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Some matrix inequalities related to norm and singular values

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Abstract: In this short note, we presented a new proof of a weak log-majorization inequality for normal matrices and obtained a singular value inequality related to positive semi-definite matrices. What's more, we also gave an example to show that some conditions in an existing norm inequality are necessary.

Keywords: singular values; weak log-majorization; normal matrices
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1. Introduction

Let M_n be the space of $n \times n$ complex matrices. If A is a Hermitian element of M_n, then we enumerate its eigenvalues as $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ (see [1] for more details). The singular values of $A \in M_n$ are defined to be the nonnegative square roots of the eigenvalues of A^*A, where A^* denotes the conjugate transpose of a matrix A, i.e., $s_i(A) = \lambda_i(|A|)$ $(1 \leq i \leq n)$ for $|A| = (A^*A)^{\frac{1}{2}}$. The notation $A \geq (>) \, 0$ is used to mean that A is positive semi-definite (positive definite).

Given a real vector $x = (x_1, x_2, \cdots, x_n) \in R^n$, we rearrange its components as $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. For $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in R^n$, we say x is weakly majorized by y ($x \prec_w y$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \cdots, n$. If $x \prec_w y$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ hold, we say x is majorized by y and denote $x \prec y$.

For $x = (x_1, x_2, \cdots, x_n)$ with $x_i \geq 0$ for $1 \leq i \leq n$, we write $x \in R_+^n$. Let $x, y \in R_+^n$. The weak log-majorization $x \prec_{w\log} y$ can be defined as

$$\prod_{i=1}^{k} x_{[i]} \leq \prod_{i=1}^{k} y_{[i]}, k = 1, 2, \cdots, n.$$

The log-majorization $x \prec_{\log} y$ holds if, and only if, $x \prec_{w\log} y$ and $\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i$.

Let $A, B \geq 0$. The weak log-majorization inequality

$$s(A + zB) \prec_{w\log} s(A + |z| B) \tag{1.1}$$

for any complex number z was proved by Zhan [2].

Next, in [3], the inequality (1.1) was extended to the form

$$s\left(\sum_{i=1}^{m} A_i\right) \prec_{w\log} s\left(\sum_{i=1}^{m} |A_i|\right) \tag{1.2}$$

where A_i are normal matrices, $i = 1, 2, \cdots, m$.

For $t \in [0, 1]$, the t-geometric mean of $A, B \in M_n$ with A, B are positive definite and defined as $A\#_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}$ ([4]). Their geometric mean is $A\#B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}$ and a matrix Cauchy-Schwarz inequality for positive definite matrices A_i and B_i ($i = 1, 2, \cdots, n$) is

$$\sum_{i=1}^{n} A_i \# B_i \leq \left(\sum_{i=1}^{n} A_i\right) \# \left(\sum_{i=1}^{n} B_i\right) \tag{1.3}$$

also, see [5].

A norm $\|\cdot\|$ on M_n is called unitarily invariant if $\|UAV\| = \|A\|$ for any $A \in M_n$ and any unitary $U, V \in M_n$. Fan's dominance principle [5] illustrates the relevance of majorization in matrix theory: For A, B in M_n, the weak majorization $s(A) \prec_w s(B)$ means $\|A\| \leq \|B\|$ for all unitarily invariant norms $\|\cdot\|$ (see [4] for more details).

Norm inequality for sums of positive semi-definite matrices shown by M. Hayajneh, S. Hayajneh, and F. Kittaneh [6] can be stated as follows:

$$\left\|\left(\sum_{i=1}^{m} A_i^{\frac{1}{2}} B_i^{\frac{1}{2}}\right)^2\right\| \leq \left\|\left(\sum_{i=1}^{m} A_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} B_i\right)\left(\sum_{i=1}^{m} A_i\right)^{\frac{1}{2}}\right\| \tag{1.4}$$

where $A_i, B_i \in M_n$ ($i = 1, 2, \cdots, m$) are positive semi-definite matrices and A_i commutes B_i for each i. Inequality (1.4) is a refinement of the following inequality obtained by Audenaert [7]:

$$\left\|\left(\sum_{i=1}^{m} A_i^{\frac{1}{2}} B_i^{\frac{1}{2}}\right)^2\right\| \leq \left\|\left(\sum_{i=1}^{m} A_i\right)\left(\sum_{i=1}^{m} B_i\right)\right\|. \tag{1.5}$$

Zhao and Jiang [8] derived a generalization of inequality (1.4),

$$s^r\left(\sum_{i=1}^{m} (A_i B_i)^{\frac{1}{2}}\right) \prec_{w\log} s\left(\left(\sum_{i=1}^{m} A_i\right)^{\frac{r}{4}} \left(\sum_{i=1}^{m} B_i\right)^{\frac{r}{2}} \left(\sum_{i=1}^{m} A_i\right)^{\frac{r}{4}}\right) \tag{1.6}$$

where $r \geq 1$.

