Research article

On the $A_{\alpha^{-}}$-spectra of graphs and the relation between $A_{\alpha}$- and $A_{\alpha^{-}}$-spectra

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Abstract: Let $G$ be a graph with adjacency matrix $A(G)$, and let $D(G)$ be the diagonal matrix of the degrees of $G$. For any real number $\alpha \in [0, 1]$, Nikiforov defined the $A_{\alpha}$-matrix of $G$ as

$$A_{\alpha}(G) = \alpha D(G) + (1 - \alpha)A(G).$$

The eigenvalues of the matrix $A_{\alpha}(G)$ form the $A_{\alpha}$-spectrum of $G$. The $A_{\alpha}$-spectral radius of $G$ is the largest eigenvalue of $A_{\alpha}(G)$ denoted by $\rho_{\alpha}(G)$. In this paper, we propose the $A_{\alpha^{-}}$-matrix of $G$ as

$$A_{\alpha^{-}}(G) = \alpha D(G) + (\alpha - 1)A(G), \quad 0 \leq \alpha \leq 1.$$

Let the $A_{\alpha^{-}}$-spectral radius of $G$ be denoted by $\lambda_{\alpha^{-}}(G)$, and let $S_{\beta}^{\alpha}(G)$ and $S_{\beta}^{\alpha^{-}}(G)$ be the sum of the $\beta$th powers of the $A_{\alpha}$ and $A_{\alpha^{-}}$ eigenvalues of $G$, respectively. We determine the $A_{\alpha^{-}}$-spectra of some graphs and obtain some bounds of the $A_{\alpha^{-}}$-spectral radius. Moreover, we establish a relationship between the $A_{\alpha}$-spectral radius and $A_{\alpha^{-}}$-spectral radius. Indeed, for $\alpha \in (\frac{1}{2}, 1)$, we show that $\lambda_{\alpha^{-}} \leq \rho_{\alpha}$, and we prove that if $G$ is connected, then the equality holds if and only if $G$ is bipartite. Employing this relation, we obtain some upper bounds of $\lambda_{\alpha}(G)$, and we prove that the $A_{\alpha^{-}}$-spectrum and $A_{\alpha}$-spectrum are equal if and only if $G$ is a bipartite connected graph. Furthermore, we generalize the relation established by S. Akbari et al. in (2010) as follows: for $\alpha \in [\frac{1}{2}, 1)$, if $0 < \beta \leq 1$ or $2 \leq \beta \leq 3$, then $S_{\beta}^{\alpha}(G) \geq S_{\beta}^{\alpha^{-}}(G)$, and if $1 \leq \beta \leq 2$, then $S_{\beta}^{\alpha}(G) \leq S_{\beta}^{\alpha^{-}}(G)$, where the equality holds if and only if $G$ is a bipartite graph such that $\beta \notin \{1, 2, 3\}$.

Keywords: Laplacian; singless Laplacian; $A_{\alpha^{-}}$-spectral radius; $A_{\alpha}$-matrix; sum of powers of $A_{\alpha}$-eigenvalues

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1. Introduction

Let

\[ G = (V(G), E(G)) \]

be a simple undirected graph on \( n \) vertices such that \( V(G) \) and \( E(G) \) denote the vertex set and edge set of \( G \), respectively. Let \( V(G) = \{v_1, v_2, \cdots, v_n\} \), for \( v_i \in V(G) \), and \( d_G(v_i) = d(v_i) = d_i \) denotes the degree of vertex \( v_i \) and the set of vertices adjacent to \( v_i \), denoted by \( N(v_i) \), refers to the neighborhood of \( v_i \). The maximum and the minimum degree of \( G \) are denoted by \( \Delta = \Delta(G) \) and \( \delta = \delta(G) \), respectively. \( \lambda(M) \), \( \lambda_k(M) \) and \( \lambda_{\min}(M) \) denote the largest eigenvalue, the \( k \)-th largest eigenvalue, and the smallest eigenvalue of a matrix \( M \), respectively. The set of all eigenvalues for a matrix \( M \) together with their multiplicities is called the \( M \)-spectrum. A real symmetric matrix \( M \) is called positive semidefinite if \( \lambda_{\min}(M) \geq 0 \). Denote by \( K_n \), \( P_n \), \( C_n \) and \( K_{s,n-s} \) the complete graph, path, cycle, and complete bipartite graph with \( n \) vertices, respectively. Let \( A(G) \) and \( D(G) \) be the adjacency matrix and the diagonal matrix of the degrees of \( G \), respectively. The signless Laplacian \( Q(G) \) of \( G \) is defined as

\[ Q(G) = D(G) + A(G). \]

The Laplacian \( L(G) \) of \( G \) is defined as

\[ L(G) = D(G) - A(G). \]

We denote the eigenvalues of \( Q(G) \) and \( L(G) \) by \( \rho_1(G) \) and \( \lambda_k(G) \), respectively, where \( 1 \leq k \leq n \) (we drop \( G \) when it is clear from the context). In particular, the largest eigenvalue of \( Q(G) \) is called the signless Laplacian spectral radius of \( G \), denoted by \( \rho(G) \), and the largest eigenvalue of \( L(G) \) is called the Laplacian spectral radius of \( G \), denoted by \( \lambda(G) \).

In [1], Nikiforov introduced the \( A_\alpha \)-matrix of \( G \), which is the convex linear combination of \( A(G) \) and \( D(G) \) defined by

\[ A_\alpha(G) = \alpha D(G) + (1 - \alpha) A(G), \quad 0 \leq \alpha \leq 1. \]

Clearly,

\[ A_0(G) = A(G), \quad A_1(G) = D(G) \quad \text{and} \quad 2A_{1/2}(G) = Q(G). \]

Thus, the matrix \( A_\alpha(G) \) is a generalization of the adjacency matrix and the signless Laplacian matrix. The largest eigenvalue of \( A_\alpha(G) \) is called the \( A_\alpha \)-spectral radius of \( G \), denoted by \( \rho_\alpha(G) \). Nikiforov [1] had studied the matrix \( A_\alpha(G) \), and he has investigated many properties on \( A_\alpha(G) \), including bounds on the \( k \)-th largest (especially, the largest, the second largest, and the smallest) eigenvalue of \( A_\alpha(G) \), the positive semidefiniteness of \( A_\alpha(G) \), etc. From then on, the study of \( A_\alpha \)-spectra and the \( A_\alpha \)-spectral radius for the graph has attracted the attention of many researchers. In [2], Nikiforov et al. gave an upper bound of the spectral radius of \( A_\alpha(T_\Delta) \), where \( T_\Delta \) is the tree of maximal degree \( \Delta \). Also, he obtained several bounds on the spectral radius of \( A_\alpha \) of general graphs. The graphs with

\[ \lambda_k(A_\alpha(G)) = \alpha n - 1, \quad 2 \leq k \leq n \]

had been characterized by Lin et al. in [3]. In [4], Guo and Zhou gave upper bounds for the \( A_\alpha \)-spectral radius for some graphs under some conditions. Guo and Zhang [5]; obtained a sharp upper bound on the \( A_\alpha \)-spectral radius for \( \alpha \in [\frac{1}{2}, 1) \), and proved that for two connected graphs \( G_1 \) and \( G_2 \),
ρ_{\alpha}(G_1) > \rho_{\alpha}(G_2) \text{ under certain conditions. For further related studies, one may see [6–8].}

