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Research article

New soliton solutions of the conformal time derivative generalized q-deformed sinh-Gordon equation

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Abstract: In this article, our main purpose was to study the soliton solutions of conformal time derivative generalized q-deformed sinh-Gordon equation. New soliton solutions have been obtained by the complete discrimination system for the polynomial method. The solutions we obtained mainly included hyperbolic function solutions, solitary wave solutions, Jacobi elliptic function solutions, trigonometric function solutions and rational function solutions. The results showed abundant traveling wave patterns of conformal time derivative generalized q-deformed sinh-Gordon equation.

Keywords: generalized q-deformed sinh-Gordon equation; conformal time derivative definition; complete discrimination system for the polynomial method
Mathematics Subject Classification: 35C05, 35C07, 35R11

1. Introduction

Nonlinear partial differential equations arised from many scientific fields, such as physics, biology and so on. In recent decades, scientists devoted themselves to formulate methods to find the solutions to the nonlinear problems [1–8]. It is well known that the sinh-Gordon equation is one of the essential nonlinear equations in integrable quantum field theory, kink dynamics, and fluid dynamics [9–14]. The sinh-Gordon equation arises as a special case of the Toda lattice equation, a well-known soliton equation in one space and one time dimension, which can be used to model the interaction of neighboring particles of equal mass in a lattice formation with a crystal.

The sinh-Gordon equation [15] is

$$\frac{\partial^2\Omega}{\partial x^2} - \frac{\partial^2\Omega}{\partial t^2} = [\sinh(\Omega)]. \tag{1.1}$$

In recent years, many powerful mathematical methods have been proposed to derive soliton solutions for the sinh-Gordon equation, such as the Tanh method [16], bifurcation method [17], (G'/G)-expansion method and Exp-function method [18], F-expansion method [19], and the homogeneous balance method [20].

When the q-deformed hyperbolic function, proposed in the 19th century by Arai, was included in the dynamical system, the symmetry of the system was destroyed and, consequently as was the symmetry of the solution [21, 22]. Recently, there have been several solutions generated for the Schrödinger equation and Dirac equation with q-deformed hyperbolic potential. q-deformed functions show promise in modeling atom-trapping potentials or statistical distributions in Bose-Einstein condensates, as well as exploring vibrational spectra of diatomic molecules [23, 24]. The generalized q-deformed sinh-Gordon equation is

$$\frac{\partial^2\Omega}{\partial x^2} - \frac{\partial^2\Omega}{\partial t^2} = [\sinh_q(\Omega^\theta)]^l - \omega. \tag{1.2}$$

This equation was introduced and studied in this form for the first time in 2018 by Eleuch [25]. Nauman et al. studied the equation via the $exp(-\phi(\xi))$-expansion method [26]. Ali et al. employed the extended tanh method to obtain the exact solutions of the equation [27].

In this paper, we shall consider the conformal time derivative generalized q-deformed sinh-Gordon equation [28]:

$$\frac{\partial^2\Omega}{\partial x^2} - \frac{\partial^{2\alpha}\Omega}{\partial t^{2\alpha}} = [\sinh_q(\Omega^\theta)]^l - \omega, \tag{1.3}$$

where θ, l are constants, $\alpha \in (0, 1]$, and $q \in (0, 1)$. $q = 1$ gives the standard sinh-Gordon equation. In addition, we get the generalized q-deformed sinh-Gordon equation at $\alpha = 1$. The equation also appears in problems varying from fluid flow to quantum field theory. In this work, we analyze the conformal time derivative generalized q-deformed sinh-Gordon equation with the aid of a conformable derivative operator to find solitons using the complete discrimination system for the polynomial method.

This article is arranged as follows: In Section 2, we review the definition of conformable derivative and introduce the complete discrimination system for constructing the exact traveling wave solutions of fractional partial differential equation. In Section 3, we will apply this method to solve the conformal time derivative generalized q-deformed sinh-Gordon equation. In Section 4, we draw the numerical simulations. In Section 5, we present the concluding remarks.

2. Mathematical analysis

The conformable derivative of order α is defined as

$$D_t^\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{2.1}$$

for all $t > 0$, $\alpha \in (0, 1]$.

