Research article

Covering cross-polytopes with smaller homothetic copies

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Abstract: Let $C_n$ be an $n$-dimensional cross-polytope and $\Gamma_p(C_n)$ be the smallest positive number $\gamma$ such that $C_n$ can be covered by $p$ translates of $\gamma C_n$. We obtain better estimates of $\Gamma_2(C_n)$ for small $n$ and a universal upper bound of $\Gamma_2(C_n)$ for all positive integers $n$.

Keywords: convex body; covering functional; Hadwiger’s covering conjecture; homothetic copy

Mathematics Subject Classification: 52A20, 52C17, 52A15

1. Introduction

Let $K$ be a convex body in $\mathbb{R}^n$, i.e., a compact convex set having interior points. The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$, and the set of convex bodies that are centrally symmetric is denoted by $C^n$. For each $x \in \mathbb{R}^n$ and each $\lambda > 0$, the set

$$x + \lambda K := \{x + \lambda y \mid y \in K\}$$

is called a homothetic copy of $K$; when $\lambda \in (0, 1)$, it is called a smaller homothetic copy of $K$. For each $K \in \mathcal{K}^n$, we denote by $c(K)$ the least number of translates of int $K$ needed to cover $K$. Concerning the least upper bound of $c(K)$ in $\mathcal{K}^n$, there is a long-standing conjecture (see [1–6] for the origin, history, and classical known results concerning this conjecture):

**Conjecture 1.** *(Hadwiger’s covering conjecture [4])* For each $K \in \mathcal{K}^n$, we have

$$c(K) \leq 2^n,$$

and the equality holds if and only if $K$ is a parallelotope.

Although many people have conducted in-depth research, this conjecture is confirmed completely only for the planar case [7]. In [8], Chuanming Zong proposed a four-step program to attack this
conjecture. In this program, it is important to estimate

\[ \Gamma_m(K) := \inf \left\{ \gamma > 0 \mid \exists \{c_i \mid i \in [m]\} \subseteq \mathbb{R}^n \text{ s.t. } K \subseteq \bigcup_{i \in [m]} (c_i + \gamma K) \right\}, \]

i.e., \( \Gamma_m(K) \) is the smallest positive number \( \gamma \) such that \( K \) can be covered by \( m \) translates of \( \gamma K \). The map \( \Gamma_m(\cdot) : \mathcal{K}^n \rightarrow [0, 1], K \mapsto \Gamma_m(K) \) is called the covering functional with respect to \( m \), where \([m] := \{ i \in \mathbb{Z}^+ \mid 1 \leq i \leq m\}\). Clearly, \( c(K) \leq m \) if and only if \( \Gamma_m(K) < 1 \). For each \( m \in \mathbb{Z}^+ \), \( \Gamma_m(\cdot) \) is an affine invariant. More precisely,

\[ \Gamma_m(K) = \Gamma_m(T(K)), \forall T \in \mathcal{A}^n, \]

where \( \mathcal{A}^n \) is the set of non-degenerate affine transformations from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

A compact convex set \( K \) is said to be an \( d \)-dimensional cross-polytope if there exist \( d \) linearly independent vectors \( v_1, \ldots, v_d \) such that

\[ K = \text{conv}\{\pm v_1, \ldots, \pm v_d\}. \]

Clearly, any \( d \)-dimensional cross-polytope is the image of \( C_d = \left\{ (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d \mid \sum_{i=1}^d |\alpha_i| \leq 1 \right\} \) under a non-degenerate affine transformation. Therefore, \( \Gamma_m(K) = \Gamma_m(C_d) \) holds for each pair of positive integers \( m \) and \( d \). In a recent work [9], Xia Li et al. obtained some estimations of \( \Gamma_m(C_d) \) for large \( d \). Moreover, they showed that, if \( P \in C^n \) is a convex polytope with \( 2d \) vertices, then

\[ \Gamma_m(P) \leq \Gamma_m(C_d), \] (1.1)

which shows the importance of estimating \( \Gamma_m(C_d) \).

