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Research article

The power sum of balancing polynomials and their divisible properties

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Abstract: In recent years, many scholars have studied the division properties of polynomials and sequence power sums. In this paper, we use Girard-Waring formula and combinatorial method to study the power sum problem of balancing polynomials and Lucas-balancing polynomials, and then study the division of balancing polynomials and Lucas-balancing polynomials by mathematical induction and the properties of polynomials.

Keywords: balancing polynomials; Lucas-balancing polynomials; power sum problem; divisible properties
Mathematics Subject Classification: 11B39, 11B37

1. Introduction

Behera and Panda [1] introduced the concept of balancing numbers B_n, a positive integer n is a balancing number if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),\ n,\ r \in N^*.$$

That r is the balancer corresponding to the balancing number n. The balancing numbers B_n satisfy the relation $B_{n+1} = 6B_n - B_{n-1}$, $n \geq 1$ with $B_0 = 0,\ B_1 = 1$. The sequence $C_n = \sqrt{8B_n^2 + 1}$ is called a Lucas-balancing number. The Lucas-balancing number satisfies same relation $C_{n+1} = 6C_n - C_{n-1}$, $n \geq 1$ with $C_0 = 1,\ C_1 = 3$. Some conclusions about these two sequences can be found in the references [2,3]. The balancing polynomial and the Lucas-balancing polynomial are natural extensions of balancing numbers and Lucas-balancing numbers.

For any integer $n \geq 0$, the balancing polynomials $B_n(x)$ and Lucas-balancing polynomials $C_n(x)$ are defined as follows (see Frontczak and Goy [4]):

$$B_n(x) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n-1-k}{k} (6x)^{n-1-2k},$$

$$C_n(x) = \frac{n}{2}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k}\binom{n-k}{k}(6x)^{n-2k},$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

$B_n(x)$ and $C_n(x)$ are the second-order linear recurrence polynomials, they satisfy the recurrence formulae (see Frontczak and Goy [4]):

$B_{n+1}(x) = 6xB_n(x) - B_{n-1}(x)$ for all $n \geq 1$, with $B_0(x) = 0$, $B_1(x) = 1$,

$C_{n+1}(x) = 6xC_n(x) - C_{n-1}(x)$ for all $n \geq 1$, with $C_0(x) = 1$, $C_1(x) = 3x$.

The closed forms which are also called Binets formulas for balancing polynomials and Lucas-balancing polynomials are given by

$$B_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{2\sqrt{9x^2-1}},\ \mathrm{C}_n(x) = \frac{\alpha^n(x) + \beta^n(x)}{2},$$

where $\alpha(x) = 3x + \sqrt{9x^2-1}$, $\beta(x) = 3x - \sqrt{9x^2-1}$. The relations $B_n(-x) = (-1)^{n+1}B_n(x)$ and $C_n(-x) = (-1)^n C_n(x)$ follow from $\alpha(-x) = -\beta(x)$ and $-\alpha(x) = \beta(x)$.

If we take $x = 1$, then $\{B_n(x)\}$ becomes balancing sequences $\{B_n\}$, and $\{C_n(x)\}$ becomes Lucas-balancing sequences $\{C_n\}$. Such balancing numbers and balancing polynomials have been widely studied in recent years. Frontczak [5] proves the sum of powers of balancing polynomials and Lucas balancing polynomials:

$$B_n^{2m+1}(x) = 2^{-2m}\left(9x^2-1\right)^{-m}\sum_{k=0}^{m}\binom{2m+1}{m-k}(-1)^{m-k}B_{(2k+1)n}(x),$$

$$C_n^{2m+1}(x) = 2^{-2m}\sum_{k=0}^{m}\binom{2m+1}{m-k}C_{(2k+1)n}(x).$$

Kim and Kim [6] used nine orthogonal polynomials to represent the sum of the finite product of balancing polynomials to obtained the following result:

$$\begin{aligned}&\sum\nolimits_{i_1+i_2+\cdots+i_{r+1}=n} B_{i_1+1}(x)B_{i_2+1}(x)\cdots B_{i_{r+1}+1}(x)\\ =\ &\frac{(-2)^n}{r!}\sum\nolimits_{k=0}^{n}\frac{(-2)^k\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)}\sum\nolimits_{l=0}^{\left[\frac{n-k}{2}\right]}\frac{\left(-\frac{1}{4}\right)^l(n+r-l)!}{l!\,(n-k-2l)!}\\ &\times {}_2F_1(k+2l-n, k+\beta+1; 2k+\alpha+\beta+2; 2)\,P_k^{(\alpha,\beta)}(3x).\end{aligned}$$

Ray [7] studied the divisible property of balancing numbers and Lucas-balancing number obtained the congruence:

$$B_{2mn+k} \equiv (-1)^n B_k\,(\mathrm{mod} C_m),\quad C_{2mn+k} \equiv (-1)^n C_k\,(\mathrm{mod} C_m).$$

For any integer $n \geq 0$, the famous Fibonacci polynomials $F_n(x)$ and Lucas polynomials $L_n(x)$ are defined as follows (see Wang and Zhang [8]) :

$F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, with $F_0(x) = 0$, $F_1(x) = 1$,

$L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$, with $L_0(x) = 2$, $L_1(x) = x$.

