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Research article

Answers to questions on Kannan's fixed point theorem in strong b-metric spaces

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Abstract: Our purpose of this paper is to answer several open questions posed by Doan (AIMS Math., 6 (2021), 7895–7908). First, we present two fixed point theorems, which are positive answers to Doan's questions. Second, we establish a new type of Riech's fixed point theorem to improve a result of Doan. Finally, we offer a straightforward example illustrating that a set-valued mapping satisfying the conditions of our fixed point theorem may has more than one fixed point.

Keywords: fixed point; strong b-metric space; Geraghty functions; Kannan mapping; Riech's fixed point theorem; multi-valued mapping
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

Fixed point theory is one of the most powerful and fundamental tools of modern mathematics and may be considered a core subject of nonlinear analysis. The theory has developed rapidly since Banach's contraction principle [1] was introduced. There are many theorems that have the same conclusion as the contraction principle but with different sufficient conditions. For example, Kannan [2], Chatterjea [3], Geraghty [4], and Ćirić [5]. Next, we recall the concept of Kannan mapping.

Let (X, d) be a metric space, $T : X \to X$ is said to be a Kannan mapping if there exists a constant $\lambda \in [0, \frac{1}{2})$ such that

$$d(x, y) \leq \lambda(d(x, Tx) + d(y, Ty)),$$

for all $x, y \in X$. Kannan proved that every Kannan mapping in a complete metric space has a unique fixed point [2]. In our view, Kannan's fixed point theorem is very important because Subrahmanyam [6] proved that a metric space X is complete if and only if every Kannan mapping has a fixed point. Thereafter, Suzuki [8–10] further generalized this conclusion. In recent years, Lu [11] introduced the best area of Kannan system with degree s in b-metric spaces with constant s. Futhermore, Berinde and Pacurar [12] presented the concept of enriched Kannan mappings. Mohapatra et al. [13] defined the new concepts of mutual Kannan contractivity and mutual contractivity that generalized the Kannan mapping and contraction. In [14], Debnath generalized Kannan's fixed point Theorem and used it to solve a particular type of integral equation. For more conclusions on Geraghty type contractions, see [4, 16, 18, 19, 25]. About multi-valued mappings, see [15, 26–30].

On the other hand, in 2018, Górnicki [7] proved some extensions of Kannan's fixed point theorem in the framework of metric space. In 2021, Doan [17] extended a result of [7] and proved some generalizations of Kannan-type fixed point theorems for singlevalued and multivalued mappings defined on a complete strong b-metric space. On this basis, Doan raised two open questions. Our main purpose of this paper is to give positive answers to those two questions and establish a new type of Riech's fixed point theorem to improve results of Doan.

Kirk and Shahzad [20] introduced the notion of strong b-metric space. Some deep results about strong b-metric spaces are obtained in [21–24].

Definition 1.1. *[20] Let X be a nonempty set, $K \geq 1$, $D : X \times X \to [0, \infty)$ be a mapping. If for all $x, y, z \in X$,*

(1) $D(x, y) = 0 \Leftrightarrow x = y$;
(2) $D(x, y) = D(y, x)$;
(3) $D(x, y) \leq KD(x, z) + D(z, y)$.

Then D is called a strong b-metric on X and (X, D, K) is called a strong b-metric space.

Remark 1.2. *Let (X, D, K) be a strong b-metric space. From Definition 1.1, we can derive the inequality,*

$$D(x, y) \leq D(x, z) + KD(z, y), \text{ for all } x, y, z \in X.$$

In fact, for all $x, y, z \in X$, we have

$$D(x, y) = D(y, x) \leq KD(y, z) + D(z, x) = D(x, z) + KD(z, y).$$

Therefore, for every strong b-metric D with constant K, it implies that

$$D(x, y) \leq \min\{\, KD(x, z) + D(z, y), D(x, z) + KD(z, y) \,\},$$

refer to [21].

It is obvious that if (X, D) is a metric space, then it is a strong b-metric space.

Definition 1.3. *[20] Let (X, D, K) be a strong b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then*

(1) $\{x_n\}$ is said to converge to x if $\lim_{n\to\infty} D(x_n, x) = 0$;
(2) $\{x_n\}$ is called Cauchy if $\lim_{n,m\to\infty} D(x_n, x_m) = 0$;
(3) (X, D, K) is said to be complete if every Cauchy sequence converges.

Throughout this paper, we denote $\mathbb{N}^*$ as the set of all positive integers. Let (X, D) be a metric space. We denote by $CB(X)$ the collection of all nonempty bounded closed subsets of (X, D). Let $T : X \to CB(X)$ be a multi-valued mapping, we say that x is a fixed point of T if $x \in Tx$. Let $H : CB(X) \times CB(X) \to [0, \infty)$ be the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) := \max\{\sup_{x\in B} d(x, A), \sup_{x\in A} d(x, B)\},$$

where $A, B \in CB(X)$ and $d(x, A) := \inf_{y\in A} D(x, y)$.