Let $A, B \in M_n$ be positive semi-definite and suppose that $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, $a \in (0, 1)$. Wu proved in [9] that if $r \geq \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, then

$$\left\||AB|^{2r}\right\| \leq \left[\frac{1}{4a(1-a)}\right]^r \left\|(aA + (1-a)B)^{2rp}\right\|^{\frac{1}{p}} \left\|((1-a)A + aB)^{2rq}\right\|^{\frac{1}{q}}. \tag{1.7}$$

It is natural to raise the question that if $r < \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, then does inequality (1.7) hold or not?

Zhang [3] utilized the compound matrix technique to derive inequality (1.2). In this paper, we present a new proof of inequality (1.2). We also give a generalization of inequality (1.6). Finally, we present some numerical examples to show that inequality (1.7) is not always true when $r < \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

2. Main results

We begin this section with the following lemmas, which play an important role in our discussion.

Lemma 1. *[5] Let $A \in M_n$, then*

$$\prod_{j=1}^{k} s_j(A) = \max |\det W^* A W| \tag{2.1}$$

where the maximum is taken over all $n \times k$ matrices W such that $W^*W = I$.

Lemma 2. *[8] Let $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$, then*

$$|\det(X)| \leq \det\left(A^{\frac{1}{2}} B^{\frac{1}{2}}\right).$$

Lemma 3. *[10] Let $p > 0$, $t \in [0,1]$, then*

$$\lambda^p(A \#_t B) \prec_{w\log} \lambda\left(B^{\frac{pt}{2}} A^{(1-t)p} B^{\frac{pt}{2}}\right). \tag{2.2}$$

We give a new proof of inequality (1.2).

Theorem 4. *Let $A_i \in M_n$ be normal matrices A_i ($i = 1, 2, \cdots, m$), then*

$$s\left(\sum_{i=1}^{m} A_i\right) \prec_{w\log} s\left(\sum_{i=1}^{m} |A_i|\right).$$

Proof. An application of the polar decomposition reveals $\begin{pmatrix} |A_i^*| & A_i \\ A_i^* & |A_i| \end{pmatrix} \geq 0$ for any i. Hence,

$$\begin{pmatrix} \sum_{i=1}^m |A_i^*| & \sum_{i=1}^m A_i \\ \sum_{i=1}^m A_i^* & \sum_{i=1}^m |A_i| \end{pmatrix} = \sum_{i=1}^{m} \begin{pmatrix} |A_i^*| & A_i \\ A_i^* & |A_i| \end{pmatrix}$$

is positive semi-definite. It follows from $|A_i^*| = |A_i|$ that $\begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i \\ \sum_{i=1}^m A_i^* & \sum_{i=1}^m |A_i| \end{pmatrix} \geq 0$.

For all $n \times k$ matrices W with $W^*W = I$,

$$\begin{pmatrix} W^*\left(\sum_{i=1}^m |A_i|\right) W & W^*\left(\sum_{i=1}^m A_i\right) W \\ W^*\left(\sum_{i=1}^m A_i^*\right) W & W^*\left(\sum_{i=1}^m |A_i|\right) W \end{pmatrix} \geq 0.$$

Using Lemmas 1 and 2, we obtain

$$\prod_{j=1}^{k} s_j\left(\sum_{i=1}^{m} A_i\right) = \max\left|\det W^*\left(\sum_{i=1}^{m} A_i\right) W\right|$$

$$\leq \max\left|\det W^*\left(\sum_{i=1}^{m}|A_i|\right)W\right|$$
$$= \prod_{j=1}^{k} s_j\left(\sum_{i=1}^{m}|A_i|\right).$$

□

Next, we give a generalization of inequality (1.6).

Theorem 5. *Let A_i, $B_i \in M_n$ be positive semi-definite matrices, then*

$$\prod_{j=1}^{k} s_j^r\left(\sum_{i=1}^{m} A_i\#B_i\right) \leq \prod_{j=1}^{k} s_j\left(\left(\sum_{i=1}^{m} A_i\right)^{\frac{r}{4}}\left(\sum_{i=1}^{m} B_i\right)^{\frac{r}{2}}\left(\sum_{i=1}^{m} A_i\right)^{\frac{r}{4}}\right)$$

for $r > 0$.

Proof. We first consider the case $A_i, B_i > 0$ $(i = 1, 2, \cdots, m)$.

Using inequality (1.3), we get

$$\prod_{j=1}^{k} s_j^r\left(\sum_{i=1}^{m} A_i\#B_i\right) \leq \prod_{j=1}^{k} s_j^r\left(\left(\sum_{i=1}^{m} A_i\right)\#\left(\sum_{i=1}^{m} B_i\right)\right)$$

for $k = 1, 2, \cdots, n$.