These polynomials and sequences have some similarities in structure and properties. Kim [9–13] obtained many meaningful results by studying connections between polynomials. Mathematics has a wide range of applications in other disciplines, see [14–16]. We can obtain some divisible properties of polynomials and sequences in references [17–19]. For example, Wang and Zhang [8] proved the congruence of the sum of powers of Fibonacci numbers. That is

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} L_{2k}^{2m+1} \equiv 0 \bmod \left(L_{2n+1} - 1\right).$$

In this paper, we use the properties of balancing polynomials and Lucas balancing polynomials to study the divisible properties of $\sum_{m=0}^{h} B_{2^s ml}^{2n+1}(x)$ and $\sum_{m=0}^{h} C_{2^s ml}^{2n+1}(x)$ to get more general results. That is, we shall prove the following two theorems.

Theorem 1. Let n and h be non-negative integer with $h \geq 1$, s and l be positive integers. Then we have the congruence

$$\begin{aligned} & 2^{2n+1}\left(9x^2-1\right)^{n+1} B_{2^{s-1}l}(x)\, B_{2^{s-1}3l}(x) \cdots B_{2^{s-1}(2n+1)l}(x) \sum_{m=0}^{h} B_{2^s ml}^{2n+1}(x) \\ \equiv\ & 0 \bmod \left(C_{2^{s-1}l(2h+1)}(x) - C_{2^{s-1}l}(x)\right). \end{aligned}$$

Theorem 2. Let n and h be non-negative integers with $h \geq 1$, s and l be positive integers. Then we have the congruence

$$\begin{aligned} & 2^{2n+1} B_{2^{s-1}l}(x)\, B_{2^{s-1}3l}(x) \cdots B_{2^{s-1}(2n+1)l}(x) \sum_{m=0}^{h} C_{2^s ml}^{2n+1}(x) \\ \equiv\ & 0 \bmod \left(B_{2^{s-1}l(2h+1)}(x) + B_{2^{s-1}l}(x)\right). \end{aligned}$$

From the two theorems, we can obtain the following corollaries.

Corollary 1. For any non-negative integers n and h with $h \geq 1$, we have

$$2^{2n+1}\left(9x^2-1\right)^{n+1} B_1(x)\, B_3(x) \cdots B_{(2n+1)}(x) \sum_{m=0}^{h} B_{2m}^{2n+1}(x) \equiv 0 \bmod \left(C_{2h+1}(x) - 3x\right).$$

Corollary 2. For any non-negative integers n and h with $h \geq 1$, we have

$$2^{2n+1} B_2(x)\, B_6(x) \cdots B_{2(2n+1)}(x) \sum_{m=0}^{h} C_{4m}^{2n+1}(x) \equiv 0 \bmod \left(B_{2(2h+1)}(x) + 6x\right).$$

Corollary 3. For any non-negative integers n and h with $h \geq 1$, and s and l be positive integers, we have

$$2^{5n+4} B_{2^{s-1}l} B_{2^{s-1}3l} \cdots B_{2^{s-1}(2n+1)l} \sum_{m=0}^{h} B_{2^s ml}^{2n+1} \equiv 0 \bmod \left(C_{2^{s-1}l(2h+1)} - C_{2^{s-1}l}\right).$$

Corollary 4. For any non-negative integers n and h with $h \geq 1$, and s and l be positive integers, we have

$$2^{2n+1} B_{2^{s-1}l} B_{2^{s-1}3l} \cdots B_{2^{s-1}(2n+1)l} \sum_{m=0}^{h} C_{2^s ml}^{2n+1} \equiv 0 \bmod \left(B_{2^{s-1}l(2h+1)} + B_{2^{s-1}l}\right).$$

For Chebyshev polynomials of the first kind $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ with $T_0(x) = 1$, $T_1(x) = x$ and Chebyshev polynomials of the second kind $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ with $U_0(x) = 1$, $U_1(x) = 2x$. The balancing polynomials possess a simple connection to Chebyshev polynomials of the first and second kind $T_n(x)$ and $U_n(x)$, specifically $B_n(x) = U_{n-1}(3x)$ and $C_n(x) = T_n(3x)$.