In order to characterize the open problems posed by Doan [17]. We will use the following class of functions

$$\Psi_q = \{\psi : (0, \infty) \to [0, q) \mid \psi(t_n) \to q \text{ implies } t_n \to 0\},$$

where $q \in (0, \frac{1}{2})$. We call Ψ_q the class of Geraghty functions. We next introduce the two questions raised by Doan.

Theorem 1.4. *[17, Theorem 2.4] Let (X, D, K) be a complete strong b-metric space, $T : X \to X$ be a mapping, $q \in (0, \frac{1}{2})$. If there exists $\psi \in \Psi_q$ satisfying for all $x, y \in X$ with $x \neq y$,*

$$\frac{1}{K+1} D(x, Tx) \leq D(x, y),$$

implies

$$D(Tx, Ty) \leq \psi(D(x, y))(D(x, Tx) + D(y, Ty)).$$

Then, T has a unique fixed point $x^* \in X$.

Question 1.5. *Does there exist $q = \frac{1}{2}$ such that the above theorem holds?*

For brevity, we denote $\Psi_{\frac{1}{2}} := \{\psi : (0, \infty) \to [0, \frac{1}{2}) \mid \psi(t_n) \to \frac{1}{2} \text{ implies } t_n \to 0\}$.

Theorem 1.6. *[17, Theorem 3.3] Let (X, D, K) be a complete strong b-metric space and $T : X \to CB(X)$ be a multi-valued mapping. Suppose there exists $s \in (0, k)$ with $0 < k < \frac{1}{2}$ satisfying*

$$\frac{1}{K+1} d(x, Tx) \leq D(x, y) \text{ implies } H(Tx, Ty) \leq s(d(x, Tx) + d(y, Ty)),$$

for each $x, y \in X$. Then T has a fixed point.

Question 1.7. *Does there exist $k = \frac{1}{2}$ such that mapping T in Theorem 1.6 has a fixed point free?*

2. Answer to questions

2.1. Answer to question 1

In this section, we answer question 1, and first we give the following lemma.

Lemma 2.1. *Let (X, D, K) be a strong b-metric space, $T : X \to X$ be a mapping. If there exists $q \in (0, \frac{1}{2}]$ and $\psi \in \Psi_q$ satisfying for all $x, y \in X$ with $x \neq y$,*

$$\frac{1}{K+1} D(x, Tx) \leq D(x, y),$$

implies

$$D(Tx, Ty) \leq \psi(D(x, y))(D(x, Tx) + D(y, Ty)).$$

Then,

(1) $D(Tx, T^2x) \leq D(x, Tx)$, for each $x \in X$;
(2) for all $x, y \in X$, either $\frac{1}{K+1} D(x, Tx) \leq D(x, y)$ or $\frac{1}{K+1} D(Tx, T^2x) \leq D(Tx, y)$.

Proof. (1) Let $x \in X$ be an arbitrary point. Without loss of generality, we can suppose that $x \neq Tx$. From $\frac{1}{K+1}D(x, Tx) \leq D(x, Tx)$, we have

$$\begin{aligned} D(Tx, T(Tx)) &\leq \psi(D(x, Tx))(D(x, Tx) + D(Tx, T(Tx))) \\ &< \frac{1}{2}(D(x, Tx) + D(Tx, T(Tx))), \end{aligned}$$

which implies that

$$D(Tx, T^2x) \leq D(x, Tx), \quad \forall x \in X. \tag{2.1}$$

(2) By contradiction, assume that there exists $x', y' \in X$ such that $D(x', y') < \frac{1}{K+1}D(x', Tx')$ and $D(Tx', y') < \frac{1}{K+1}D(Tx', T^2x')$. Using the triangle inequality and (2.1), we have

$$\begin{aligned} D(x', Tx') &\leq D(x', y') + KD(y', Tx') \\ &< \frac{1}{K+1}D(x', Tx') + \frac{K}{K+1}D(Tx', T^2x') \\ &\leq \frac{1}{K+1}D(x', Tx') + \frac{K}{K+1}D(x', Tx') \\ &= D(x', Tx'), \end{aligned}$$

which contradicts the fact that $D(x', Tx') > 0$ (because $D(x', Tx') > (K + 1)D(x', y') \geq 0$). Thus, we proved (2). □

Theorem 2.2. *Let (X, D, K) be a complete strong b-metric space, $T : X \to X$ be a mapping. If there exists $\psi \in \Psi_{\frac{1}{2}}$ satisfying for all $x, y \in X$ with $x \neq y$,*

$$\frac{1}{K+1}D(x, Tx) \leq D(x, y),$$

implies

$$D(Tx, Ty) \leq \psi(D(x, y))(D(x, Tx) + D(y, Ty)).$$

Then, T has a unique fixed point $x^* \in X$.

