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Research article

Sturmian comparison theorem for hyperbolic equations on a rectangular prism

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Abstract: In this paper, new Sturmian comparison results were obtained for linear and nonlinear hyperbolic equations on a rectangular prism. The results obtained for linear equations extended those given by Kreith [Sturmian theorems on hyperbolic equations, *Proc. Amer. Math. Soc.*, **22** (1969), 277–281] in which the Sturmian comparison theorem for linear equations was obtained on a rectangular region in the plane. For the purpose of verification, an application was described using an eigenvalue problem.

Keywords: hyperbolic equation; Sturm comparison; rectangular prism; oscillation; eigenvalue problem; hyperrectangle
Mathematics Subject Classification: 34C10, 35L10, 35L20, 35L70

1. Introduction

The classical Sturmian comparison theory of second-order ordinary differential equations and their oscillatory behaviors is acknowledged as the foundation for investigating many fundamental features of their solutions [1, 2]. The theory of Sturm comparison on partial differential equations has grown rapidly in the last few decades. Comparison theorems have been explored on various types of partial differential equations and have made significant contributions to the literature [3, 4]. Some of remarkable contributions can be counted as for elliptic-type self-adjoint equations by Hartman et al. [5], second-order elliptic equations by Shimoda [6], fourth-order elliptic systems by Kusano et al. [7], a genus of higher-order elliptic systems by Yoshida [8], a genus of second-order half-linear partial differential equations by Kusano et al. [9], a genus of half-linear elliptic equations by Yoshida [10], quasilinear elliptic equations with mixed nonlinearities via Picone-type inequality by Yoshida [11], half-linear elliptic operators with $p(x)$-Laplacians by Yoshida [12], quasilinear elliptic operators with

$p(x)$-Laplacians by Yoshida [13], and a genus of partial differential equations of order $4m$ by Jaroš [14] (see also [15, 16]). We direct the readers to the monograph by Yoshida [17] for historical development in the oscillation and comparison theory of partial differential equations.

In 1969, Kreith [18] considered Sturmian comparison theory on hyperbolic differential equations, which was the prime and very likely the sole publication on the topic; see also [19, pp. 24–26]. He considered the pair of hyperbolic equations, which are inspired by the simplistic harmonic motion perception of the Sturm comparison theorem as

$$u_{tt} - u_{xx} + p(x,t)u = 0 \tag{1.1}$$

and

$$v_{tt} - v_{xx} + q(x,t)v = 0 \tag{1.2}$$

illustrating throwing two vibrating strings with the same density and elastic constant movement as they oscillate, regarding the equilibrium lines $v = 0$ and $u = 0$ under the effect of continuous restorative forces $p(x,t)$ and $q(x,t)$, respectively. Presume that if

$$q(x,t) \geq p(x,t),$$

then in some sense Eq (1.2) should oscillate faster than Eq (1.1). Experimentation with straightforward cases that allow variable separation indicates that in the absence of an auxiliary condition, this is not the case. Physically, it is important to analyze finite strings that are elastically bounded at the ends, with the string that is to oscillate more faster being firmly bound. When viewed mathematically, Kreith established an analogue of the Sturm comparison theorem for the pair hyperbolic initial value problems of the form

$$\begin{aligned} &u_{tt} - u_{xx} + p(x,t)u = 0, \\ &u_x(x_k,t) + (-1)^k \sigma_k(t) u(x_k,t) = 0 \quad (k = 1,2) \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &v_{tt} - v_{xx} + q(x,t)v = 0, \\ &v_x(x_k,t) + (-1)^k \tau_k(t) v(x_k,t) = 0 \quad (k = 1,2) \end{aligned} \tag{1.4}$$

on the rectangular domain:

$$\mathcal{D} = \{(x,t) : x \in (x_1,x_2),\ t \in (t_1,t_2)\};$$

see [18, Theorem 1] and [19, Theorem 3.8].

