrow{r}}(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})$ is the collection of all tempered distributions f on $\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R}$ satisfying

$$\|f\|_{\dot{F}_p^{\varepsilon,\overrightarrow{r}}(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})} = \left\| \left(\sum_{j,k\in\mathbb{Z}} 2^{j\gamma\varepsilon} 2^{k\nu\varepsilon} \left|(\psi_j \otimes \phi_k) * f\right|^\varepsilon \right)^{1/\varepsilon} \right\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})} < \infty$$

where $\widehat{\psi_j}(x) = 2^{-j\kappa} \mathcal{I}_\kappa(2^{-j}x)$ for $j \in \mathbb{Z}$, $\widehat{\phi_k}(y) = 2^{-k\tau} \mathcal{I}_\tau(2^{-k}y)$ for $k \in \mathbb{Z}$, and the radial functions $\mathcal{I}_\kappa \in C_0^\infty(\mathbb{R}^\kappa)$, $\mathcal{I}_\tau \in C_0^\infty(\mathbb{R}^\tau)$ satisfy the following:

(1) $0 \le \mathcal{I}_\kappa \le 1, \quad 0 \le \mathcal{I}_\tau \le 1$,

(2) $supp\,(\mathcal{I}_\kappa) \subset \left\{x : \frac{1}{2} \le |x| \le 2\right\}, \quad supp\,(\mathcal{I}_\tau) \subset \left\{y : \frac{1}{2} \le |y| \le 2\right\}$,

(3) there exists $C > 0$ such that $\mathcal{I}_\kappa(x), \mathcal{I}_\tau(y) \ge C$ for all $|x|, |y| \in [\frac{3}{5}, \frac{5}{3}]$,

(4) $\sum_{j\in\mathbb{Z}} \mathcal{I}_\kappa(2^{-j}x) = 1$ with $x \neq 0$ and $\sum_{k\in\mathbb{Z}} \mathcal{I}_\tau(2^{-k}y) = 1$ with $y \neq 0$.

The authors of [33] pointed out that the following properties hold:

(i) The Schwartz space $\mathcal{S}(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})$ is dense in $\dot{F}_p^{\varepsilon,\overrightarrow{r}}(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})$,

(ii) $\dot{F}_p^{2,\overrightarrow{0}}(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R}) = L^p(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})$ for $1 < p < \infty$,

(iii) $\dot{F}_p^{\varepsilon_1,\overrightarrow{r}}(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R}) \subseteq \dot{F}_p^{\varepsilon_2,\overrightarrow{r}}(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})$ if $\varepsilon_1 \le \varepsilon_2$.

For $\mu \ge 2$ and an appropriate function Θ on $\mathbb{R}_+ \times \mathbb{R}_+$, define the family of measures $\Upsilon_{\Theta,\mho,h,t,s} := \{\Upsilon_{t,s} : t, s \in \mathbb{R}_+\}$ and its corresponding maximal operators Υ_h^* and $\mathrm{M}_{h,\mu}$ on $\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R}$ by

$$\iiint_{\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R}} f d\Upsilon_{t,s} = \frac{1}{t^{\eta_1} s^{\eta_2}} \int_{1/2t \le |x| \le t} \int_{1/2s \le |y| \le s} f(x, y, \Theta(|x|, |y|)) \mathcal{K}_{\mho,h}(x, y) dx dy,$$

$$\Upsilon_h^*(f)(x, y, z) = \sup_{t,s\in\mathbb{R}_+} ||\Upsilon_{t,s}| * f(x, y, z)|,$$

and

$$\mathrm{M}_{h,\mu}(f)(x, y, z) = \sup_{j,k\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \int_{\mu^k}^{\mu^{k+1}} ||\Upsilon_{t,s}| * f(x, y, z)| \frac{dt ds}{ts}$$

where $|\Upsilon_{t,s}|$ is defined similarly to $\Upsilon_{t,s}$, but with replacing $\mho h$ by $|\mho h|$.

We shall need the following two lemmas from [22].