Taking $x = \frac{1}{3}x$ in Theorem 1, we can get the following,

Corollary 5. For any non-negative integers n and h with $h \geq 1$, and s and l be positive integers, we have

$$2^{2n+1}\left(x^2-1\right)^{n+1} U_{2^{s-1}l-1}(x)\, U_{2^{s-1}3l-1}(x) \cdots U_{2^{s-1}(2n+1)l-1}(x) \sum_{m=0}^{h} U_{2^s ml-1}^{2n+1}(x)$$
$$\equiv 0 \bmod \left(T_{2^{s-1}l(2h+1)}(x) - T_{2^{s-1}l}(x)\right).$$

Taking $s = 1$ and $x = \frac{1}{3}x$ in Theorem 2, we can get the following,

Corollary 6. For any non-negative integers n and h with $h \geq l$, and l be positive integers, we have

$$2^{2n+1} U_{l-1}(x)\, U_{3l-1}(x) \cdots U_{(2n+1)l-1}(x) \sum_{m=0}^{h} T_{2ml}^{2n+1}(x) \equiv 0 \bmod \left(U_{l(2h+1)-1}(x) + U_{l-1}(x)\right).$$

2. Some lemmas

In the following, we use the properties of balancing polynomials and Lucas-balancing polynomials to prove our next several lemmas, which will help us better complete the proofs of the theorems.

Lemma 1. Let s and h be positive integers. Then, for any integers n and l, we have the identity

$$C_{2^{s-1}l(2n+1)(2\text{h}+1)}(x) - C_{2^{s-1}l(2n+1)}(x) \equiv 0 \bmod \left(C_{2^{s-1}l(2h+1)}(x) - C_{2^{s-1}l}(x)\right).$$

Proof. We prove this polynomial congruence by complete induction for $n \geq 0$. It is clear that Lemma 1 is true for $n = 0$. If $n = 1$, then note $C_{2^{s-1}3l(2\text{h}+1)}(x) = 4C_{2^{s-1}l(2h+1)}^3(x) - 3C_{2^{s-1}l(2h+1)}(x)$, we have

$$\begin{aligned} & C_{2^{s-1}3l(2\text{h}+1)}(x) - C_{2^{s-1}3l}(x) \\ = \;& 4C_{2^{s-1}l(2h+1)}^3(x) - 3C_{2^{s-1}l(2\text{h}+1)}(x) - 4C_{2^{s-1}l}^3(x) + 3C_{2^{s-1}l}(x) \\ = \;& \left(C_{2^{s-1}l(2\text{h}+1)}(x) - C_{2^{s-1}l}(x)\right)\Big(4C_{2^{s-1}l(2h+1)}^2(x) \\ & +4C_{2^{s-1}l(2\text{h}+1)}(x)\, C_{2^{s-1}l}(x) + 4C_{2^{s-1}l}^2(x) - 3\Big) \\ \equiv \;& 0 \bmod \left(C_{2^{s-1}l(2\text{h}+1)}(x) - C_{2^{s-1}l}(x)\right). \end{aligned}$$

That is to say, Lemma 1 is true for $n = 1$.

Suppose that Lemma 1 is true for all positive integers $0 \leq n \leq j$. That is,

$$C_{2^{s-1}l(2n+1)(2\text{h}+1)}(x) - C_{2^{s-1}l(2n+1)}(x) \equiv 0 \bmod \left(C_{2^{s-1}l(2h+1)}(x) - C_{2^{s-1}l}(x)\right), \tag{2.1}$$

for all $0 \leq n \leq j$.

Then, for $n = j + 1 \geq 2$, we have

$$C_{2^{s-1}2l(2h+1)}(x)\, C_{2^{s-1}l(2n+1)(2h+1)}(x)$$

$$= \frac{1}{4}\left(\alpha^{2^{s-1}2l(2h+1)}(x)+\beta^{2^{s-1}2l(2h+1)}(x)\right)\left(\alpha^{2^{s-1}l(2n+1)(2h+1)}(x)+\beta^{2^{s-1}l(2n+1)(2h+1)}(x)\right)$$
$$= \frac{1}{4}\left(\alpha^{2^{s-1}l(2h+1)(2n+3)}(x)+\beta^{2^{s-1}l(2h+1)(2n-1)}(x)+\alpha^{2^{s-1}l(2h+1)(2n-1)}(x)+\beta^{2^{s-1}l(2h+1)(2n+3)}(x)\right)$$
$$= \frac{1}{2}\left(C_{2^{s-1}l(2h+1)(2n+3)}(x)+C_{2^{s-1}l(2h+1)(2n-1)}(x)\right)$$

and

$$\begin{aligned} C_{2^{s-1}2l(2h+1)}(x) &= \frac{1}{2}\left(\alpha^{2^{s-1}2l(2h+1)}(x)+\beta^{2^{s-1}2l(2h+1)}(x)\right) \\ &= \frac{1}{2}\left(\alpha^{2^{s-1}l(2h+1)}(x)+\beta^{2^{s-1}l(2h+1)}(x)\right)^2-1 \\ &\equiv 2C^2_{2^{s-1}l}(x)-1 \mod \left(C_{2^{s-1}l(2h+1)}(x)-C_{2^{s-1}l}(x)\right). \end{aligned}$$