The following transformation is used to determine the traveling wave solution of Eq (1.3):

$$\Omega(x, t) = v(\xi),\ \xi = \rho x - \kappa \frac{t^\alpha}{\alpha}, \tag{2.2}$$

and κ denotes the speed of the traveling wave.

Using Eq (2.2), Eq (1.3) becomes

$$(\rho^2 - \kappa^2)v''(\xi) + \xi - [\sinh_q(v(\xi)^\theta)]^l = 0. \tag{2.3}$$

Now, we take two cases for Eq (2.3).

Case one: Suppose $l = \theta = 1,\ \omega = 0$. Then, Eq (2.3) can be written as

$$(\rho^2 - \kappa^2)v^{''}(\xi) - \sinh_q(v(\xi)) = 0. \tag{2.4}$$

We can multiply both sides of Eq (2.4) by $v^{'}(\xi)$ and, after the integration, we get [28]

$$\frac{1}{2}(\rho^2 - \kappa^2)(v^{'}(\xi))^2 - \cosh_q(v(\xi)) - m = 0, \tag{2.5}$$

where m is the integration constant.

Let

$$v(\xi) = \ln(u(\xi)), \tag{2.6}$$

and Eq (2.5) becomes

$$(\rho^2 - \kappa^2)u^{'}(\xi)^2 + 2mu^2(\xi) + qu(\xi) + u^3(\xi) = 0. \tag{2.7}$$

Thus, we can solve Eq (2.7) using our method, and from Eqs (2.2) and (2.6), we can get the solution of Eq (1.3) in the first case.

Case two: Suppose $\theta = 1,\ l = 2,\ \omega = -\frac{q}{2}$. In this case, Eq (2.3) can be expressed as [28]:

$$(\rho^2 - \kappa^2)v^{''}(\xi) - (\sinh_q v(\xi))^2 - \frac{q}{2} = 0. \tag{2.8}$$

After simplifying Eq (2.8), we get

$$(\rho^2 - \kappa^2)v^{''}(\xi) - \frac{1}{2}\cosh_{q^2}(2v(\xi)) = 0. \tag{2.9}$$

Let

$$v(\xi) = \frac{1}{2}\ln(u(\xi)), \tag{2.10}$$

and Eq (2.10) becomes

$$2(\rho^2 - \kappa^2)u^{'}(\xi)^2 - 2(\rho^2 - \kappa^2)u(\xi)u^{''}(\xi) + q^2u(\xi) + u^3(\xi) = 0. \tag{2.11}$$

Thus, we can solve Eq (2.11) using our method and from Eqs (2.2) and (2.10) we can get the solution of Eq (1.3) in the second case.

Equations (2.7) and (2.10) are usually reduced to

$$(u^{'}(\xi))^2 = G(u), \tag{2.12}$$

where $G(u)$ is a polynomial, then, integrating Eq (2.12) once, we can obtain

$$\pm(\xi - \xi_0) = \int \frac{du}{\sqrt{G(u)}}, \tag{2.13}$$

where ξ_0 is an integral constant.

According to the above procedures, recent results have been reported via the complete discrimination system [29, 30].

3. Applications

In this section, the polynomial method applies to find the analytic solution to the two cases that were imposed for Eq (1.3).

• The analytical solution of case one at $l = \theta = 1,\ \omega = 0$:

Based on Eq (2.7), we can get

$$(u')^2 = \frac{1}{\rho^2-\kappa^2}u^3 + \frac{2m}{\rho^2-\kappa^2}u^2 + \frac{q}{\rho^2-\kappa^2}u. \tag{3.1}$$

Making the transformation $\psi = (\frac{1}{\rho^2-\kappa^2})^{\frac{1}{3}}u$, $\xi_1 = (\frac{1}{\rho^2-\kappa^2})^{\frac{1}{3}}\xi$, Eq (3.1) becomes

$$(\psi')^2 = \psi^3 + p_1\psi^2 + p_2\psi, \tag{3.2}$$

where $p_1 = 2m(\frac{1}{\rho^2-\kappa^2})^{\frac{1}{3}}$, $p_2 = q(\frac{1}{\rho^2-\kappa^2})^{\frac{2}{3}}$.

Integrating Eq (3.2), we have

$$\pm(\xi_1 - \xi_0) = \int \frac{du}{\sqrt{\psi^3 + p_1\psi^2 + p_2\psi}}. \tag{3.3}$$

According to the complete discrimination system, we give the corresponding single traveling wave solutions to Eq (2.7).