Lemma 2.1. *Let* $\mho \in L^q\left(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1}\right)$ *with* $1 < q \le 2$ *and* $h \in \nabla_\ell\,(\mathbb{R}_+ \times \mathbb{R}_+)$ *with* $\ell > 1$. *Assume that* Θ *belongs to the class* **W**. *Then the inequalities*

$$\|\Upsilon_h^*(f)\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})} \le C_{p,\mho,h} \|f\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})} \tag{2.1}$$

and

$$\|\mathrm{M}_{h,\mu}(f)\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})} \le C_{p,\mho,h} (\ln \mu)^2 \|f\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})} \tag{2.2}$$

hold for all $f \in L^p(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})$ *with* $p \in (\ell', \infty)$.

Lemma 2.2. *Let h, $\mho$ and Θ be given as in Lemma 2.1. Then the following are satisfied:*

$$\left\|\Upsilon_{t,s}\right\| \;\leq\; C_{p,\mho,h}, \tag{2.3}$$

$$\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\widehat{\Upsilon}_{t,s}(\zeta,\xi,\omega)\right|^2\frac{dtds}{ts} \;\leq\; C^2_{p,\mho,h}(\ln\mu)^2\left|\mu^k\zeta\right|^{\pm\frac{2\theta}{\ln(\mu)}}\left|\mu^j\xi\right|^{\pm\frac{2\theta}{\ln(\mu)}} \tag{2.4}$$

where $\theta < 1/(2q')$ and $\left\|\Upsilon_{t,s}\right\|$ is the total variation of $\Upsilon_{t,s}$.

Lemma 2.3. *Let $h \in \nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)$ and $\mho \in L^q\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1}\right)$ with $1 < \ell, q \leq 2$. Assume that $1 < \varepsilon \leq \ell'$ and that Θ belongs to the class* **W**. *Then the estimate*

$$\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{t,s}*\mathcal{H}_{j,k}\right|^\varepsilon\frac{dtds}{ts}\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \leq C_{p,\mho,h}(\ln\mu)^{2/\varepsilon}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}\right|^\varepsilon\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \tag{2.5}$$

holds for all $p \in (\frac{\varepsilon\ell'}{\varepsilon+\ell'-1}, \frac{\varepsilon'\ell}{\varepsilon'-\ell})$, where $\{\mathcal{H}_{j,k}(\cdot,\cdot,\cdot),\ j,k\in\mathbb{Z}\}$ is any set of functions on $\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R}$.

Proof. We shall follow a similar argument as that in [29]. We need to consider three cases:
Case 1. $p \in (\varepsilon, \frac{\varepsilon'\ell}{\varepsilon'-\ell})$. As $p/\varepsilon > 1$, by duality there exists a nonnegative function $\rho \in L^{(p/\varepsilon)'}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})$ with $\|\rho\|_{L^{(p/\varepsilon)'}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \leq 1$ and satisfies

$$\begin{aligned}&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{t,s}*\mathcal{H}_{j,k}\right|^\varepsilon\frac{dtds}{ts}\right)^{1/\varepsilon}\right\|^\varepsilon_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\\ &= \iiint_{\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R}}\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{t,s}*\mathcal{H}_{j,k}(x,y,z)\right|^\varepsilon\frac{dtds}{ts}\rho(x,y,z)dxdydz. \end{aligned}\tag{2.6}$$

By applying Hölder's inequality, we have

$$\begin{aligned}\left|\Upsilon_{t,s}*\mathcal{H}_{j,k}(x,y,z)\right|^\varepsilon \;\leq\;& C\,\|\mho\|^{(\varepsilon/\varepsilon')}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}\|h\|^{(\varepsilon/\varepsilon')}_{\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)}\int_{s/2}^{s}\int_{t/2}^{t}\iint_{\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1}}|\mho(u,v)|\\ &\times\;\left|\mathcal{H}_{j,k}(x-lu,y-rv,z-\Theta(l,r))\right|^\varepsilon d\sigma_\kappa(u)d\sigma_\tau(v)\,|h(l,r)|^{\varepsilon-\frac{\varepsilon\ell}{\varepsilon'}}\frac{dldr}{lr}.\end{aligned}\tag{2.7}$$