Theorem 1.1. *Let u be a positive solution of problem* (1.3) *satisfying the boundary conditions*

$$u(x,t_1) = u(x,t_2) = 0, \quad x_1 \leq x \leq x_2$$

on $[x_1,x_2] \times (t_1,t_2)$. If $q(x,t) \geq p(x,t)$ on $\mathcal{D}$ and $\tau_k(t) \geq \sigma_k(t)$ $(k = 1,2)$ for $t \in [t_1,t_2]$, then every solution v of problem (1.4) *has a zero in*

$$\bar{\mathcal{D}} = \{(x,t) : x \in [x_1,x_2],\ t \in [t_1,t_2]\}.$$

For our purpose, we fix $x_0, y_0, t_0 \in \mathbb{R}$. Let $\mathcal{I} = (x_1, x_2) \subset [x_0, \infty)$, $\mathcal{J} = (y_1, y_2) \subset [y_0, \infty)$ and $\mathcal{K} = (t_1, t_2) \subset [t_0, \infty)$ be three nondegenerate intervals and define the domain (a rectangular prism) as

$$\Omega = \mathcal{I} \times \mathcal{J} \times \mathcal{K}. \tag{1.5}$$

In this paper, we are attempting to arrange some analogical comparison results for the continuous solutions of a couple of hyperbolic equations

$$u_{tt} - \Delta u + f(x, y, t)u = 0, \tag{1.6}$$

$$v_{tt} - \Delta v + g(x, y, t)v = 0 \tag{1.7}$$

for $(x, y, t) \in \Omega$, satisfying the initial conditions

$$\begin{array}{ll} u_x(x_k, y, t) + (-1)^k r_k(t) u(x_k, y, t) = 0, & (y, t) \in \bar{\mathcal{J}} \times \bar{\mathcal{K}}, \\ u_y(x, y_k, t) + (-1)^k r_{k+2}(t) u(x, y_k, t) = 0, & (x, t) \in \bar{\mathcal{I}} \times \bar{\mathcal{K}} \end{array} \tag{1.8}$$

and

$$\begin{array}{ll} v_x(x_k, y, t) + (-1)^k s_k(t) v(x_k, y, t) = 0, & (y, t) \in \bar{\mathcal{J}} \times \bar{\mathcal{K}}, \\ v_y(x, y_k, t) + (-1)^k s_{k+2}(t) v(x, y_k, t) = 0, & (x, t) \in \bar{\mathcal{I}} \times \bar{\mathcal{K}} \end{array} \tag{1.9}$$

for $k = 1, 2$, respectively, where $f, g \in \mathrm{C}\left(\bar{\Omega}, \mathbb{R}\right)$, $r_k, s_k \in \mathrm{C}\left(\bar{\mathcal{K}}, \mathbb{R}\right)$ $(k = 1, 2, 3, 4)$, and Δ is the usual Laplace operator in $\mathbb{R}^2$, i.e.,

$$\Delta = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}.$$

A nontrivial function $z(x, y, t)$ is claimed to be the solution of problem (1.6)-(1.8) if

i. $z : \bar{\Omega} \to \mathbb{R} \in \mathrm{C}\left(\bar{\Omega}, \mathbb{R}\right)$;
ii. for each $(x, y, t) \in \Omega$, it has second-order partial derivatives z_{xx}, z_{yy}, z_{tt}, and z satisfies Eq (1.6);
iii. z satisfies the initial conditions (1.8) on $\bar{\Omega}$.

Solution $z(x, y, t) = 0$ of problem (1.6)-(1.8) has a zero at $t = t^*$ if $z(x, y, t^*) = 0$. A solution z of problem (1.6)-(1.8) is said to be an oscillatory if there exists a sequence $\{\zeta_n\}_{n=1}^{\infty}$ of real numbers such that $z(x, y, \zeta_n) = 0$ with

$$\lim_{n\to\infty} \zeta_n = \infty.$$

Otherwise, $z(x, y, t)$ is said to be nonoscillatory. Moreover, problem (1.6)-(1.8) is said to be oscillatory if all the solutions of it are oscillatory. The similar definition and properties given above are also valid for the solution v of problem (1.7)-(1.9).

Motivated by the Kreith's comparison result obtained on the rectangular domain in the plane (i.e., Theorem 1.1), we attempt to give an analogous result for continuous solutions of the pair of hyperbolic initial value problems (1.6)-(1.8) and (1.7)-(1.9) on a rectangular prism. The main findings for the linear hyperbolic problems (1.6)-(1.8) and (1.7)-(1.9) are extended to nonlinear hyperbolic problems in Section 3.

The paper is structured as follows: Sections 2 and 3 are devoted the Sturmian comparison results for linear and nonlinear hyperbolic initial value problems, respectively. The last section deals with an interesting Sturm oscillation result via separation of variables under the assumption that one of the corresponding ordinary differential equations is oscillatory.

