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Research article

Four symmetric systems of the matrix equations with an application over the Hamilton quaternions

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Abstract: In this paper, we established some necessary and sufficient conditions for the four symmetric systems to be consistent. Moreover, we derived the expressions of their general solutions when they were solvable. As an application, we investigated the solvability conditions of matrix equations involving η-Hermicity matrices. Finally, we presented an example to illustrate the main results of this paper.

Keywords: matrix equation; quaternion; Moore-Penrose; rank; η-Hermitian matrix
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1. Introduction

In this paper, we establish the following four symmetric quaternion matrix systems:

$$\begin{cases} A_{11}X_1 = B_{11}, C_{11}X_1D_{11} = E_{11}, \\ X_2A_{22} = B_{22}, C_{22}X_2D_{22} = E_{22}, \\ F_{11}X_1H_{11} + X_2F_{22} = G_{11}, \end{cases} \tag{1.1}$$

$$\begin{cases} A_{11}X_1 = B_{11}, C_{11}X_1D_{11} = E_{11}, \\ X_2A_{22} = B_{22}, C_{22}X_2D_{22} = E_{22}, \\ F_{11}X_1 + H_{11}X_2F_{22} = G_{11}, \end{cases} \tag{1.2}$$

$$\begin{cases} A_{11}X_1 = B_{11}, C_{11}X_1D_{11} = E_{11}, \\ A_{22}X_2 = B_{22}, C_{22}X_2D_{22} = E_{22}, \\ F_{11}X_1 + H_{11}X_2F_{22} = G_{11}, \end{cases} \tag{1.3}$$

$$\begin{cases} A_{11}X_1 = B_{11}, C_{11}X_1D_{11} = E_{11}, \\ A_{22}X_2 = B_{22}, C_{22}X_2D_{22} = E_{22}, \\ \quad F_{11}X_1 + X_2F_{22} = G_{11}, \end{cases} \tag{1.4}$$

where A_{ii}, B_{ii}, C_{ii}, D_{ii}, E_{ii}, $F_{ii}(i = \overline{1,2})$, H_{11}, and G_{11} are known matrices, while $X_i(i = \overline{1,2})$ are unknown.

In this paper, $\mathbb{R}$ and $\mathbb{H}^{m\times n}$ denote the real number field and the set of all quaternion matrices of order $m \times n$, respectively.

$$\mathbb{H} = \{v_0 + v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}|\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, v_0, v_1, v_2, v_3 \in \mathbb{R}\}.$$

Moreover, $r(A)$, 0 and I represent the rank of matrix A, the zero matrix of suitable size, and the identity matrix of suitable size, respectively. The conjugate transpose of A is A^*. For any matrix A, if there exists a unique solution X such that

$$AXA = A, XAX = X, (AX)^* = AX, (XA)^* = XA,$$

then X is called the Moore-Penrose $(M - P)$ inverse. It should be noted that $A^\dagger$ is used to represent the $M - P$ inverse of A. Additionally, $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ denote two projectors toward A.

$\mathbb{H}$ is known to be an associative noncommutative division algebra over $\mathbb{R}$ with extensive applications in computer science, orbital mechanics, signal and color image processing, control theory, and so on (see [1–4]).

Matrix equations, significant in the domains of descriptor systems control theory [5], nerve networks [6], back feed [7], and graph theory [8], are one of the key research topics in mathematics.

The study of matrix equations in $\mathbb{H}$ has garnered the attention of various researchers; consequently they have been analyzed by many studies (see, e.g., [9–12]). Among these the system of symmetric matrix equations is a crucial research object. For instance, Mahmoud and Wang [13] established some necessary and sufficient conditions for the three symmetric matrix systems in terms of $M - P$ inverses and rank equalities:

$$\begin{cases} A_1V = C_1,\ VB_1 = C_2, \\ A_3X + YB_3 = C_3, \\ A_2Y + ZB_2 + A_5VB_5 = C_5, \\ A_4W + ZB_4 = C_4, \end{cases} \begin{cases} A_1V = C_1,\ VB_1 = C_2, \\ A_3X + YB_3 = C_3, \\ A_2Z + YB_2 + A_5VB_5 = C_5, \\ A_4Z + WB_4 = C_4, \end{cases} \begin{cases} A_1V = C_1,\ VB_1 = C_2, \\ A_3X + YB_3 = C_3, \\ A_2Y + ZB_2 + A_5VB_5 = C_5, \\ A_4Z + WB_4 = C_4. \end{cases} \tag{1.5}$$

Wang and He [14] established the sufficient and necessary conditions for the existence of solutions to the following three symmetric coupled matrix equations and the expressions for their general solutions:

$$\begin{cases} A_1X + YB_1 = C_1, \\ A_2Y + ZB_2 = C_2, \\ A_3W + ZB_3 = C_3, \end{cases} \begin{cases} A_1X + YB_1 = C_1, \\ A_2Z + YB_2 = C_2, \\ A_3Z + WB_3 = C_3, \end{cases} \begin{cases} A_1X + YB_1 = C_1, \\ A_2Y + ZB_2 = C_2, \\ A_3Z + WB_3 = C_3. \end{cases} \tag{1.6}$$

It is noteworthy that the following matrix equation plays an important role in the analysis of the solvability conditions of systems (1.1)–(1.4):

$$A_1U + VB_1 + A_2XB_2 + A_3YB_3 + A_4ZB_4 = B. \tag{1.7}$$

Liu et al. [15] derived some necessary and sufficient conditions to solve the quaternion matrix equation (1.7) using the ranks of coefficient matrices and $M-P$ inverses. Wang et al. [16] derived the following quaternion equations after obtaining some solvability conditions for the quaternion equation presented in Eq (1.8) in terms of $M-P$ inverses:

$$\begin{cases} A_{11}X_1 = B_{11}, C_{11}X_1D_{11} = E_{11}, \\ X_2A_{22} = B_{22}, C_{22}X_2D_{22} = E_{22}, \\ \quad F_{11}X_1 + X_2F_{22} = G_{11}. \end{cases} \tag{1.8}$$

To our knowledge, so far, there has been little information on the solvability conditions and an expression of the general solution to systems (1.1)–(1.4).

In mathematical research and applications, the concept of η-Hermitian matrices has gained significant attention [17]. An η-Hermitian matrix, for $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, is defined as a matrix A such that $A = A^{\eta^*}$, where $A^{\eta^*} = -\eta A^* \eta$. These matrices have found applications in various fields including linear modeling and the statistics of random signals [1, 17]. As an application of (1.1), this paper establishes some necessary and sufficient conditions for the following matrix equation:

$$\begin{cases} A_{11}X_1 = B_{11}, C_{11}X_1C_{11}^{\eta^*} = E_{11}, \\ F_{11}X_1F_{11}^{\eta^*} + (F_{22}X_1)^{\eta^*} = G_{11} \end{cases} \tag{1.9}$$

to be solvable.

Motivated by the study of Systems (1.8), symmetric matrix equations, η-Hermitian matrices, and the widespread use of matrix equations and quaternions as well as the need for their theoretical advancements, we examine the solvability conditions of the quaternion systems presented in systems (1.1)–(1.4) by utilizing the rank equalities and the $M-P$ inverses of coefficient matrices. We then obtain the general solutions for the solvable quaternion equations in systems (1.1)–(1.4). As an application of (1.1), we utilize the $M-P$ inverse and the rank equality of matrices to investigate the necessary and sufficient conditions for the solvability of quaternion matrix equations involving η-Hermicity matrices. It is evident that System (1.8) is a specific instance of System (1.1).

The remainder of this article is structured as follows. Section 2 outlines the basics. Section 3 examines some solvability conditions of the quaternion equation presented in System (1.1) using the $M-P$ inverses and rank equalities of the matrices, and derives the solution of System (1.1). Section 4 establishes some solvability conditions for matrix systems (1.2)–(1.4) to be solvable. Section 5 investigates some necessary and sufficient conditions for matrix equation (1.9) to have common solutions. Section 6 concludes the paper.

2. Preliminaries

Marsaglia and Styan [18] presented the following rank equality lemma over the complex field, which can be generalized to $\mathbb{H}$.

Lemma 2.1. *[18] Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$, $C \in \mathbb{H}^{l\times n}$, $D \in \mathbb{H}^{j\times k}$, and $E \in \mathbb{H}^{l\times i}$ be given. Then, the following rank equality holds:*

$$r\begin{pmatrix} A & BL_D \\ R_EC & 0 \end{pmatrix} = r\begin{pmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{pmatrix} - r\left(D\right) - r\left(E\right).$$

Lemma 2.2. *[19] Let $A \in \mathbb{H}^{m\times n}$ be given. Then,*

$$\begin{aligned}
&(1)\quad (A^{\eta})^{\dagger} = (A^{\dagger})^{\eta}, (A^{\eta^*})^{\dagger} = (A^{\dagger})^{\eta^*};\\
&(2)\quad r(A) = r(A^{\eta^*}) = r(A^{\eta});\\
&(3)\quad (L_A)^{\eta^*} = -\eta(L_A)\eta = (L_A)^{\eta} = L_{A^{\eta^*}} = R_{A^{\eta^*}},\\
&(4)\quad (R_A)^{\eta^*} = -\eta(R_A)\eta = (R_A)^{\eta} = R_{A^{\eta^*}} = L_{A^{\eta^*}};\\
&(5)\quad (AA^{\dagger})^{\eta^*} = (A^{\dagger})^{\eta^*}A^{\eta^*} = (A^{\dagger}A)^{\eta} = A^{\eta}(A^{\dagger})^{\eta};\\
&(6)\quad (A^{\dagger}A)^{\eta^*} = A^{\eta^*}(A^{\dagger})^{\eta^*} = (AA^{\dagger})^{\eta} = (A^{\dagger})^{\eta}A^{\eta}.
\end{aligned}$$

Lemma 2.3. *[20] Let A_1 and A_2 be given quaternion matrices with adequate shapes. Then, the equation $A_1X = A_2$ is solvable if, and only if, $A_2 = A_1A_1^{\dagger}A_2$. In this case, the general solution to this equation can be expressed as*

$$X = A_1^{\dagger}A_2 + L_{A_1}U_1,$$

where U_1 is any matrix with appropriate size.

Lemma 2.4. *[20] Let A_1 and A_2 be given quaternion matrices with adequate shapes. Then, the equation $XA_1 = A_2$ is solvable if, and only if, $A_2 = A_2A_1^{\dagger}A_1$. In this case, the general solution to this equation can be expressed as*

$$X = A_2A_1^{\dagger} + U_1R_{A_1},$$

where U_1 is any matrix with appropriate size.

Lemma 2.5. *[21] Let A, B, and C be known quaternion matrices with appropriate sizes. Then, the matrix equation*

$$AXB = C$$

is consistent if, and only if,

$$R_AC = 0, CL_B = 0.$$

In this case, the general solution to this equation can be expressed as

$$X = A^{\dagger}CB^{\dagger} + L_AU + VR_B,$$

where U and V are any quaternion matrices with appropriate sizes.