The research on \( L(G) \) has shown that it is a remarkable matrix in many respects. Since; \( L(G) \) is just the difference between \( D(G) \) and \( A(G) \), then to understand to what extent each of the summands \( -A(G) \) and \( D(G) \) determines the properties of \( L(G) \), and motivated by the above works, we introduce the \( A_{\alpha}^- \)-matrix defined by

\[
A_{\alpha}^-(G) = \alpha D(G) + (\alpha - 1)A(G), \quad 0 \leq \alpha \leq 1.
\]  

(1.1)

Through our study, several facts indicate that the study of the \( A_{\alpha}^- \)-family is of a unique importance. First of all, obviously

\[
A(G) = -A_0^-(G), \quad D(G) = A_1^-(G) \quad \text{and} \quad L(G) = 2A_{\frac{1}{2}}^-(G).
\]

Since \( A_{\frac{1}{2}}^- \) is essentially equivalent to \( L(G) \), then the matrix \( A_{\alpha}^- \) can be regarded as a generalization of the Laplacian matrix \( L(G) \). Since

\[
A_1(G) = A_{1^-}(G) = D(G);
\]

unless otherwise specified, we only consider the case of \( 0 \leq \alpha < 1 \) in this paper.

In this paper, we study some properties on \( A_{\alpha}^- \) and obtain some bounds of the largest eigenvalue of \( A_{\alpha}^- \). For the relation between the \( Q \)-spectrum and \( L \)-spectrum of a connected graph \( G \), it is well known that \( G \) is bipartite if and only if \( Q(G) \) and \( L(G) \) has the same spectrum ([9, Proposition 1.3.10]), and we extend this relation on the \( A_{\alpha}^- \) and \( A_{\alpha}^- \)-spectrum. In [10], Akbari et al. proved that, for a graph \( G \) of order \( n \) and any real number \( \beta \), if \( 0 < \beta \leq 1 \) or \( 2 \leq \beta \leq 3 \), then

\[
\sum_{k=1}^{n} \rho_k^\beta \geq \sum_{k=1}^{n} \lambda_k^\beta;
\]

and if \( 1 \leq \beta \leq 2 \), then

\[
\sum_{k=1}^{n} \rho_k^\beta \leq \sum_{k=1}^{n} \lambda_k^\beta.
\]

In this work we present the relation between the sum of powers of \( A_{\alpha}^- \) and \( A_{\alpha}^- \) eigenvalues, which is a generalization of the relation between the sum of powers of \( Q(G) \) and \( L(G) \) eigenvalues in [10].

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and lemmas used later. In Section 3, we derive new basic properties of \( A_{\alpha}^- \). In Section 4, we present the \( A_{\alpha}^- \)-spectra of the complete graphs and the complete split graphs. Section 5 gives some bounds of the largest eigenvalue of \( A_{\alpha}^- \). In Section 6, we deduce the relation between the \( A_{\alpha}^- \) and \( A_{\alpha}^- \)-spectral radius. Finally, in Section 7, we prove the relation between the sum of powers of \( A_{\alpha}^- \) and \( A_{\alpha}^- \)-eigenvalues.

2. Preliminaries

Although Weyl’s inequalities have been known for almost a century (see, e.g., [11]), their equality case was first established by So in [12], and for convenience we state Weyl and So’s complete theorem as follows:
Theorem 2.1. Let $A$ and $B$ be Hermitian matrices of order $n$, and let $1 \leq i \leq n$ and $1 \leq j \leq n$. Then,

$$
\lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-n}(A + B), \quad \text{if} \quad i + j \geq n + 1,
$$

(2.1)

$$
\lambda_i(A) + \lambda_j(B) \geq \lambda_{i+j-1}(A + B), \quad \text{if} \quad i + j \leq n + 1.
$$

(2.2)

In either of these inequalities, the equality holds if and only if there exists a nonzero $n$-vector that is an eigenvector to each of the three eigenvalues involved.

A simplified version of (2.1) and (2.2) gives

$$
\lambda_k(A) + \lambda_{\min}(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda(B).
$$

(2.3)

We shall need the following lemmas for our new results.

Lemma 2.2. [3] Let $G$ be a graph of order $n$. If $e \in E(G)$ and $\alpha \geq \frac{1}{2}$, then,

$$
\rho_i(A_{\alpha}(G)) \geq \rho_i(A_{\alpha}(G - e)), \quad \text{for} \quad 1 \leq i \leq n.
$$

Lemma 2.3. [13] Let $B$ be a real symmetric nonnegative irreducible matrix and $\lambda$ be the largest eigenvalue of $B$. $Z \in \mathbb{R}^n$. If $Z^tBZ = \lambda$ and $|Z| = 1$, then $BZ = \lambda Z$.

Lemma 2.4. [5] Let $G$ be a connected graph with $n \geq 4$ vertices and $m$ edges. If $\alpha \in \left[ \frac{1}{2}, 1 \right)$, then,

$$
\rho_{\alpha}(G) \leq \max \left\{ \alpha \Delta(G), (1 - \alpha)(m - \frac{n - 1}{2}) \right\} + 2\alpha.
$$

Equality holds if and only if $\alpha = \frac{n-1}{n+1}$ and $G = K_n$.

3. Basic properties of $A_{\alpha}$-$(G)$

For a graph $G$ of order $n$, suppose that $\lambda$ is an eigenvalue of $A_{\alpha}$-$(G)$ and $x$ is an eigenvector of $A_{\alpha}$-$(G)$ with respect to $\lambda$. We use $x(v)$ to denote the entry of $x$ corresponding to the vertex $v \in V(G)$. It is clear that the system of eigenequations for the matrix $A_{\alpha}$-$(G)$ is

$$
\lambda x(v) = \alpha d_G(v)x(v) + (\alpha - 1) \sum_{u \in N(v)} x(u).
$$

(3.1)

If $G$ is a graph of order $n$ with $A_{\alpha}$-$(G) = A_{\alpha}$, and $x$ is a real vector, the quadratic form $\langle A_{\alpha} - x, x \rangle$ can be represented in several equivalent ways:

$$
\langle A_{\alpha} - x, x \rangle = \sum_{uv \in E(G)} (\alpha x(u)^2 + 2(\alpha - 1)x(u)x(v) + \alpha x(v)^2),
$$

(3.2)

$$
\langle A_{\alpha} - x, x \rangle = (2\alpha - 1) \sum_{u \in V(G)} x(u)^2 d_G(u) + (1 - \alpha) \sum_{uv \in E(G)} (x(u) - x(v))^2,
$$

(3.3)

$$
\langle A_{\alpha} - x, x \rangle = \alpha \sum_{u \in V(G)} x(u)^2 d_G(u) + 2(\alpha - 1) \sum_{uv \in E(G)} x(u)x(v).
$$

(3.4)

Each of these representations can be useful in proofs.