Hence, by (2.6) and (2.7) and Hölder's inequality, we get

$$\begin{aligned}&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{t,s}*\mathcal{H}_{j,k}\right|^\varepsilon\frac{dtds}{ts}\right)^{1/\varepsilon}\right\|^\varepsilon_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \leq C\,\|h\|^{(\varepsilon/\varepsilon')}_{\nabla_1(\mathbb{R}_+\times\mathbb{R}_+)}\|\mho\|^{(\varepsilon/\varepsilon')}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}\\ &\times\;\iiint_{\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R}}\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}(x,y,z)\right|^\varepsilon\right)\mathrm{M}_{|h|^{\varepsilon-\frac{\varepsilon\ell}{\varepsilon'}},\mu}(\overline{\rho})(-x,-y,-z)dxdydz\\ &\leq\; C\,\|h\|^{(\varepsilon/\varepsilon')}_{\nabla_1(\mathbb{R}_+\times\mathbb{R}_+)}\|\mho\|^{(\varepsilon/\varepsilon')}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}\left\|\mathrm{M}_{|h|^{\frac{\varepsilon(\varepsilon'-\ell)}{\varepsilon'}},\mu}(\overline{\rho})\right\|_{L^{(p/\varepsilon)'}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\left\|\sum_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}\right|^\varepsilon\right\|_{L^{(p/\varepsilon)}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\end{aligned}$$

where $\overline{\rho}(x,y,z)=\rho(-x,-y,-z)$. Since $|h|^{\frac{\varepsilon(\varepsilon'-\ell)}{\varepsilon'}}\in\nabla_{\frac{\varepsilon'\ell}{\varepsilon(\varepsilon'-\ell)}}(\mathbb{R}_+\times\mathbb{R}_+)$, we directly deduce that

$$\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{t,s}*\mathcal{H}_{j,k}\right|^{\varepsilon}\frac{dtds}{ts}\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\leq C_{p,\mho,h}(\ln\mu)^{2/\varepsilon}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}\right|^{\varepsilon}\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\tag{2.8}$$

for all $p\in(\varepsilon,\frac{\varepsilon'\ell}{\varepsilon'-\ell})$.

Case 2. $p=\varepsilon$. By employing (2.7) and Hölder's inequality, we get

$$\begin{aligned}
&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{t,s}*\mathcal{H}_{j,k}\right|^{\varepsilon}\frac{dtds}{ts}\right)^{1/\varepsilon}\right\|^{\varepsilon}_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\leq C\,\|h\|^{(\varepsilon/\varepsilon')}_{\nabla_1(\mathbb{R}_+\times\mathbb{R}_+)}\,\|\mho\|^{(\varepsilon/\varepsilon')}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}\\
&\times\sum_{j,k\in\mathbb{Z}}\iiint_{\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\int_{s/2}^{s}\int_{t/2}^{t}\iint_{\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1}}\left|\mathcal{H}_{j,k}(x-lu,y-rv,z-\Theta(l,r))\right|^{\varepsilon}\\
&\times\;|\mho(u,v)|\,|h(l,r)|^{\frac{\varepsilon(\varepsilon'-\ell)}{\varepsilon'}}\,d\sigma_\kappa(u)d\sigma_\tau(v)\frac{dldr}{lr}\frac{dtds}{ts}dxdydz\\
&\leq\;C(\ln\mu)^2\,\|h\|^{(\varepsilon/\varepsilon')+1}_{\nabla_1(\mathbb{R}_+\times\mathbb{R}_+)}\,\|\mho\|^{(\varepsilon/\varepsilon')+1}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}\iiint_{\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R}}\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}(x,y,z)\right|^{\varepsilon}\right)dxdydz.
\end{aligned}\tag{2.9}$$