Lemma 2.6. *[15] Let C_i, D_i, and Z ($i = \overline{1,4}$) be known quaternion matrices with appropriate sizes.*

$$C_1X_1 + X_2D_1 + C_2Y_1D_2 + C_3Y_2D_3 + C_4Y_3D_4 = Z. \tag{2.1}$$

Denote

$$\begin{aligned}
&R_{C_1}C_2 = C_{12}, R_{C_1}C_3 = C_{13}, R_{C_1}C_4 = C_{14}, D_2L_{D_1} = D_{21}, D_{31}L_{D_{21}} = N_{32}, D_3L_{D_1} = D_{31},\\
&D_4L_{D_1} = D_{41}, R_{C_{12}}C_{13} = M_{23}, S_{12} = C_{13}L_{M_{23}}, R_{C_1}ZL_{D_1} = T_1, C_{32} = R_{M_{23}}R_{C_{12}}, A_1 = C_{32}C_{14},\\
&A_2 = R_{C_{12}}C_{14}, A_3 = R_{C_{13}}C_{14}, A_4 = C_{14}, D_{13} = L_{D_{21}}L_{N_{32}}, B_1 = D_{41}, B_2 = D_{41}L_{D_{31}},
\end{aligned}$$

$$B_3 = D_{41}L_{D_{21}}, B_4 = D_{41}D_{13}, E_1 = C_{32}T_1, E_2 = R_{C_{12}}T_1L_{D_{31}}, E_3 = R_{C_{13}}T_1L_{D_{21}}, E_4 = T_1D_{13},$$
$$A_{24} = (L_{A_2}, L_{A_4}), B_{13} = \begin{pmatrix} R_{B_1} \\ R_{B_3} \end{pmatrix}, A_{11} = L_{A_1}, B_{22} = R_{B_2}, A_{33} = L_{A_3}, B_{44} = R_{B_4}, E_{11} = R_{A_{24}}A_{11},$$
$$E_{22} = R_{A_{24}}A_{33}, E_{33} = B_{22}L_{B_{13}}, E_{44} = B_{44}L_{B_{13}}, N = R_{E_{11}}E_{22}, M = E_{44}L_{E_{33}}, K = K_2 - K_1,$$
$$E = R_{A_{24}}KL_{B_{13}}, S = E_{22}L_N, K_{11} = A_2L_{A_1}, G_1 = E_2 - A_2A_1^\dagger E_1B_1^\dagger B_2, K_{22} = A_4L_{A_3},$$
$$G_2 = E_4 - A_4A_3^\dagger E_3B_3^\dagger B_4, K_1 = A_1^\dagger E_1B_1^\dagger + L_{A_1}A_2^\dagger E_2B_2^\dagger, K_2 = A_3^\dagger E_3B_3^\dagger + L_{A_3}A_4^\dagger E_4B_4^\dagger.$$

Then, the following statements are equivalent:

(1) *Equation* (2.1) *is consistent.*

(2)

$$R_{A_i}E_i = 0, E_iL_{B_i} = 0(i = \overline{1,4}), R_{E_{11}}EL_{E_{44}} = 0.$$

(3)

$$r\begin{pmatrix} Z & C_2 & C_3 & C_4 & C_1 \\ D_1 & 0 & 0 & 0 & 0 \end{pmatrix} = r(D_1) + r(C_2, C_3, C_4, C_1),$$
$$r\begin{pmatrix} Z & C_2 & C_4 & C_1 \\ D_3 & 0 & 0 & 0 \\ D_1 & 0 & 0 & 0 \end{pmatrix} = r(C_2, C_4, C_1) + r\begin{pmatrix} D_3 \\ D_1 \end{pmatrix},$$
$$r\begin{pmatrix} Z & C_3 & C_4 & C_1 \\ D_2 & 0 & 0 & 0 \\ D_1 & 0 & 0 & 0 \end{pmatrix} = r(C_3, C_4, C_1) + r\begin{pmatrix} D_2 \\ D_1 \end{pmatrix},$$
$$r\begin{pmatrix} Z & C_4 & C_1 \\ D_2 & 0 & 0 \\ D_3 & 0 & 0 \\ D_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} D_2 \\ D_3 \\ D_1 \end{pmatrix} + r(C_4, C_1),$$
$$r\begin{pmatrix} Z & C_2 & C_3 & C_1 \\ D_4 & 0 & 0 & 0 \\ D_1 & 0 & 0 & 0 \end{pmatrix} = r(C_2, C_3, C_1) + r\begin{pmatrix} D_4 \\ D_1 \end{pmatrix},$$
$$r\begin{pmatrix} Z & C_2 & C_1 \\ D_3 & 0 & 0 \\ D_4 & 0 & 0 \\ D_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} D_3 \\ D_4 \\ D_1 \end{pmatrix} + r(C_2, C_1),$$
$$r\begin{pmatrix} Z & C_3 & C_1 \\ D_2 & 0 & 0 \\ D_4 & 0 & 0 \\ D_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} D_2 \\ D_4 \\ D_1 \end{pmatrix} + r(C_3, C_1),$$

$$r\begin{pmatrix} Z & C_1 \\ D_2 & 0 \\ D_3 & 0 \\ D_4 & 0 \\ D_1 & 0 \end{pmatrix} = r\begin{pmatrix} D_2 \\ D_3 \\ D_4 \\ D_1 \end{pmatrix} + r(C_1),$$

$$r\begin{pmatrix} Z & C_2 & C_1 & 0 & 0 & 0 & C_4 \\ D_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ D_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Z & C_3 & C_1 & C_4 \\ 0 & 0 & 0 & D_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_1 & 0 & 0 & 0 \\ D_4 & 0 & 0 & D_4 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} D_3 & 0 \\ D_1 & 0 \\ 0 & D_2 \\ 0 & D_1 \\ D_4 & D_4 \end{pmatrix} + r\begin{pmatrix} C_2 & C_1 & 0 & 0 & C_4 \\ 0 & 0 & C_3 & C_1 & C_4 \end{pmatrix}.$$

Under these conditions, the general solution to the matrix equation (2.1) *is*

$$\begin{aligned}
X_1 &= C_1^{\dagger}(Z - C_2Y_1D_2 - C_3Y_2D_3 - C_4Y_3D_4) - C_1^{\dagger}U_1D_1 + L_{C_1}U_2,\\
X_2 &= R_{C_1}(Z - C_2Y_1D_2 - C_3Y_2D_3 - C_4Y_3D_4)D_1^{\dagger} + C_1C_1^{\dagger}U_1 + U_3R_{D_1},\\
Y_1 &= C_{12}^{\dagger}TD_{21}^{\dagger} - C_{12}^{\dagger}C_{13}M_{23}^{\dagger}TD_{21}^{\dagger} - C_{12}^{\dagger}S_{12}C_{13}^{\dagger}TN_{32}^{\dagger}D_{31}D_{21}^{\dagger}\\
&\quad - C_{12}^{\dagger}S_{12}U_4R_{N_{32}}D_{31}D_{21}^{\dagger} + L_{C_{12}}U_5 + U_6R_{D_{21}},\\
Y_2 &= M_{23}^{\dagger}TD_{31}^{\dagger} + S_{12}^{\dagger}S_{12}C_{13}^{\dagger}TN_{32}^{\dagger} + L_{M_{23}}L_{S_{12}}U_7 + U_8R_{D_{31}} + L_{M_{23}}U_4R_{N_{32}},\\
Y_3 &= K_1 + L_{A_2}V_1 + V_2R_{B_1} + L_{A_1}V_3R_{B_2},\ or\ Y_3 = K_2 - L_{A_4}W_1 - W_2R_{B_3} - L_{A_3}W_3R_{B_4},
\end{aligned}$$

where $T = T_1 - C_4Y_3D_4, U_i(i = \overline{1,8})$ *are arbitrary matrices with appropriate sizes over* $\mathbb{H}$*,*

$$\begin{aligned}
V_1 &= (I_m, 0)[A_{24}^{\dagger}(K - A_{11}V_3B_{22} - A_{33}W_3B_{44}) - A_{24}^{\dagger}U_{11}B_{13} + L_{A_{24}}U_{12}],\\
W_1 &= (0, I_m)[A_{24}^{\dagger}(K - A_{11}V_3B_{22} - A_{33}W_3B_{44}) - A_{24}^{\dagger}U_{11}B_{13} + L_{A_{24}}U_{12}],\\
W_2 &= [R_{A_{24}}(K - A_{11}V_3B_{22} - A_{33}W_3B_{44})B_{13}^{\dagger} + A_{24}A_{24}^{\dagger}U_{11} + U_{21}R_{B_{13}}]\begin{pmatrix} 0 \\ I_n \end{pmatrix},\\
V_2 &= [R_{A_{24}}(K - A_{11}V_3B_{22} - A_{33}W_3B_{44})B_{13}^{\dagger} + A_{24}A_{24}^{\dagger}U_{11} + U_{21}R_{B_{13}}]\begin{pmatrix} I_n \\ 0 \end{pmatrix},\\
V_3 &= E_{11}^{\dagger}KE_{33}^{\dagger} - E_{11}^{\dagger}E_{22}N^{\dagger}KE_{33}^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}KM^{\dagger}E_{44}E_{33}^{\dagger}\\
&\quad - E_{11}^{\dagger}SU_{31}R_ME_{44}E_{33}^{\dagger} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},\\
W_3 &= N^{\dagger}KE_{44} + S^{\dagger}SE_{22}^{\dagger}KM^{\dagger} + L_NL_SU_{41} + L_NU_{31}R_M - U_{42}R_{E_{44}},
\end{aligned}$$

$U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41},$ *and* U_{42} *are arbitrary quaternion matrices with appropriate sizes, and* m *and* n *denote the column number of* C_4 *and the row number of* D_4*, respectively.*

3. Solvability conditions and expression of general solution for system (1.1)

Some necessary and sufficient conditions for System (1.1) to be solvable will be established in this section. The general solution of System (1.1) will also be derived in this section. Moreover, we provide an example to illustrate our main results.

Theorem 3.1. *Let $A_{ii}, B_{ii}, C_{ii}, D_{ii}, E_{ii}, F_{ii}, H_{11}$, and G_{11} (i=1,2) be given quaternion matrices. Put*

$$\begin{cases} A_1 = C_{11}L_{A_{11}}, P_1 = E_{11} - C_{11}A_{11}^{\dagger}B_{11}D_{11}, B_2 = R_{A_{22}}D_{22}, \\ P_2 = E_{22} - C_{22}B_{22}A_{22}^{\dagger}D_{22}, \hat{B}_1 = R_{B_2}R_{A_{22}}F_{22}, \hat{A}_2 = F_{11}L_{A_{11}}L_{A_1}, \\ \hat{A}_3 = F_{11}L_{A_{11}}, \hat{B}_3 = R_{D_{11}}H_{11}, \hat{A}_4 = L_{C_{22}}, \hat{B}_4 = R_{A_{22}}F_{22}, H_{11}L_{\hat{B}_1} = \hat{B_{11}}, \\ P = G_{11} - F_{11}A_{11}^{\dagger}B_{11}H_{11} - F_{11}L_{A_{11}}A_1^{\dagger}P_1D_{11}^{\dagger}H_{11} - B_{22}A_{22}^{\dagger}F_{22} - C_{22}^{\dagger}P_2B_2^{\dagger}R_{A_{22}}F_{22}, \end{cases} \tag{3.1}$$

$$\begin{cases} \hat{B_{22}}L_{\hat{B_{11}}} = N_1, \hat{B}_3L_{\hat{B}_1} = \hat{B_{22}}, \hat{B}_4L_{\hat{B}_1} = \hat{B_{33}}, R_{\hat{A}_2}\hat{A}_3 = \hat{M}_1, S_1 = \hat{A}_3L_{\hat{M}_1}, T_1 = PL_{\hat{B}_1}, \\ C = R_{\hat{M}_1}R_{\hat{A}_2}, C_1 = C\hat{A}_4, C_2 = R_{\hat{A}_2}\hat{A}_4, C_3 = R_{\hat{A}_3}\hat{A}_4, C_4 = \hat{A}_4, D = L_{\hat{B_{11}}}L_{N_1}, \\ D_1 = \hat{B_{33}}, D_2 = \hat{B_{33}}L_{\hat{B_{22}}}, D_4 = \hat{B_{33}}D, E_1 = CT_1, E_2 = R_{\hat{A}_2}T_1L_{\hat{B_{22}}}, \\ E_3 = R_{\hat{A}_3}T_1L_{\hat{B_{11}}}, E_4 = T_1D, \hat{C_{11}} = (L_{C_2}, L_{C_4}), D_3 = \hat{B_{33}}L_{\hat{B_{11}}}, \hat{D_{11}} = \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix}, \\ \hat{C_{22}} = L_{C_1}, \hat{D_{22}} = R_{D_2}, \hat{C_{33}} = L_{C_3}, \hat{D_{33}} = R_{D_4}, \hat{E_{11}} = R_{\hat{C_{11}}}\hat{C_{22}}, \\ \hat{E_{22}} = R_{\hat{C_{11}}}\hat{C_{22}}, \hat{E_{33}} = \hat{D_{22}}L_{\hat{D_{11}}}, \hat{E_{44}} = \hat{D_{33}}L_{\hat{D_{11}}}, M = R_{\hat{E_{11}}}\hat{E_{22}}, \\ N = \hat{E_{44}}L_{\hat{E_{33}}}, F = F_2 - F_1, E = R_{\hat{C_{11}}}FL_{\hat{D_{11}}}, S = \hat{E_{22}}L_M, \hat{F_{11}} = C_2L_{C_1}, \\ G_1 = E_2 - C_2C_1^{\dagger}E_1D_1^{\dagger}D_2, \hat{F_{22}} = C_4L_{C_3}, G_2 = E_4 - C_4C_3^{\dagger}E_3D_3^{\dagger}D_4, \\ F_1 = C_1^{\dagger}E_1D_1^{\dagger} + L_{C_1}C_2^{\dagger}E_2D_2^{\dagger}, F_2 = C_3^{\dagger}E_3D_3^{\dagger} + L_{C_3}C_4^{\dagger}E_4D_4^{\dagger}. \end{cases} \tag{3.2}$$