Applying inductive hypothesis (2.1), we have

$$\begin{aligned} & C_{2^{s-1}l(2n+1)(2h+1)}(x)-C_{2^{s-1}l(2n+1)}(x) \\ =\ & C_{2^{s-1}l(2j+3)(2h+1)}(x)-C_{2^{s-1}l(2j+3)}(x) \\ =\ & 2C_{2^{s-1}2l(2h+1)}(x)\,C_{2^{s-1}l(2j+1)(2h+1)}(x)-C_{2^{s-1}l(2j-1)(2h+1)}(x) \\ & -2C_{2^{s-1}2l}(x)\,C_{2^{s-1}l(2j+1)}(x)+C_{2^{s-1}l(2j-1)}(x) \\ =\ & 2\left(2C^2_{2^{s-1}l(2h+1)}(x)-1\right)C_{2^{s-1}l(2j+1)(2h+1)}(x)-C_{2^{s-1}l(2j-1)(2h+1)}(x) \\ & -2\left(2C^2_{2^{s-1}l}(x)-1\right)C_{2^{s-1}l(2j+1)}(x)+C_{2^{s-1}l(2j-1)}(x) \\ \equiv\ & 2\left(2C^2_{2^{s-1}l}(x)-1\right)\left(C_{2^{s-1}l(2j+1)(2h+1)}(x)-C_{2^{s-1}l(2j+1)}(x)\right) \\ & -\left(C_{2^{s-1}l(2j-1)(2h+1)}(x)-C_{2^{s-1}l(2j-1)}(x)\right) \\ \equiv\ & 0 \mod \left(C_{2^{s-1}l(2h+1)}(x)-C_{2^{s-1}l}(x)\right). \end{aligned}$$

That is to say, the Lemma 1 is true for $n = j + 1$.

Now Lemma 1 follows from complete induction. □

Lemma 2. Let s and h be positive integers. Then, for any integers n and l, we have the identity

$$B_{2^{s-1}l(2n+1)(2h+1)}(x)+B_{2^{s-1}l(2n+1)}(x)\equiv 0 \mod \left(B_{2^{s-1}l(2h+1)}(x)+B_{2^{s-1}l}(x)\right).$$

Proof. We can also prove Lemma 2 by complete induction. If $n = 0$, then it is clear that Lemma 2 is true. If $n = 1$, then note

$$B_{2^{s-1}3l(2h+1)}(x)=4\left(9x^2-1\right)B^3_{2^{s-1}l(2h+1)}(x)+3B_{2^{s-1}l(2h+1)}(x),$$

we have

$$\begin{aligned} & B_{2^{s-1}3l(2h+1)}(x)+B_{2^{s-1}3l}(x) \\ =\ & 4\left(9x^2-1\right)B^3_{2^{s-1}l(2h+1)}(x)+3B_{2^{s-1}l(2h+1)}(x) \\ & +4\left(9x^2-1\right)B^3_{2^{s-1}l}(x)+3B_{2^{s-1}l}(x) \end{aligned}$$

$$\begin{aligned}
&= 4\left(9x^2-1\right)\left(B_{2^{s-1}l(2\mathrm{h}+1)}(x)+B_{2^{s-1}l}(x)\right)\left(B^2_{2^{s-1}l(2h+1)}(x)+B^2_{2^{s-1}l}(x)\right.\\
&\quad \left.-B_{2^{s-1}l(2\mathrm{h}+1)}(x)\,B_{2^{s-1}l}(x)\right)+3\left(B_{2^{s-1}l(2\mathrm{h}+1)}(x)+B_{2^{s-1}l}(x)\right)\\
&\equiv 0 \bmod \left(B_{2^{s-1}l(2\mathrm{h}+1)}(x)+B_{2^{s-1}l}(x)\right).
\end{aligned}$$

So Lemma 2 is true for $n = 1$. Suppose that Lemma 2 is true for positive integers $0 \leq n \leq j$. That is,

$$B_{2^{s-1}l(2n+1)(2\mathrm{h}+1)}(x)+B_{2^{s-1}l(2n+1)}(x) \equiv 0 \bmod \left(B_{2^{s-1}l(2h+1)}(x)+B_{2^{s-1}l}(x)\right) \tag{2.2}$$

for all $0 \leq n \leq j$.

Then, for $n = k + 1$, note the identities

$$\begin{aligned}
&2C_{2^{s-1}2l(2h+1)}(x)\,B_{2^{s-1}l(2n+1)(2h+1)}(x)\\
&= \frac{1}{2\sqrt{9x^2-1}}\left(\alpha^{2^{s-1}2l(2h+1)}(x)+\beta^{2^{s-1}2l(2h+1)}(x)\right)\left(\alpha^{2^{s-1}l(2n+1)(2h+1)}(x)-\beta^{2^{s-1}l(2n+1)(2h+1)}(x)\right)\\
&= \frac{1}{2\sqrt{9x^2-1}}(\alpha^{2^{s-1}l(2h+1)(2n+3)}(x)-\beta^{2^{s-1}l(2h+1)(2n-1)}(x)+\alpha^{2^{s-1}l(2h+1)(2n-1)}(x)-\beta^{2^{s-1}l(2h+1)(2n+3)}(x))\\
&= B_{2^{s-1}l(2n+3)(2h+1)}(x)+B_{2^{s-1}l(2n-1)(2h+1)}(x)
\end{aligned}$$