2. Linear comparison results

In this section we provide Sturm comparison results for linear hyperbolic initial value problems. The first linear comparison consequence of this section is as follows.

Theorem 2.1. *(Sturm comparison theorem) Let $u > 0$ be a solution of problem* (1.6)-(1.8) *satisfying the boundary conditions*

$$u(x, y, t_1) = u(x, y, t_2) = 0, \quad (x, y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}} \tag{2.1}$$

on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$. If the inequalities

$$g(x, y, t) \geq f(x, y, t), \quad (x, y, t) \in \Omega \tag{2.2}$$

and

$$s_k(t) \geq r_k(t), \quad t \in \bar{\mathcal{K}} \quad (k = 1, 2, 3, 4) \tag{2.3}$$

hold, then every solution v of problem (1.7)-(1.9) *has a zero in $\bar{\Omega}$.*

Proof. Assume that a solution v of problem (1.7)-(1.9) has no zero in $\bar{\Omega}$, then without loss of generality, we may assume that $v > 0$ in $\bar{\Omega}$. Multiplying Eqs (1.6) and (1.7) by v and u, respectively, and then subtracting, it can be verified that the identity

$$[uv_x - vu_x]_x + [uv_y - vu_y]_y + [vu_t - uv_t]_t = [g(x, y, t) - f(x, y, t)]uv \tag{2.4}$$

holds for all $(x, y, t) \in \bar{\Omega}$.

Integrating both sides of identity (2.4) over Ω, we obtain

$$\iiint_\Omega [g(x, y, t) - f(x, y, t)]uv\mathrm{d}V = \iiint_\Omega \left\{[uv_x - vu_x]_x + [uv_y - vu_y]_y + [vu_t - uv_t]_t\right\} \mathrm{d}V. \tag{2.5}$$

Note that Ω is a simple, solid region with the piece-wise smooth boundary $\mathcal{S}$, so by applying divergence theorem to the smooth vector field $\mathbf{F}$ on Ω defined by

$$\mathbf{F}(x, y, t) := (uv_x - vu_x)\mathbf{i} + (uv_y - vu_y)\mathbf{j} + (vu_t - uv_t)\mathbf{k}, \tag{2.6}$$

the righthand side of (2.5) turns out to be

$$\begin{aligned} &\iiint_\Omega \left\{[uv_x - vu_x]_x + [uv_y - vu_y]_y + [vu_t - uv_t]_t\right\} \mathrm{d}V \\ &= \iiint_\Omega \nabla \bullet \mathbf{F}\mathrm{d}V \quad \left(= \iiint_\Omega \operatorname{div} \mathbf{F}\mathrm{d}V\right) \\ &= \oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\mathrm{d}S, \end{aligned} \tag{2.7}$$

where $\hat{\mathbf{N}}$ is the unit outward normal on the surface $\mathcal{S}$ $(= \partial\Omega)$ and the ∇ is the usual nabla (gradient) operator defined by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial t}.$$

Note that $\mathcal{S}$ is the union of six rectangular regions, that is

$$\mathcal{S} = \bigcup_{j=1}^{6} \mathcal{S}_j,$$

where each $\mathcal{S}_j$ are disjoint, oriented, closed surfaces and defined by

$$\begin{aligned}
\mathcal{S}_1 &= \{(x, y, t) : x = x_1, (y, t) \in \mathcal{J} \times \mathcal{K}\},\\
\mathcal{S}_2 &= \{(x, y, t) : x = x_2, (y, t) \in \mathcal{J} \times \mathcal{K}\},\\
\mathcal{S}_3 &= \{(x, y, t) : y = y_1, (x, t) \in \mathcal{I} \times \mathcal{K}\},\\
\mathcal{S}_4 &= \{(x, y, t) : y = y_2, (x, t) \in \mathcal{I} \times \mathcal{K}\},\\
\mathcal{S}_5 &= \{(x, y, t) : t = t_1, (x, y) \in \mathcal{I} \times \mathcal{J}\}
\end{aligned}$$

and

$$\mathcal{S}_6 = \{(x, y, t) : t = t_2, (x, y) \in \mathcal{I} \times \mathcal{J}\}.$$