Proof. Let x be an arbitrary point in X. Let $x_n = T^n x$, $n \in \mathbb{N}^*$. If for some $n_0 \in \mathbb{N}^*$, $x_{n_0} = x_{n_0+1}$, then x_{n_0} will be a fixed point of T. So, we can suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}^*$. From Lemma 2.1, for all $n \in \mathbb{N}^*$, we have

$$D(x_{n+1}, x_{n+2}) = D(Tx_n, T^2x_n) \leq D(x_n, Tx_n) = D(x_n, x_{n+1}).$$

Therefore, $\{D(x_n, x_{n+1})\}_{n=1}^{\infty}$ is a decreasing sequence of nonnegative real numbers, which implies that it has a limit. Let $\lim_{n\to\infty} D(x_n, x_{n+1}) = t \geq 0$. In order to prove that $t = 0$, suppose that $t > 0$. In such a case, since $0 < \frac{1}{K+1}D(x_n, x_{n+1}) \leq D(x_n, x_{n+1})$, for all $n \in \mathbb{N}^*$, we have

$$D(x_{n+1}, x_{n+2}) \leq \psi(D(x_n, x_{n+1}))(D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2})).$$

Then

$$\frac{D(x_{n+1}, x_{n+2})}{D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2})} \leq \psi(D(x_n, x_{n+1})) < \frac{1}{2}.$$

Passing to the limit as $n \to \infty$, we get $\lim_{n\to\infty} \psi(D(x_n, x_{n+1})) = \frac{1}{2}$, which implies that $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$, which is a contradiction. Therefore, $t = 0$ and $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$.

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}^*$ such that

$$D(x_{n-1}, x_n) < \frac{\varepsilon}{K+1}, \quad \forall n > N.$$

From Lemma 2.1, for all $n, m \in \mathbb{N}^*$ with $m > n > N$, either $\frac{1}{K+1} D(x_{n-1}, Tx_{n-1}) \leq D(x_{n-1}, x_{m-1})$ or $\frac{1}{K+1} D(Tx_{n-1}, T^2 x_{n-1}) \leq D(Tx_{n-1}, x_{m-1})$. We consider two cases.
Case 1. If $\frac{1}{K+1} D(x_{n-1}, Tx_{n-1}) \leq D(x_{n-1}, x_{m-1})$. In this case, notice that $D(x_{n-1}, Tx_{n-1}) = D(x_{n-1}, x_n) > 0$, we have

$$\begin{aligned} D(x_n, x_m) = D(Tx_{n-1}, Tx_{m-1}) &\leq \psi(D(x_{n-1}, x_{m-1}))(D(x_{n-1}, x_n) + D(x_{m-1}, x_m)) \\ &< \frac{1}{2}(D(x_{n-1}, x_n) + D(x_{m-1}, x_m)) \leq \max\{D(x_{n-1}, x_n), D(x_{m-1}, x_m)\} \\ &< \frac{\varepsilon}{K+1} < \varepsilon. \end{aligned}$$

Case 2. If $\frac{1}{K+1} D(Tx_{n-1}, T^2 x_{n-1}) \leq D(Tx_{n-1}, x_{m-1})$. In this case, notice that $D(Tx_{n-1}, T^2 x_{n-1}) = D(x_n, x_{n+1}) > 0$, we have

$$\begin{aligned} D(x_n, x_m) &\leq KD(x_n, x_{n+1}) + D(Tx_n, Tx_{m-1}) \\ &\leq KD(x_n, x_{n+1}) + \psi(D(x_n, x_{m-1}))(D(x_n, x_{n+1}) + D(x_{m-1}, x_m)) \\ &< KD(x_n, x_{n+1}) + \max\{D(x_n, x_{n+1}), D(x_{m-1}, x_m)\} \\ &< K\frac{\varepsilon}{K+1} + \frac{\varepsilon}{K+1} = \varepsilon. \end{aligned}$$

Thus, combining all the cases we have

$$D(x_n, x_m) < \varepsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in (X, D, K). As it is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Since $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$, for all $\varepsilon' > 0$, there exists $N' \in \mathbb{N}^*$ such that

$$D(x^*, Tx_n) < \frac{\varepsilon'}{4K} \text{ and } D(x_n, x_{n+1}) < \frac{\varepsilon'}{2}, \quad n > N'. \tag{2.2}$$

Obviously, the sequence $\{x_n\}$ has an infinite number of terms not equal to x^*. By Lemma 2.1, for all x_n, where $x_n \neq x^*$ and $n > N'$, either $\frac{1}{K+1} D(x_n, Tx_n) \leq D(x_n, x^*)$ or $\frac{1}{K+1} D(Tx_n, T^2 x_n) \leq D(Tx_n, x^*)$. Clearly, there exists x_{n_0}, where $x_{n_0} \neq x^*$ and $n_0 > N'$, such that $\frac{1}{K+1} D(x_{n_0}, Tx_{n_0}) \leq D(x_{n_0}, x^*)$. Then

$$\begin{aligned} D(x^*, Tx^*) &\leq KD(x^*, Tx_{n_0}) + D(Tx_{n_0}, Tx^*) \\ &\leq KD(x^*, Tx_{n_0}) + \psi(D(x_{n_0}, x^*))(D(x_{n_0}, x_{n_0+1}) + D(x^*, Tx^*)) \\ &< KD(x^*, Tx_{n_0}) + \frac{1}{2}(D(x_{n_0}, x_{n_0+1}) + D(x^*, Tx^*)). \end{aligned}$$

From (2.2), we have

$$D(x^*, Tx^*) \leq 2KD(x^*, Tx_{n_0}) + D(x_{n_0}, x_{n_0+1}) < 2K \cdot \frac{\varepsilon'}{4K} + \frac{\varepsilon'}{2} = \varepsilon'.$$

Then, $D(x^*, Tx^*) = 0$, x^* is a fixed point of T.