It is well known that \( \Gamma_2([-1, 1]^n) = 1/2, \forall n \geq 2 \). It is interesting to ask whether there exists a universal upper bound for \( \Gamma_2(C_n) \). In this paper, by using elementary yet interesting observations and refining techniques used in the recent works [9, 10], we get better estimates of \( \Gamma_2(C_n) \). Based on this, we present the first nontrivial universal upper bound of \( \Gamma_2(C_n) \) for all positive \( n \). By (1.1), results mentioned above yield also estimates of covering functionals of convex polytopes with few vertices.

Throughout this paper, the dimension \( n \) of the underlying space is at least 3.

2. Covering functionals of cross-polytopes

For each \( n, k \in \mathbb{Z}^+ \), we put

\[ M(n, k) = \left\{ (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n \mid \sum_{i=1}^n |\alpha_i| \leq k \right\}. \]

It is known that (cf. [11] or [12])

\[ \#M(n, k) = \sum_{i=n-k}^n 2^{n-i} \binom{n}{i} \binom{k}{n-i}. \]
Lemma 1. Let \( n, k \in \mathbb{Z}^+ \). If \( k \leq \frac{n}{2} \), then
\[
(n+k)C_n \subseteq nC_n + S_k,
\]
where
\[
S_k = \left\{ (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n \mid \sum_{i \in [n]} |\alpha_i| = k \right\} \cup \{o\}.
\]
Moreover,
\[
\#S_k = \sum_{i=1}^{k} 2^i \binom{n}{i} \left( \frac{k-1}{i-1} \right) + 1.
\]
Proof. Let \((\alpha_1, \cdots, \alpha_n)\) be an arbitrary point in \((n+k)C_n\). Then
\[
\sum_{i \in [n]} |\alpha_i| \leq n+k.
\]
If \(\sum_{i \in [n]} |\alpha_i| \leq n\), then \((\alpha_1, \cdots, \alpha_n) \in nC_n \subseteq nC_n + S_k\). Otherwise, there exists \(m \in [k]\) such that
\[
n + m - 1 < \sum_{i \in [n]} |\alpha_i| \leq n + m.
\]
On the one hand, since \(\sum_{i \in [n]} (|\alpha_i| - |\alpha_i|) < n\), we have \(\sum_{i \in [n]} |\alpha_i| \geq m\). Then there exist integers \(\beta_1, \cdots, \beta_n \geq 0\) such that
\[
\beta_i = |\alpha_i|, \forall i \in [n] \quad \text{and} \quad \sum_{i \in [n]} \beta_i = m.
\]
Clearly, we have
\[
\sum_{i \in [n]} |\alpha_i - \text{sgn} \cdot \beta_i| = \sum_{i \in [n]} (|\alpha_i| - \beta_i) \leq n.
\]
Therefore,
\[
(\alpha_1, \cdots, \alpha_n) = (\alpha_1 - \text{sgn} \cdot \beta_1, \cdots, \alpha_n - \text{sgn} \cdot \beta_n) + (\text{sgn} \cdot \beta_1, \cdots, \text{sgn} \cdot \beta_n) \in nC_n + S_m.
\]
On the other hand, set
\[
m_i = \begin{cases} 
|\alpha_i|, & \text{if } |\alpha_i| - |\alpha_i| < \frac{1}{2}, \\
|\alpha_i| + 1, & \text{if } |\alpha_i| - |\alpha_i| \geq \frac{1}{2},
\end{cases} \quad \forall i \in [n].
\]
We have
\[
(\alpha_1, \cdots, \alpha_n) = (\alpha_1 - \text{sgn} \cdot m_1, \cdots, \alpha_n - \text{sgn} \cdot m_n) + (\text{sgn} \cdot m_1, \cdots, \text{sgn} \cdot m_n).
\]
By the Triangle Inequality, we have
\[
n - \sum_{i \in [n]} m_i < \sum_{i \in [n]} |\alpha_i| - \sum_{i \in [n]} m_i \leq \sum_{i \in [n]} |\alpha_i - \text{sgn} \cdot m_i| = \sum_{i \in [n]} |\alpha_i - m_i| \leq \frac{n}{2}.
\]
Thus,
\[ m_1 + \cdots + m_n > \frac{n}{2} \geq k. \]

Without loss of generality, assume that \( \alpha_1, \cdots, \alpha_n \geq 0, \) and
\[ \alpha_1, \cdots, \alpha_{n_0}^c \geq 1, \quad \alpha_{n_0+1}', \cdots, \alpha_m \in \left[ \frac{1}{2}, 1 \right), \quad \alpha_{m_0+1}', \cdots, \alpha_n \in \left[ 0, \frac{1}{2} \right). \]