The last (surface) integral in (2.7) can be expressed as

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S = \sum_{j=1}^{6} \iint_{\mathcal{S}_j} \mathbf{F} \bullet \hat{\mathbf{N}}_j \mathrm{d}S, \tag{2.8}$$

where the vectors $\hat{\mathbf{N}}_j$ are the unit outward normal vectors on the surfaces $\mathcal{S}_j$, $j = 1, \ldots, 6$, and, hence, we have

$$\begin{aligned}
\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S = &\iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}_1 \mathrm{d}S + \iint_{\mathcal{S}_2} \mathbf{F} \bullet \hat{\mathbf{N}}_2 \mathrm{d}S + \iint_{\mathcal{S}_3} \mathbf{F} \bullet \hat{\mathbf{N}}_3 \mathrm{d}S\\
&+ \iint_{\mathcal{S}_4} \mathbf{F} \bullet \hat{\mathbf{N}}_4 \mathrm{d}S + \iint_{\mathcal{S}_5} \mathbf{F} \bullet \hat{\mathbf{N}}_5 \mathrm{d}S + \iint_{\mathcal{S}_6} \mathbf{F} \bullet \hat{\mathbf{N}}_6 \mathrm{d}S.
\end{aligned} \tag{2.9}$$

Since $\hat{\mathbf{N}}_1 = -\mathbf{i}$, $\hat{\mathbf{N}}_2 = \mathbf{i}$, $\hat{\mathbf{N}}_3 = -\mathbf{j}$, $\hat{\mathbf{N}}_4 = \mathbf{j}$, $\hat{\mathbf{N}}_5 = -\mathbf{k}$ and $\hat{\mathbf{N}}_6 = \mathbf{k}$, the integrals on the righthand side of (2.9) become

$$\iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}_1 \mathrm{d}S = -\iint_{\mathcal{S}_1} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S = -\int_{t_1}^{t_2} \int_{y_1}^{y_2} [uv_x - vu_x](x_1, y, t) \mathrm{d}y \mathrm{d}t, \tag{2.10}$$

$$\iint_{\mathcal{S}_2} \mathbf{F} \bullet \hat{\mathbf{N}}_2 \mathrm{d}S = \iint_{\mathcal{S}_2} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S = \int_{t_1}^{t_2} \int_{y_1}^{y_2} [uv_x - vu_x](x_2, y, t) \mathrm{d}y \mathrm{d}t, \tag{2.11}$$

$$\iint_{\mathcal{S}_3} \mathbf{F} \bullet \hat{\mathbf{N}}_3 \mathrm{d}S = -\iint_{\mathcal{S}_3} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S = -\int_{x_1}^{x_2} \int_{t_1}^{t_2} [uv_y - vu_y](x, y_1, t) \mathrm{d}t \mathrm{d}x, \tag{2.12}$$

$$\iint_{\mathcal{S}_4} \mathbf{F} \bullet \hat{\mathbf{N}}_4 \mathrm{d}S = \iint_{\mathcal{S}_4} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S = \int_{x_1}^{x_2} \int_{t_1}^{t_2} [uv_y - vu_y](x, y_2, t) \mathrm{d}t \mathrm{d}x, \tag{2.13}$$

$$\iint_{\mathcal{S}_5} \mathbf{F} \bullet \hat{\mathbf{N}}_5 \mathrm{d}S = -\iint_{\mathcal{S}_5} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, t_1) \mathrm{d}x \mathrm{d}y \tag{2.14}$$

and

$$\iint_{S_6} \mathbf{F} \bullet \hat{\mathbf{N}}_6 \mathrm{d}S = \iint_{S_6} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, t_2) \mathrm{d}x\mathrm{d}y. \tag{2.15}$$

Imposing the initial conditions (1.8) and (1.9) in the integrals on the righthand sides of (2.10)–(2.13), we get that

$$\iint_{S_1} \mathbf{F} \bullet \hat{\mathbf{N}}_1 \mathrm{d}S = \int_{t_1}^{t_2} \int_{y_1}^{y_2} [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \mathrm{d}y\mathrm{d}t, \tag{2.16}$$

$$\iint_{S_2} \mathbf{F} \bullet \hat{\mathbf{N}}_2 \mathrm{d}S = \int_{t_1}^{t_2} \int_{y_1}^{y_2} [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \mathrm{d}y\mathrm{d}t, \tag{2.17}$$

$$\iint_{S_3} \mathbf{F} \bullet \hat{\mathbf{N}}_3 \mathrm{d}S = \int_{x_1}^{x_2} \int_{t_1}^{t_2} [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \mathrm{d}t\mathrm{d}x \tag{2.18}$$

and

$$\iint_{S_4} \mathbf{F} \bullet \hat{\mathbf{N}}_4 \mathrm{d}S = \int_{x_1}^{x_2} \int_{t_1}^{t_2} [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \mathrm{d}t\mathrm{d}x. \tag{2.19}$$