Now, we give some of the spectral properties of the $A_{\alpha}$-matrix. Let us call the largest eigenvalue of $A_{\alpha}$-$(G)$ the $A_{\alpha}$-spectral radius of $G$, and denote it as $\lambda_{\alpha}$-$(G)$. Let us also denote the smallest eigenvalue of $A_{\alpha}$-$(G)$ as $\mu_{\alpha}$-$(G)$. Since $A_{\alpha}$-$(G)$ is a real symmetric matrix, and by using Rayleigh’s principle, the following result holds:
Proposition 3.1. If $\alpha \in [0, 1)$ and $G$ is a graph of order $n$, then

$$\lambda_{\alpha^{-}}(G) = \max_{\|x\|=1} \langle A_{\alpha^{-}}(G)x, x \rangle \quad \text{and} \quad \mu_{\alpha^{-}}(G) = \min_{\|x\|=1} \langle A_{\alpha^{-}}(G)x, x \rangle.$$  \hfill (3.5)

Moreover, if $x$ is a unit $n$-vector, then,

$$\lambda_{\alpha^{-}}(G) = \langle A_{\alpha^{-}}(G)x, x \rangle$$

if and only if $x$ is an eigenvector to $\lambda_{\alpha^{-}}(G)$, and

$$\mu_{\alpha^{-}}(G) = \langle A_{\alpha^{-}}(G)x, x \rangle$$

if and only if $x$ is an eigenvector to $\mu_{\alpha^{-}}(G)$.

By using these relations, the following result is evident:

Proposition 3.2. If $\alpha \in [0, 1)$ and $G$ is a graph, then

$$\lambda_{\alpha^{-}}(G) = \max \{ \lambda_{\alpha^{-}}(H) : H \text{ is a component of } G \},$$

$$\mu_{\alpha^{-}}(G) = \min \{ \mu_{\alpha^{-}}(H) : H \text{ is a component of } G \}.$$

It is clear that if $G$ is a $d$-regular graph of order $n$, then

$$A_{\alpha^{-}}(G) = \alpha d I_n + (\alpha - 1) A(G),$$

and so there is a linear correspondence between the spectra of $A_{\alpha^{-}}(G)$ and of $A(G)$,

$$\lambda_k(A_{\alpha^{-}}(G)) = \alpha d + (\alpha - 1) \lambda_{n-k+1}(A(G)), \quad 1 \leq k \leq n.$$  \hfill (3.6)

In particular, if $k = n$, then

$$\mu_{\alpha^{-}}(G) = (2\alpha - 1)d$$

for any $\alpha \in [0, 1]$. As a consequence of Weyl’s inequality (2.3), the following result is immediate:

Proposition 3.3. If $\alpha \in [0, 1)$ and $G$ is a graph with

$$A(G) = A \quad \text{and} \quad A_{\alpha^{-}}(G) = A_{\alpha^{-}},$$

then,

$$\alpha \delta + (\alpha - 1) \lambda_{n-k+1}(A) \leq \lambda_k(A_{\alpha^{-}}) \leq \alpha \Delta + (\alpha - 1) \lambda_{n-k+1}(A).$$

An important property of the Laplacian $L(G)$ is that it is positive semidefinite. This is certainly not true for $A_{\alpha^{-}}(G)$ if $\alpha < \frac{1}{2}$ and $G$ is a regular graph, but if $\alpha \geq \frac{1}{2}$, then $A_{\alpha^{-}}(G)$ is like $L(G)$. We give this result in the following:

Proposition 3.4. If $\alpha \geq \frac{1}{2}$, and $G$ is a graph, then $A_{\alpha^{-}}(G)$ is positive semidefinite, and if $\alpha > \frac{1}{2}$ and $G$ has no isolated vertices, then $A_{\alpha^{-}}(G)$ is positive definite. Moreover, if $\alpha < \frac{1}{2}$ and $G$ is a regular graph, then $A_{\alpha^{-}}(G)$ is not positive semidefinite.
Proof. Let $x$ be a nonzero vector. If $\alpha \leq \frac{1}{2}$, then for any edge $uv \in E(G)$, we see that
\[
\langle A_{\alpha^2}(G)x, x \rangle \geq (2\alpha - 1)(x(u)^2 + x(v)^2) + (1 - \alpha)(x(u) - x(v))^2 \geq 0.
\]
Hence, $A_{\alpha^2}(G)$ is positive semidefinite. Now, assume that $G$ has no isolated vertices. Choose a vertex $u$ with $x(u) \neq 0$ and let $uv \in E(G)$. Then, (3.7) becomes a strict inequality, and so $A_{\alpha^2}(G)$ is positive definite. Finally, suppose that $G$ is a $d$–regular graph, $\alpha < \frac{1}{2}$, and let $A$ be its adjacency matrix, then $\lambda(A) = d$. Thus
\[
\mu_{\alpha^2}(G) = (2\alpha - 1)d < 0.
\]

By the above proposition we get the following lemma which gives a relation between the $A_{\alpha^2}$-eigenvalues of $G$ and the $A_{\alpha^2}$-eigenvalues of spanning subgraphs of $G$.

Lemma 3.5. Let $G$ be a graph of order $n$ and let $\alpha \in [\frac{1}{2}, 1)$. If $G' = G - e$, where $e \in E(G)$, then,
\[
\lambda_i(G') \leq \lambda_i(G) \text{ for } 1 \leq i \leq n.
\]
Proof. Let $e = uv$ such that $u, v \in V(G)$. It is easy to see that
\[
A_{\alpha^2}(G) = A_{\alpha^2}(G') + M,
\]
where $M$ is the matrix of order $n$ indexed by the vertices of $G$ having $(u, v)^{th}$ and $(v, u)^{th}$ entries both equal to $\alpha - 1$, and the $(u, u)^{th}$ and $(v, v)^{th}$ entries both equal to $\alpha$ and all other entries equal to zero, hence $M$ is an $A_{\alpha^2}$-matrix of a graph containing only one edge. Since $\alpha \in [\frac{1}{2}, 1)$, then $A_{\alpha^2}(G), A_{\alpha^2}(G')$ and $M$ are positive semidefinite and Weyl's inequalities (2.3) imply that $\lambda_i(G') \leq \lambda_i(G)$.