Case 3. $p\in(\frac{\varepsilon\ell'}{\varepsilon+\ell'-1},\varepsilon)$. Define the linear operator $\mathcal{I}$ on an arbitrary function $\mathcal{H}=\mathcal{H}_{j,k}(x,y,z)$ by $\mathcal{I}(\mathcal{H})=\Upsilon_{\mu^k t,\mu^j s}*\mathcal{H}_{j,k}(x,y,z)$. Thus, we have

$$\left\|\left\|\|\mathcal{I}(\mathcal{H})\|_{L^1([1,\mu)\times[1,\mu)),\frac{dtds}{ts}}\right\|_{l^1(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^1(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\leq C(\ln\mu)^2\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}\right|\right)\right\|_{L^1(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}.\tag{2.10}$$

In addition, the inequality (2.1) gives

$$\begin{aligned}
\left\|\sup_{j,k\in\mathbb{Z}}\sup_{(t,s)\in[1,\mu]\times[1,\mu]}\left|\Upsilon_{\mu^k t,\mu^j s}*\mathcal{H}_{j,k}\right|\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}&\leq\left\|\Upsilon^*_h\left(\sup_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}\right|\right)\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\\
&\leq C_{p,\mho,h}\left\|\sup_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}\right|\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}
\end{aligned}$$

for all $p\in(\ell',\infty)$, which in turn implies that

$$\left\|\left\|\|\Upsilon_{\mu^k t,\mu^j s}*\mathcal{H}_{j,k}\|_{L^\infty([1,\mu]\times[1,\mu],\frac{dtds}{ts})}\right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\leq C_{p,\mho,h}\left\|\left\|\mathcal{H}_{j,k}\right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}.\tag{2.11}$$

Consequently, by interpolating (2.10) with (2.11), the estiamte (2.5) is satisfied for $p\in(\frac{\varepsilon\ell'}{\varepsilon+\ell'-1},\varepsilon)$. □

Lemma 2.4. *Let $h\in\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)$ with $2<\ell<\infty$ and $\mho\in L^q\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1}\right)$ with $1<q\leq 2$. Assume that $1<\varepsilon\leq\ell'$ and that Θ belongs to the class* **W**. *Then the estimate*

$$\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{t,s}*\mathcal{H}_{j,k}\right|^{\varepsilon}\frac{dtds}{ts}\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\leq C_{p,\mho,h}(\ln\mu)^{2/\varepsilon}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}\right|^{\varepsilon}\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\tag{2.12}$$

holds for all $p\in(1,\varepsilon)$, where $\{\mathcal{H}_{j,k}(\cdot,\cdot,\cdot),\ \ j,k\in\mathbb{Z}\}$ is any set of functions on $\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R}$.

Proof. Thanks to the duality, there exists a collection of functions $\{X_{j,k}(x,y,z,t,s)\}$ defined on $\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+$ with $\left\|\left\|\|X_{j,k}\|_{L^{\varepsilon'}([\mu^k,\mu^{k+1}]\times[\mu^j,\mu^{j+1}],\frac{dtds}{ts})}\right\|_{l^{\varepsilon'}(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^{p'}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\le 1$ and

$$\begin{aligned}
&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{t,s}*\mathcal{H}_{j,k}\right|^{\varepsilon}\frac{dtds}{ts}\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\\
=&\iiint_{\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R}}\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left(\Upsilon_{t,s}*\mathcal{H}_{j,k}(x,y,z)\right)X_{j,k}(x,y,z,t,s)\frac{dtds}{ts}dxdydz\\
\le&\ C(\ln\mu)^{2/\varepsilon}\left\|(\Psi(X)^{1/\varepsilon'}\right\|_{L^{p'}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}\right|^{\varepsilon}\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}
\end{aligned}\tag{2.13}$$

where

$$\Psi(X)(x,y,z)=\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{t,s}*X_{j,k}(x,y,z,t,s)\right|^{\varepsilon'}\frac{dtds}{ts}.$$