Now, suppose that y^* is another fixed point of T such that $y^* \neq x^*$. Since $\frac{1}{K+1}D(x^*, Tx^*) \leq D(x^*, y^*)$, we have

$$D(x^*, y^*) = D(Tx^*, Ty^*) \leq \psi(D(x^*, y^*))(D(x^*, Tx^*) + D(y^*, Ty^*)) = 0,$$

which is a contradiction. Therefore, T has a unique fixed point x^* and $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$. □

Remark 2.3. *Theorem 1.4 is a corollary of Theorem 2.2.*

Proof. Let (X, D, K) be a complete strong b-metric space, $q \in (0, \frac{1}{2})$, $T : X \to X$ be a mapping, which satisfying the condition of Theorem 1.4 with $\psi \in \Psi_q$. It is not difficult to observe that the function $\varphi : (0, \infty) \to [0, q)$ defined by

$$\varphi(t) = \frac{\psi(t)}{2q}, \quad t \in (0, \infty),$$

belongs to $\Psi_{\frac{1}{2}}$. For all $x, y \in X$ with $x \neq y$, if $\frac{1}{K+1}D(x, Tx) \leq D(x, y)$, then

$$\begin{aligned} D(Tx, Ty) &\leq \psi(D(x, y))(D(x, Tx) + D(y, Ty)) \\ &\leq \frac{\psi(D(x, y))}{2q}(D(x, Tx) + D(y, Ty)) \\ &= \varphi(D(x, y))(D(x, Tx) + D(y, Ty)). \end{aligned}$$

According to Theorem 2.2, T has a unique fixed point. □

Corollary 2.4. *[17, Theorem 2.1] Let (X, D, K) be a complete strong b-metric space, $T : X \to X$ be a mapping. If there exists $\psi \in \Psi_{\frac{1}{2}}$ satisfying for all $x, y \in X$,*

$$D(Tx, Ty) \leq \psi(D(x, y))(D(x, Tx) + D(y, Ty)).$$

Then, T has a unique fixed point $x^* \in X$.

2.2. Answer to question 2

In order to answer question 2, we first need a couple of lemmas.

Lemma 2.5. *[17] Let (X, D, K) be a strong b-metric space and $A, B \in CB(X)$. If $H(A, B) > 0$ then for all $h > 1$ and $a \in A$, there exists $b \in B$ such that*

$$D(a, b) < h \cdot H(A, B).$$

Lemma 2.6. *[24] Let (X, D, K) be a strong b-metric space and let $\{x_n\}$ be a sequence in X. Assume that there exists $\lambda \in [0, 1)$ satisfying*

$$D(x_{n+1}, x_{n+2}) \leq \lambda D(x_n, x_{n+1}),$$

for any $n \in \mathbb{N}^*$. Then $\{x_n\}$ is Cauchy.

Lemma 2.7. *[26] Let (X, D, K) be a strong b-metric space, then for all $a \in X$ and $A, B \in CB(X)$*

$$d(a, A) \leq Kd(a, B) + H(A, B).$$

Proof. Let $a \in X$, $A, B \in CB(X)$. Using the triangular inequality, for all $y \in B$, we have

$$\begin{aligned} d(a,A) &= \inf_{x\in A} D(a,x) \\ &\leq \inf_{x\in A}(KD(a,y) + D(y,x)) \\ &= KD(a,y) + \inf_{x\in A} D(y,x) \\ &= KD(a,y) + d(y,A) \\ &\leq KD(a,y) + H(A,B). \end{aligned}$$

Hence, we have

$$\begin{aligned} d(a,A) &\leq \inf_{y\in B} KD(a,y) + H(A,B) \\ &= Kd(a,B) + H(A,B). \end{aligned}$$

The proof is complete. □

Theorem 2.8. *Let (X, D, K) be a complete strong b-metric space and $T : X \to CB(X)$ be a multi-valued mapping. Suppose there exists $s \in (0, \frac{1}{2})$ satisfying*

$$\frac{1}{K+1} d(x, Tx) \leq D(x,y) \text{ implies } H(Tx,Ty) \leq s(d(x,Tx) + d(y,Ty)),$$

for each $x, y \in X$. Then T has at least one fixed point.