4. The $A_{\alpha^2}$-spectra of some graphs

Equality (3.6) and the fact the eigenvalues of $A(K_n)$ are $\{n - 1, -1, \cdots, -1\}$ give the spectrum of $A_{\alpha^2}(K_n)$ as follows:

Proposition 4.1. The eigenvalues of $A_{\alpha^2}(K_n)$ are
\[
\mu_{\alpha^2}(K_n) = (2\alpha - 1)(n - 1) \text{ and } \lambda_k(A_{\alpha^2}(K_n)) = \alpha(n - 2) + 1 \text{ for } 1 \leq k \leq n - 1.
\]
If $S \subseteq V(G)$, then we use $G[S]$ to denote the subgraph of $G$ induced by $S$. Recall that $G[S]$ is an independent set if no two vertices of $S$ are adjacent and $G[S]$ is a clique if it is a complete subgraph of $G$. The graph $K_r \lor (n - r)K_1$ is called a complete split graph, denoted by $CS_{r,n-r}$. The work in the following proposition is motivated by the proof of [3, Proposition 2.4].

Proposition 4.2. Let $G$ be a graph with $A_{\alpha^2}(G)$, and $\alpha \in [0, 1)$. Let $S \subseteq V(G)$ and $|S| = k$. Suppose that $d_G(u) = d$ for each vertex $u \in S$, and $N(v) \setminus \{w\} = N(w) \setminus \{v\}$ for any two vertices $v, w \in S$. Then, we have the following statements:
(i) If $G[S]$ is a clique, then $\alpha(d - 1) + 1$ is an eigenvalue of $A_{\alpha^2}(G)$ with multiplicity at least $k - 1$.
(ii) If $G[S]$ is an independent set, then $\alpha d$ is an eigenvalue of $A_{\alpha^2}(G)$ with multiplicity at least $k - 1$. 
Proof. Let \( S = \{v_1, v_2, \cdots, v_k\} \). Clearly, \( d_1 = \cdots = d_k = d \). Let \( z_1, z_2, \cdots, z_{k-1} \) be vectors such that
\[
\begin{align*}
  z_i(v_1) &= 1, \\
  z_i(v_{i+1}) &= -1, \\
  z_i(v) &= 0, \quad \text{if } v \in V(G) \setminus \{v_1, v_{i+1}\},
\end{align*}
\]
for \( i = 1, \cdots, k - 1 \). Assume that \( G[S] \) is a clique. It is easy to obtain that
\[
A_{\alpha}(G)z_i = (\alpha(d_1 - 1) + 1, 0, \cdots, 0, \alpha(1 - d_i) - 1, 0, \cdots, 0)' = (\alpha(d - 1) + 1)z_i,
\]
for \( i = 1, \cdots, k - 1 \). Hence, \( \alpha(d - 1) + 1 \) is an eigenvalue of \( A_{\alpha}(G) \) and \( z_1, z_2, \cdots, z_{k-1} \) are eigenvectors of \( A_{\alpha}(G) \) corresponding to \( \alpha(d - 1) + 1 \). In addition, since \( z_1, z_2, \cdots, z_{k-1} \) are linearly independent, the multiplicity of \( \alpha(d - 1) + 1 \) is at least \( k - 1 \). Now, supposing that \( G[S] \) is an independent set, we have
\[
A_{\alpha}(G)z_i = (\alpha d_1, 0, \cdots, 0, -\alpha d_i, 0, \cdots, 0)' = \alpha dz_i,
\]
for \( i = 1, \cdots, k - 1 \). Since \( z_1, z_2, \cdots, z_{k-1} \) are linearly independent, it follows that \( \alpha d \) is an eigenvalue of \( A_{\alpha}(G) \) with multiplicity at least \( k - 1 \). Thus, the proof is completed. \( \square \)

Consider an \( n \times n \) real symmetric matrix
\[
S = \begin{pmatrix}
S_{11} & S_{12} & \cdots & S_{1t} \\
S_{21} & S_{22} & \cdots & S_{2t} \\
\vdots & \vdots & \ddots & \vdots \\
S_{t1} & S_{t2} & \cdots & S_{tt}
\end{pmatrix},
\]
whose rows and columns are partitioned according to a partitioning \( P_1, P_2, \cdots, P_t \) of \( \{1, 2, \cdots, n\} \). The quotient matrix \( B \) of the matrix \( S \) is the \( t \times t \) matrix whose entries are the average row sums of the blocks \( S_{ij} \) of \( S \). The partition is equitable if each block \( S_{ij} \) of \( S \) has constant row sum.

Lemma 4.3. [14] Let \( B \) be an equitable quotient matrix of a symmetric real matrix \( S \). If \( \lambda \) is an eigenvalue of \( B \), then \( \lambda \) is also an eigenvalue of \( S \).

Now, we can determine all \( A_{\alpha} \)-eigenvalues of \( CS_{r,n-r} \) as follows:

Proposition 4.4. The \( A_{\alpha} \)-spectrum of \( CS_{r,n-r} \) contains \( \alpha(n - 2) + 1 \) with multiplicity \( r - 1 \), \( ar \) with multiplicity \( n - r - 1 \), and the remaining two \( A_{\alpha} \)-eigenvalues are
\[
\frac{\alpha(n + 2(r - 1)) + 1 \pm \sqrt{(\alpha(n + 2(r - 1)) + 1)^2 + 4r(1 - 2\alpha)(n - r + \alpha(r - 1))}}{2}.
\]

Proof. We can write \( A_{\alpha}(CS_{r,n-r}) \) as
\[
A_{\alpha}(CS_{r,n-r}) = \begin{pmatrix}
(1 - \alpha)I_{r,r} + (an - 1)I_r & (1 - \alpha)\alpha I_{n-r,n-r} \\
(1 - \alpha)\alpha I_{r,n-r} & \alpha r I_{n-r,n-r}
\end{pmatrix}.
\]
Then, the quotient matrix of \( A_{\alpha}(CS_{r,n-r}) \) is equitable and it can be written in the form
\[
B(A_{\alpha}(CS_{r,n-r})) = \begin{pmatrix}
(n-r)\alpha + r - 1 & (1 - \alpha)(n - r) \\
(1 - \alpha)r & ar
\end{pmatrix}.
\]
Thus, by Lemma 4.3, the eigenvalues of \( B(A_{\alpha}(CS_{r,n-r})) \) are eigenvalues of \( A_{\alpha}(CS_{r,n-r}) \), and according to Proposition 4.2, we get the result. \( \square \)
5. The largest eigenvalue $\lambda_\alpha(G)$

In this section we give a few general bounds on $\lambda_\alpha(G)$.