Case 1.1. $\Delta = 0$, $D_1 < 0$, and we have $F(\psi) = (\psi-\lambda_1)^2(\psi-\lambda_2)$, $\lambda_1 \neq \lambda_2$.

When $\psi > \lambda_2$, we have

$$\pm(\xi_1 - \xi_0) = \int \frac{d\psi}{(\psi-\lambda_1)\sqrt{\psi-\lambda_2}} = \begin{cases} \frac{1}{\sqrt{\lambda_1-\lambda_2}} \ln \mid \frac{\sqrt{\psi-\lambda_1}-\sqrt{\lambda_1-\lambda_2}}{\sqrt{\psi-\lambda_2}+\sqrt{\lambda_1-\lambda_2}} & (\lambda_1 > \lambda_2), \\ \frac{2}{\sqrt{\lambda_2-\lambda_1}} \arctan \sqrt{\frac{\psi-\lambda_2}{\lambda_1-\lambda_2}} & (\lambda_1 < \lambda_2); \end{cases} \tag{3.4}$$

the corresponding solutions of Eq (1.3) are

$$\Omega_1(x,t) = \frac{1}{3}\ln(\rho^2-\kappa^2) + \ln\{(\lambda_1-\lambda_2)\tanh^2[\frac{\sqrt{\lambda_1-\lambda_2}}{2}(\frac{1}{\rho^2-\kappa^2})^{\frac{1}{3}}(\xi-\xi_0)] + \lambda_2\} \quad (\lambda_1 > \lambda_2); \tag{3.5}$$

$$\Omega_2(x,t) = \frac{1}{3}\ln(\rho^2-\kappa^2) + \ln\{(\lambda_1-\lambda_2)\coth^2[\frac{\sqrt{\lambda_1-\lambda_2}}{2}(\frac{1}{\rho^2-\kappa^2})^{\frac{1}{3}}(\xi-\xi_0)] + \lambda_2\} \quad (\lambda_1 > \lambda_2); \tag{3.6}$$

$$\Omega_3(x,t) = \frac{1}{3}\ln(\rho^2-\kappa^2) + \ln\{(\lambda_2-\lambda_1)\tan^2[\frac{\sqrt{\lambda_2-\lambda_1}}{2}(\frac{1}{\rho^2-\kappa^2})^{\frac{1}{3}}(\xi-\xi_0)] + \lambda_2\} \quad (\lambda_1 > \lambda_2). \tag{3.7}$$

Case 1.2. $\Delta = 0$, $D_1 = 0$, then $F(\psi) = (\omega-\lambda)^3$ and the corresponding solutions of Eq (1.3) are

$$\Omega_4(x,t) = \ln[4(\frac{1}{\rho^2-\kappa^2})^{-\frac{2}{3}}(\xi-\xi_0)^{-2} + \lambda]. \tag{3.8}$$

Case 1.3. $\Delta > 0$, $D_1 < 0$, then $F(\psi) = (\psi-\lambda_1)(\psi-\lambda_3)(\psi-\lambda_3)$, with $\lambda_1 < \lambda_2 < \lambda_3$. Therefore, we have

$$\pm(\xi_1 - \xi_0) = \int \frac{du}{\sqrt{(\psi-\lambda_1)(\psi-\lambda_1)(\psi-\lambda_3)}}. \tag{3.9}$$

When $\lambda_1 < \psi < \lambda_3$, the corresponding solutions of Eq (1.3) are

$$\Omega_5(x,t) = \frac{1}{3}\ln(\rho^2-\kappa^2) + \ln[\lambda_1 + (\lambda_2-\lambda_1)sn^2(\frac{\sqrt{\lambda_3-\lambda_1}}{2})(\frac{1}{\rho^2-\kappa^2})^{-\frac{1}{3}}(\xi-\xi_0), m)]. \tag{3.10}$$

When $\psi > \lambda_3$, the corresponding solutions of Eq (1.3) are

$$\Omega_6(x,t) = \frac{1}{3}\ln(\rho^2-\kappa^2) + \ln[\frac{\lambda_3 - \lambda_2 sn^2(\frac{\sqrt{\lambda_3-\lambda_1}}{2}(\frac{1}{\rho^2-\kappa^2})^{-\frac{1}{3}}(\xi-\xi_0), m)}{cn^2(\frac{\sqrt{\lambda_3-\lambda_1}}{2}(\frac{1}{\rho^2-\kappa^2})^{-\frac{1}{3}}(\xi-\xi_0), m)}], \tag{3.11}$$

where $m^2 = \frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}$.