Then, the following statements are equivalent:

(1) *System* (1.1) *is solvable.*

(2)

$$R_{A_{11}}B_{11} = 0, R_{A_1}P_1 = 0, P_1L_{D_{11}} = 0, B_{22}L_{A_{22}} = 0, R_{C_{22}}P_2 = 0,$$
$$P_2L_{B_2} = 0, R_{C_i}E_i = 0, E_iL_{D_i} = 0(i = \overline{1,4}), R_{\hat{E_{11}}}EL_{\hat{E_{44}}} = 0.$$

(3)

$$r(B_{11}, A_{11}) = r(A_{11}), r\begin{pmatrix} E_{11} & C_{11} \\ B_{11}D_{11} & A_{11} \end{pmatrix} = r\begin{pmatrix} C_{11} \\ A_{11} \end{pmatrix}, \tag{3.3}$$

$$r\begin{pmatrix} E_{11} \\ D_{11} \end{pmatrix} = r(D_{11}), r\begin{pmatrix} B_{22} \\ A_{22} \end{pmatrix} = r(A_{22}), \tag{3.4}$$

$$r(E_{22}, C_{22}) = r(C_{22}), r\begin{pmatrix} E_{22} & C_{22}B_{22} \\ D_{22} & A_{22} \end{pmatrix} = r(D_{22}, A_{22}), \tag{3.5}$$

$$r\begin{pmatrix} F_{22} & 0 & D_{22} & A_{22} \\ B_{11}H_{11} & A_{11} & 0 & 0 \\ C_{22}G_{11} & C_{22}F_{11} & E_{22} & C_{22}B_{22} \end{pmatrix} = r(F_{22}, D_{22}, A_{22}) + r\begin{pmatrix} A_{11} \\ C_{22}F_{11} \end{pmatrix}, \tag{3.6}$$

$$r\begin{pmatrix} H_{11} & 0 & -D_{11} & 0 & 0 \\ F_{22} & 0 & 0 & D_{22} & A_{22} \\ 0 & C_{11} & E_{11} & 0 & 0 \\ 0 & A_{11} & B_{11}D_{11} & 0 & 0 \\ C_{22}G_{11} & C_{22}F_{11} & 0 & E_{22} & C_{22}B_{22} \end{pmatrix} \tag{3.7}$$

$$= r\begin{pmatrix} C_{11} \\ A_{11} \\ C_{22}F_{11} \end{pmatrix} + r\begin{pmatrix} H_{11} & D_{11} & 0 & 0 \\ F_{22} & 0 & D_{22} & A_{22} \end{pmatrix},$$

$$r\begin{pmatrix} H_{11} & 0 & 0 & 0 \\ F_{22} & 0 & D_{22} & A_{22} \\ 0 & A_{11} & 0 & 0 \\ C_{22}G_{11} & C_{22}F_{11} & E_{22} & C_{22}B_{22} \end{pmatrix} = r\begin{pmatrix} H_{11} & 0 & 0 \\ F_{22} & D_{22} & A_{22} \end{pmatrix} + r\begin{pmatrix} A_{11} \\ C_{22}F_{11} \end{pmatrix}, \tag{3.8}$$

$$r\begin{pmatrix} H_{11} & 0 & 0 \\ F_{22} & D_{22} & A_{22} \\ C_{22}G_{11} & E_{22} & C_{22}B_{22} \end{pmatrix} = r\begin{pmatrix} H_{11} & 0 & 0 \\ F_{22} & D_{22} & A_{22} \end{pmatrix}, \tag{3.9}$$

$$r\begin{pmatrix} G_{11} & F_{11} & B_{22} \\ F_{22} & 0 & A_{22} \\ B_{11}H_{11} & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} F_{11} \\ A_{11} \end{pmatrix} + r(F_{22}, A_{22}), \tag{3.10}$$

$$r\begin{pmatrix} G_{11} & F_{11} & 0 & B_{22} \\ H_{11} & 0 & -D_{11} & 0 \\ F_{22} & 0 & 0 & A_{22} \\ 0 & C_{11} & E_{11} & 0 \\ 0 & A_{11} & B_{11}D_{11} & 0 \end{pmatrix} = r\begin{pmatrix} H_{11} & D_{11} & 0 \\ F_{22} & 0 & A_{22} \end{pmatrix} + r\begin{pmatrix} F_{11} \\ C_{11} \\ A_{11} \end{pmatrix}, \tag{3.11}$$

$$r\begin{pmatrix} G_{11} & F_{11} & B_{22} \\ H_{11} & 0 & 0 \\ F_{22} & 0 & A_{22} \\ 0 & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} H_{11} & 0 \\ F_{22} & A_{22} \end{pmatrix} + r\begin{pmatrix} F_{11} \\ A_{11} \end{pmatrix}, \tag{3.12}$$

$$r\begin{pmatrix} G_{11} & B_{22} \\ H_{11} & 0 \\ F_{22} & A_{22} \end{pmatrix} = r\begin{pmatrix} H_{11} & 0 \\ F_{22} & A_{22} \end{pmatrix}, \tag{3.13}$$

$$\begin{aligned} & r\begin{pmatrix} H_{11} & 0 & 0 & 0 & 0 & 0 & 0 & D_{11} & 0 \\ F_{22} & 0 & 0 & 0 & 0 & D_{22} & A_{22} & 0 & 0 \\ 0 & 0 & H_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F_{22} & D_{22} & A_{22} & 0 & 0 & 0 & 0 \\ F_{22} & 0 & F_{22} & 0 & 0 & 0 & 0 & 0 & A_{22} \\ 0 & C_{11} & 0 & 0 & 0 & 0 & 0 & -E_{11} & 0 \\ 0 & A_{11} & 0 & 0 & 0 & 0 & 0 & -B_{11}D_{11} & 0 \\ C_{22}G_{11} & C_{22}F_{11} & 0 & 0 & 0 & E_{22} & C_{22}B_{22} & 0 & 0 \end{pmatrix} \\ = {} & r\begin{pmatrix} H_{11} & 0 & 0 & 0 & 0 & 0 & D_{11} & 0 \\ F_{22} & 0 & 0 & 0 & D_{22} & A_{22} & 0 & 0 \\ 0 & H_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{22} & D_{22} & A_{22} & 0 & 0 & 0 & 0 \\ F_{22} & F_{22} & 0 & 0 & 0 & 0 & 0 & A_{22} \end{pmatrix} + r\begin{pmatrix} C_{11} \\ A_{11} \\ C_{22}F_{11} \end{pmatrix}. \end{aligned} \tag{3.14}$$

Proof. (1) $\Leftrightarrow$ (2): The System (1.1) can be written as follows.

$$A_{11}X_1 = B_{11},\ X_2A_{22} = B_{22}, \tag{3.15}$$

$$C_{11}X_1D_{11} = E_{11},\ C_{22}X_2D_{22} = E_{22}, \tag{3.16}$$

and

$$F_{11}X_1H_{11} + X_2F_{22} = G_{11}. \tag{3.17}$$

Next, the solvability conditions and the expression for the general solutions of Eq (3.15) to Eq (3.17) are given by the following steps:
Step 1: According to Lemma 2.3 and Lemma 2.4, the system (3.15) is solvable if, and only if,

$$R_{A_{11}}B_{11} = 0,\ B_{22}L_{A_{22}} = 0. \tag{3.18}$$

When condition (3.18) holds, the general solution of System (3.15) is

$$X_1 = A_{11}^{\dagger}B_{11} + L_{A_{11}}U_1,\ X_2 = B_{22}A_{22}^{\dagger} + U_2R_{A_{22}}. \tag{3.19}$$

Step 2: Substituting (3.19) into (3.16) yields,

$$A_1U_1D_{11} = P_1,\ C_{22}U_2B_2 = P_2, \tag{3.20}$$

where A_1, P_1, B_2, P_2 are defined by (3.1). By Lemma 2.5, the system (3.20) is consistent if, and only if,

$$R_{A_1}P_1 = 0,\ P_1L_{D_{11}} = 0,\ R_{C_{22}}P_2 = 0,\ P_2L_{B_2} = 0. \tag{3.21}$$

When (3.21) holds, the general solution to System (3.20) is

$$U_1 = A_1^{\dagger}P_1D_{11}^{\dagger} + L_{A_1}W_1 + W_2R_{D_{11}}, U_2 = C_{22}^{\dagger}P_2B_2^{\dagger} + L_{C_{22}}W_3 + W_4R_{B_2}. \tag{3.22}$$

Comparing (3.22) and (3.19), hence,

$$\begin{aligned} X_1 &= A_{11}^{\dagger}B_{11} + L_{A_{11}}A_1^{\dagger}P_1D_{11}^{\dagger} + L_{A_{11}}L_{A_1}W_1 + L_{A_{11}}W_2R_{D_{11}}, \\ X_2 &= B_{22}A_{22}^{\dagger} + C_{22}^{\dagger}P_2B_2^{\dagger}R_{A_{22}} + L_{C_{22}}W_3R_{A_{22}} + W_4R_{B_2}R_{A_{22}}. \end{aligned} \tag{3.23}$$

Step 3: Substituting (3.23) into (3.17) yields

$$W_4\hat{B}_1 + \hat{A}_2W_1H_{11} + \hat{A}_3W_2\hat{B}_3 + \hat{A}_4W_3\hat{B}_4 = P, \tag{3.24}$$

where $\hat{B}_i, \hat{A}_j (i = \overline{1,4}, j = \overline{2,4})$ are defined by (3.1). It follows from Lemma 2.6 that Eq (3.24) is solvable if, and only if,

$$R_{C_i}E_i = 0, E_iL_{D_i} = 0 (i = \overline{1,4}), R_{\hat{E_{11}}}EL_{\hat{E_{44}}} = 0. \tag{3.25}$$

When (3.25) holds, the general solution to matrix equation (3.24) is

$$\begin{aligned} W_1 &= \hat{A_2}^{\dagger}T\hat{B_{11}}^{\dagger} - \hat{A_2}^{\dagger}\hat{A}_3\hat{M_1}^{\dagger}T\hat{B_{11}}^{\dagger} - \hat{A_2}^{\dagger}S_1\hat{A_3}^{\dagger}TN_1^{\dagger}\hat{B_{22}}\hat{B_{11}}^{\dagger} \\ &\quad - \hat{A_2}^{\dagger}S_1V_4R_{N_1}\hat{B_{22}}\hat{B_{11}}^{\dagger} + L_{\hat{A_2}}V_5 + V_6R_{\hat{B_{11}}}, \\ W_2 &= \hat{M_1}^{\dagger}T\hat{B_{22}}^{\dagger} + S_1^{\dagger}S_1\hat{A_3}^{\dagger}TN_1^{\dagger} + L_{\hat{M_1}}L_{S_1}V_7 + V_8R_{\hat{B_{22}}} + L_{\hat{M_1}}V_4R_{N_1}, \\ W_3 &= F_1 + L_{C_2}\hat{V}_1 + \hat{V}_2R_{D_1} + L_{C_1}\hat{V}_3R_{D_2},\ or\ W_3 = F_2 - L_{C_4}V_1 - V_2R_{D_3} - L_{C_3}V_3R_{D_4}, \\ W_4 &= (P - \hat{A}_2W_1H_{11} - \hat{A}_3W_2\hat{B}_3 - \hat{A}_4W_3\hat{B}_4)\hat{B_1}^{\dagger} + V_3R_{\hat{B}_1}, \end{aligned}$$

where $C_i, E_i, D_i (i = \overline{1,4}), \hat{E_{11}}, \hat{E_{44}}$ are defined as (3.2), $T = T_1 - \hat{A}_4W_3\hat{B}_4, V_i (i = \overline{1,8})$ are arbitrary matrices with appropriate sizes over $\mathbb{H}$,

$$\hat{V}_1 = (I_m, 0)[\hat{C_{11}}^{\dagger}(F - \hat{C_{22}}V_3\hat{D_{22}} - \hat{C_{33}}\hat{V}_3\hat{D_{33}}) - \hat{C_{11}}^{\dagger}U_{11}\hat{D_{11}} + L_{\hat{C_{11}}}U_{12}],$$

$$V_1 = (0, I_m)[\hat{C_{11}}^{\dagger}(F - \hat{C_{22}}V_3\hat{D_{22}} - \hat{C_{33}}\hat{V}_3\hat{D_{33}}) - \hat{C_{11}}^{\dagger}U_{11}\hat{D_{11}} + L_{\hat{C_{11}}}U_{12}],$$
$$V_2 = [R_{\hat{C_{11}}}(F - \hat{C_{22}}V_3\hat{D_{22}} - \hat{C_{33}}\hat{V}_3\hat{D_{33}})\hat{D_{11}}^{\dagger} + \hat{C_{11}}\hat{C_{11}}^{\dagger}U_{11} + U_{21}R_{\hat{D_{11}}}]\begin{pmatrix} 0 \\ I_n \end{pmatrix},$$
$$\hat{V}_2 = [R_{\hat{C_{11}}}(F - \hat{C_{22}}V_3\hat{D_{22}} - \hat{C_{33}}\hat{V}_3\hat{D_{33}})\hat{D_{11}}^{\dagger} + \hat{C_{11}}\hat{C_{11}}^{\dagger}U_{11} + U_{21}R_{\hat{D_{11}}}]\begin{pmatrix} I_n \\ 0 \end{pmatrix},$$
$$\hat{V}_3 = \hat{E_{11}}^{\dagger}F\hat{E_{33}}^{\dagger} - \hat{E_{11}}^{\dagger}\hat{E_{22}}M^{\dagger}F\hat{E_{33}}^{\dagger} - \hat{E_{11}}^{\dagger}S\hat{E_{22}}^{\dagger}FN^{\dagger}\hat{E_{44}}\hat{E_{33}}^{\dagger} - \hat{E_{11}}^{\dagger}SU_{31}R_N\hat{E_{44}}\hat{E_{33}}^{\dagger}$$
$$+ L_{\hat{E_{11}}}U_{32} + U_{33}R_{\hat{E_{33}}},$$
$$V_3 = M^{\dagger}F\hat{E_{44}}^{\dagger} + S^{\dagger}S\hat{E_{22}}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{\hat{E_{44}}},$$

$U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$, and U_{42} are any quaternion matrices with appropriate sizes, and m and n denote the column number of C_{22} and the row number of A_{22}, respectively. We summarize that System (1.1) has a solution if, and only if, (3.18), (3.21), and (3.25) hold, i.e., the System (1.1) has a solution if, and only if, (2) holds.