**Proposition 5.1.** Let $G$ be a graph, with

$$\Delta(G) = \Delta, \quad A(G) = A \quad \text{and} \quad D(G) = D.$$ 

The following inequalities hold for $\lambda_\alpha(G)$:

$$\lambda_\alpha(G) \geq \alpha \Delta + (\alpha - 1) \lambda(A), \quad (5.1)$$

$$\lambda_\alpha(G) \geq (2\alpha - 1) \lambda(A). \quad (5.2)$$

**Proof.** Inequality (5.1) follows by Weyl’s inequalities (2.3) because

$$\alpha \Delta + (\alpha - 1) \lambda(A) = \lambda(\alpha D) + (\alpha - 1) \lambda(A) \leq \lambda(\alpha D + (\alpha - 1) A) = \lambda_\alpha(G).$$

To prove the inequality (5.2), let $H$ be a component of $G$ such that $\lambda(A) = \lambda(A(H))$. Let $x$ be a positive unit vector to $\lambda(A(H))$. For every edge $uv \in E(H)$, the AM-GM inequality implies that

$$2x(u)x(v) = 2\alpha x(u)x(v) + 2(1 - \alpha)x(u)x(v) \leq \alpha \frac{(x(u) + x(v))^2}{2} + 2(1 - \alpha)x(u)x(v) = \frac{1}{2}(\alpha x(u)^2 + 2(\alpha - 1)x(u)x(v) + \alpha x(v)^2) + (3 - 2\alpha)x(u)x(v).$$

Summing this inequality over all edges $uv \in E(H)$, and using (3.2), we get

$$\lambda(A) = \lambda(A(H)) = \langle A(H)x, x \rangle$$

$$\leq \frac{1}{2} \langle A_\alpha'(H)x, x \rangle + \frac{1}{2}(3 - 2\alpha) \lambda(A);$$

and then we get

$$2\lambda(A) - (3 - 2\alpha) \lambda(A) \leq \langle A_\alpha'(H)x, x \rangle \leq \lambda_\alpha(G);$$

hence

$$(2\alpha - 1) \lambda(A) \leq \lambda_\alpha(G). \quad \Box$$

Having inequality (5.2) in hand, if $\alpha \geq \frac{1}{2}$, then every lower bound of $\lambda(A)$ gives a lower bound on $\lambda_\alpha(G)$, but if $\alpha < \frac{1}{2}$, then every upper bound of $\lambda(A)$ gives a lower bound on $\lambda_\alpha(G)$. We mention just two such bounds.

**Corollary 5.2.** Let $G$ be a graph such that $\alpha \geq \frac{1}{2}$. If $G$ is of order $n$ and has $m$ edges, then

$$\lambda_\alpha(G) \geq (2\alpha - 1) \frac{2m}{n}.$$
Corollary 5.3. Let $G$ be a connected graph such that $\alpha < \frac{1}{2}$. If $G$ is of order $n$ and has $m$ edges, then
$$\lambda_{\alpha-}(G) \geq (2\alpha - 1) \sqrt{2m - n + 1}.$$ 

Proposition 5.4. Let $G$ be a graph of order $n$. If $\alpha \in (\frac{1}{2}, 1]$, then,
$$\lambda_{\alpha-}(G) \leq \alpha(n - 2) + 1.$$
Moreover,
$$\lambda_{\alpha-}(G) = \alpha(n - 2) + 1$$
with multiplicity $k - 1$ if $G$ has $k$ vertices of degree $n - 1$.

Proof. Applying Lemma 3.5 leads to
$$\lambda_{\alpha-}(G) \leq \lambda_{\alpha-}(K_n) = \alpha(n - 2) + 1.$$ 
If $G$ has $k$ vertices of degree $n - 1$, then it follows from Proposition 4.2 that $\alpha(n - 2) + 1$ is an eigenvalue of $A_{\alpha-}(G)$ with multiplicity at least $k - 1$, and since
$$\lambda_{\alpha-}(G) \leq \alpha(n - 2) + 1,$$
we get
$$\lambda_{\alpha-}(G) = \alpha(n - 2) + 1$$
with multiplicity $k - 1$. \qed

6. The relationship between $A_{\alpha-}$ and $A_{\alpha}$-spectral radius

Let $G$ be a connected graph. Merris [15] pointed out $\lambda(L(G)) \leq \rho(Q(G))$, and the equality holds if $G$ is a bipartite graph. In the next result, we generalize this result to the $A_{\alpha-}$- and $A_{\alpha}$-spectral radius of a connected graph.

Theorem 6.1. Let $G$ be a graph of order $n$, $\alpha \in (0, 1)$, $\lambda_{\alpha-}(G) = \lambda_{\alpha}$ and $\rho_{\alpha}(G) = \rho_{\alpha}$. We have $\lambda_{\alpha-} \leq \rho_{\alpha}$. Moreover, if $G$ is connected, then the equality holds if and only if $G$ is bipartite.

Proof. Let
$$V(G) = \{v_1, v_2, \ldots, v_n\}, \quad x = (x_1, x_2, \ldots, x_n)^t \in \mathbb{R}^n$$
be an arbitrary vector such that $\|x\| = 1$. Let
$$y = (y_1, y_2, \ldots, y_n)^t \in \mathbb{R}^n$$
be a unit eigenvector of $A_{\alpha-}(G)$ belonging to $\lambda_{\alpha-}$ and
$$y' = (y'_1, y'_2, \ldots, y'_n)^t \in \mathbb{R}^n$$
be a unit eigenvector of $A_{\alpha}(G)$ belonging to $\rho_{\alpha}$. Let
$$|y| = (|y_1|, |y_2|, \ldots, |y_n|)^t.$$
First, we prove that $\lambda_{\alpha^-} \leq \rho_{\alpha}$. 

\[
\lambda_{\alpha^-} = \max x' A_{\alpha^-} x \\
= \max x' (\alpha D + (\alpha - 1) A) x \\
= \max \left( \alpha \sum_{i=1}^{n} x_i^2 d_i + 2(\alpha - 1) \sum_{v \neq v' \in E(G)} x_i x_j \right) \\
= y' A_{\alpha^-} y \\
= \alpha \sum_{i=1}^{n} y_i^2 d_i + 2(\alpha - 1) \sum_{v \neq v' \in E(G)} y_i y_j
\]

and

\[
\rho_{\alpha} = \max x' A_{\alpha} x \\
= \max x' (\alpha D + (1 - \alpha) A) x \\
= \max \left( \alpha \sum_{i=1}^{n} x_i^2 d_i + 2(1 - \alpha) \sum_{v \neq v' \in E(G)} x_i x_j \right) \\
= y' A_{\alpha} y' \\
= \alpha \sum_{i=1}^{n} y_i^2 d_i + 2(1 - \alpha) \sum_{v \neq v' \in E(G)} y_i y_j'.
\]

Thus,

\[
\lambda_{\alpha^-} = \alpha \sum_{i=1}^{n} y_i^2 d_i + 2(\alpha - 1) \sum_{v \neq v' \in E(G)} y_i y_j \leq \alpha \sum_{i=1}^{n} y_i^2 d_i + 2(1 - \alpha) \sum_{v \neq v' \in E(G)} |y_i y_j| = |y'A_{\alpha}y| \\
\leq \max x' A_{\alpha} x = \rho_{\alpha}.
\] (6.1)

Now, if $G$ is bipartite, then the matrix $A_{\alpha^-}$ and the matrix $A_{\alpha}$ are similar by a diagonal matrix $D'$ with diagonal entries $\pm 1$, that is, $A_{\alpha} = D'A_{\alpha^-} D'^{-1}$. Therefore, $A_{\alpha^-}$ and $A_{\alpha}$ have the same spectrum, and thus we get $\lambda_{\alpha^-} = \rho_{\alpha}$. Finally, when $G$ is connected and $\lambda_{\alpha^-} = \rho_{\alpha}$, all inequalities (6.1) must be equalities.