Since $\varepsilon\le\ell'<2<\ell$, we deduce by Hölder's inequality that

$$\begin{aligned}
\left|\Upsilon_{t,s}*X_{j,k}(x,y,z)\right|^{\varepsilon'}&\le C\,\|\mho\|^{(\varepsilon'/\varepsilon)}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}\|h\|^{(\varepsilon'/\varepsilon)}_{\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)}\\
\times\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}&\iint_{\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1}}\left|X_{j,k}(x-lu,y-rv,z-\Theta(l,r),t,s)\right|^{\varepsilon'}|\mho(u,v)|\,d\sigma_\kappa(u)d\sigma_\tau(v)\frac{dldr}{lr}
\end{aligned}\tag{2.14}$$

and since $(p'/\varepsilon')>1$, we deduce that there is a function Q belonging to the space $L^{(p'/\varepsilon')'}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})$ such that

$$\begin{aligned}
\|\Psi(X)\|_{L^{(p'/\varepsilon')}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}=&\sum_{j,k\in\mathbb{Z}}\iiint_{\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{t,s}*X_{j,k}(x,y,z,t,s)\right|^{\varepsilon'}\frac{dtds}{ts}\\
&\times Q(x,y,z)dxdydz
\end{aligned}$$

which gives, by a simple change of variable along with Lemma 2.1 and (2.14), that

$$\begin{aligned}
\|\Psi(X)\|_{L^{(p'/\varepsilon')}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\le&\ C\,\|\mho\|^{(\varepsilon'/\varepsilon)}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}\|h\|^{(\varepsilon')}_{\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)}\|\Upsilon^*(Q)\|_{L^{(p'/\varepsilon')'}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\\
&\times\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|X_{j,k}(\cdot,\cdot,\cdot,t,s)\right|^{\varepsilon'}\frac{dtds}{ts}\right)\right\|_{L^{(p'/\varepsilon')}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\\
\le&\ C\,\|h\|^{\varepsilon'}_{\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)}\|\mho\|^{(\varepsilon'/\varepsilon)+1}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}\|Q\|_{L^{(p'/\varepsilon')'}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}.
\end{aligned}\tag{2.15}$$

Consequently, by (2.13) and (2.15), the estimate (2.12) is satisfied for all $p\in(1,\varepsilon)$. The proof of Lemma 2.4 is finished. □

Lemma 2.5. *Let $\mho$, Θ, and $\{\mathcal{H}_{j,k}(\cdot,\cdot,\cdot),\ j,k\in\mathbb{Z}\}$ be given as in Lemma 2.3. Suppose that $h\in\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\ell\in(1,\infty)$ and that $\varepsilon\ge\ell'$. Then there is a constant $C_{p,\mho,h}>0$ such that*

$$\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{t,s}*\mathcal{H}_{j,k}\right|^{\varepsilon}\frac{dtds}{ts}\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\le C_{p,\mho,h}(\ln\mu)^{2/\varepsilon}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}\right|^{\varepsilon}\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}\tag{2.16}$$

for all $p \in (\ell', \infty)$.

Proof. It is clear that the inequality (2.1) leads to

$$\begin{aligned} \left\| \sup_{j,k\in\mathbb{Z}} \sup_{(t,s)\in[1,\mu]\times[1,\mu]} \left|\Upsilon_{\mu^k r,\mu^j s} * \mathcal{H}_{j,k}\right| \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} &\leq \left\| \Upsilon_h^* \left(\sup_{j,k\in\mathbb{Z}} \left|\mathcal{H}_{j,k}\right| \right) \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \\ &\leq C_{p,\mho,h} \left\| \sup_{j,k\in\mathbb{Z}} \left|\mathcal{H}_{j,k}\right| \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \end{aligned} \tag{2.17}$$

for all $p \in (\ell', \infty)$. Thus,

$$\begin{aligned} &\left\| \left\| \|\Upsilon_{\mu^k t,\mu^j s} * \mathcal{H}_{j,k}\|_{L^\infty([1,\mu]\times[1,\mu],\frac{dtds}{ts})} \right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})} \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \\ \leq\ & C_{p,\mho,h} \left\| \left\| \mathcal{H}_{j,k} \right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})} \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}. \end{aligned} \tag{2.18}$$