By (2.1), we have
\[ \beta_i \leq \lfloor \alpha_i \rfloor \leq \lceil \alpha_i \rceil, \quad \forall i \in [n_0], \]
\[ \beta_i = 0 < 1 = m_i, \quad \forall i \in [n_0] \setminus [n_0], \]
\[ \beta_i = m_i = 0, \quad \forall i \in [n] \setminus [n_0]. \]

Then there exist integers \( m_i', \cdots, m_n' \) such that
\[ \beta_i \leq m_i' \leq m_i, \quad \forall i \in [n] \quad \text{and} \quad \sum_{i \in [n]} m_i' = k. \]

Set, for each \( i \in [n], f_i(\lambda) = |\alpha_i - \lambda|. \) Then \( f_i \) is decreasing on \([\beta_i, [\alpha_i]].\) We claim that
\[ f_i(\beta_i) \geq f_i(m_i'), \quad \forall i \in [n]. \tag{2.2} \]

The case when \( m_i' \in [\beta_i, [\alpha_i]] \) is clear. If \( m_i' > [\alpha_i], \) then \( m_i' = m_i = [\alpha_i] + 1 \) and \( 1/2 \leq \alpha_i - [\alpha_i] < 1. \) Thus
\[ f_i(\beta_i) \geq f_i([\alpha_i]) = \alpha_i - [\alpha_i] \geq \frac{1}{2} \geq 1 - (\alpha_i - [\alpha_i]) = f_i([\alpha_i] + 1) = f_i(m_i'). \]

Hence (2.2) holds as claimed. It follows that
\[ \sum_{i \in [n]} |\alpha_i - m_i'| = \sum_{i \in [n]} f_i(m_i') \leq \sum_{i \in [n]} f_i(\beta_i) \leq n. \]

Therefore,
\[ (\alpha_1, \cdots, \alpha_n) = (\alpha_1 - m_1', \cdots, \alpha_n - m_n') + (m_1', \cdots, m_n') \in nC_n + S_k. \]

Moreover,
\[
\#S_k = \#M_2(n, k) - \#M_2(n, k_1) + 1
\]
\[
= \sum_{i=0}^{n-k} 2^{n-i} \binom{n}{i} \binom{k}{i} - \sum_{i=n-k+1}^{n} 2^{n-i} \binom{n}{i} \binom{k-1}{n-i} + 1
\]
\[
= 2^k \left( \binom{n}{n-k} \binom{k}{k} + 2^{k-1} \binom{n}{n-k+1} \binom{k}{k-1} + \cdots + \binom{n}{n} \binom{k}{0} \right)
\]
\[
- 2^{k-1} \binom{n}{n-k} \binom{k-1}{k-1} - \cdots - \binom{n}{n} \binom{k-1}{0} + 1
\]
\[
= 2^k \left( \binom{n}{n-k} \sum_{i=1}^{k-1} 2^i \binom{n}{n-i} \binom{k-1}{i} \right) + 1
\]
\[
= 2^k \left( \binom{n}{n-k} \binom{k-1}{k-1} \right) + \sum_{i=1}^{k-1} 2^i \binom{n}{n-i} \binom{k-1}{i-1} + 1
\]
\[
= \sum_{i=1}^{k} 2^i \binom{n}{n-i} \binom{k-1}{i-1} + 1 = \sum_{i=1}^{k} 2^i \binom{n}{n-i} \binom{k-1}{i-1} + 1. \]

For each \( n \in \mathbb{Z}^+ \), let \( k_1(n) \) be the nonnegative integer satisfying
\[
\sum_{i=1}^{k_1(n)} 2^i \binom{n}{i} (k_1(n) - 1) + 1 \leq 2^n < \sum_{i=1}^{k_1(n)+1} 2^i \binom{n}{i} (k_1(n)) + 1.
\]
It is easy to prove that \( k_1(n) \leq \frac{n}{2} \).