Case 1.4. $\Delta < 0$, then, $F(\psi) = (\psi-\lambda)(\psi^2+b_1\psi+b_2)$, $b_1^2-4b_2 < 0$, and the corresponding integral becomes

$$\pm(\xi_1-\xi_0) = \int \frac{d\psi}{(\psi-\lambda)(\psi^2+b_1\psi+b_2)}. \tag{3.12}$$

When $\psi > \lambda$, the corresponding solutions of Eq (1.3) are

$$\Omega_7(x,t) = \frac{1}{3}\ln(\rho^2-\kappa^2) + \ln[\lambda + \frac{2\sqrt{\lambda^2+b_1\lambda+b_2}}{1+cn(\lambda^2+b_1\lambda+b_2)^{\frac{1}{4}}(\frac{1}{\rho^2-\kappa^2})^{-\frac{1}{3}}(\xi_1-\xi_0), m)} - \sqrt{\lambda^2+b_1\lambda+b_2}], \tag{3.13}$$

where $m^2 = \frac{1}{2}(1-\frac{\lambda+\frac{b_1}{2}}{\sqrt{\lambda^2+b_1\lambda+b_2}})$.

- The analytical solution of case two at $l=2,\ \theta=1,\ \omega=-\frac{q}{2}$.

By our trial equation method, we have the trial equation

$$u'' = a_0 + a_1u + a_2u^2. \tag{3.14}$$

Integrating Eq (3.14), we have

$$(u')^2 = \frac{2}{3}a_2u^3 + a_1u^2 + a_0u + d. \tag{3.15}$$

Substituting Eqs (3.14) and (3.15) into Eq (2.11),

$$r_3u^3 + r_2u^2 + r_1u + r_0 = 0, \tag{3.16}$$

where $r_3 = \frac{2}{3}(\rho^2-\kappa^2)a_2 + 1$, $r_2 = 4(\rho^2-\kappa^2)a_1$, $r_1 = 2(\rho^2-\kappa^2)a_0 + q^2$, $r_0 = 2(\rho^2-\kappa^2)d$.

Thus, we have a system of algebraic equations from the coefficients of polynomial u. Solving the algebraic equation system yields the following: $a_2 = \frac{3}{2(\rho^2-\kappa^2)}$, a_1 is the constant, $a_0 = -\frac{q^2}{2(\rho^2-\kappa^2)}$, $d = 0$.

Making the transformation $u = (\frac{2}{3}a_2)^{-\frac{1}{3}}\phi$, $\xi_1 = (\frac{2}{3}a_2)^{-\frac{1}{3}}\xi$, we have

$$\pm(\xi_1-\xi_0) = \int \frac{d\phi}{\sqrt{\phi[\phi^2+q_1\phi+q_2]}}, \tag{3.17}$$

where ξ_0 is the integration constant.

Suppose that $\Delta = q_1^2-4q_2$, $F(\phi) = \phi^2+q_1\phi+q_2$, and there are four cases for the solutions of Eq (1.3).

Case 2.1. $\Delta = 0$. Since $\phi > 0$, we have

$$\pm(\xi_1 - \xi_0) = \int \frac{d\phi}{\sqrt{\phi}(\phi + \frac{q_1}{2})}. \tag{3.18}$$

If $q_1 < 0$, Eq (3.18) becomes

$$\pm(\xi_1 - \xi_0) = \sqrt{\frac{2}{-q_1}} \ln | \frac{\sqrt{2\phi} - \sqrt{-q_1}}{\sqrt{2\phi} + \sqrt{-q_1}} |, \tag{3.19}$$

and the traveling wave solutions of Eq (1.3) are

$$\Omega_8(x,t) = \frac{1}{3} \ln \frac{a_1}{2}(\rho^2 - \kappa^2) + \ln \tanh(\frac{1}{8} \sqrt{a_1(\rho^2 - \kappa^2)^{\frac{2}{3}}}(\xi - \xi_0)), \tag{3.20}$$

$$\Omega_9(x,t) = \frac{1}{3} \ln \frac{a_1}{2}(\rho^2 - \kappa^2) + \ln \coth(\frac{1}{8} \sqrt{a_1(\rho^2 - \kappa^2)^{\frac{2}{3}}}(\xi - \xi_0)). \tag{3.21}$$