(2) $\Leftrightarrow$ (3): We prove the equivalence in two parts. In the first part, we want to show that (3.18) and (3.21) are equivalent to (3.3) to (3.5), respectively. In the second part, we want to show that (3.25) is equivalent to (3.6) to (3.14). It is easy to know that there exist X_1^0, X_2^0, U_1^0, and U_2^0 such that

$$A_{11}X_1^0 = B_{11},\ X_2^0A_{22} = B_{22},$$
$$A_1U_1^0D_{11} = P_1,\ C_{22}U_2^0B_2 = P_2$$

holds, where

$$X_1^0 = A_{11}^{\dagger}B_{11}, U_1^0 = A_1^{\dagger}P_1D_{11}^{\dagger},$$
$$X_2^0 = B_{22}A_{22}^{\dagger}, U_2^0 = C_{22}^{\dagger}P_2B_2^{\dagger},$$

$P_1 = E_{11} - C_{11}X_1^0D_{11}, P_2 = E_{22} - C_{22}X_2^0D_{22}$, and $P = G_{11} - F_{11}X_1^0H_{11} - F_{11}L_{A_{11}}U_1^0H_{11} - X_2^0F_{22} - U_2^0R_{A_{22}}F_{22}$.

Part 1: We want to show that (3.18) and (3.21) are equivalent to (3.3) to (3.5), respectively. It follows from Lemma 2.1 and elementary transformations that

$$(3.18) \Leftrightarrow r(R_{A_{11}}B_{11}) = 0 \Leftrightarrow r(B_{11}, A_{11}) = r(A_{11}) \Leftrightarrow (3.3),$$
$$(3.21) \Leftrightarrow r(R_{A_1}P_1) = 0 \Leftrightarrow r(P_1, A_1) = r(A_1) \Leftrightarrow r(E_{11} - C_{11}A_{11}^{\dagger}B_{11}D_{11}, C_{11}L_{A_{11}})$$
$$= r(C_{11}L_{A_{11}}) \Leftrightarrow r\begin{pmatrix} E_{11} & C_{11} \\ B_{11}D_{11} & A_{11} \end{pmatrix} = r\begin{pmatrix} C_{11} \\ A_{11} \end{pmatrix} \Leftrightarrow (3.3),$$
$$(3.21) \Leftrightarrow r(P_1L_{D_{11}}) = 0 \Leftrightarrow r\begin{pmatrix} P_1 \\ D_{11} \end{pmatrix} = r(D_{11}) \Leftrightarrow r\begin{pmatrix} E_{11} - C_{11}A_{11}^{\dagger}B_{11}D_{11} \\ D_{11} \end{pmatrix}$$
$$= r(D_{11}) \Leftrightarrow r\begin{pmatrix} E_{11} \\ D_{11} \end{pmatrix} = r(D_{11}) \Leftrightarrow (3.4),$$
$$(3.18) \Leftrightarrow r(B_{22}L_{A_{22}}) = 0 \Leftrightarrow r\begin{pmatrix} B_{22} \\ A_{22} \end{pmatrix} = r(A_{22}) \Leftrightarrow (3.4).$$

Similarly, we can show that (3.21) is equivalent to (3.5). Hence, (3.18) and (3.21) are equivalent to (3.3) and (3.5), respectively.

Part 2: In this part, we want to show that (3.25) is equivalent to (3.6) and (3.14). According to Lemma 2.6, we have that (3.25) is equivalent to the following:

$$r\begin{pmatrix} P & \hat{A}_2 & \hat{A}_3 & \hat{A}_4 \\ \hat{B}_1 & 0 & 0 & 0 \end{pmatrix} = r(\hat{B}_1) + r(\hat{A}_2, \hat{A}_3, \hat{A}_4), \tag{3.26}$$

$$r\begin{pmatrix} P & \hat{A}_2 & \hat{A}_4 \\ \hat{B}_3 & 0 & 0 \\ \hat{B}_1 & 0 & 0 \end{pmatrix} = r(\hat{A}_2, \hat{A}_4) + r\begin{pmatrix} \hat{B}_3 \\ \hat{B}_1 \end{pmatrix}, \tag{3.27}$$

$$r\begin{pmatrix} P & \hat{A}_3 & \hat{A}_4 \\ H_{11} & 0 & 0 \\ \hat{B}_1 & 0 & 0 \end{pmatrix} = r(\hat{A}_3, \hat{A}_4) + r\begin{pmatrix} H_{11} \\ \hat{B}_1 \end{pmatrix}, \tag{3.28}$$

$$r\begin{pmatrix} P & \hat{A}_4 \\ H_{11} & 0 \\ \hat{B}_3 & 0 \\ \hat{B}_1 & 0 \end{pmatrix} = r\begin{pmatrix} H_{11} \\ \hat{B}_3 \\ \hat{B}_1 \end{pmatrix} + r(\hat{A}_4), \tag{3.29}$$

$$r\begin{pmatrix} P & \hat{A}_2 & \hat{A}_3 \\ \hat{B}_4 & 0 & 0 \\ \hat{B}_1 & 0 & 0 \end{pmatrix} = r(\hat{A}_2, \hat{A}_3) + r\begin{pmatrix} \hat{B}_4 \\ \hat{B}_1 \end{pmatrix}, \tag{3.30}$$

$$r\begin{pmatrix} P & \hat{A}_2 \\ \hat{B}_3 & 0 \\ \hat{B}_4 & 0 \\ \hat{B}_1 & 0 \end{pmatrix} = r\begin{pmatrix} \hat{B}_3 \\ \hat{B}_4 \\ \hat{B}_1 \end{pmatrix} + r(\hat{A}_2), \tag{3.31}$$

$$r\begin{pmatrix} P & \hat{A}_3 \\ H_{11} & 0 \\ \hat{B}_4 & 0 \\ \hat{B}_1 & 0 \end{pmatrix} = r\begin{pmatrix} H_{11} \\ \hat{B}_4 \\ \hat{B}_1 \end{pmatrix} + r(\hat{A}_3), \tag{3.32}$$

$$r\begin{pmatrix} P \\ H_{11} \\ \hat{B}_3 \\ \hat{B}_4 \\ \hat{B}_1 \end{pmatrix} = r\begin{pmatrix} H_{11} \\ \hat{B}_3 \\ \hat{B}_4 \\ \hat{B}_1 \end{pmatrix}, \tag{3.33}$$

$$r\begin{pmatrix} P & \hat{A}_2 & 0 & 0 & \hat{A}_4 \\ \hat{B}_3 & 0 & 0 & 0 & 0 \\ \hat{B}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -P & \hat{A}_3 & \hat{A}_4 \\ 0 & 0 & H_{11} & 0 & 0 \\ 0 & 0 & \hat{B}_1 & 0 & 0 \\ \hat{B}_4 & 0 & \hat{B}_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} \hat{B}_3 & 0 \\ \hat{B}_1 & 0 \\ 0 & H_{11} \\ 0 & \hat{B}_1 \\ \hat{B}_4 & \hat{B}_4 \end{pmatrix} + r\begin{pmatrix} \hat{A}_2 & 0 & \hat{A}_4 \\ 0 & \hat{A}_3 & \hat{A}_4 \end{pmatrix}, \tag{3.34}$$

respectively. Hence, we only prove that (3.26)–(3.34) are equivalent to (3.6)–(3.14) when we prove that (3.25) is equivalent to (3.6)–(3.14). Now, we prove that (3.26)–(3.34) are equivalent to (3.6)–(3.14). In fact, we only prove that (3.26), (3.30), and (3.34) are equivalent to (3.6), (3.10), and (3.14); the remaining part can be proved similarly. According to Lemma 2.1 and elementary transformations, we have that

$$(3.26) = r\begin{pmatrix} P & \hat{A}_2 & \hat{A}_3 & \hat{A}_4 \\ \hat{B}_1 & 0 & 0 & 0 \end{pmatrix} = r(\hat{B}_1) + r(\hat{A}_2, \hat{A}_3, \hat{A}_4)$$

$$\Leftrightarrow r\begin{pmatrix} G_{11} - F_{11}X_1^0H_{11} - F_{11}L_{A_{11}}U_1^0H_{11} - X_2^0F_{22} - U_2^0R_{A_{22}}F_{22} & F_{11}L_{A_{11}}L_{A_1} & F_{11}L_{A_{11}} & L_{C_{22}} \\ R_{B_2}R_{A_{22}}F_{22} & 0 & 0 & 0 \end{pmatrix}$$

$$= r(R_{B_2}R_{A_{22}}F_{22}) + r(F_{11}L_{A_{11}}L_{A_1}, F_{11}L_{A_{11}}, L_{C_{22}})$$

$$\Leftrightarrow$$

$$r\begin{pmatrix} G_{11} - F_{11}X_1^0H_{11} - X_2^0F_{22} - U_2^0R_{A_{22}}F_{22} & F_{11} & I & 0 \\ R_{A_{22}}F_{22} & 0 & 0 & B_2 \\ 0 & A_{11} & 0 & 0 \\ 0 & 0 & C_{22} & 0 \end{pmatrix} = r(R_{A_{22}}F_{22}, B_2) + r\begin{pmatrix} F_{11} & I \\ A_{11} & 0 \\ 0 & C_{22} \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} G_{11} & F_{11} & I & U_2^0B_2 & 0 \\ F_{22} & 0 & 0 & B_2 & A_{22} \\ B_{11}H_{11} & A_{11} & 0 & 0 & 0 \\ C_{22}X_2^0F_{22} & 0 & C_{22} & 0 & 0 \end{pmatrix} = r(F_{22}, D_{22}, A_{22}) + r\begin{pmatrix} F_{11} & I \\ A_{11} & 0 \\ 0 & C_{22} \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} F_{22} & 0 & D_{22} & A_{22} \\ B_{11}H_{11} & A_{11} & 0 & 0 \\ C_{22}G_{11} & C_{22}F_{11} & E_{22} & C_{22}B_{22} \end{pmatrix} = r(F_{22}, D_{22}, A_{22}) + r\begin{pmatrix} A_{11} \\ F_{11}C_{22} \end{pmatrix} \Leftrightarrow (3.6).$$

Similarly, we have that (3.27) $\Leftrightarrow$ (3.7), (3.28) $\Leftrightarrow$ (3.8), (3.29) $\Leftrightarrow$ (3.9).