By Lemma 2.3 and the equality

\[|y'A_{\alpha}y| = \rho_{\alpha};\]

we know that $|y|$ is an eigenvector of $A_{\alpha}$ belonging to $\rho_{\alpha}$. So, $|y| = \pm y'$. Using the Perron-Frobenius’ theorem for $A_{\alpha}(G)$, we have $y' > 0$, $|y| = y'$, and $|y_i| > 0$, $i = 1, 2, \ldots, n$.

Since

\[
\alpha \sum_{i=1}^{n} y_i^2 d_i + 2(\alpha - 1) \sum_{v \neq v' \in E(G)} y_i y_j = \alpha \sum_{i=1}^{n} y_i^2 d_i + 2(1 - \alpha) \sum_{v \neq v' \in E(G)} |y_i y_j|;
\]

\[A_{\alpha'} A_{\alpha} x = \rho_{\alpha} x;\]
we get
\[-\sum_{v_i,v_j \in E(G)} y_i y_j = \sum_{v_i,v_j \in E(G)} |y_i y_j|,
\]
hence, \( |y_i y_j| = -y_i y_j \) when \( v_i v_j \in E(G) \). Therefore, \( y_i y_j < 0 \) if \( v_i v_j \in E(G) \).

Let \( U = \{ v_i : y_i > 0 \} \) and \( W = \{ v_j : y_j < 0 \} \).

For each edge \( e = v_i v_j \), we have \( y_i y_j < 0 \), one of the vertices of edge \( e \) is in \( U \), and the other is in \( W \).

So, \( G \) is a bipartite graph. \( \square \)

**Remark 6.2.** If \( G \) is not bipartite and is not connected, and \( \lambda \) is the largest eigenvalue of \( A_{\alpha^-} \) for a bipartite connected component of \( G \), then the equality in Theorem 6.1 still holds.

**Example 6.3.** Take
\[
G = C_3 \cup C_4.
\]
Then, \( G \) is not bipartite and not connected. Now, \( A_{\frac{1}{2}}(G) \) has a spectrum \( 2^{[1]}, 1.25^{[2]}, 0.5^{[2]}, -1^{[2]} \) (where \( \lambda^{[i]} \) means the eigenvalue \( \lambda \) is repeated \( i \) times in the spectrum). On the other hand, \( A_{\frac{1}{2}}(G) \) has a spectrum \( 2^{[2]}, 0.5^{[2]}, -0.25^{[2]}, -1^{[1]} \). Then, we have
\[
\lambda_{\frac{1}{2}}(G) = \rho_{\frac{1}{2}}(G) = 2.
\]

Note that 2 is the largest eigenvalue of \( A_{\frac{1}{2}}(C_4) \) as well.

Now, we introduce the relationship between the \( A_{\alpha^-} \) and \( A_{\alpha^-} \)-spectra of bipartite graphs, which is a generalization [9, Proposition 1.3.10], and it follows from the proof of the above theorem.

**Corollary 6.4.** Let \( G \) be a connected graph. Then, \( G \) is bipartite if and only if the \( A_{\alpha^-} \)-spectrum and \( A_{\alpha^-} \)-spectrum are equal.

**Remark 6.5.** In fact, if \( G \) is bipartite and is not connected, the \( A_{\alpha^-} \)-spectrum still equals the \( A_{\alpha^-} \)-spectrum.

**Example 6.6.** Take \( G = P_3 \cup C_4 \). Then \( G \) is bipartite and not connected. We have that \( A_{\frac{1}{2}}(G) \) has a spectrum \( 2^{[1]}, 1.64039^{[1]}, 1.5^{[2]}, 1^{[1]}, 0.75^{[1]}, 0.609612^{[1]} \). In contrast, \( A_{\frac{1}{2}}(G) \) has the same spectrum as \( A_{\frac{1}{2}}(G) \), although \( G \) is not connected.

According to Corollary 6.4 and [1, Propositions 38 and 39], we get the following two results:

**Corollary 6.7.** The \( A_{\alpha^-} \)-spectrum and the \( A_{\alpha^-} \)-spectrum of the complete bipartite graph \( K_{a,b} \) are equal, that is, if \( a \geq b \geq 1 \) and \( \alpha \in (0, 1) \), the eigenvalues of \( A_{\alpha^-}(K_{a,b}) \) are
\[
\lambda_{\alpha^-}(K_{a,b}) = \frac{1}{2} \left( \alpha(a + b) + \sqrt{\alpha^2(a + b)^2 + 4ab(1 - 2\alpha)} \right),
\]
\[
\mu_{\alpha^-}(K_{a,b}) = \frac{1}{2} \left( \alpha(a + b) - \sqrt{\alpha^2(a + b)^2 + 4ab(1 - 2\alpha)} \right),
\]
\[
\lambda_k(A_{\alpha^-}(K_{a,b})) = \alpha a \quad \text{for} \quad 1 < k \leq b,
\]
\[
\lambda_k(A_{\alpha^-}(K_{a,b})) = \alpha b \quad \text{for} \quad b < k < a + b.
\]
Corollary 6.8. The $A_{\alpha}$-spectrum and the $A_{\alpha}$-spectrum of the star $K_{1,n-1}$ are equal, that is, the eigenvalues of $A_{\alpha}(K_{1,n-1})$ are

$$
\lambda_{\alpha}(K_{1,n-1}) = \frac{1}{2} \left( \alpha n + \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha)} \right),
$$

$$
\mu_{\alpha}(K_{1,n-1}) = \frac{1}{2} \left( \alpha n - \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha)} \right),
$$

$$\lambda_1(A_{\alpha}(K_{1,n-1})) = \alpha \text{ for } 1 < k < n.$$

Indeed, many practical results of Theorem 6.1 can be deduced, and here are some of them.