If $q_1 > 0$, Eq (3.18) becomes

$$\pm(\xi_1 - \xi_0) = -\sqrt{\frac{q_1}{2}} \arctan \sqrt{\frac{2\phi}{q_1}}, \tag{3.22}$$

and the traveling wave solutions of Eq (1.3) are

$$\Omega_{10}(x,t) = \frac{1}{2} \ln \frac{a_1}{2}(\rho^2 - \kappa^2) + \ln \tan[\sqrt{\frac{a_1}{2}(\rho^2 - \kappa^2)^{\frac{2}{3}}}(\xi - \xi_0)]. \tag{3.23}$$

If $q_1 = 0$, Eq (3.18) becomes

$$\pm(\xi_1 - \xi_0) = \frac{-1}{\sqrt{\phi}}, \tag{3.24}$$

and the traveling wave solutions of Eq (1.3) are

$$\Omega_{11}(x,t) = \frac{1}{2} \ln \frac{4}{(\xi - \xi_0)^2}. \tag{3.25}$$

Case 2.2. $\Delta > 0$, $q_2 = 0$. Since $\phi > -q_1$, Eq (3.17) can be written as

$$\pm(\xi_1 - \xi_0) = \int \frac{d\phi}{\phi \sqrt{\phi + q_1}}. \tag{3.26}$$

If $q_1 > 0$, then, Eq (3.26) becomes

$$\pm(\xi_1 - \xi_0) = \sqrt{\frac{2}{q_1}} \ln | \frac{\sqrt{2(\phi + q_1)} - \sqrt{q_1}}{\sqrt{2(\phi + q_1) + \sqrt{q_1}}} |, \tag{3.27}$$

and the traveling wave solutions of Eq (1.3) are

$$\Omega_{12}(x,t) = \frac{1}{2} \ln \frac{a_1}{2}(\rho^2 - \kappa^2) + \ln \tanh[\sqrt{\frac{a_1}{2}(\rho^2 - \kappa^2)^{\frac{2}{3}}}(\xi - \xi_0)] - a_1(\rho^2 - \kappa^2)^{\frac{2}{3}}], \tag{3.28}$$

$$\Omega_{13}(x,t)=\frac{1}{2}\ln\frac{a_1}{2}(\rho^2-\kappa^2)+\ln\coth[\sqrt{\frac{a_1}{2}(\rho^2-\kappa^2)^{\frac{2}{3}}}(\xi-\xi_0)]-a_1(\rho^2-\kappa^2)^{\frac{2}{3}}]. \tag{3.29}$$

If $q_1<0$, then, Eq (3.26) becomes

$$\pm(\xi_1-\xi_0)=-2\sqrt{-\frac{2}{q_1}}\arctan\sqrt{\frac{2(\phi+q_1)}{-q_1}}, \tag{3.30}$$

and the traveling wave solutions of Eq (1.3) are

$$\Omega_{14}(x,t)=\frac{1}{2}\ln\{-\frac{a_1}{2}(\rho^2-\kappa^2)[\sqrt{\frac{a_1}{2}(\rho^2-\kappa^2)^{\frac{2}{3}}}(\xi-\xi_0)]-a_1(\rho^2-\kappa^2)^{\frac{2}{3}}\}. \tag{3.31}$$

Case 2.3. $\Delta>0$, $q_2\neq 0$. Suppose $\alpha_1<\alpha_2<\alpha_3$, one of them is zero and the other two are the roots of $F(\phi)$.