$$(3.30) = r\begin{pmatrix} P & \hat{A}_2 & \hat{A}_3 \\ \hat{B}_4 & 0 & 0 \\ \hat{B}_1 & 0 & 0 \end{pmatrix} = r(\hat{A}_2, \hat{A}_3) + r\begin{pmatrix} \hat{B}_4 \\ \hat{B}_1 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} G_{11} - F_{11}X_1^0H_{11} - F_{11}L_{A_{11}}U_1^0H_{11} - X_2^0F_{22} - U_2^0R_{A_{22}}F_{22} & F_{11}L_{A_{11}}L_{A_1} & F_{11}L_{A_1} \\ R_{A_{22}}F_{22} & 0 & 0 \\ R_{B_2}R_{A_{22}}F_{22} & 0 & 0 \end{pmatrix}$$

$$= r(F_{11}L_{A_{11}}L_{A_1}, F_{11}L_{A_{11}}) + r\begin{pmatrix} R_{A_{22}}F_{22} \\ R_{B_2}R_{A_{22}}F_{22} \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} G_{11} - F_{11}X_1^0H_{11} & F_{11} & B_{22} \\ F_{22} & 0 & A_{22} \\ 0 & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} F_{11} \\ A_{11} \end{pmatrix} + r(F_{22}, A_{22})$$

$$\Leftrightarrow r\begin{pmatrix} G_{11} & F_{11} & B_{22} \\ F_{22} & 0 & A_{22} \\ B_{11}H_{11} & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} F_{11} \\ A_{11} \end{pmatrix} + r(F_{22}, A_{22}) \Leftrightarrow (3.10).$$

Similarly, we have that (3.31) $\Leftrightarrow$ (3.11), (3.32) $\Leftrightarrow$ (3.12), (3.33) $\Leftrightarrow$ (3.13).

$$(3.34) = r\begin{pmatrix} P & \hat{A}_2 & 0 & 0 & \hat{A}_4 \\ \hat{B}_3 & 0 & 0 & 0 & 0 \\ \hat{B}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -P & \hat{A}_3 & \hat{A}_4 \\ 0 & 0 & H_{11} & 0 & 0 \\ 0 & 0 & \hat{B}_1 & 0 & 0 \\ \hat{B}_4 & 0 & \hat{B}_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} \hat{B}_3 & 0 \\ \hat{B}_1 & 0 \\ 0 & H_{11} \\ 0 & \hat{B}_1 \\ \hat{B}_4 & \hat{B}_4 \end{pmatrix} + r\begin{pmatrix} \hat{A}_2 & 0 & \hat{A}_4 \\ 0 & \hat{A}_3 & \hat{A}_4 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} P & F_{11}L_{A_{11}}L_{A_1} & 0 & 0 & L_{C_{22}} \\ R_{D_{11}}H_{11} & 0 & 0 & 0 & 0 \\ R_{B_2}R_{A_{22}}F_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & -P & F_{11}L_{A_{11}} & L_{C_{22}} \\ 0 & 0 & H_{11} & 0 & 0 \\ 0 & 0 & R_{B_2}R_{A_{22}}F_{22} & 0 & 0 \\ R_{A_{22}}F_{22} & 0 & R_{A_{22}}F_{22} & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} R_{D_{11}}H_{11} & 0 \\ R_{B_2}R_{A_{22}}F_{22} & 0 \\ 0 & H_{11} \\ 0 & R_{B_2}R_{A_{22}}F_{22} \\ R_{A_{22}F_{22}} & R_{A_{22}}F_{22} \end{pmatrix} + r\begin{pmatrix} F_{11}L_{A_{11}}L_{A_1} & 0 & L_{C_{22}} \\ 0 & F_{11}L_{A_{11}} & L_{C_{22}} \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} P & F_{11}L_{A_{11}} & 0 & 0 & L_{C_{22}} & 0 & 0 & 0 \\ H_{11} & 0 & 0 & 0 & 0 & D_{11} & 0 & 0 \\ R_{A_{22}}F_{22} & 0 & 0 & 0 & 0 & 0 & B_2 & 0 \\ 0 & 0 & -G_{11} + X_2^0F_{22} + U_2^0R_{A_{22}}F_{22} & F_{11}L_{A_{11}} & L_{C_{22}} & 0 & 0 & 0 \\ 0 & 0 & H_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{A_{22}}F_{22} & 0 & 0 & 0 & 0 & B_2 \\ R_{A_{22}}F_{22} & 0 & R_{A_{22}}F_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} H_{11} & 0 & D_{11} & 0 & 0 \\ R_{A_{22}} & 0 & 0 & B_2 & 0 \\ 0 & H_{11} & 0 & 0 & 0 \\ 0 & R_{A_{22}}F_{22} & 0 & 0 & B_2 \\ R_{A_{22}}F_{22} & R_{A_{22}}F_{22} & 0 & 0 & 0 \end{pmatrix} + r\begin{pmatrix} F_{11}L_{A_{11}} & 0 & L_{C_{22}} \\ 0 & F_{11}L_{A_{11}} & L_{C_{22}} \\ A_1 & 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} H_{11} & 0 & 0 & 0 & 0 & 0 & 0 & D_{11} & 0 \\ F_{22} & 0 & 0 & 0 & 0 & D_{22} & A_{22} & 0 & 0 \\ 0 & 0 & H_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F_{22} & D_{22} & A_{22} & 0 & 0 & 0 & 0 \\ F_{22} & 0 & F_{22} & 0 & 0 & 0 & 0 & 0 & A_{22} \\ 0 & C_{11} & 0 & 0 & 0 & 0 & 0 & -E_{11} & 0 \\ 0 & A_{11} & 0 & 0 & 0 & 0 & 0 & -B_{11}D_{11} & 0 \\ C_{22}G_{11} & C_{22}F_{11} & 0 & 0 & 0 & E_{22} & C_{22}B_{22} & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} H_{11} & 0 & 0 & 0 & 0 & 0 & D_{11} & 0 \\ F_{22} & 0 & 0 & 0 & D_{22} & A_{22} & 0 & 0 \\ 0 & H_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{22} & D_{22} & A_{22} & 0 & 0 & 0 & 0 \\ F_{22} & F_{22} & 0 & 0 & 0 & 0 & 0 & A_{22} \end{pmatrix} + r\begin{pmatrix} C_{11} \\ A_{11} \\ C_{22}F_{11} \end{pmatrix} \Leftrightarrow (3.14).$$

□

Theorem 3.2. *Let System* (1.1) *be solvable. Then, the general solution of System* (1.1) *is*

$$\begin{aligned} X_1 &= A_{11}^{\dagger}B_{11} + L_{A_{11}}A_1^{\dagger}P_1D_{11}^{\dagger} + L_{A_{11}}L_{A_1}W_1 + L_{A_{11}}W_2R_{D_{11}}, \\ X_2 &= B_{22}A_{22}^{\dagger} + C_{22}^{\dagger}P_2B_2^{\dagger}R_{A_{22}} + L_{C_{22}}W_3R_{A_{22}} + W_4R_{B_2}R_{A_{22}}, \end{aligned}$$

where

$$\begin{aligned} W_1 &= \widehat{A_2}^{\dagger}T\widehat{B_{11}}^{\dagger} - \widehat{A_2}^{\dagger}\widehat{A_3}\widehat{M_1}^{\dagger}T\widehat{B_{11}}^{\dagger} - \widehat{A_2}^{\dagger}S_1\widehat{A_3}^{\dagger}TN_1^{\dagger}\widehat{B_{22}}\widehat{B_{11}}^{\dagger} \\ &\quad - \widehat{A_2}^{\dagger}S_1V_4R_{N_1}\widehat{B_{22}}\widehat{B_{11}}^{\dagger} + L_{\widehat{A_2}}V_5 + V_6R_{\widehat{B_{11}}}, \\ W_2 &= \widehat{M_1}^{\dagger}T\widehat{B_{22}}^{\dagger} + S_1^{\dagger}S_1\widehat{A_3}^{\dagger}TN_1^{\dagger} + L_{\widehat{M_1}}L_{S_1}V_7 + V_8R_{\widehat{B_{22}}} + L_{\widehat{M_1}}V_4R_{N_1}, \\ W_3 &= F_1 + L_{C_2}\hat{V}_1 + \hat{V}_2R_{D_1} + L_{C_1}\hat{V}_3R_{D_2}, \text{ or } W_3 = F_2 - L_{C_4}V_1 - V_2R_{D_3} - L_{C_3}V_3R_{D_4}, \\ W_4 &= (P - \hat{A}_2W_1H_{11} - \hat{A}_3W_2\hat{B}_3 - \hat{A}_4W_3\hat{B}_4)\hat{B}_1^{\dagger} + V_3R_{\hat{B}_1}, \\ \hat{V}_1 &= (I_m, 0)[\widehat{C_{11}}^{\dagger}(F - \widehat{C_{22}}V_3\widehat{D_{22}} - \widehat{C_{33}}\hat{V}_3\widehat{D_{33}}) - \widehat{C_{11}}^{\dagger}U_{11}\widehat{D_{11}} + L_{\widehat{C_{11}}}U_{12}], \\ V_1 &= (0, I_m)[\widehat{C_{11}}^{\dagger}(F - \widehat{C_{22}}V_3\widehat{D_{22}} - \widehat{C_{33}}\hat{V}_3\widehat{D_{33}}) - \widehat{C_{11}}^{\dagger}U_{11}\widehat{D_{11}} + L_{\widehat{C_{11}}}U_{12}], \\ V_2 &= [R_{\widehat{C_{11}}}(F - \widehat{C_{22}}V_3\widehat{D_{22}} - \widehat{C_{33}}\hat{V}_3\widehat{D_{33}})\widehat{D_{11}}^{\dagger} + \widehat{C_{11}}\widehat{C_{11}}^{\dagger}U_{11} + U_{21}R_{\widehat{D_{11}}}]\begin{pmatrix} 0 \\ I_n \end{pmatrix}, \\ \hat{V}_2 &= [R_{\widehat{C_{11}}}(F - \widehat{C_{22}}V_3\widehat{D_{22}} - \widehat{C_{33}}\hat{V}_3\widehat{D_{33}})\widehat{D_{11}}^{\dagger} + \widehat{C_{11}}\widehat{C_{11}}^{\dagger}U_{11} + U_{21}R_{\widehat{D_{11}}}]\begin{pmatrix} I_n \\ 0 \end{pmatrix}, \\ \hat{V}_3 &= \widehat{E_{11}}^{\dagger}F\widehat{E_{33}}^{\dagger} - \widehat{E_{11}}^{\dagger}\widehat{E_{22}}M^{\dagger}F\widehat{E_{33}}^{\dagger} - \widehat{E_{11}}^{\dagger}S\widehat{E_{22}}^{\dagger}FN^{\dagger}\widehat{E_{44}}\widehat{E_{33}}^{\dagger} - \widehat{E_{11}}^{\dagger}SU_{31}R_N\widehat{E_{44}}\widehat{E_{33}}^{\dagger} \\ &\quad + L_{\widehat{E_{11}}}U_{32} + U_{33}R_{\widehat{E_{33}}}, \\ V_3 &= M^{\dagger}F\widehat{E_{44}}^{\dagger} + S^{\dagger}S\widehat{E_{22}}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{\widehat{E_{44}}}, \end{aligned}$$

$T = T_1 - \hat{A}_4W_3\hat{B}_4$, $V_i(i = \overline{4,8})$ *are arbitrary matrices with appropriate sizes over* $\mathbb{H}$, $U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$, *and* U_{42} *are any quaternion matrices with appropriate sizes, and m and n denote the column number of* C_{22} *and the row number of* A_{22}*, respectively.*

Next, we consider a special case of the System (1.1).