On the other hand, boundary conditions (2.1) imply that (2.14) and (2.15) reduce to

$$\iint_{S_5} \mathbf{F} \bullet \hat{\mathbf{N}}_5 \mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} v(x, y, t_1) u_t(x, y, t_1) \mathrm{d}x\mathrm{d}y \tag{2.20}$$

and

$$\iint_{S_6} \mathbf{F} \bullet \hat{\mathbf{N}}_6 \mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} v(x, y, t_2) u_t(x, y, t_2) \mathrm{d}x\mathrm{d}y. \tag{2.21}$$

Since u and v are positive solutions on $\bar{\Omega}$, conditions (2.3) of the theorem imply that (2.9) turns out to be

$$\oiint_{S} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S \leq \int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ v(x, y, t_2) u_t(x, y, t_2) - v(x, y, t_1) u_t(x, y, t_1) \Big\} \mathrm{d}x\mathrm{d}y. \tag{2.22}$$

Since $u(x, y, t_1) = u(x, y, t_2) = 0$ and $u > 0$ on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$, we have that $u_t(x, y, t_1) \geq 0$ and $u_t(x, y, t_2) \leq 0$ for all $(x, y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}}$. This implies that the righthand side of (2.22) is nonpositive, and we have

$$\oiint_{S} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S \leq 0. \tag{2.23}$$

Finally, (2.5), (2.7) and inequality (2.23) imply that

$$\iiint_{\Omega} [g(x, y, t) - f(x, y, t)] uv \mathrm{d}V \leq 0 \tag{2.24}$$

which contradicts with condition (2.2). This contradiction yields that v cannot be a positive solution of problem (1.7)-(1.9) on $\bar{\Omega}$. The same proof can be repeated under the assumption that $v < 0$ on $\bar{\Omega}$. Therefore, v has a zero in $\bar{\Omega}$. Theorem 2.1 has been proved. □

Remark 2.2. *If inequalities* (2.2) *and* (2.3) *in Theorem 2.1 are replaced by the strict ones*

$$g(x, y, t) > f(x, y, t), \quad (x, y, t) \in \Omega \tag{2.25}$$

and

$$s_k(t) > r_k(t), \quad t \in \bar{\mathcal{K}} \quad (k = 1, 2, 3, 4), \tag{2.26}$$

then it can be easily proved that v has a zero in interior of $\bar{\Omega}$.

Proposition 2.3. *(Sturm comparison theorem) Let* $u > 0$ *be a solution of problem* (1.6)-(1.8) *satisfying the boundary condition* (2.1) *on* $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$*. If inequalities* (2.25) *and* (2.26) *hold, then every solution v of problem* (1.7)-(1.9) *has a zero in* Ω.

Remark 2.4. *Inequalities* (2.25) *and* (2.26) *in Proposition 2.3 can be weakened and Proposition 2.3 can be commuted by the following conclusion.*

Proposition 2.5. *(Sturm comparison theorem) Let* $u > 0$ *be a solution of problem* (1.6)-(1.8) *satisfying the boundary condition* (2.1) *on* $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$*, and assume that inequalities* (2.2) *and* (2.3) *hold.*

If either

$$\text{meas}\{(x, y, t) \in \Omega : g(x, y, t) - f(x, y, t) > 0\} > 0 \tag{2.27}$$

or

$$\text{meas}\{t \in \bar{\mathcal{K}} : s_k(t) - r_k(t) > 0, k = 1, 2, 3, 4\} > 0, \tag{2.28}$$

then every solution v of problem (1.7)-(1.9) *has a zero in* Ω.

The following oscillation criterion is immediate.