and

$$\begin{aligned}
C_{2^{s-1}2l(2h+1)}(x) &= \frac{1}{2}\left(\alpha^{2^{s-1}2l(2h+1)}(x)+\beta^{2^{s-1}2l(2h+1)}(x)\right)\\
&= \frac{1}{2}\left(\alpha^{2^{s-1}l(2h+1)}(x)-\beta^{2^{s-1}l(2h+1)}(x)\right)^2+1\\
&= 2\left(9x^2-1\right)B^2_{2^{s-1}l(2h+1)}(x)+1\\
&\equiv 2\left(9x^2-1\right)B^2_{2^{s-1}l}(x)+1 \mod \left(B_{2^{s-1}l(2h+1)}(x)+B_{2^{s-1}l}(x)\right),
\end{aligned}$$

applying inductive hypothesis (2.2), we have

$$\begin{aligned}
&B_{2^{s-1}l(2n+1)(2\mathrm{h}+1)}(x)+B_{2^{s-1}l(2\mathrm{n}+1)}(x)\\
&= B_{2^{s-1}l(2j+3)(2\mathrm{h}+1)}(x)+B_{2^{s-1}l(2j+3)}(x)\\
&= 2C_{2^{s-1}2l(2\mathrm{h}+1)}(x)\,B_{2^{s-1}l(2j+1)(2\mathrm{h}+1)}(x)-B_{2^{s-1}l(2j-1)(2\mathrm{h}+1)}(x)\\
&\quad +2C_{2^{s-1}2l}(x)\,B_{2^{s-1}l(2j+1)}(x)-B_{2^{s-1}l(2j\text{-}1)}(x)\\
&= \left[4\left(9x^2-1\right)B^2_{2^{s-1}l(2h+1)}(x)+2\right]B_{2^{s-1}l(2j+1)(2\mathrm{h}+1)}(x)-B_{2^{s-1}l(2j-1)(2\mathrm{h}+1)}(x)\\
&\quad +\left[4\left(9x^2-1\right)B^2_{2^{s-1}l}(x)+2\right]B_{2^{s-1}l(2j+1)}(x)-B_{2^{s-1}l(2j-1)}(x)\\
&\equiv 2\left[2\left(9x^2-1\right)B^2_{2^{s-1}l}(x)+1\right]\left(B_{2^{s-1}l(2j+1)(2\mathrm{h}+1)}(x)+B_{2^{s-1}l(2j+1)}(x)\right)\\
&\quad -\left(B_{2^{s-1}l(2j-1)(2\mathrm{h}+1)}(x)+B_{2^{s-1}l(2j-1)}(x)\right)\\
&\equiv 0 \bmod \left(B_{2^{s-1}l(2h+1)}(x)+B_{2^{s-1}l}(x)\right).
\end{aligned}$$

This completes the proof of Lemma 2. □

Lemma 3. For all non-negative integers r and real numbers w, v, we have the identity

$$\sum_{k=0}^{\left[\frac{r}{2}\right]}(-1)^k\frac{r}{r-k}\binom{r-k}{k}(w+v)^{r-2k}(wv)^k = w^r+v^r,$$

in which $[x]$ denotes the greatest integer $\leq x$.
Proof. The formula due to E. Waring [20] and can be found in H. W. Gould [21]. □

3. Proofs of the theorems

We shall prove our theorems by mathematical induction. Taking $w = \alpha^{2^s ml}(x)$, $v = -\beta^{2^s ml}(x)$ and $r = 2n+1$ in Lemma 3, we notice that $wv = -1$, from the expression of $B_n(x)$ we have

$$\begin{aligned} & B_{2^s ml(2n+1)}(x) \\ = & \sum_{k=0}^{n} (-1)^k 2^{2n-2k} \frac{2n+1}{2n+1-k} \binom{2n+1-k}{k} B_{2^s ml}^{2n+1-2k}(x) \left(\sqrt{9x^2-1}\right)^{2n-2k} (-1)^k \\ = & \sum_{k=0}^{n} 2^{2n-2k} \left(\sqrt{9x^2-1}\right)^{2n-2k} \frac{2n+1}{2n+1-k} \binom{2n+1-k}{k} B_{2^s ml}^{2n+1-2k}(x). \end{aligned} \tag{3.1}$$

For any integer $h \geq 1$, from (3.1) we have

$$\begin{aligned} & \sum_{m=0}^{h} \left(B_{2^s ml(2n+1)}(x) - (2n+1) B_{2^s ml}(x)\right) \\ = & \sum_{k=0}^{n-1} \frac{2n+1}{2n+1-k} 2^{2n-2k} \left(9x^2-1\right)^{n-k} \binom{2n+1-k}{k} \sum_{m=0}^{h} B_{2^s ml}^{2n+1-2k}(x). \end{aligned} \tag{3.2}$$