Proof. First, we construct a sequence $\{x_n\} \subseteq X$ such that for each $n \in \mathbb{N}^*$, $x_n \in Tx_{n-1}$ and

$$D(x_n, x_{n+1}) < hH(Tx_{n-1}, Tx_n), \tag{2.3}$$

where $h = \frac{1}{4s} + \frac{1}{2} > 1$. Let $x_0 \in X$ and $x_1 \in Tx_0$. If $H(Tx_0, Tx_1) = 0$, which implies that $Tx_0 = Tx_1$, then $x_1 \in Tx_0 = Tx_1$ and x_1 is a fixed point of T. So, let us suppose that $H(Tx_0, Tx_1) > 0$. From Lemma 2.5, for $h = \frac{1}{4s} + \frac{1}{2} > 1$ and $x_1 \in Tx_0$, there exists $x_2 \in Tx_1$ such that

$$D(x_1, x_2) < hH(Tx_0, Tx_1).$$

Similarly, let us suppose that $H(Tx_1, Tx_2) > 0$, by Lemma 2.5, there exists $x_3 \in Tx_2$ such that

$$D(x_2, x_3) < hH(Tx_1, Tx_2).$$

Suppose that $H(Tx_{n-1}, Tx_n) > 0$, for each $n \in \mathbb{N}^*$. Using Lemma 2.5 and proceeding inductively, we can obtain a sequence $\{x_n\}$ such that $x_n \in Tx_{n-1}$ and (2.3) holds for each $n \in \mathbb{N}^*$.

Since $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}^*$, then $\frac{1}{K+1} d(x_{n-1}, Tx_{n-1}) \leq D(x_{n-1}, x_n)$. Hence, we have

$$\begin{aligned} H(Tx_{n-1}, Tx_n) &\leq s(d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)) \\ &\leq s(D(x_{n-1}, x_n) + D(x_n, x_{n+1})). \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we get

$$D(x_n, x_{n+1}) < hs(D(x_{n-1}, x_n) + D(x_n, x_{n+1})).$$

Therefore, for all $n \in \mathbb{N}^*$, we have

$$D(x_n, x_{n+1}) < \lambda D(x_{n-1}, x_n), \tag{2.5}$$

where $\lambda = \frac{hs}{1-hs} = \frac{1+2s}{3-2s} \in (\frac{1}{3}, 1)$. According to Lemma 2.6, $\{x_n\}$ is Cauchy. Since (X, D, K) complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

We claim that for all $n \in \mathbb{N}^*$, either $\frac{1}{K+1} d(x_n, Tx_n) \le D(x_n, x^*)$, or $\frac{1}{K+1} d(x_{n+1}, Tx_{n+1}) \le D(x_{n+1}, x^*)$. In order to prove our claim, we argue by contradiction. If there exists $n_0 \in \mathbb{N}^*$ such that $D(x_{n_0}, x^*) < \frac{1}{K+1} d(x_{n_0}, Tx_{n_0})$ and $D(x_{n_0+1}, x^*) < \frac{1}{K+1} d(x_{n_0+1}, Tx_{n_0+1})$. By (2.5), we have

$$\begin{aligned}
D(x_{n_0}, x_{n_0+1}) &\le KD(x_{n_0}, x^*) + D(x^*, x_{n_0+1}) \\
&< \frac{K}{K+1} d(x_{n_0}, Tx_{n_0}) + \frac{1}{K+1} d(x_{n_0+1}, Tx_{n_0+1}) \\
&\le \frac{K}{K+1} D(x_{n_0}, x_{n_0+1}) + \frac{1}{K+1} D(x_{n_0+1}, x_{n_0+2}) \\
&\le \frac{K}{K+1} D(x_{n_0}, x_{n_0+1}) + \frac{\lambda}{K+1} D(x_{n_0}, x_{n_0+1}) \\
&< D(x_{n_0}, x_{n_0+1}).
\end{aligned}$$

On the other hand, since $H(Tx_{n_0}, Tx_{n_0+1}) > 0$, then $Tx_{n_0} \ne Tx_{n_0+1}$. Hence, $D(x_{n_0}, x_{n_0+1}) > 0$. This contradiction guarantees that our claim holds.

Without loss of the generality, we may assume that $\frac{1}{K+1} d(x_n, Tx_n) \le D(x_n, x^*)$ holds for infinity positive integers n. Then, there exists $\{x_{n_i}\}_{i=1}^{\infty} \subseteq \{x_n\}$ such that

$$\frac{1}{K+1} d(x_{n_i}, Tx_{n_i}) \le D(x_{n_i}, x^*), \quad i \in \mathbb{N}^*.$$

By Lemma 2.7, for each $i \in \mathbb{N}^*$, we have

$$\begin{aligned}
d(x^*, Tx^*) &\le Kd(x^*, Tx_{n_i}) + H(Tx_{n_i}, Tx^*) \\
&\le Kd(x^*, Tx_{n_i}) + s(d(x_{n_i}, Tx_{n_i}) + d(x^*, Tx^*)).
\end{aligned}$$