If $\alpha_1<\phi<\alpha_3$, and take $\phi=\alpha_1+(\alpha_2-\alpha_1)\sin\theta$, then, Eq (3.17) can be rewritten as

$$\pm(\xi_1-\xi_0)=\frac{2}{\alpha_3-\alpha_1}\int\frac{d\phi}{\sqrt{1-m^2\sin^2\theta}}. \tag{3.32}$$

Here, $m^2=\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}$. According to the definition of Jacobian elliptic function sn, we obtain the solution of Eq (1.3) in the following form:

$$\Omega_{15}(x,t)=\frac{1}{2}\ln\{(\rho^2-\kappa^2)[\alpha_1+(\alpha_2-\alpha_1)sn^2(\sqrt{\alpha_3-\alpha_1}(\xi-\xi_0),m)]\}. \tag{3.33}$$

If $\phi>\alpha_3$, and take $\psi=\frac{-\alpha_2\sin^2\varphi+\alpha_3}{\cos^2\varphi}$, the solution of Eq (1.3) can be constructed as follows:

$$\Omega_{16}(x,t)=\frac{1}{2}\ln(\rho^2-\kappa^2)\frac{-\beta sn(\sqrt{\alpha_3-\alpha_1}(\xi_1-\xi_0),m)+\alpha_3}{cn(\sqrt{\alpha_3-\alpha_1}(\xi_1-\xi_0),m)}. \tag{3.34}$$

Case 2.4. $\Delta<0$, $\phi>0$, and take the transformation

$$\phi=\sqrt{q_2}\tan^2\frac{\varphi}{2}. \tag{3.35}$$

Substituting Eq (3.35) into Eq (3.17),

$$\pm(\xi_1-\xi_0)=q_2^{-\frac{1}{4}}\int\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \tag{3.36}$$

where $k^2=\frac{1}{2}(1-\frac{q_1}{2\sqrt{q_2}})$. From the definition of Jacobian elliptic function cn, we obtain

$$cn(2(q_2^{\frac{1}{4}})(\xi_1-\xi_0),m)=\cos\varphi. \tag{3.37}$$

From Eq (3.35), we get

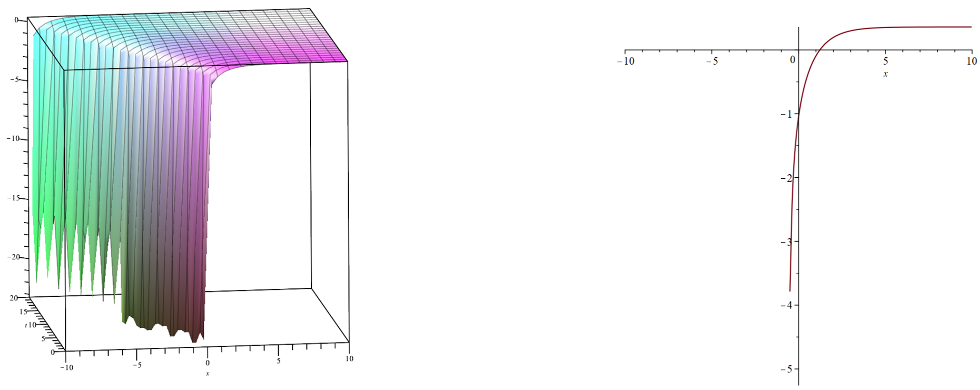
$$\cos\varphi=\frac{2\sqrt{q_2}}{\phi+\sqrt{q_2}}-1. \tag{3.38}$$

By using Eqs (3.37) and (3.38), the traveling wave solution of Eq (1.3) can be derived in the following form:

$$\Omega_{17}(x,t)=\frac{1}{2}\ln[\frac{2q_2}{1+cn(2\sqrt{q_2}((\rho^2-\kappa^2)^{-\frac{1}{6}})(\xi-\xi_0),m)}-q_2(\rho^2-\kappa^2)^{\frac{1}{3}}]. \tag{3.39}$$

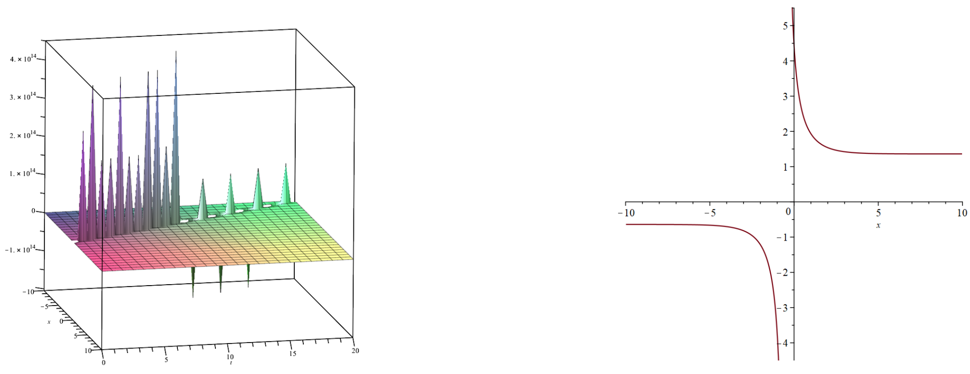
4. Graphical representation of the obtained solutions