Note the identities

$$\begin{aligned} & \sum_{m=0}^{h} B_{2^s ml(2n+1)}(x) = \sum_{m=0}^{h} \frac{1}{2\sqrt{9x^2-1}} \left(\alpha^{2^s ml(2n+1)}(x) - \beta^{2^s ml(2n+1)}(x)\right) \\ = & \frac{1}{2\sqrt{9x^2-1}} \left(\frac{1-\alpha^{2^s l(2n+1)(h+1)}(x)}{1-\alpha^{2^s l(2n+1)}(x)} - \frac{1-\beta^{2^s l(2n+1)(h+1)}(x)}{1-\beta^{2^s l(2n+1)}(x)} \right) \\ = & \frac{1}{2\sqrt{9x^2-1}} \left(\frac{\alpha^{2^{s-1} l(2n+1)(2h+1)}(x) - \beta^{2^{s-1} l(2n+1)}(x)}{\alpha^{2^{s-1} l(2n+1)}(x) - \beta^{2^{s-1} l(2n+1)}(x)} \right. \\ & \left. - \frac{\alpha^{2^{s-1} l(2n+1)}(x) - \beta^{2^{s-1} l(2n+1)(2h+1)}(x)}{\alpha^{2^{s-1} l(2n+1)}(x) - \beta^{2^{s-1} l(2n+1)}(x)} \right) \\ = & \frac{C_{2^{s-1} l(2h+1)(2n+1)}(x) - C_{2^{s-1} l(2n+1)}(x)}{2\left(9x^2-1\right) B_{2^{s-1} l(2n+1)}(x)}. \end{aligned} \tag{3.3}$$

And

$$\sum_{m=0}^{h} C_{2^s ml(2n+1)}(x) = \frac{1}{2} \frac{B_{2^{s-1} l(2n+1)(2h+1)}(x) + B_{2^{s-1} l(2n+1)}(x)}{B_{2^{s-1} l(2n+1)}(x)}. \tag{3.4}$$

Combining (3.2) and (3.3) we have

$$\sum_{m=0}^{h} \left(B_{2^s ml(2n+1)}(x) - (2n+1) B_{2^s ml}(x)\right)$$

$$\begin{aligned}
&= \frac{C_{2^{s-1}l(2h+1)(2n+1)}(x) - C_{2^{s-1}l(2n+1)}(x)}{2(9x^2-1)B_{2^{s-1}l(2n+1)}(x)} \\
&\quad -(2n+1)\frac{C_{2^{s-1}l(2h+1)}(x) - C_{2^{s-1}l}(x)}{2(9x^2-1)B_{2^{s-1}l}(x)} \\
&= \sum_{k=0}^{n-1}\frac{2n+1}{2n+1-k}2^{2n-2k}\left(9x^2-1\right)^{n-k}\binom{2n+1-k}{k}\sum_{m=0}^{h}B_{2^sml}^{2n+1-2k}(x). \qquad (3.5)
\end{aligned}$$

Now we apply (3.5) and mathematical induction to prove Theorem 1. If $n = 1$, then from (3.5) we have

$$\begin{aligned}
&2\left(9x^2-1\right)B_{2^{s-1}l}(x)B_{2^{s-1}3l}(x)\left(\frac{C_{2^{s-1}3l(2h+1)}(x) - C_{2^{s-1}3l}(x)}{2(9x^2-1)B_{2^{s-1}3l}(x)}\right. \\
&\left.-3\frac{C_{2^{s-1}l(2h+1)}(x) - C_{2^{s-1}l}(x)}{2(9x^2-1)B_{2^{s-1}l}(x)}\right) \\
&= 8\left(9x^2-1\right)^2 B_{2^{s-1}l}(x)B_{2^{s-1}3l}(x)\sum_{m=0}^{h}B_{2^sml}^{3}(x). \qquad (3.6)
\end{aligned}$$

From Lemma 1 we know that

$$\begin{aligned}
&2\left(9x^2-1\right)B_{2^{s-1}l}(x)B_{2^{s-1}3l}(x)\left(\frac{C_{2^{s-1}3l(2h+1)}(x) - C_{2^{s-1}3l}(x)}{2(9x^2-1)B_{2^{s-1}3l}(x)}\right. \\
&\left.-3\frac{C_{2^{s-1}l(2h+1)}(x) - C_{2^{s-1}l}(x)}{2(9x^2-1)B_{2^{s-1}l}(x)}\right) \\
&\equiv 0 \bmod \left(C_{2^{s-1}l(2h+1)}(x) - C_{2^{s-1}l}(x)\right). \qquad (3.7)
\end{aligned}$$

Combining (3.6) and (3.7) we know that Theorem 1 is true for $n = 1$.