Again, by the duality a function $\varphi \in L^{(p/\ell')'}(\mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R})$ exists such that $\|\varphi\|_{L^{(p/\ell')'}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \leq 1$ and

$$\begin{aligned} &\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_1^\mu \int_1^\mu \left|\Upsilon_{\mu^k r,\mu^j s} * \mathcal{H}_{j,k}\right|^{\ell'} \frac{dtds}{ts} \right)^{1/\ell'} \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})}^{\ell'} \\ =\ & \iiint_{\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R}} \sum_{j,k\in\mathbb{Z}} \int_1^\mu \int_1^\mu \left|\Upsilon_{\mu^k r,\mu^j s} * \mathcal{H}_{j,k}\right|^{\ell'} \frac{dtds}{ts} \varphi(x,y,z) dxdydz \\ \leq\ & C \|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}^{(\ell'/\ell)} \|h\|_{\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)}^{\ell'} \\ \times\ & \iiint_{\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R}} \left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{H}_{j,k}(x,y,z)\right|^{\ell'} \right) \Upsilon_h^*(\overline{\varphi})(-x,-y,-z) dxdydz \\ \leq\ & C(\ln\mu)^2 \|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\tau-1})}^{(\ell'/\ell)} \|h\|_{\nabla_\ell(\mathbb{R}_+\times\mathbb{R}_+)}^{(\ell')} \left\| \sum_{j,k\in\mathbb{Z}} \left|\mathcal{H}_{j,k}\right|^{\ell'} \right\|_{L^{(p/\ell')}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \\ \times\ & \left\| \Upsilon_h^*(\overline{\varphi}) \right\|_{L^{(p/\ell')'}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \end{aligned} \tag{2.19}$$

where $\overline{\varphi}(x,y,z) = \psi(-x,-y,-z)$. Let $\mathcal{I}$ be the linear operator, which is defined in the proof of Lemma 2.3. Then by combining (2.18) with (2.19), we get

$$\begin{aligned} &\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \int_{\mu^k}^{\mu^{k+1}} \left|\Upsilon_{t,s} * \mathcal{H}_{j,k}\right|^{\varepsilon} \frac{dtds}{ts} \right)^{1/\varepsilon} \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \\ \leq\ & C \left\| \left(\sum_{j,k\in\mathbb{Z}} \int_1^\mu \int_1^\mu \left|\Upsilon_{\mu^k t,\mu^j s} * \mathcal{H}_{j,k}\right|^{\varepsilon} \frac{dtds}{ts} \right)^{1/\varepsilon} \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \\ \leq\ & C_{p,\mho,h} (\ln\mu)^{2/\varepsilon} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{H}_{j,k}\right|^{\varepsilon} \right)^{1/\varepsilon} \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \end{aligned}$$

for all $p \in (\ell', \infty)$ with $\ell' < \varepsilon$. This finishes the proof of Lemma 2.5. □

3. Proof of the main results

Proof of Theorem 1.1. Let $\Theta \in \mathbf{W}$ and $\varepsilon > 1$. Assume that $h \in \nabla_{\ell}(\mathbb{R}_+ \times \mathbb{R}_+)$ and $\mho \in L^q\left(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\tau-1}\right)$ with $\ell, q \in (1, 2]$. By Minkowski's inequality we have

$$\begin{aligned}
\mathcal{G}^{(\varepsilon)}_{\Theta,\mho,h}(f)(x,y,z) &= \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left| \sum_{j,k=0}^{\infty} \frac{1}{t^{\eta_1} s^{\eta_2}} \int_{2^{-j-1}s<|v|\le 2^{-j}s} \int_{2^{-k-1}t<|u|\le 2^{-k}t} \mathcal{K}_{\mho,h}(u,v) \right.\right. \\
&\times \left.\left. f(x-u, y-v, z-\Theta(|u|,|v|))\ dudv \right|^{\varepsilon} \frac{dtds}{ts}\right)^{1/\varepsilon} \\
&\le \sum_{j,k=0}^{\infty} \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left| \frac{1}{t^{\eta_1} s^{\eta_2}} \int_{2^{-j-1}s<|v|\le 2^{-j}s} \int_{2^{-k-1}t<|u|\le 2^{-k}t} \mathcal{K}_{\mho,h}(u,v) \right.\right. \\
&\times \left.\left. f(x-u, y-v, z-\Theta(|u|,|v|))dudv \right|^{\varepsilon} \frac{dtds}{ts}\right)^{1/\varepsilon} \\
&\le C\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|\Upsilon_{t,s} * f(x,y,z)\right|^{\varepsilon} \frac{dtds}{ts}\right)^{1/\varepsilon}. \qquad (3.1)
\end{aligned}$$