In this section, the exact solutions of the conformal time derivative generalized q-deformed sinh-Gordon equation are given. Through the above results, we get some new exact solutions, such as solitary wave solutions $\Omega_1(x,t), \Omega_2(x,t), \Omega_8(x,t), \Omega_9(x,t), \Omega_{12}(x,t), \Omega_{13}(x,t)$; rational function solutions $\Omega_4(x,t), \Omega_{11}(x,t), \Omega_{14}(x,t)$; trigonometric function solutions $\Omega_3(x,t), \Omega_{10}(x,t)$; and Jacobi elliptic function double periodic solutions $\Omega_5(x,t), \Omega_6(x,t), \Omega_7(x,t), \Omega_{15}(x,t), \Omega_{16}(x,t), \Omega_{17}(x,t)$. Furthermore, $\Omega_1(x,t), \Omega_8(x,t), \Omega_{12}(x,t)$ are bounded solutions and $\Omega_2(x,t), \Omega_9(x,t), \Omega_{13}(x,t)$ are unbounded solutions. Using the mathematical software Maple, we plot some of these obtained solutions, which are shown in Figures 1–3.



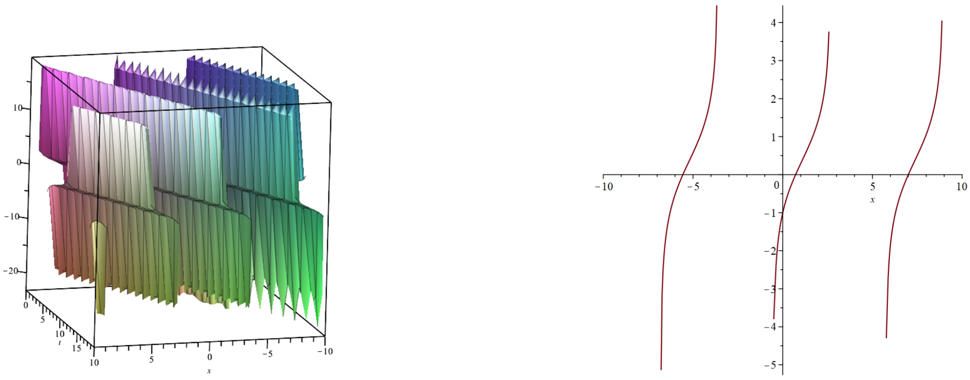
(a) (b)

Figure 1. The solitary wave solution $\Omega_8(x,t)$ for Eq (1.1) with $\rho = 1,\ \kappa = \frac{1}{2},\ a_1 = -1,\ \xi_0 = 0$. (a) Perspective view of the wave. (b) The wave along the z-axis.



(a) (b)

Figure 2. The solitary wave solution $\Omega_9(x,t)$ for Eq (1.1) with $\rho = \frac{2}{3},\ \kappa = \frac{1}{2},\ a_1 = -1,\ \xi_0 = 0$. (a) Perspective view of the wave. (b) The wave along the z-axis.



(a) (b)

Figure 3. The solitary wave solution $\Omega_{10}(x, t)$ for Eq (1.1) with $\rho = 1,\ \kappa = \frac{1}{2},\ a_1 = 1,\ \xi_0 = 0$. (a) Perspective view of the wave. (b) The wave along the z-axis.

5. Conclusions

In this paper, the conformal time derivative generalized q-deformed sinh-Gordon equation has been investigated via the complete discriminant system method. A range of new traveling wave solutions is obtained, such as periodic solutions, rational wave solutions, Jacobi elliptic solutions, triangular functions solutions, and hyperbolic function solutions. Comparing with other works [28], these solutions have not been reported in the former literature. This also indicates that the complete discrimination system for polynomial method is powerful in nonlinear analysis.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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