Suppose that Theorem 1 is true for all integers $1 \le n \le j$. Then, for $n = j + 1$, from (3.5) we have

$$\begin{aligned}
&\frac{C_{2^{s-1}l(2h+1)(2j+3)}(x) - C_{2^{s-1}l(2j+3)}(x)}{2(9x^2-1)B_{2^{s-1}l(2j+3)}(x)} - (2j+3)\frac{C_{2^{s-1}l(2h+1)}(x) - C_{2^{s-1}l}(x)}{2(9x^2-1)B_{2^{s-1}l}(x)} \\
&= (2j+3)\sum_{k=0}^{j}\frac{1}{2j+3-k}\binom{2j+3-k}{k}2^{2j+2-2k}\left(9x^2-1\right)^{j+1-k}\sum_{m=0}^{h}B_{2^sml}^{2j+3-2k}(x) \\
&= (2j+3)\sum_{k=1}^{j}\frac{1}{2j+3-k}\binom{2j+3-k}{k}2^{2j+2-2k}\left(9x^2-1\right)^{j+1-k}\sum_{m=0}^{h}B_{2^sml}^{2j+3-2k}(x) \\
&\quad +2^{2j+2}\left(9x^2-1\right)^{j+1}\sum_{m=0}^{h}B_{2^sml}^{2j+3}(x). \qquad (3.8)
\end{aligned}$$

From Lemma 1 we have

$$\begin{aligned}
&2\left(9x^2-1\right)B_{2^{s-1}\cdot l}(x)B_{2^{s-1}\cdot 3l}(x)\cdots B_{2^{s-1}\cdot(2n+1)l}(x) \\
&\frac{C_{2^{s-1}l(2h+1)(2n+1)}(x) - C_{2^{s-1}l(2n+1)}(x)}{2(9x^2-1)B_{2^{s-1}l(2n+1)}(x)} \\
&\equiv 0 \bmod \left(C_{2^{s-1}l(2h+1)}(x) - C_{2^{s-1}l}(x)\right). \qquad (3.9)
\end{aligned}$$

Applying inductive hypothesis (3.8), we have

$$\begin{aligned} & B_{2^{s-1}l}(x)\, B_{2^{s-1}3l}(x) \cdots B_{2^{s-1}(2n+1)l}(x) \sum_{k=1}^{j} \frac{2j+3}{2j+3-k} \binom{2j+3-k}{k} \\ & \times 2^{2j+3-2k} \left(9x^2-1\right)^{j+2-k} \sum_{m=0}^{h} B_{2^s ml}^{2j+3-2k}(x) \\ \equiv\ & 0 \bmod \left(C_{2^{s-1}l(2h+1)}(x) - C_{2^{s-1}l}(x)\right). \end{aligned} \tag{3.10}$$

Combining (3.7)–(3.10) and Lemma 1, we have the congruence

$$\begin{aligned} & 2^{2n+1}\left(9x^2-1\right)^{n+1} B_{2^{s-1}l}(x)\, B_{2^{s-1}3l}(x) \cdots B_{2^{s-1}(2n+1)l}(x) \sum_{m=0}^{h} B_{2^s ml}^{2n+1}(x) \\ \equiv\ & 0 \bmod \left(C_{2^{s-1}l(2h+1)}(x) - C_{2^{s-1}l}(x)\right). \end{aligned}$$

This completes the proof of Theorem 1 by mathematical induction.
Now we prove Theorem 2, we have

$$\begin{aligned} & \sum_{m=0}^{h} \left(C_{2^s ml(2n+1)}(x) - (-1)^n (2n+1)\, C_{2^s ml}(x)\right) \\ =\ & \frac{B_{2^{s-1}l(2h+1)(2n+1)}(x) + B_{2^{s-1}l(2n+1)}(x)}{2B_{2^{s-1}l(2n+1)}(x)} - (-1)^n (2n+1) \\ & \frac{B_{2^{s-1}l(2h+1)}(x) + B_{2^{s-1}l}(x)}{2B_{2^{s-1}l}(x)} \\ =\ & \sum_{k=0}^{n-1} (-1)^k \frac{2n+1}{2n+1-k} 2^{2n-2k} \binom{2n+1-k}{k} \sum_{m=0}^{h} C_{2^s ml}^{2n+1-2k}(x). \end{aligned} \tag{3.11}$$

Applying (3.11), Lemma 2 and the method of proving Theorem 1, we can deduce the congruence

$$\begin{aligned} & 2^{2n+1} B_{2^{s-1}l}(x)\, B_{2^{s-1}3l}(x) \cdots B_{2^{s-1}(2n+1)l}(x) \sum_{m=0}^{h} C_{2^s ml}^{2n+1}(x) \\ \equiv\ & 0 \bmod \left(B_{2^{s-1}l(2h+1)}(x) + B_{2^{s-1}l}(x)\right). \end{aligned}$$

□

4. Conclusions

In this paper, we study the divisible property of the general power sum of balancing polynomials and Lucas-balancing polynomials. By taking specific values for s and l in the Theorems 1 and 2, similar results can be obtained as studied in the literature. In this paper, we take $x = 1$ and obtain the divisible property of the sequence $\sum_{m=0}^{h} B_{2^s ml}^{2n+1}$ and $\sum_{m=0}^{h} C_{2^s ml}^{2n+1}$. We apply a simple relation between the balancing polynomials and the Chebyshev polynomials to further obtain the divisibility properties of $\sum_{m=0}^{h} U_{2^s ml-1}^{2n+1}(x)$ and $\sum_{m=0}^{h} T_{2ml}^{2n+1}(x)$ in the Corollaries 5 and 6. This paper can help us to investigate the properties of polynomials and explore further relations between polynomials.