Corollary 2.6. *(Sturm oscillation theorem) If the inequalities*

$$g(x, y, t) \geq f(x, y, t), \quad (x, y, t) \in \mathcal{I} \times \mathcal{J} \times (t_*, \infty) \tag{2.29}$$

and

$$s_k(t) \geq r_k(t), \quad t \in [t_*, \infty) \quad (k = 1, 2, 3, 4) \tag{2.30}$$

hold for every $t_* \geq t_0$*, then every solution of problem* (1.7)-(1.9) *is oscillatory whenever problem* (1.6)-(1.8) *is oscillatory.*

3. Nonlinear comparison results

The results obtained for linear equations in the previous section can be extended to the nonlinear hyperbolic equations of the form

$$u_{tt} - \Delta u + \mathcal{F}(u, x, y, t) = 0, \quad (x, y, t) \in \Omega \tag{3.1}$$

and

$$v_{tt} - \Delta v + \mathcal{G}(v, x, y, t) = 0, \quad (x, y, t) \in \Omega \tag{3.2}$$

satisfying the initial conditions (1.8) and (1.9), respectively. The functions $r_k(t)$ and $s_k(t)$, $k = 1, 2, 3, 4$, are as previously defined, and $\mathcal{F}, \mathcal{G} : \mathbb{R} \times \bar{\Omega} \to \mathbb{R}$ are continuous functions satisfying

$$\mu\mathcal{F}(\mu, x, y, t) \leq p(t)\mu^2; \quad (\mu, x, y, t) \in \mathbb{R} \times \bar{\Omega}$$

and

$$\mu\mathcal{G}(\mu, x, y, t) \geq q(t)\mu^2; \quad (\mu, x, y, t) \in \mathbb{R} \times \bar{\Omega}$$

for which $p, q : \bar{\mathcal{K}} \to \mathbb{R}$ are continuous functions.

The second primary conclusion of this paper is as follows.

Theorem 3.1. *(Sturm comparison theorem) Let $u > 0$ be a solution of problem* (1.8)-(3.1) *satisfying the boundary condition* (2.1) *on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$. If the inequalities*

$$q(t) \geq p(t) \quad \text{and} \quad s_k(t) \geq r_k(t) \quad (k = 1, 2, 3, 4) \tag{3.3}$$

hold for $t \in \bar{\mathcal{K}}$, then every solution v of problem (1.9)-(3.2) *has a zero in $\bar{\Omega}$.*

The proof of Theorem 3.1 is based on the inequality

$$\begin{aligned} &\left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t \\ =&\left[u\mathcal{G}(v, x, y, t) - v\mathcal{F}(u, x, y, t)\right] \\ \geq&\left[q(t) - p(t)\right]uv \end{aligned}$$

for $u \in C(\overline{\Omega}, \mathbb{R}^+ \cup \{0\})$, $v \in \mathrm{C}\left(\bar{\Omega}, \mathbb{R}^+\right)$, and can be done following the same steps as those in Theorem 2.1. Therefore, it is left to the reader.

Remark 3.2. *If the inequalities given in* (3.3) *are replaced by the strict ones*

$$q(t) > p(t) \quad \text{and} \quad s_k(t) > r_k(t) \quad (k = 1, 2, 3, 4), \tag{3.4}$$

then the comparison conclusion is as follows.

Proposition 3.3. *(Sturm comparison theorem) Let $u > 0$ be a solution of problem* (1.8)-(3.1) *satisfying the boundary condition* (2.1) *$\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$. If the inequalities in* (3.4) *hold for $t \in \bar{\mathcal{K}}$, then every solution v of problem* (1.9)-(3.2) *has a zero in Ω.*

As mentioned in Remark 2.4, Proposition 3.3 can be alternated by the following result.

Proposition 3.4. *(Sturm comparison theorem) Let $u > 0$ be a solution of problem* (1.8)-(3.1) *satisfying the boundary condition* (2.1) *on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$, and assume that the inequalities in* (3.3) *hold for $t \in \bar{\mathcal{K}}$.*

If either

$$\operatorname{meas}\{t \in \bar{\mathcal{K}} : q(t) - p(t) > 0\} > 0$$

or

$$\operatorname{meas}\{t \in \bar{\mathcal{K}} : s_k(t) - r_k(t) > 0, k = 1, 2, 3, 4\} > 0,$$

then every solution v of problem (1.9)-(3.2) *has a zero in Ω.*

The following oscillation criteria is immediate.