Then, from (2.5), we get

$$\begin{aligned}
d(x^*, Tx^*) &\le \frac{K}{1-s} d(x^*, Tx_{n_i}) + \frac{s}{1-s} d(x_{n_i}, Tx_{n_i}) \\
&\le 2KD(x^*, x_{n_i+1}) + D(x_{n_i}, x_{n_i+1}) \\
&< 2KD(x^*, x_{n_i+1}) + \lambda D(x_{n_i-1}, x_{n_i}) \\
&< \cdots \\
&\le 2KD(x^*, x_{n_i+1}) + \lambda^{n_i} D(x_0, x_1),
\end{aligned}$$

where $\lambda \in (\frac{1}{3}, 1)$. Letting $i \to \infty$ in the above inequality, we obtain $d(x^*, Tx^*) = 0$. Then x^* is a fixed point of T. □

Remark 2.9. *Notice that the Hausdorff semidistance is utilized in the fixed point theorems for multi-valued mappings, for example [31–33]. It is obvious that the Hausdorff semidistance $e(A, B)$ and the Hausdorff distance $H(A, B)$ are distinct. However, we can demonstrate that Lemma 2.5, Lemma 2.7, and Theorem 2.8 hold, if replacing "$H(A, B)$" with "$e(A, B)$", "$e(B, A)$", and "$e(A, B)$", respectively.*

Remark 2.10. *It is evident to see that Theorem 1.6 can be obtained from Theorem 2.8.*

Corollary 2.11. *[15] Let (X, d) be a complete metric space, $0 \leq s < \frac{1}{2}$. Suppose $T : X \to CB(X)$ is a continuous multi-valued mapping satisfying*

$$H(Tx, Ty) \leq s(d(x, Tx) + d(y, Ty)), \quad \text{for all } x, y \in X,$$

then T has at least one fixed point.

We give an example of a multi-valued mapping T that satisfies the conditions of Theorem 2.8. It is worth noting that all points in X are fixed points of T.

Example 2.12. *Let $X = \mathbb{N}^*$, $D : X \times X \to [0, \infty)$ defined by $D(x, y) = |x - y|$, for all $x, y \in X$. It is easy to verify that $(X, D, 1)$ is a complete strong b-metric space. Let $T : X \to CB(X)$ defined by*

$$Tx \equiv X, \quad \text{for all } x \in X.$$

Then it is clear that $d(x, Tx) = 0$ and $H(Tx, Ty) = 0$ for each $x, y \in X$. By Theorem 2.8, T has at least one fixed point. Furthermore, it is easy to see that any point in X is an fixed point of T.

3. A new type of Riech's fixed point theorem

Lemma 3.1. *Let (X, D, K) be a strong b-metric space, $T : X \to X$ be a mapping. If there exists $\varphi \in \Psi_{\frac{1}{3}}$ satisfying for all $x, y \in X$ with $x \neq y$,*

$$\frac{1}{K+1} D(x, Tx) \leq D(x, y),$$

implies

$$D(Tx, Ty) \leq \varphi(D(x, y))(D(x, Tx) + D(y, Ty) + D(x, y)).$$

Then,

(1) $D(Tx, T^2x) \leq D(x, Tx)$, for each $x \in X$;
(2) for all $x, y \in X$, either $\frac{1}{K+1} D(x, Tx) \leq D(x, y)$ or $\frac{1}{K+1} D(Tx, T^2x) \leq D(Tx, y)$.

Proof. For any $x \in X$, without loss of generality, we may consider $x \neq Tx$. By $\frac{1}{K+1} D(x, Tx) \leq D(x, Tx)$, we have

$$\begin{aligned} D(Tx, T(Tx)) &\leq \varphi(D(x, Tx))(D(x, Tx) + D(Tx, T(Tx)) + D(x, Tx)) \\ &< \frac{2}{3} D(x, Tx) + \frac{1}{3} D(Tx, T(Tx)). \end{aligned}$$

Thus, $D(Tx, T^2x) \leq D(x, Tx)$ for all $x \in X$. The proof of the second part of this Lemma follows in a similar manner as Lemma 2.1 and so is omitted. □

Theorem 3.2. *Let (X, D, K) be a complete strong b-metric space, $T : X \to X$ be a mapping. If there exists $\varphi \in \Psi_{\frac{1}{3}}$ satisfying for all $x, y \in X$ with $x \neq y$,*

$$\frac{1}{K+1} D(x, Tx) \leq D(x, y),$$

implies

$$D(Tx, Ty) \leq \varphi(D(x, y))(D(x, Tx) + D(y, Ty) + D(x, y)).$$

Then, T has a unique fixed point $x^* \in X$.

Proof. Let $x \in X$ be an arbitrary point and $\{x_n\}$ be a sequence defined by $x_n = T^n x$ for all $n \in \mathbb{N}^*$, suppose that every $D(x_n, x_{n+1}) > 0$. By Lemma 3.1,

$$D(x_{n+1}, x_{n+2}) = D(Tx_n, T^2x_n) \leq D(x_n, Tx_n) = D(x_n, x_{n+1}), \quad n \in \mathbb{N}^*.$$

Then, $\{D(x_n, x_{n+1})\}_{n=1}^{\infty}$ is monotonically decreasing with a lower bound. Hence, $\{D(x_n, x_{n+1})\}$ converges. For each $n \in \mathbb{N}^*$, since $D(x_n, x_{n+1}) > 0$ and $\frac{1}{K+1}D(x_n, Tx_n) \leq D(x_n, x_{n+1})$, we get

$$D(Tx_n, Tx_{n+1}) \leq \varphi(D(x_n, x_{n+1}))(2D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2})).$$

Then

$$\frac{D(x_{n+1}, x_{n+2})}{2D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2})} \leq \varphi(D(x_n, x_{n+1})) < \frac{1}{3}.$$

Suppose that $\lim_{n\to\infty} D(x_n, x_{n+1}) > 0$. Letting $n \to \infty$, we obtain $\varphi(D(x_n, x_{n+1})) \to \frac{1}{3}$, which implies $D(x_n, x_{n+1}) \to 0$. This contradiction guarantees that $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$.