Use of AI tools declaration

The author declares he/she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

References

1. A. Behera, G. K. Panda, On the square roots of triangular numbers, *Fibonacci Quart.,* **37** (1999), 98–105.
2. G. K. Panda, Some fascinating properties of balancing numbers, *Fibonacci Numbers Appl.,* **194** (2009), 185–189.
3. S. G. Rayaguru, G. K. Panda, Sum formulas involving powers of balancing and Lucas-balancing numbers-II, *Notes Number Theory,* **25** (2019), 102–110. http://dx.doi.org/10.7546/nntdm.2019.25.3.102-110
4. R. Frontczak, T. Goy, Additional close links between balancing and Lucas-balancing polynomials, *Adv. Stud. Contemp. Math.,* **31** (2021), 287–300. http://dx.doi.org/10.17777/ascm2021.31.3.287
5. R. Frontczak, L. B. Wrttemberg, Powers of balancing polynomials and some consequences for Fibonacci sums, *Int. J. Math. Anal.,* **13** (2019), 109–115. http://dx.doi.org/10.12988/ijma.2019.9211
6. D. S. Kim, T. Kim, On sums of finite products of balancing polynomials, *J. Comput. Appl. Math.,* **377** (2020), 112913. http://dx.doi.org/10.1016/j.cam.2020.112913
7. P. K. Ray, Some congruences for balancing and Lucas-Balancing numbers and their applications, *Integers,* **14** (2014), A8.
8. T. T. Wang, W. P. Zhang, Some identities involving Fibonacci, Lucas polynomials and their applications, *B. Math. Soc. Sci. Math.,* **103** (2012), 95–103.
9. T. Kim, D. S. Kim, D. V. Dolgy, J. Kwon, A note on sums of finite products of Lucas-balancing polynomials, *Proc. Jangjeon Math. Soc.,* **23** (2020), 1–22. http://dx.doi.org/10.17777/pjms2020.23.1.1
10. T. Kim, C. S. Ryoo, D. S. Kim, J. Kwon, A difference of sums of finite products of Lucas-balancing polynomials, *Adv. Stud. Contemp. Math.,* **30** (2020), 121–134. http://dx.doi.org/10.17777/ascm2020.30.1.121

11. D. S. Kim, T. K. Kim, Normal ordering associated with λ-Whitney numbers of the first kind in λ-shift algebra, *Russ. J. Math. Phys.,* **30** (2023), 310–319. http://dx.doi.org/10.1134/S1061920823030044

12. T. Kim, D. S. Kim, D. V. Dolgy, J. W. Park, Sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials, *J. Inequal. Appl.,* **148** (2018), 1–14. http://dx.doi.org/10.1186/s13660-018-1744-5

13. T. Kim, D. S. Kim, D. V. Dolgy, J. Kwon, Representing sums of finite products of Chebyshev polynomials of the first kind and Lucas polynomials by Chebyshev polynomials, *Mathematics,* **7** (2019). http://dx.doi.org/10.3390/math7010026

14. C. F. Wei, New solitary wave solutions for the fractional Jaulent-Miodek hierarchy model, *Fractals,* **31** (2023), 2350060. http://dx.doi.org/10.1142/S0218348X23500603

15. R. A. Attia, X. Zhang, M. M. Khater, Analytical and hybrid numerical simulations for the (2+ 1)-dimensional Heisenberg ferromagnetic spin chain, *Results Phys.,* **43** (2022), 106045. http://dx.doi.org/10.1016/j.rinp.2022.106045

16. K. Wang, Fractal travelling wave solutions for the fractal-fractional Ablowitz-Kaup-Newell-Segur model, *Fractals,* **30** (2022), 2250171. http://dx.doi.org/10.1142/S0218348X22501717

17. R. S. Melham, Some conjectures concerning sums of odd powers of Fibonacci and Lucas numbers, *Fibonacci Quart.,* **46** (2009), 312–315.

18. L. Chen, X. Wang, The power sums involving Fibonacci polynomials and their applications, *Symmetry*, **11** (2019), 635. http://dx.doi.org/10.3390/sym11050635

19. L. Chen, W. P. Zhang, Chebyshev polynomials and their some interesting applications, *Adv. Differ. Equ.,* **303** (2017), 1–9. http://dx.doi.org/10.1186/s13662-017-1365-1

20. E. Waring, *Miscellanea analytica de aequationibus algebraicis et curvarum proprietatibus*, USA: Academic Press, 2010.

21. H. W. Gould, The Girard-Waring power sum formulas for symmetric functions and Fibonacci sequences, *Fibonacci Quart.,* **37** (1999), 135–140.

22. E. Waring, *Miscellanea analytica de aequationibus algebraicis et curvarum proprietatibus*, USA: Academic Press, 2010.

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