Corollary 3.3. *[16] Let* $A_{ii}, B_{ii}, C_{ii}, D_{ii}, E_{ii}, F_{ii}(i = 1, 2)$, *and* G_{11} *be given matrices with appropriate dimensions over* $\mathbb{H}$*. Denote*

$$\begin{aligned} &T = C_{11}L_{A_{11}}, K = R_{A_{22}}D_{22},\ B_1 = R_KR_{A_{22}}F_{22}, A_1 = F_{11}L_{A_{11}}L_T, C_3 = F_{11}L_{A_{11}}, D_3 = R_{D_{11}}, \\ &C_4 = L_{C_{22}}, D_4 = R_{A_{22}}F_{22}, A_{\alpha} = R_{A_1}C_3, B_{\beta} = D_3L_{B_1}, C_c = R_{A_{\alpha}}C_4, D_d = D_4L_{B_1}, E = R_{A_1}E_1L_{B_1}, \\ &A = A_{11}^{\dagger}B_{11} + L_{A_{11}}T^{\dagger}(E_{11} - C_{11}A_{11}^{\dagger}B_{11}D_{11})D^{\dagger}, B = B_{22}A_{22}^{\dagger} + C_{22}^{\dagger}(E_{22} - C_{22}B_{22}A_{22}^{\dagger}D_{22})K^{\dagger}R_{A_{22}}, \end{aligned}$$

$E_1 = G_{11} - F_{11}A - BF_{22}, M = R_{A_\alpha}C_c, N = D_dL_{B_\beta}, S = C_cL_M.$

Then, the following statements are equivalent:

(1) *Equation* (1.8) *is consistent.*

(2)

$$\begin{aligned}
&R_{A_{11}}B_{11} = 0, B_{22}L_{A_{22}} = 0, R_{C_{22}}E_{22} = 0, E_{11}L_{D_{11}} = 0,\\
&R_T(E_{11} - C_{11}A_{11}^{\dagger}B_{11}D_{11}) = 0, (E_{22} - C_{22}B_{22}A_{22}^{\dagger}D_{22})L_K = 0,\\
&R_MR_{A_\alpha}E = 0, EL_{B_\beta}L_N = 0, R_{A_\alpha}EL_{D_d} = 0, R_{C_c}EL_{B_\beta} = 0.
\end{aligned}$$

(3)

$$\begin{aligned}
&r(B_{11}, A_{11}) = r(A_{11}), r\begin{pmatrix} E_{11} & C_{11}\\ B_{11}D_{11} & A_{11}\end{pmatrix} = r\begin{pmatrix} C_{11}\\ A_{11}\end{pmatrix},\\
&r\begin{pmatrix} E_{11}\\ D_{11}\end{pmatrix} = r(D_{11}), r\begin{pmatrix} B_{22}\\ A_{22}\end{pmatrix} = r(A_{22}),\\
&r(E_{22}, C_{22}) = r(C_{22}), r\begin{pmatrix} E_{22} & C_{22}B_{22}\\ D_{22} & A_{22}\end{pmatrix} = r(D_{22}, A_{22}),\\
&r\begin{pmatrix} F_{22} & 0 & D_{22} & A_{22}\\ B_{11} & A_{11} & 0 & 0\\ C_{22}G_{11} & C_{22}F_{11} & E_{22} & C_{22}B_{22}\end{pmatrix} = r(F_{22}, D_{22}, A_{22}) + r\begin{pmatrix} A_{11}\\ C_{22}F_{11}\end{pmatrix},\\
&r\begin{pmatrix} 0 & F_{22}D_{11} & D_{22} & A_{22}\\ C_{11} & E_{11} & 0 & 0\\ A_{11} & B_{11}D_{11} & 0 & 0\\ C_{22}F_{11} & C_{22}G_{11}D_{11} & E_{22} & C_{22}B_{22}\end{pmatrix} = r\begin{pmatrix} C_{11}\\ A_{11}\\ C_{22}F_{11}\end{pmatrix} + r(F_{22}D_{11}, D_{22}, A_{22}),\\
&r\begin{pmatrix} G_{11} & F_{11} & B_{22}\\ F_{22} & 0 & A_{22}\\ B_{11} & A_{11} & 0\end{pmatrix} = r\begin{pmatrix} F_{11}\\ A_{11}\end{pmatrix} + r(F_{22}, A_{22}),\\
&r\begin{pmatrix} F_{11} & G_{11}D_{11} & B_{22}\\ 0 & F_{22}D_{11} & A_{22}\\ C_{11} & E_{11} & 0\\ A_{11} & B_{11}D_{11} & 0\end{pmatrix} = r(F_{22}D_{11}, A_{22}) + r\begin{pmatrix} F_{11}\\ C_{11}\\ A_{11}\end{pmatrix}.
\end{aligned}$$

Finally, we provide an example to illustrate the main results of this paper.

Example 3.4. *Conside the matrix equation* (1.1)

$$\begin{aligned}
&A_{11} = \begin{pmatrix} a_{111}\\ a_{121}\end{pmatrix}, B_{11} = \begin{pmatrix} b_{111} & b_{112}\\ b_{121} & b_{122}\end{pmatrix}, C_{11} = \begin{pmatrix} c_{111}\\ c_{121}\end{pmatrix}, D_{11} = \begin{pmatrix} d_{111}\\ d_{121}\end{pmatrix},\\
&E_{11} = \begin{pmatrix} e_{111}\\ e_{121}\end{pmatrix}, A_{22} = \begin{pmatrix} a_{211} & a_{212}\end{pmatrix}, B_{22} = \begin{pmatrix} b_{211} & b_{212}\\ b_{221} & b_{222}\end{pmatrix},\\
&C_{22} = \begin{pmatrix} c_{211} & c_{212}\\ c_{221} & c_{222}\end{pmatrix}, D_{22} = \begin{pmatrix} d_{211}\end{pmatrix}, E_{22} = \begin{pmatrix} e_{211}\\ e_{221}\end{pmatrix}, F_{11} = \begin{pmatrix} f_{111}\\ f_{121}\end{pmatrix},
\end{aligned}$$

$$H_{11}=\begin{pmatrix}h_{111} & h_{112}\\ h_{121} & h_{122}\end{pmatrix}, F_{22}=\begin{pmatrix}f_{211} & f_{212}\end{pmatrix}, G_{11}=\begin{pmatrix}g_{111} & g_{112}\\ g_{121} & g_{122}\end{pmatrix},$$

where

$$\begin{aligned}
&a_{111}=0.9787+0.5005\boldsymbol{i}+0.0596\boldsymbol{j}+0.0424\boldsymbol{k}, a_{121}=0.7127+0.4711\boldsymbol{i}+0.6820\boldsymbol{j}+0.0714\boldsymbol{k},\\
&b_{111}=0.5216+0.8181\boldsymbol{i}+0.7224\boldsymbol{j}+0.6596\boldsymbol{k}, b_{112}=0.9730+0.8003\boldsymbol{i}+0.4324\boldsymbol{j}+0.0835\boldsymbol{k},\\
&b_{121}=0.0967+0.8175\boldsymbol{i}+0.1499\boldsymbol{j}+0.5186\boldsymbol{k}, b_{122}=0.6490+0.4538\boldsymbol{i}0.8253\boldsymbol{j}+0.1332\boldsymbol{k},\\
&c_{111}=0.1734+0.8314\boldsymbol{i}+0.0605\boldsymbol{j}+0.5269\boldsymbol{k}, c_{121}=0.3909+0.8034\boldsymbol{i}+0.3993\boldsymbol{j}+0.4168\boldsymbol{k},\\
&d_{111}=0.6569+0.2920\boldsymbol{i}+0.0159\boldsymbol{j}+0.1671\boldsymbol{k}, d_{121}=0.6280+0.4317\boldsymbol{i}+0.9841\boldsymbol{j}+0.1062\boldsymbol{k},\\
&e_{111}=0.3724+0.4897\boldsymbol{i}+0.9516\boldsymbol{j}+0.0527\boldsymbol{k}, e_{121}=0.1981+0.3395\boldsymbol{i}+0.9203\boldsymbol{j}+0.7379\boldsymbol{k},\\
&a_{211}=0.2691+0.4228\boldsymbol{i}+0.5479\boldsymbol{j}+0.9427\boldsymbol{k}, a_{212}=0.4177+0.9831\boldsymbol{i}+0.3015\boldsymbol{j}+0.7011\boldsymbol{k},\\
&b_{211}=0.6663+0.6981\boldsymbol{i}+0.1781\boldsymbol{j}+0.9991\boldsymbol{k}, b_{212}=0.0326+0.8819\boldsymbol{i}+0.1904\boldsymbol{j}+0.4607\boldsymbol{k},\\
&b_{221}=0.5391+0.6665\boldsymbol{i}+0.1280\boldsymbol{j}+0.1711\boldsymbol{k}, b_{222}=0.5612+0.6692\boldsymbol{i}+0.3689\boldsymbol{j}+0.9816\boldsymbol{k},\\
&c_{211}=0.1564+0.6448\boldsymbol{i}+0.1909\boldsymbol{j}+0.4820\boldsymbol{k}, c_{212}=0.5895+0.3846\boldsymbol{i}+0.2518\boldsymbol{j}+0.6171\boldsymbol{k},\\
&c_{221}=0.8555+0.3763\boldsymbol{i}+0.4283\boldsymbol{j}+0.1206\boldsymbol{k}, c_{222}=0.2262+0.5830\boldsymbol{i}+0.2904\boldsymbol{j}+0.2653\boldsymbol{k},\\
&d_{211}=0.8244+0.9827\boldsymbol{i}+0.7302\boldsymbol{j}+0.3439\boldsymbol{k}, e_{211}=0.5847+0.9063\boldsymbol{i}+0.8178\boldsymbol{j}+0.5944\boldsymbol{k},\\
&e_{221}=0.1078+0.8797\boldsymbol{i}+0.2607\boldsymbol{j}+0.0225\boldsymbol{k}, f_{111}=0.4253+0.1615\boldsymbol{i}+0.4229\boldsymbol{j}+0.5985\boldsymbol{k},\\
&f_{121}=0.3127+0.1788\boldsymbol{i}+0.0942\boldsymbol{j}+0.4709\boldsymbol{k}, h_{111}=0.6959+0.6385\boldsymbol{i}+0.0688\boldsymbol{j}+0.5309\boldsymbol{k},\\
&h_{112}=0.4076+0.7184\boldsymbol{i}+0.5313\boldsymbol{j}+0.1056\boldsymbol{k}, h_{121}=0.6999+0.0336\boldsymbol{i}+0.3196\boldsymbol{j}+0.6544\boldsymbol{k},\\
&h_{122}=0.8200+0.9686\boldsymbol{i}+0.3251\boldsymbol{j}+0.6110\boldsymbol{k}, f_{211}=0.7788+0.4235\boldsymbol{i}+0.0908\boldsymbol{j}+0.2665\boldsymbol{k},\\
&f_{212}=0.1537+0.2810\boldsymbol{i}+0.4401\boldsymbol{j}+0.5271\boldsymbol{k}, g_{111}=0.4574+0.5181\boldsymbol{i}+0.6377\boldsymbol{j}+0.2407\boldsymbol{k},\\
&g_{112}=0.2891+0.6951\boldsymbol{i}+0.2548\boldsymbol{j}+0.6678\boldsymbol{k}, g_{121}=0.8754+0.9436\boldsymbol{i}+0.9577\boldsymbol{j}+0.6761\boldsymbol{k},\\
&g_{122}=0.6718+0.0680\boldsymbol{i}+0.2240\boldsymbol{j}+0.8444\boldsymbol{k}.
\end{aligned}$$

Computing directly yields the following:

$$r\begin{pmatrix}B_{11} & A_{11}\end{pmatrix}=r\begin{pmatrix}A_{11}\end{pmatrix}=2, r\begin{pmatrix}E_{11} & C_{11}\\ B_{11}D_{11} & A_{11}\end{pmatrix}=r\begin{pmatrix}C_{11}\\ A_{11}\end{pmatrix}=2,$$

$$r\begin{pmatrix}E_{11}\\ D_{11}\end{pmatrix}=r\begin{pmatrix}D_{11}\end{pmatrix}=1, r\begin{pmatrix}B_{22}\\ A_{22}\end{pmatrix}=r\begin{pmatrix}A_{22}\end{pmatrix}=2,$$

$$r\begin{pmatrix}E_{22} & C_{22}\end{pmatrix}=r\begin{pmatrix}C_{22}\end{pmatrix}=2, r\begin{pmatrix}E_{22} & C_{22}B_{22}\\ D_{22} & A_{22}\end{pmatrix}=r\begin{pmatrix}D_{22} & A_{22}\end{pmatrix}=3,$$

$$r\begin{pmatrix}F_{22} & 0 & D_{22} & A_{22}\\ B_{11}H_{11} & A_{11} & 0 & 0\\ C_{22}G_{11} & C_{22}F_{11} & E_{22} & C_{22}B_{22}\end{pmatrix}=r\begin{pmatrix}F_{22} & D_{22} & A_{22}\end{pmatrix}+r\begin{pmatrix}A_{11}\\ C_{22}F_{11}\end{pmatrix}=5,$$

$$r\begin{pmatrix}H_{11} & 0 & -D_{11} & 0 & 0\\ F_{22} & 0 & 0 & D_{22} & A_{22}\\ 0 & C_{11} & E_{11} & 0 & 0\\ 0 & A_{11} & B_{11}D_{11} & 0 & 0\\ C_{22}G_{11} & C_{22}F_{11} & 0 & E_{22} & C_{22}B_{22}\end{pmatrix}=r\begin{pmatrix}C_{11}\\ A_{11}\\ C_{22}F_{11}\end{pmatrix}+r\begin{pmatrix}H_{11} & D_{11} & 0 & 0\\ F_{22} & 0 & D_{22} & A_{22}\end{pmatrix}=7,$$