According to Lemma 3.1, for each $p, q \in \mathbb{N}^*$, either $0 < \frac{1}{K+1}D(x_p, Tx_p) \leq D(x_p, x_q)$ or $0 < \frac{1}{K+1}D(Tx_p, T^2x_p) \leq D(Tx_p, x_q)$. Let $M(p, q) = (K + \frac{K+1}{3})D(x_p, x_{p+1}) + \frac{1}{3}D(x_q, x_{q+1}) + \frac{1}{3}D(x_p, x_q)$, where $p, q \in \mathbb{N}^*$. We claim that

$$D(Tx_p, Tx_q) \leq M(p, q), \quad p, q \in \mathbb{N}^*. \tag{3.1}$$

Now there are the following two cases.
Case 1. If $0 < \frac{1}{K+1}D(x_p, Tx_p) \leq D(x_p, x_q)$. In this case, we have

$$\begin{aligned} D(Tx_p, Tx_q) &\leq \varphi(D(x_p, x_q))(D(x_p, x_{p+1}) + D(x_q, x_{q+1}) + D(x_p, x_q)) \\ &< \frac{1}{3}(D(x_p, x_{p+1}) + D(x_q, x_{q+1}) + D(x_p, x_q)) \\ &\leq M(p, q). \end{aligned}$$

Case 2. If $0 < \frac{1}{K+1}D(Tx_p, T^2x_p) \leq D(Tx_p, x_q)$. In this case, by Lemma 3.1, we have

$$\begin{aligned} D(Tx_p, Tx_q) &\leq KD(Tx_p, T^2x_p) + D(T^2x_p, Tx_q) \\ &\leq KD(Tx_p, T^2x_p) + \varphi(D(Tx_p, x_q))(D(Tx_p, T^2x_p) + D(x_q, Tx_q) + D(Tx_p, x_q)) \\ &\leq (K + \frac{1}{3})D(Tx_p, T^2x_p) + \frac{1}{3}D(x_q, Tx_q) + \frac{K}{3}D(Tx_p, x_p) + \frac{1}{3}D(x_p, x_q) \\ &\leq (K + \frac{1+K}{3})D(x_p, Tx_p) + \frac{1}{3}D(x_q, Tx_q) + \frac{1}{3}D(x_p, x_q) \\ &= M(p, q). \end{aligned}$$

Therefore, we obtain (3.1).

Next, we demonstrate that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. If not, it is easy to show that there exists $\varepsilon_0 > 0$ and two subsequence $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that for each $k \in \mathbb{N}^*$, we have

$$D(x_{n_k}, x_{m_k}) \geq \varepsilon_0 \text{ and } D(x_{n_k}, x_{m_k-1}) < \varepsilon_0. \tag{3.2}$$

From $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$, there exists $N \in \mathbb{N}^*$ such that $D(x_n, x_{n+1}) < \frac{\varepsilon_0}{7K+2}$ for each $n \geq N$. For all $k > N$, since $\min\{n_k, m_k, m_k - 1\} \geq K - 1 \geq N$, then

$$\max\{D(x_{n_k}, x_{n_k+1}), D(x_{m_k}, x_{m_k+1}), D(x_{m_k-1}, x_{m_k})\} < \frac{\varepsilon_0}{7K+2}.$$

By (3.1) and (3.2), we have

$$\begin{aligned}
D(Tx_{n_k}, Tx_{m_k}) &\leq D(x_{n_k+1}, x_{m_k}) + KD(x_{m_k} + x_{m_k+1}) \\
&\leq M(n_k, m_k - 1) + KD(x_{m_k} + x_{m_k+1}) \\
&= (K + \frac{K+1}{3})D(x_{n_k}, x_{n_k+1}) + \frac{1}{3}D(x_{m_k-1}, x_{m_k}) + KD(x_{m_k} + x_{m_k+1}) + \frac{1}{3}D(x_{n_k}, x_{m_k-1}) \\
&\leq (2K + \frac{K+2}{3}) \max\{D(x_{n_k}, x_{n_k+1}), D(x_{m_k-1}, x_{m_k}), D(x_{m_k} + x_{m_k+1})\} + \frac{1}{3}D(x_{n_k}, x_{m_k-1}) \\
&< (2K + \frac{K+2}{3}) \cdot \frac{\varepsilon_0}{7K+2} + \frac{\varepsilon_0}{3} = \frac{2\varepsilon_0}{3}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
D(x_{n_k}, x_{m_k}) &\leq KD(x_{n_k}, x_{n_k+1}) + D(x_{n_k+1} + x_{m_k}) \\
&\leq KD(x_{n_k}, x_{n_k+1}) + KD(x_{m_k} + x_{m_k+1}) + D(x_{m_k+1} + x_{n_k+1}) \\
&\leq 2K \max\{D(x_{n_k}, x_{n_k+1}), D(x_{m_k} + x_{m_k+1})\} + \frac{2\varepsilon_0}{3} \\
&< 2K \cdot \frac{\varepsilon_0}{7K+2} + \frac{2\varepsilon_0}{3} < \frac{\varepsilon_0}{3} + \frac{2\varepsilon_0}{3} = \varepsilon_0,
\end{aligned}$$