**Corollary 2.** For each \( n \in \mathbb{Z}^+ \), we have
\[
\Gamma_{2n}(C_n) \leq \frac{n}{n + k_1(n)}.
\]

**Remark 3.** It can be verified that
\[
\sum_{i=1}^{k} 2^i \binom{n}{i} (k - 1) + 1 \leq 2^n \sum_{i=1}^{k} \binom{n}{i} (k - 1) = 2^n \sum_{i=1}^{k} \frac{n}{n - i} (k - 1)
\]
\[
= 2^n \left( \binom{n}{k - 1} \binom{0}{k - 1} + \cdots + \binom{n}{n - k} \binom{k - 1}{k - 1} \right)
\]
\[
= 2^n \left( \frac{n + k - 1}{n - 1} \right) \leq 2^n \left( \frac{n + k}{n} \right).
\]
For \( x \in (0, +\infty) \), we define
\[
g(x) = \frac{2^x (1 + x)^{1+x}}{x^x}.
\]
Clearly, \( g \) is strictly increasing on \((0, +\infty)\), and \( \lim_{x \to 0^+} g(x) = 1 \). For each \( t \in (1, +\infty) \), let \( b(t) \) be the solution to the equation \( g(x) = t \). Numerical calculation shows that \( b(2) \approx 0.205597 \). We can easily prove that, if \( k_2(n) \) is the integer satisfying
\[
2^{k_1(n)} \binom{n + k_2(n)}{n} \leq 2^n < 2^{k_2(n)+1} \binom{n + k_2(n) + 1}{n},
\]
then we have \( \lim_{n \to \infty} \frac{k_2(n)}{n} = b(2) \) \([9, 10]\). It can be verified that
\[
\lim_{n \to \infty} \frac{k_1(n)}{n} > b(2).
\]
Therefore, the estimate in Corollary 2 is slightly better than that given by \([9, \text{Proposition 5}]\) in the asymptotical sense, and it is much better for particular choices of small \( n \). For example, we have \( k_1(7) = 2 \) and \( k_2(7) = 1 \). It follows that
\[
\Gamma_{128}(C_7) \leq \frac{n}{n + k_1(n)} = \frac{7}{7 + 2} \approx 0.78,
\]
which is better than \( \Gamma_{128}(C_7) \leq \frac{n}{n + k_2(n)} = \frac{7}{7 + 1} \leq 0.875 \) \([9]\). See Table 1 for more examples.
Table 1. Comparison of estimates of $\Gamma_{2^n}(C_n)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k_2(n)$</th>
<th>$\frac{n}{n+k_2(n)}$</th>
<th>$k_1(n)$</th>
<th>$\frac{n}{n+k_1(n)}$</th>
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<td>2</td>
<td>0.778</td>
</tr>
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<td>0.846</td>
<td>3</td>
<td>0.786</td>
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<tr>
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<td>4</td>
<td>0.8</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>0.833</td>
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<td>0.8</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
<td>0.833</td>
<td>6</td>
<td>0.806</td>
</tr>
</tbody>
</table>

**Theorem 4.** For each $n \geq 3$, we have $\Gamma_{2^n}(C_n) \leq \frac{6}{7}$.

**Proof.** By numerical calculations, for each $3 \leq n \leq 49$, we have $\Gamma_{2^n}(C_n) \leq \frac{6}{7}$. Set $c = b(2) - 0.02$. Then for each $n \geq 50$, we have $cn \leq b(2)n - 1$, which shows that $(1 + c)n \leq n + \lfloor b(2)n \rfloor$. Therefore,

$$\Gamma_{2^n}(C_n) \leq \frac{n}{n + \lfloor b(2)n \rfloor} \leq \frac{n}{(1 + c)n} \approx 0.8435.$$

Thus, for each $n \geq 3$, we have $\Gamma_{2^n}(C_n) \leq \max\{\frac{6}{7}, 0.8435\} = \frac{6}{7}$. □

3. Conclusions

By refining techniques used in the recent works [9, 10], we get better estimates of $\Gamma_{2^n}(C_n)$ and the first nontrivial universal upper bound of $\Gamma_{2^n}(C_n)$. It is natural to find universal bounds of $\Gamma_{2^n}(B^n_p)$ for fixed $p \in (1, \infty)$, where $B^n_p$ is the closed unit ball of $(\mathbb{R}^n, \|\cdot\|_p)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There is no conflicts of interest between all authors.

References


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