$$r\begin{pmatrix} H_{11} & 0 & 0 & 0 \\ F_{22} & 0 & D_{22} & A_{22} \\ 0 & A_{11} & 0 & 0 \\ C_{22}G_{11} & C_{22}F_{11} & E_{22} & C_{22}B_{22} \end{pmatrix} = r\begin{pmatrix} H_{11} & 0 & 0 \\ F_{22} & D_{22} & A_{22} \end{pmatrix} + r\begin{pmatrix} A_{11} \\ C_{22}F_{11} \end{pmatrix} = 6,$$

$$r\begin{pmatrix} H_{11} & 0 & 0 \\ F_{22} & D_{22} & A_{22} \\ C_{22}G_{11} & E_{22} & C_{22}B_{22} \end{pmatrix} = r\begin{pmatrix} H_{11} & 0 & 0 \\ F_{22} & D_{22} & A_{22} \end{pmatrix} = 5,$$

$$r\begin{pmatrix} G_{11} & F_{11} & B_{22} \\ F_{22} & 0 & A_{22} \\ B_{11}H_{11} & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} F_{11} \\ A_{11} \end{pmatrix} + r\begin{pmatrix} F_{22}, A_{22} \end{pmatrix} = 5,$$

$$r\begin{pmatrix} G_{11} & F_{11} & 0 & B_{22} \\ H_{11} & 0 & -D_{11} & 0 \\ F_{22} & 0 & 0 & A_{22} \\ 0 & C_{11} & E_{11} & 0 \\ 0 & A_{11} & B_{11}D_{11} & 0 \end{pmatrix} = r\begin{pmatrix} H_{11} & D_{11} & 0 \\ F_{22} & 0 & A_{22} \end{pmatrix} + r\begin{pmatrix} F_{11} \\ C_{11} \\ A_{11} \end{pmatrix} = 6,$$

$$r\begin{pmatrix} G_{11} & F_{11} & B_{22} \\ H_{11} & 0 & 0 \\ F_{22} & 0 & A_{22} \\ 0 & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} H_{11} & 0 \\ F_{22} & A_{22} \end{pmatrix} + r\begin{pmatrix} F_{11} \\ A_{11} \end{pmatrix} = 5,\ r\begin{pmatrix} G_{11} & B_{22} \\ H_{11} & 0 \\ F_{22} & A_{22} \end{pmatrix} = r\begin{pmatrix} H_{11} & 0 \\ F_{22} & A_{22} \end{pmatrix} = 4,$$

$$r\begin{pmatrix} H_{11} & 0 & 0 & 0 & 0 & 0 & 0 & D_{11} & 0 \\ F_{22} & 0 & 0 & 0 & 0 & D_{22} & A_{22} & 0 & 0 \\ 0 & 0 & H_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F_{22} & D_{22} & A_{22} & 0 & 0 & 0 & 0 \\ F_{22} & 0 & F_{22} & 0 & 0 & 0 & 0 & 0 & A_{22} \\ 0 & C_{11} & 0 & 0 & 0 & 0 & 0 & -E_{11} & 0 \\ 0 & A_{11} & 0 & 0 & 0 & 0 & 0 & -B_{11}D_{11} & 0 \\ C_{22}G_{11} & C_{22}F_{11} & 0 & 0 & 0 & E_{22} & C_{22}B_{22} & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} H_{11} & 0 & 0 & 0 & 0 & 0 & D_{11} & 0 \\ F_{22} & 0 & 0 & 0 & D_{22} & A_{22} & 0 & 0 \\ 0 & H_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{22} & D_{22} & A_{22} & 0 & 0 & 0 & 0 \\ F_{22} & F_{22} & 0 & 0 & 0 & 0 & 0 & A_{22} \end{pmatrix} + r\begin{pmatrix} C_{11} \\ A_{11} \\ C_{22}F_{11} \end{pmatrix} = 11.$$

All rank equations in (3.3) *to* (3.14) *hold. So, according to Theorem 3.1, the system of matrix equation* (1.1) *has a solution. By Theorem 3.2, the solution of System* (1.1) *can be expressed as*

$$X_1 = \begin{pmatrix} 0.4946 + 0.1700\boldsymbol{i} - 0.1182\boldsymbol{j} - 0.3692\boldsymbol{k} & 0.4051 - 0.0631\boldsymbol{i} - 0.2403\boldsymbol{j} - 0.1875\boldsymbol{k} \end{pmatrix},$$

$$X_2 = \begin{pmatrix} -0.0122 + 0.2540\boldsymbol{i} - 0.3398\boldsymbol{j} - 0.3918\boldsymbol{k} \\ 0.7002 - 0.3481\boldsymbol{i} - 0.2169\boldsymbol{j} + 0.0079\boldsymbol{k} \end{pmatrix}.$$

4. The solvability conditions and the general solutions to the Systems (1.2)–(1.4)

In this section, we use the same method and technique as in Theorem 3.1 to study the three systems of Eqs (1.2)–(1.4). We only present their results and omit their proof.

Theorem 4.1. *Consider the matrix equation* (1.2) *over* $\mathbb{H}$*, where* $A_{ii}, B_{ii}, C_{ii}, D_{ii}, E_{ii}, F_{ii}, G_{11}$*, and* $H_{11} (i = \overline{1,2})$ *are given. Put*

$$\begin{aligned}
&A_1 = C_{11}L_{A_{11}}, P_1 = E_{11} - C_{11}A_{11}^{\dagger}B_{11}D_{11}, B_2 = R_{A_{22}}D_{22}, P_2 = E_{22} - C_{22}B_{22}A_{22}^{\dagger}D_{22},\\
&\hat{A}_1 = F_{11}L_{A_{11}}L_{A_1}, \hat{A}_2 = F_{11}L_{A_1}, \hat{B}_2 = R_{D_{11}}, \hat{A}_3 = H_{11}L_{C_{22}}, \hat{B}_3 = R_{A_{22}}F_{22}, \hat{B}_4 = R_{B_2}R_{A_{22}}F_{22},\\
&B = G_{11} - F_{11}A_{11}^{\dagger}B_{11} - F_{11}L_{A_{11}}A_1^{\dagger}P_1D_{11}^{\dagger} - H_{11}B_{22}A_{22}^{\dagger}F_{22} - H_{11}C_{22}^{\dagger}P_2B_2^{\dagger}R_{A_{22}}F_{22}, R_{\hat{A}_1}\hat{A}_2 = A_{12},\\
&R_{\hat{A}_1}\hat{A}_3 = A_{13}, R_{\hat{A}_1}H_{11} = A_{14}, \hat{B}_3L_{\hat{B}_2} = N_1, R_{A_{12}}A_{13} = M_1, S_1 = A_{13}L_{M_1}, R_{\hat{A}_1}B = T_1,\\
&C = R_{M_1}R_{A_{12}}, \hat{C}_1 = CA_{14}, \hat{C}_2 = R_{A_{12}}A_{14}, \hat{C}_3 = R_{A_{13}}A_{14}, \hat{C}_4 = A_{14}, D = L_{\hat{B}_2}L_{N_1}, \hat{D}_1 = \hat{B}_4,\\
&\hat{D}_2 = \hat{B}_4L_{\hat{B}_3}, \hat{D}_3 = \hat{B}_4L_{\hat{B}_2}, \hat{D}_4 = \hat{B}_4D, \hat{E}_1 = CT_1, \hat{E}_2 = R_{A_{12}}T_1L_{\hat{B}_3}, \hat{E}_3 = R_{A_{13}}T_1L_{\hat{B}_2}, \hat{E}_4 = T_1D,\\
&C_{24} = (L_{\hat{C}_2}, L_{\hat{C}_4}), D_{13} = \begin{pmatrix} R_{\hat{D}_1} \\ R_{\hat{D}_3} \end{pmatrix}, C_{12} = L_{\hat{C}_1}, D_{12} = R_{\hat{D}_2}, C_{33} = L_{\hat{C}_3}, D_{33} = R_{\hat{D}_4}, E_{24} = R_{C_{24}}C_{12},\\
&E_{13} = R_{C_{24}}C_{33}, E_{33} = D_{12}L_{D_{13}}, E_{44} = D_{33}L_{D_{13}}, M = R_{E_{24}}E_{13}, N = E_{44}L_{E_{33}}, F = F_2 - F_1,\\
&E = R_{C_{24}}FL_{D_{13}}, S = E_{13}L_M, \hat{F_{11}} = \hat{C}_2L_{\hat{C}_1}, \hat{G}_1 = \hat{E}_2 - \hat{C}_2\hat{C}_1^{\dagger}\hat{E}_1\hat{D}_1^{\dagger}\hat{D}_2, F_{33} = \hat{C}_4L_{\hat{C}_3},\\
&\hat{G}_2 = \hat{E}_4 - \hat{C}_4\hat{C}_3^{\dagger}\hat{E}_3\hat{D}_3^{\dagger}\hat{D}_4, F_1 = \hat{C}_1^{\dagger}\hat{E}_1\hat{D}_1^{\dagger} + L_{\hat{C}_1}\hat{C}_2^{\dagger}\hat{E}_2\hat{D}_2^{\dagger}, F_2 = \hat{C}_3^{\dagger}\hat{E}_3\hat{D}_3^{\dagger} + L_{\hat{C}_3}\hat{C}_4^{\dagger}\hat{E}_4\hat{D}_4^{\dagger}.
\end{aligned}$$

Then, the following statements are equivalent:

(1) *System* (1.2) *is consistent.*

(2)

$$\begin{aligned}
&R_{A_{11}}B_{11} = 0, R_{A_1}P_1 = 0, P_1L_{D_{11}} = 0, B_{22}L_{A_{22}} = 0, R_{C_{22}}P_2 = 0,\\
&P_2L_{B_2} = 0, R_{\hat{C}_i}\hat{E}_i = 0, \hat{E}_iL_{\hat{D}_i} = 0 (i = \overline{1,4}), R_{E_{24}}EL_{E_{44}} = 0.
\end{aligned}$$

(3)

$$\begin{aligned}
&r(B_{11}, A_{11}) = r(A_{11}), r\begin{pmatrix} E_{11} & C_{11} \\ B_{11}D_{11} & A_{11} \end{pmatrix} = r\begin{pmatrix} C_{11} \\ A_{11} \end{pmatrix}, r\begin{pmatrix} E_{11} \\ D_{11} \end{pmatrix} = r(D_{11}),\\
&r(B_{22}, A_{22}) = r(A_{22}), r(E_{22}, C_{22}) = r(C_{22}), r\begin{pmatrix} E_{22} & C_{22}B_{22} \\ D_{22} & A_{22} \end{pmatrix} = r(D_{22}, A_{22}),\\
&r\begin{pmatrix} G_{11}D_{11} & F_{11} & H_{11} \\ E_{11} & C_{11} & 0 \\ B_{11}D_{11} & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} F_{11} & H_{11} \\ C_{11} & 0 \\ A_{11} & 0 \end{pmatrix},\\
&r\begin{pmatrix} G_{11}D_{11} & F_{11} & H_{11} & 0 \\ F_{22}D_{11} & 0 & 0 & A_{22} \\ E_{11} & C_{11} & 0 & 0 \\ B_{11}D_{11} & A_{11} & 0 & 0 \end{pmatrix} = r(F_{22}, A_{22}) + r\begin{pmatrix} F_{11} & H_{11} \\ C_{11} & 0 \\ A_{11} & 0 \end{pmatrix},
\end{aligned}$$

$$r\begin{pmatrix} H_{11} & F_{11} & G_{11}D_{11} \\ 0 & C_{11} & E_{11} \\ 0 & A_{11} & B_{11}D_{11} \end{pmatrix} = r\begin{pmatrix} H_{11} & F_{11} \\ 0 & C_{11} \\ 0 & A_{11} \end{pmatrix},$$

$$r\begin{pmatrix} H_{11} & F_{11} & 0 & G_{11}D_{11} \\ 0 & 0 & A_{22} & F_{22}D_{11} \\ 0 & C_{11} & 0 & E_{11} \\ 0 & A_{11} & 0 & B_{11}D_{11} \end{pmatrix} = r(F_{22}D_{11}, A_{22}) + r\begin{pmatrix} H_{11} & F_{11} \\ 0 & C_{11} \\ 0 & A_{11} \end{pmatrix},$$

$$r\begin{pmatrix} G_{11}D_{11} & F_{11} & H_{11} & 0 & 0 \\ F_{22}D_{11} & 0 & 0 & D_{22} & A_{22} \\ E_{11} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & C_{22} & -E_{22} & -C_{22}B_{22} \\ B_{11}D_{11} & A_{11} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_{11} & H_{11} \\ C_{11} & 0 \\ 0 & C_{22} \\ A_{11} & 0 \end{pmatrix} + r(F_{22}, D_{22}, A_{22}),$$