Corollary 3.5. *(Sturm oscillation theorem) If the inequalities given in* (3.4) *hold for* $t \in [t^*, \infty)$ *for every* $t^* \geq t_0$*, then every solution of problem* (1.9)-(3.2) *is oscillatory whenever problem* (1.8)-(3.1) *is oscillatory.*

4. An application

Consider the hyperbolic equation (1.6) with only the time dependent potential

$$w_{tt} - \Delta w + \hat{f}(t)w = 0, \quad (x, y, t) \in \mathcal{I} \times \mathcal{J} \times (\hat{t}, \infty) \tag{4.1}$$

satisfying the initial conditions

$$\begin{aligned} &w_x(x_k, y, t) + (-1)^k \alpha_k w(x_k, y, t) = 0, \qquad &(y, t) \in \bar{\mathcal{J}} \times [\hat{t}, \infty), \\ &w_y(x, y_k, t) + (-1)^k \alpha_{k+2} w(x, y_k, t) = 0, \qquad &(x, t) \in \bar{\mathcal{I}} \times [\hat{t}, \infty) \end{aligned} \tag{4.2}$$

for $k = 1, 2$, where $\hat{t} \in [t_0, \infty)$ and α_k's are real constants.

As Eq (4.1) allows a separation of variables, we set $w(x, y, t) = H(x, y)T(t)$. Solving (4.1), we get the eigenvalue problem

$$\Delta H = \lambda H, \quad (x, y) \in \mathcal{I} \times \mathcal{J} \tag{4.3}$$

with the initial conditions

$$\begin{aligned} &H_x(x_k, y) + (-1)^k \alpha_k H(x_k, y) = 0, \qquad &y \in \bar{\mathcal{J}}, \\ &H_y(x, y_k) + (-1)^k \alpha_{k+2} H(x, y_k) = 0, \qquad &x \in \bar{\mathcal{I}} \end{aligned} \tag{4.4}$$

for $k = 1, 2$, and

$$T'' + \hat{f}(t)T = \lambda T, \quad t \in [\hat{t}, \infty).$$

Applying Corollary 2.6, we can derive an interesting oscillation criterion for a class of hyperbolic equations.

Theorem 4.1. *(Sturm oscillation theorem) Let* w *be a nontrivial solution of problem* (4.1)-(4.2) *and assume that equation*

$$T'' + [\hat{f}(t) - \lambda_0]T = 0 \tag{4.5}$$

is oscillatory, where λ_0 *is the first eigenvalue of problem* (4.3)-(4.4).

If the inequalities

$$g(x, y, t) \geq \hat{f}(t), \quad (x, y, t) \in \mathcal{I} \times \mathcal{J} \times [\tilde{t}, \infty) \tag{4.6}$$

and

$$s_k(t) \geq \alpha_k, \quad t \in [\tilde{t}, \infty) \quad (k = 1, 2, 3, 4) \tag{4.7}$$

hold for every $\tilde{t} \geq \hat{t}$*, then every solution* v *of problem* (1.7)-(1.9) *has a zero in* $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times [\tilde{t}, \infty)$.

For the elliptic case, analogous results of Theorem 4.1 can be found in a paper by Kreith [20].

Remark 4.2. *When the potential does not depend on time variable, the technique of Theorem 4.1 also can be applied to the equation*

$$v_{tt} - \Delta v + Q(x, y)v = 0, \quad (x, y, t) \in \mathcal{I} \times \mathcal{J} \times (\hat{t}, \infty)$$

under the analogous initial conditions with (4.2).

5. Conclusions

The paper presented novel Sturmian comparison results for linear and nonlinear hyperbolic equations on a rectangular prism. The results for linear equations provided an extension to those obtained by Kreith in [18], which were founded within a rectangular region in the plane. The results were verified by considering a certain class of hyperbolic equations that were converted to an eigenvalue problem, which enabled us to draw a new and interesting oscillation criterion.

It will be of great interest for the reader to obtain all the results given in this paper on the $(n+1)$-orthotope (hyperrectangle) for the hyperbolic equations of the form

$$u_{tt} - \Delta u + \mathcal{H}(\mathbf{x}, t)u = 0, \quad (\mathbf{x}, t) \in \Gamma \subset \mathbb{R}^n \times (t_0, \infty)$$

under the proper boundary conditions, where $\mathbf{x} = (x_1, \ldots, x_n)$ and Δ is the usual Laplace operator in $\mathbb{R}^n$, i.e.,

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial^2 x_j},$$

and Γ is the hyperrectangle defined by

$$\Gamma := ((x_1)_1, (x_1)_2) \times ((x_2)_1, (x_2)_2) \times \cdots \times ((x_n)_1, (x_n)_2) \times (t_1, t_2)$$

for $(x_j)_1, (x_j)_2 \in \mathbb{R}$ are points on the x_j-axis, and $j = 1, 2, \ldots, n$. The details are left for future consideration.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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