It follows from Lemma 3 that

$$\begin{aligned}\prod_{j=1}^{k} s_j^r\left(\sum_{i=1}^{m} A_i\#B_i\right) &\leq \prod_{j=1}^{k} \lambda_j^r\left(\left(\sum_{i=1}^{m} A_i\right)\#\left(\sum_{i=1}^{m} B_i\right)\right)\\ &\leq \prod_{j=1}^{k} \lambda_j\left(\left(\sum_{i=1}^{m} A_i\right)^{\frac{r}{4}}\left(\sum_{i=1}^{m} B_i\right)^{\frac{r}{2}}\left(\sum_{i=1}^{m} A_i\right)^{\frac{r}{4}}\right)\\ &= \prod_{j=1}^{k} s_j\left(\left(\sum_{i=1}^{m} A_i\right)^{\frac{r}{4}}\left(\sum_{i=1}^{m} B_i\right)^{\frac{r}{2}}\left(\sum_{i=1}^{m} A_i\right)^{\frac{r}{4}}\right)\end{aligned}$$

for $k = 1, 2, \cdots, n$.

For the general case, by replacing A_i and B_i by $\varepsilon I_n + A_i$ and $\varepsilon I_n + B_i$ $(\varepsilon > 0)$ for $i = 1, 2, \cdots, m$, respectively, and repeating the same process as above, we obtain that

$$\begin{aligned}&\prod_{j=1}^{k} s_j\left(\sum_{i=1}^{m} (\varepsilon I_n + A_i)\#(\varepsilon I_n + B_i)\right)\\ \leq &\prod_{j=1}^{k} s_j\left(\left(\sum_{i=1}^{m} \varepsilon I_n + A_i\right)^{\frac{r}{4}}\left(\sum_{i=1}^{m} \varepsilon I_n + B_i\right)^{\frac{r}{2}}\left(\sum_{i=1}^{m} \varepsilon I_n + A_i\right)^{\frac{r}{4}}\right).\end{aligned}$$

By continuity, we get the desired inequality. □

Finally, we show that

$$\left\||AB|^{2r}\right\| \le \left[\frac{1}{4a(1-a)}\right]^r \left\|(aA+(1-a)B)^{2rp}\right\|^{\frac{1}{p}} \left\|((1-a)A+aB)^{2rq}\right\|^{\frac{1}{q}}$$

isn't always true if $r < \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Using $\lambda_j(AB) \le \lambda_j\left(\frac{A^{\frac{1}{2}}+B^{\frac{1}{2}}}{2}\right)^4$ (see [11]), we obtain

$$\sum_{j=1}^{k}\left[\lambda_j\left(AB^2A\right)\right]^r \le \sum_{j=1}^{k}\left[\lambda_j\left(\frac{A+B}{2}\right)\right]^{4r} \tag{2.3}$$

for $r = \frac{1-2\varepsilon}{2}\left(0 < \varepsilon \le \frac{1}{2}\right)$.

Inequality (2.3) is equivalent to

$$\sum_{j=1}^{k} s_j\left(BA^2B\right)^r \le \sum_{j=1}^{k} s_j\left(\frac{A+B}{2}\right)^{4r}, k = 1, 2, \cdots, n. \tag{2.4}$$

Inequality (2.4) can be rewritten as

$$\sum_{j=1}^{k} s_j\left(|AB|^{1-2\varepsilon}\right) \le \sum_{j=1}^{k} s_j\left(\left|\frac{A+B}{2}\right|^{2-4\varepsilon}\right) \tag{2.5}$$

for $1 \le k \le n$. By Ky Fan's dominance principce [5], we see inequality (2.5) is equivalent to

$$\left\||AB|^{1-2\varepsilon}\right\| \le \left\|\left(\frac{A+B}{2}\right)^{2-4\varepsilon}\right\|. \tag{2.6}$$

Inequality (2.6) implies inequality (1.7) is true if $p = q = 2$, $a = \frac{1}{2}$, and $r < \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Example 6. *Let* $A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$, $a = 0.12$, *and* $r = 0.22$ *in inequality (1.7).*

By calculating, we obtain $s_1^r\left(BA^2B\right) + s_2^r\left(BA^2B\right) \approx 4.9044$ *and*

$$\left[\frac{1}{4a(1-a)}\right]^r \left(s_1^r(aA+(1-a)B) + s_2^r(aA+(1-a)B)\right)^{\frac{1}{2}} \left(s_1^r((1-a)A+aB) + s_2^r((1-a)A+aB)\right)^{\frac{1}{2}}$$
$$\approx 2.8895.$$

Therefore, inequality (1.7) isn't true in this case.

3. Conclusions

Matrix inequalities play important roles in linear algebra and it is of interest to study the properties of Positive semi-definite matrix. In this paper, we have presented a norm inequalities related to normal matrices by using block matrix technique. Next, a weak majorization inequality for t-geometric mean was established. Lastly, a numerical example has been provided to illustrate the necessity of a condition in an existing inequality.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no competing interests.

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