Proposition 6.9. Let $G$ be a connected graph with $n \geq 4$ vertices and $m$ edges. If $\alpha \in [\frac{1}{2}, 1)$, then

$$
\lambda_{\alpha}(G) < \max \left\{ \alpha \Delta(G), (1-\alpha)(m - \frac{n-1}{2}) \right\} + 2\alpha. \quad (6.2)
$$

**Proof.** Let

$$
\lambda_{\alpha}(G) = \lambda_{\alpha} \text{ and } \rho_{\alpha}(G) = \rho_{\alpha}.
$$

Then, Theorem 6.1 and Lemma 2.4 lead to

$$
\lambda_{\alpha} \leq \max \left\{ \alpha \Delta(G), (1-\alpha)(m - \frac{n-1}{2}) \right\} + 2\alpha. \quad (6.3)
$$

Suppose that the equality in (6.3) holds, thus $\rho_{\alpha} = \lambda_{\alpha}$, and so $G$ is bipartite and

$$
\rho_{\alpha} = \max \left\{ \alpha \Delta(G), (1-\alpha)(m - \frac{n-1}{2}) \right\} + 2\alpha.
$$

Therefore, by Lemma 2.4, $G = K_n$, and thus $G$ is bipartite if and only if $n = 2$, but $n \geq 4$, and hence the inequality is strict. $\square$

Theorem 6.1 and [2, Theorem 2] lead directly to the next result.

Proposition 6.10. If $T$ is a tree of order $n$ and $\alpha \in [0, 1]$, then

$$
\lambda_{\alpha}(T) \leq \frac{n\alpha + \sqrt{n^2 \alpha^2 + 4(n-1)(1-2\alpha)}}{2}.
$$

The equality holds if and only if $T$ is the star $K_{1,n-1}$.

By Theorem 6.1 and [1, Proposition 20], we get the next result.

Proposition 6.11. If $G$ is a graph with no isolated vertices, then

$$
\lambda_{\alpha}(G) \leq \max_{u \in V(G)} \left\{ \alpha d(u) + \frac{1-\alpha}{d(u)} \sum_{v \in E(G)} d(v) \right\}.
$$

If $\alpha \in (\frac{1}{2}, 1)$ and $G$ is connected, the equality holds if and only if $G$ is a regular bipartite graph.
In this part we give two explicit expressions for the sums and the sum of squares of the eigenvalues of $A_{\alpha}$ and $A_{\alpha}$, considering [1, Propositions 34 and 35].

**Proposition 6.12.** If $G$ is a graph of order $n$ and has $m$ edges, then

\[
\sum_{i=1}^{n} \lambda_i(A_{\alpha}^{-}(G)) = \sum_{i=1}^{n} \lambda_i(A_{\alpha}(G)) = trA_{\alpha}^{-}(G) = trA_{\alpha}(G);
\]

where

\[
\sum_{i=1}^{n} \lambda_i(A_{\alpha}^{-}(G)) = \alpha \sum_{u \in V(G)} d(u) = 2\alpha m.
\]

A similar formula for the sum of the squares of the $A_{\alpha}$ and $A_{\alpha}$-eigenvalues is given as follows:

**Proposition 6.13.** If $G$ is a graph of order $n$ and has $m$ edges, then

\[
\sum_{i=1}^{n} \lambda_i^2(A_{\alpha}^{-}(G)) = \sum_{i=1}^{n} \lambda_i^2(A_{\alpha}(G)) = trA_{\alpha}^2(G) = trA_{\alpha}^2(G);
\]

where

\[
\sum_{i=1}^{n} \lambda_i^2(A_{\alpha}^{-}(G)) = \alpha^2 \sum_{u \in V(G)} d^2(u) + 2(1 - \alpha)^2 m.
\]

**Proof.** Let

\[
A_{\alpha}^{-} = A_{\alpha}^{-}(G), \quad A = A(G) \text{ and } D = D(G).
\]

Calculating the square $A_{\alpha}^2$ and taking its trace, we find that

\[
trA_{\alpha}^2 = tr(\alpha^2 D^2 + (1 - \alpha)^2 A^2 + \alpha(\alpha - 1)DA + \alpha(\alpha - 1)AD)
\]

\[
= \alpha^2 trD^2 + (1 - \alpha)^2 trA^2 - \alpha(1 - \alpha)trDA + \alpha(\alpha - 1)trAD
\]

\[
= \alpha^2 \sum_{u \in V(G)} d^2(u) + 2(1 - \alpha)^2 m.
\]

\[\square\]

7. **Sum of powers of $A_{\alpha}$ and $A_{\alpha}$ eigenvalues**

Pirzada et al. [16] introduced the sum of the $\beta^n$ powers of the $A_{\alpha}$ eigenvalues of $G$ as

\[
S_{\beta}(G) = \sum_{i=1}^{n} \rho_i^\beta.
\]

Now, we have the notation

\[
S_{\beta}^{A_{\alpha}}(G) = \sum_{i=1}^{n} \lambda_i^\beta
\]

for the sum of the $\beta^n$ powers of the $A_{\alpha}$-eigenvalues of $G$. The following theorem is a generalization [10, Theorem 2].
Theorem 7.1. Let $G$ be a graph of order $n > 1$, $\alpha \in [\frac{1}{2}, 1)$, and let $\beta$ be a real number.

(i) If $0 < \beta \leq 1$ or $2 \leq \beta \leq 3$, then

$$S_{\beta}^{\alpha}(G) \geq S_{\beta}^{\alpha -}(G).$$

(ii) If $1 \leq \beta \leq 2$, then

$$S_{\beta}^{\alpha}(G) \leq S_{\beta}^{\alpha -}(G).$$

For $\beta \in (0, 1) \cup (2, 3)$, the equality holds in (i) if and only if $G$ is a bipartite graph. Moreover, for $\beta \in (1, 2)$, the equality holds in (ii) if and only if $G$ is a bipartite graph.

Proof. We recall that, for any real number $r$, the binomial series

$$\sum_{k=0}^{\infty} \binom{r}{k} x^k$$

converges to $(1 + x)^r$ if $|x| < 1$. This also remains true for $x = -1$ if $r > 0$ (see, e.g., [17, p. 419]). By Lemma 3.5, we find that,

$$\lambda_\alpha^-(G) = \alpha(n-2) + 1 \quad \text{and} \quad \rho_\alpha(G) = \rho_\alpha(K_n) = n - 1.$$

Since $A_\alpha$ is positive semidefinite when $\alpha \in [\frac{1}{2}, 1)$, then if $\rho_i > 0$, we have

$$\left| \frac{\rho_i}{n} - 1 \right| \leq \left| \frac{n-1}{n} - 1 \right| = \left| -\frac{1}{n} \right| < 1$$

and if $\rho_i = 0$, we get

$$\frac{\rho_i}{n} - 1 = -1.$$

Therefore,

$$\frac{S_{\beta}^{\alpha}(G)}{n^\beta} = \left( \frac{\rho_1}{n} \right)^\beta + \cdots + \left( \frac{\rho_n}{n} \right)^\beta$$

$$= \sum_{k=0}^{\infty} \binom{\beta}{k} (\frac{\rho_1}{n} - 1)^k + \cdots + \sum_{k=0}^{\infty} \binom{\beta}{k} (\frac{\rho_n}{n} - 1)^k$$

$$= \sum_{k=0}^{\infty} \binom{\beta}{k} \left( \frac{1}{n} (\alpha D + (1 - \alpha)A) - I \right)^k.$$