For $k \in \mathbb{Z}$, choose a set of smooth partition of unity $\{\Omega_k\}_{-\infty}^{\infty}$ defined on $(0,\infty)$ and adapted to the interval $[\mu^{-1-k}, \mu^{1-k}]$ with the following properties:

$$\begin{aligned}
\Omega_k &\in C^{\infty},\ 0 \le \Omega_k \le 1,\ \sum_{k\in\mathbb{Z}} \Omega_k(t) = 1, \\
\text{supp}\,(\Omega_k) &\subseteq [\mu^{-1-k}, \mu^{1-k}]\ \ \textit{and}\ \ \left|\frac{d^{\alpha}\Omega_k(t)}{dt^{\alpha}}\right| \le \frac{C_{\alpha}}{t^{\alpha}}
\end{aligned}$$

where C_β does not depend on the lacunary sequence $\{\mu^k; k \in \mathbb{Z}\}$. Define the multiplier operators $\{\Lambda_{j,k}\}$ on $\mathbb{R}^{\kappa} \times \mathbb{R}^{\tau} \times \mathbb{R}$ by $(\widehat{\Lambda_{j,k}(f)})(\xi,\zeta,\omega) = \Omega_j(|\xi|)\Omega_k(|\zeta|)\hat{f}(\xi,\zeta,\omega)$. So, for any $f \in C_0^{\infty}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\tau} \times \mathbb{R})$,

$$\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|\Upsilon_{t,s} * f(x,y,z)\right|^{\varepsilon} \frac{dtds}{ts}\right)^{1/\varepsilon} \le C \sum_{n,m\in\mathbb{Z}} \mathcal{A}_{n,m}(f)(x,y,z) \qquad (3.2)$$

where

$$\mathcal{A}_{n,m}(f)(x,y,z) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|\mathcal{B}_{n,m}(f)(x,y,z,t,s)\right|^{\varepsilon} \frac{dtds}{ts}\right)^{1/\varepsilon}$$

and

$$\mathcal{B}_{n,m}(f)(x,y,z,t,s) = \sum_{j,k\in\mathbb{Z}} \Upsilon_{t,s} * \Lambda_{j+m,k+n} * f(x,y,z)\chi_{[\mu^k,\mu^{k+1})\times[\mu^j,\mu^{j+1})}(t,s).$$

Thus, to prove Theorem 1.1, it is enough to show that a constant $\beta > 0$ exists such that

$$\left\|\mathcal{A}_{n,m}(f)\right\|_{L^p(\mathbb{R}^{\kappa}\times\mathbb{R}^{\tau}\times\mathbb{R})} \le C_{p,\mho,h}(\ln\mu)^{2/\varepsilon}\ 2^{-\frac{\beta}{2}(|n|+|m|)}\,\|f\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^{\kappa}\times\mathbb{R}^{\tau}\times\mathbb{R})} \qquad (3.3)$$

for all $p \in (\frac{\varepsilon\ell'}{\varepsilon+\ell'-1}, \frac{\varepsilon'\ell}{\varepsilon'-\ell})$ with $\ell' \geq \varepsilon$, and also for all $p \in (\ell', \infty)$ with $\ell' \leq \varepsilon$.