which contradicts (3.2). This contradiction shows that $\{x_n\}$ is Cauchy. As (X, D, K) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

According to Lemma 3.1, for each $n \in \mathbb{N}^*$, either $\frac{1}{K+1}D(x_n, Tx_n) \leq D(x_n, x^*)$ or $\frac{1}{K+1}D(Tx_n, T^2x_n) \leq D(Tx_n, x^*)$. Similarly, let us consider two cases.

Case 1. If $\frac{1}{K+1}D(x_n, Tx_n) \leq D(x_n, x^*)$, since $D(x_n, Tx_n) = D(x_n, x_{n+1}) > 0$, we have

$$\begin{aligned}
D(x^*, Tx^*) &\leq KD(x^*, Tx_n) + D(Tx_n, Tx^*) \\
&\leq KD(x^*, Tx_n) + \varphi(D(x_n, x^*))(D(x_n, x_{n+1}) + D(x^*, Tx^*) + D(x_n, x^*)) \\
&\leq KD(x^*, Tx_n) + \frac{1}{3}(D(x_n, x_{n+1}) + D(x^*, Tx^*) + D(x_n, x^*)).
\end{aligned}$$

Then

$$D(x^*, Tx^*) \leq \frac{3}{2}KD(x^*, x_{n+1}) + \frac{1}{2}(D(x_n, x_{n+1}) + D(x_n, x^*)).$$

Case 2. If $\frac{1}{K+1}D(Tx_n, T^2x_n) \leq D(Tx_n, x^*)$, by $D(Tx_n, T^2x_n) = D(x_{n+1}, x_{n+2}) > 0$, we get

$$D(x^*, Tx^*) \leq KD(x^*, T^2x_n) + D(T^2x_n, Tx^*)$$

$$\leq KD(x^*, T^2x_n) + \frac{1}{3}(D(Tx_n, T^2x_n) + D(x^*, Tx^*) + D(Tx_n, x^*)).$$

Then

$$D(x^*, Tx^*) \leq \frac{3}{2}KD(x^*, x_{n+2}) + \frac{1}{2}(D(x_{n+1}, x_{n+2}) + D(x_{n+1}, x^*)).$$

Therefore, for all $n \in \mathbb{N}^*$, we have

$$D(x^*, Tx^*) \leq \max\{\frac{3}{2}KD(x^*, x_{n+1}) + \frac{1}{2}(D(x_n, x_{n+1}) + D(x_n, x^*)),\\ \frac{3}{2}KD(x^*, x_{n+2}) + \frac{1}{2}(D(x_{n+1}, x_{n+2}) + D(x_{n+1}, x^*))\}.$$

Letting $n \to \infty$ in the above inequality, we obtain $D(x^*, Tx^*) = 0$ and x^* is a fixed point of T.

Suppose that y^* is another fixed point of T and $D(y^*, x^*) > 0$. Since $D(x^*, Tx^*) = 0$, it follows that $\frac{1}{K+1}D(x^*, Tx^*) \leq D(x^*, y^*)$. Then

$$D(x^*, y^*) = D(Tx^*, Ty^*) \leq \varphi(D(x^*, y^*))(D(x^*, Tx^*) + D(y^*, Ty^*) + D(x^*, y^*)) < \frac{1}{3}D(x^*, y^*),$$

which is a contradiction with the fact that $D(x^*, y^*) > 0$. As a consequence, T has a unique fixed point x^* and $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$. □

Corollary 3.3. *[17, Theorem 2.7] Let (X, D, K) be a complete strong b-metric space, $T : X \to X$ be a mapping. If there exists $\varphi \in \Psi_{\frac{1}{3}}$ satisfying for all $x, y \in X$ with $x \neq y$,*

$$D(Tx, Ty) \leq \varphi(D(x, y))(D(x, Tx) + D(y, Ty) + D(x, y)).$$

Then, T has a unique fixed point $x^* \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to x^*.

4. Conclusions

We focus on a new type of Kannan's fixed point theorem in the setting of strong b-metric spaces. Using some useful lemmas, we derive three fixed point theorems. The first two theorems give positive answers to Questions 1.5 and 1.7, respectively. The third theorem is a new type of Reich's fixed point theorem and also a generalization of Doan's result (Theorem 2.7 in [17]).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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