Also, since $A_\alpha^-$ is positive semidefinite when $\alpha \in [\frac{1}{2}, 1)$, then if $\lambda_i > 0$,

$$\left| \frac{\lambda_i}{n} - 1 \right| \leq \left| \frac{\alpha(n-2) + 1}{n} - 1 \right| = \left| (\alpha - 1) + \frac{1 - 2\alpha}{n} \right| < 1$$

and if $\lambda_i = 0$, we have

$$\frac{\lambda_i}{n} - 1 = -1.$$

Thus, in a similar manner as above, we obtain that

$$\frac{S_{\beta}^{\alpha -}(G)}{n^\beta} = \sum_{k=0}^{\infty} \binom{\beta}{k} \left( \frac{1}{n} (\alpha D + (\alpha - 1)A) - I \right)^k.$$
We claim that

if \( k \) is even, \( \text{tr}(\alpha D + (1 - \alpha)A - nI)^k \leq \text{tr}(\alpha D + (\alpha - 1)A - nI)^k \);

if \( k \) is odd, \( \text{tr}(\alpha D + (1 - \alpha)A - nI)^k \geq \text{tr}(\alpha D + (\alpha - 1)A - nI)^k \).

When \((\alpha D - nI) + (1 - \alpha)A)^k\) and \((\alpha D - nI) + (\alpha - 1)A)^k\) are expanded in terms of the powers of \(\alpha D - nI\) and \((1 - \alpha)A\), respectively, the terms appearing in both expansions, regardless of their signs, are the same. To prove this claim, we identify the sign of each term in both expansions. Consider the terms in the expansion of \((\alpha D - nI) + (1 - \alpha)A)^k\), where there are exactly \( j \) factors equal to \(\alpha D - nI\), for some \( j = 0, 1, \ldots, k \). The sign of the trace for each of these terms is \((-1)^j\) or 0 because all entries of \(\alpha D - nI\) and \((1 - \alpha)A\) are non-positive and non-negative, respectively. On the other hand, in each term in the expansion of \((\alpha D - nI) + (\alpha - 1)A)^k\) all factors are matrices with non-positive entries, hence the sign of the trace of each term is \((-1)^k\) or 0. Therefore, the claim has been proven.

Now, note that if \( 0 < \beta < 1 \) or \( 2 < \beta < 3 \), then the sign of \( \binom{\beta}{k} \) is \((-1)^{k-1}\), except that \( \binom{\beta}{1} > 0 \), for \( 2 < \beta < 3 \). According to this, for \( 0 < \beta < 1 \) and every \( k \),

\[
\binom{\beta}{k} \text{tr}(\alpha D + (1 - \alpha)A - nI)^k \geq \binom{\beta}{k} \text{tr}(\alpha D + (\alpha - 1)A - nI)^k.
\]

This inequality remains true for \( 2 \leq \beta \leq 3 \) as

\[
\text{tr}(\alpha D + (1 - \alpha)A - nI)^2 = \text{tr}(\alpha D + (\alpha - 1)A - nI)^2,
\]

since \( \text{tr}A^2 = \text{tr}A^2 \). Thus, Part (i) is proved. For \( 1 < \beta < 2 \), the sign of \( \binom{\beta}{k} \) is \((-1)^{k-1}\), except that \( \binom{\beta}{1} > 0 \). Since \( \text{tr}A = \text{tr}A^2 \), we have

\[
\text{tr}(\alpha D + (1 - \alpha)A - nI) = \text{tr}(\alpha D + (\alpha - 1)A - nI),
\]

and so part (ii) is similarly proved.

Now, we examine the equality case. Since \( A_\alpha \) and \( A_{\alpha^{-}} \) are similar if \( G \) is bipartite, it follows that the equality holds in both (i) and (ii). Since for any positive integer \( i \), \( \text{tr}A^i \) equals the total number of closed walks of length \( i \) in \( G \), then if \( G \) is not bipartite, there exists an odd integer \( r \) such that \( \text{tr}A^r > 0 \) (see [18, Lemma 2.5]). Hence,

\[
\text{tr}(\alpha D + (1 - \alpha)A - nI)^r > \text{tr}(\alpha D + (\alpha - 1)A - nI)^r;
\]

and so the inequalities in both (i) and (ii) are strict. \( \square \)

We know that the \( A_\alpha \)-spectra and \( A_{\alpha^{-}} \)-spectra of \( K_n \) are \( \{(n - 1)^{[1]}, (an - 1)^{[n-1]}\} \) and \( \{((2\alpha - 1)(n - 1))^{[1]}, (a(n - 2) + 1)^{[n-1]}\} \), respectively. Therefore,

\[
S^\alpha_{\beta}(K_n) - S^\alpha_{\beta}(K_n) = (n - 1)^{\beta} + (n - 1)(an - 1)^{\beta} - ((2\alpha - 1)(n - 1))^{\beta} - (n - 1)(a(n - 2) + 1)^{\beta}.
\]

By Theorem 7.1 and based on our numerical experiments, we propose the following conjecture:

**Conjecture 7.2.** For every \( \alpha \in (\frac{1}{2}, 1) \) and each integer \( n \geq 3 \), we have

\[
S^\alpha_{\beta}(K_n) - S^\alpha_{\beta}(K_n) \geq 0
\]

for any \( \beta \in [0, 1] \cup [2, \infty) \) and

\[
S^\alpha_{\beta}(K_n) - S^\alpha_{\beta}(K_n) \leq 0
\]

for any \( \beta \in (-\infty, 0) \cup (1, 2) \).
8. Conclusions

In this paper, we have introduced the \( A_\alpha \)-matrix of a graph \( G \), which is a generalization of the Laplacian matrix \( L(G) \), and we have studied the basic properties of \( A_\alpha \); and derived some bounds for its spectral radius. Furthermore, we have determined the \( A_\alpha \)-spectra for the complete graph and the complete split graph. Building upon previous results, we have extended findings related to the spectral radius of \( L(G) \) and \( Q(G) \) matrices to the \( A_\alpha \)-spectral radius. Specifically, in Theorem 6.1, we have generalized Merris’ [15] observation that \( \lambda(L(G)) \leq \rho(Q(G)) \) with equality holding for bipartite graphs. Additionally, we have extended the known relation that \( G \) is bipartite if and only if \( Q(G) \) and \( L(G) \) share the same spectrum, as demonstrated in Corollary 6.4. Finally, in Theorem 7.1; we have generalized a relation established by S. Akbari et al. in [10]; which relates the sum of powers of the eigenvalues of \( Q(G) \) and \( L(G) \). In conclusion, these findings have implications for various applications involving graph analysis.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References


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