For the case $p = \varepsilon = 2$, we estimate the norm of $\mathcal{A}_{n,m}(f)$ as follows: By employing Plancherel's theorem, Fubini's theorem, and the inequality (2.4), we directly obtain

$$\begin{aligned}
&\left\|\mathcal{A}_{n,m}(f)\right\|^2_{L^2(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \\
\leq\ & \sum_{j,k\in\mathbb{Z}} \iiint_{E_{j+n,k+m}} \left(\int_{\mu^j}^{\mu^{j+1}} \int_{\mu^k}^{\mu^{k+1}} \left|\hat{\Upsilon}_{t,s}(\zeta,\xi,\omega)\right|^2 \frac{dtds}{ts}\right) \left|\hat{f}(\zeta,\xi,\omega)\right|^2 d\zeta d\xi d\omega \\
\leq\ & C_p(\ln\mu)^2 C^2_{p,\mho,h} \sum_{j,k\in\mathbb{Z}} \iiint_{E_{j+m,k+n}} \left|\mu^k\zeta\right|^{\pm\frac{2\theta}{\ln(\mu)}} \left|\mu^j\xi\right|^{\pm\frac{2\theta}{\ln(\mu)}} \left|\hat{f}(\zeta,\xi,\omega)\right|^2 d\zeta d\xi d\omega \\
\leq\ & C_p(\ln\mu)^2\, 2^{-\beta(|n|+|m|)}\, C^2_{p,\mho,h} \sum_{j,k\in\mathbb{Z}} \iiint_{E_{j+n,m+i}} \left|\hat{f}(\zeta,\xi,\omega)\right|^2 d\zeta d\xi d\omega \\
\leq\ & C_p(\ln\mu)^2\, 2^{-\beta(|n|+|m|)}\, C^2_{p,\mho,h} \|f\|^2_{L^2(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \qquad (3.4)
\end{aligned}$$

where $E_{j,k} = \left\{(\zeta,\xi,\omega) \in \mathbb{R}^\kappa \times \mathbb{R}^\tau \times \mathbb{R} : (|\zeta|, |\xi|) \in [\mu^{-1-k}, \mu^{1-k}] \times [\mu^{-1-j}, \mu^{1-j}]\right\}$ and $\beta \in (0, 1)$.

However, for the other cases, we estimate the L^p-norm of $\mathcal{A}_{n,m}(f)$ by using an argument similar to that employed in [34]. Precisely, we invoke Littlewood-Paley theory, Lemmas 2.3, 2.5, and Lemma 2.3 in [28], so we get

$$\begin{aligned}
&\left\|\mathcal{A}_{n,m}(f)\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \\
\leq\ & C \left\|\left(\sum_{j,k\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \int_{\mu^k}^{\mu^{k+1}} \left|\Upsilon_{t,s} * \Lambda_{j+m,k+n} * f\right|^\varepsilon \frac{dtds}{ts}\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \\
\leq\ & C_{p,\mho,h}(\ln\mu)^{2/\varepsilon} \left\|\left(\sum_{j,k\in\mathbb{Z}} \left|\Lambda_{j+m,k+n} * f\right|^\varepsilon\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \\
\leq\ & C_{p,\mho,h}(\ln\mu)^{2/\varepsilon} \|f\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\tau\times\mathbb{R})} \qquad (3.5)
\end{aligned}$$

for all $p \in (\frac{\varepsilon\ell'}{\varepsilon+\ell'-1}, \frac{\varepsilon'\ell}{\varepsilon'-\ell})$ with $\varepsilon \leq \ell'$, and also for all $p \in (\ell', \infty)$ with $\varepsilon \geq \ell'$. Hence, the estimate (3.3) holds by interpolating (3.4) with (3.5) and taking $\mu = 2^{q'\ell'}$. Therefore, the proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. The proof can be obtained by following the same argument employed in the proof of Theorem 1.1, except we invoke Lemma 2.4 instead of Lemma 2.3.

4. Conclusions

In this paper, we established suitable L^p estimates for several classes of generalized Marcinkiewicz operators along surfaces of revolution on product domains with rough kernels. These estimates along with Yano's extrapolation arguments confirmed the L^p boundedness of the aforementioned operators under weaker conditions on the singular kernels. Our results improve and generalize many previously known results in Marcinkiewicz and generalized Marcinkiewicz operators.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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