$$r\begin{pmatrix} G_{11}D_{11} & F_{11} & H_{11}B_{22} \\ F_{22}D_{11} & 0 & A_{22} \\ E_{11} & C_{11} & 0 \\ B_{11}D_{11} & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} F_{11} & \\ C_{11} & A_{11} \end{pmatrix} + r(F_{22}, A_{22}),$$

$$r\begin{pmatrix} H_{11} & F_{11} & 0 & 0 & G_{11}D_{11} \\ 0 & 0 & D_{22} & A_{22} & F_{22}D_{11} \\ 0 & C_{11} & 0 & 0 & E_{11} \\ 0 & A_{11} & 0 & 0 & B_{11}D_{11} \\ C_{22} & 0 & -E_{22} & -C_{22}B_{22} & 0 \end{pmatrix} = r\begin{pmatrix} H_{11} & F_{11} \\ 0 & C_{11} \\ 0 & A_{11} \\ C_{22} & 0 \end{pmatrix} + r(D_{22}, A_{22}, F_{22}D_{11}),$$

$$r\begin{pmatrix} F_{11} & H_{11}B_{22} & G_{11}D_{11} \\ 0 & A_{22} & F_{22}D_{11} \\ C_{11} & 0 & E_{11} \\ A_{11} & 0 & B_{11}D_{11} \end{pmatrix} = r\begin{pmatrix} F_{11} \\ C_{11} \\ A_{11} \end{pmatrix} + r(A_{22}, F_{22}D_{11}),$$

$$r\begin{pmatrix} G_{11} & F_{11} & 0 & 0 & H_{11} & 0 & 0 & H_5B_{22} & 0 \\ F_{22} & 0 & 0 & 0 & 0 & 0 & 0 & A_{22} & 0 \\ 0 & 0 & H_{11} & F_{11} & H_{11} & 0 & -H_{11}B_{22} & 0 & G_{11}D_{11} \\ 0 & 0 & 0 & 0 & 0 & D_{22} & A_{22} & 0 & -F_{22}D_{11} \\ 0 & 0 & C_{22} & 0 & 0 & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{11} & 0 & 0 & 0 & 0 & E_{11} \\ 0 & 0 & 0 & A_{11} & 0 & 0 & 0 & 0 & B_{11}D_{11} \\ B_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} F_{22} & 0 & 0 & A_{22} & 0 \\ 0 & D_{22} & A_{22} & 0 & F_{22}D_{11} \end{pmatrix} + r\begin{pmatrix} F_{11} & 0 & 0 & H_{11} \\ 0 & H_{11} & F_{11} & H_{11} \\ 0 & C_{22} & 0 & 0 \\ 0 & 0 & C_{11} & 0 \\ 0 & 0 & A_{11} & 0 \\ A_{11} & 0 & 0 & 0 \end{pmatrix}.$$

Under these conditions, the general solution of System (1.2) *is*

$$X_1 = A_{11}^{\dagger}B_{11} + L_{A_{11}}A_1^{\dagger}P_1D_{11}^{\dagger} + L_{A_{11}}L_{A_1}W_1 + L_{A_{11}}W_2R_{D_{11}},$$

$$X_2 = B_{22}A_{22}^{\dagger} + C_{22}^{\dagger}P_2B_2^{\dagger}R_{A_{22}} + L_{C_{22}}W_3R_{A_{22}} + W_4R_{B_2}R_{A_{22}},$$

where

$$\begin{aligned}
W_1 &= \hat{A}_1^{\dagger}(B - \hat{A}_2W_1\hat{B}_2 - \hat{A}_3W_3\hat{B}_3 - H_{11}W_4\hat{B}_4) + L_{\hat{A}_1}U_1,\\
W_2 &= A_{12}^{\dagger}T\hat{B}_2^{\dagger} - A_{12}^{\dagger}A_{13}M_1^{\dagger}T\hat{B}_2^{\dagger} - A_{12}^{\dagger}S_1A_{13}^{\dagger}TN_1^{\dagger}\hat{B}_3\hat{B}_2^{\dagger}\\
&\quad - A_{12}^{\dagger}S_1U_2R_{N_1}\hat{B}_3\hat{B}_2^{\dagger} + L_{A_{12}}U_3 + U_4R_{\hat{B}_2},\\
W_3 &= M_1^{\dagger}T\hat{B}_3^{\dagger} + S_1^{\dagger}S_1A_{13}^{\dagger}TN_1^{\dagger} + L_{M_1}L_{S_1}U_5 + U_6R_{\hat{B}_3} + L_{M_1}U_2R_{N_1},\\
W_4 &= F_1 + L_{\hat{C}_2}V_1 + V_2R_{\hat{D}_1} + L_{\hat{C}_1}V_3R_{\hat{D}_2},\ or\ W_4 = F_2 - L_{\hat{C}_4}\hat{V}_1 - \hat{V}_2R_{\hat{D}_3} - L_{\hat{C}_3}\hat{V}_3R_{\hat{D}_4},
\end{aligned}$$

where $T = T_1 - H_{11}W_4\hat{B}_4, U_i(i = \overline{1,6})$ *are arbitrary matrices with appropriate sizes over* $\mathbb{H}$,

$$\begin{aligned}
V_1 &= (I_m, 0)[C_{24}^{\dagger}(F - C_{12}V_3D_{12} - C_{33}\hat{V}_3D_{33}) - C_{24}^{\dagger}U_{11}D_{13} + L_{C_{24}}U_{12}],\\
\hat{V}_1 &= (0, I_m)[C_{24}^{\dagger}(F - C_{12}V_3D_{12} - C_{33}\hat{V}_3D_{33}) - C_{24}^{\dagger}U_{11}D_{13} + L_{C_{24}}U_{12}],\\
\hat{V}_2 &= [R_{C_{24}}(F - C_{12}V_3D_{12} - C_{33}\hat{V}_3D_{33})D_{13}^{\dagger} + C_{24}C_{24}^{\dagger}U_{11} + U_{21}R_{D_{13}}]\begin{pmatrix}0\\ I_n\end{pmatrix},\\
V_2 &= [R_{C_{24}}(F - C_{12}V_3D_{12} - C_{33}\hat{V}_3D_{33})D_{13}^{\dagger} + C_{24}C_{24}^{\dagger}U_{11} + U_{21}R_{D_{13}}]\begin{pmatrix}I_n\\ 0\end{pmatrix},\\
V_3 &= E_{24}^{\dagger}FE_{33}^{\dagger} - E_{24}^{\dagger}E_{13}M^{\dagger}FE_{33}^{\dagger} - E_{24}^{\dagger}SE_{13}^{\dagger}FN^{\dagger}E_{44}E_{33}^{\dagger}\\
&\quad - E_{24}^{\dagger}SU_{31}R_NE_{44}E_{33}^{\dagger} + L_{E_{24}}U_{32} + U_{33}R_{E_{33}},\\
\hat{V}_3 &= M^{\dagger}FE_{44}^{\dagger} + S^{\dagger}SE_{13}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{E_{44}},
\end{aligned}$$

$U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$, *and* U_{42} *are any quaternion matrices with appropriate sizes, and* m *and* n *denote the column number of* H_{11} *and the row number of* A_{22}*, respectively.*

Theorem 4.2. *Consider the matrix equation* (1.3) *over* $\mathbb{H}$*, where* $A_{ii}, B_{ii}, C_{ii}, D_{ii}, E_{ii}, F_{ii}, G_{11}\ H_{11}(i = \overline{1,2})$ *are given. Put*

$$\begin{aligned}
&A_1 = C_{11}L_{A_{11}}, P_1 = E_{11} - C_{11}A_{11}^{\dagger}B_{11}D_{11}, A_2 = C_{22}L_{A_{22}}, P_2 = E_{22} - C_{22}A_{22}^{\dagger}B_{22}D_{22},\\
&\hat{A}_1 = F_{11}L_{A_{11}}L_{A_1}, \hat{A}_2 = F_{11}L_{A_{11}}, \hat{B}_2 = R_{D_{11}}, \hat{A_{11}} = H_{11}L_{A_{22}}L_{A_2}, \hat{A_{22}} = H_{11}L_{A_{22}}, \hat{B}_4 = R_{D_{22}}F_{22},\\
&B = G_{11} - F_{11}A_{11}^{\dagger}B_{11} - F_{11}L_{A_{11}}A_1^{\dagger}P_1D_{11}^{\dagger} - H_{11}A_{22}^{\dagger}B_{22}F_{22} - H_{11}L_{A_{22}}A_2^{\dagger}P_2D_{22}^{\dagger}F_{22},\\
&R_{\hat{A}_1}\hat{A}_2 = A_{12}, R_{\hat{A}_1}\hat{A_{11}} = A_{13}, R_{\hat{A}_1}\hat{A_{22}} = A_{33}, F_{22}L_{\hat{B}_2} = N_1, R_{A_{12}}A_{13} = M_1, S_1 = A_{13}L_{M_1},\\
&R_{\hat{A}_1}B = T_1, C = R_{M_1}R_{A_{12}}, \hat{C}_1 = CA_{33}, \hat{C}_2 = R_{A_{12}}A_{33}, \hat{C_{11}} = R_{A_{13}}A_{33}, \hat{C_{22}} = A_{33},\\
&D = L_{\hat{B}_2}L_{N_1}, \hat{D}_1 = \hat{B}_4, \hat{D}_2 = \hat{B}_4L_{F_{22}}, \hat{D_{11}} = \hat{B}_4L_{\hat{B}_2}, \hat{D_{22}} = \hat{B}_4D, \hat{E}_1 = CT_1,\\
&\hat{E}_2 = R_{A_{12}}T_1L_{F_{22}}, \hat{E_{11}} = R_{A_{13}}T_1L_{\hat{B}_2}, \hat{E}_4 = T_1D, C_{24} = (L_{\hat{C}_2}, L_{\hat{C_{22}}}), D_{13} = \begin{pmatrix}R_{\hat{D}_1}\\ R_{\hat{D_{11}}}\end{pmatrix},\\
&C_{21} = L_{\hat{C}_1}, D_{12} = R_{\hat{D}_2}, C_{33} = L_{\hat{C_{11}}}, D_{33} = R_{\hat{D_{22}}}, E_{11} = R_{C_{24}}C_{21}, E_{22} = R_{C_{24}}C_{33},\\
&E_{33} = D_{12}L_{D_{13}}, E_{44} = D_{33}L_{D_{13}}, M = R_{E_{11}}E_{22}, N = E_{44}L_{E_{33}},\\
&F = F_2 - F_1, E = R_{C_{24}}FL_{D_{13}}, S = E_{22}L_M, \hat{F_{11}} = \hat{C}_2L_{\hat{C}_1},\\
&\hat{G}_1 = \hat{E}_2 - \hat{C}_2\hat{C}_1^{\dagger}\hat{E}_1\hat{D}_1^{\dagger}\hat{D}_2, \hat{F_{22}} = \hat{C_{22}}L_{\hat{C_{11}}}, \hat{G}_2 = \hat{E}_4 - \hat{C_{22}}\hat{C_{11}}^{\dagger}\hat{E_{11}}\hat{D_{11}}^{\dagger}\hat{D_{22}},
\end{aligned}$$

$$F_1 = \hat{C_1}^{\dagger}\hat{E}_1\hat{D_1}^{\dagger} + L_{\hat{C_1}}\hat{C_2}^{\dagger}\hat{E}_2\hat{D_2}^{\dagger}, F_2 = \hat{C_{11}}^{\dagger}\hat{E_{11}}\hat{D_{11}}^{\dagger} + L_{\hat{C_{11}}}\hat{C_{22}}^{\dagger}\hat{E}_4\hat{D_{22}}^{\dagger}.$$

Then, the following statements are equivalent:
(1) *System* (1.3) *is consistent.*
(2)

$$\begin{aligned}
&R_{A_{11}}B_{11} = 0, R_{A_1}P_1 = 0, P_1L_{D_{11}} = 0, R_{A_{22}}B_{22} = 0, R_{A_2}P_2 = 0, P_2L_{D_{22}} = 0,\\
&R_{\hat{C}_i}\hat{E}_i = 0, R_{\hat{C_{11}}}\hat{E_{11}} = 0, R_{\hat{C_{22}}}\hat{E}_4 = 0, \hat{E}_iL_{\hat{D}_i} = 0 (i = \overline{1,2}),\\
&\hat{E_{11}}L_{\hat{D_{11}}} = 0, \hat{E}_4L_{\hat{D_{22}}} = 0, R_{E_{11}}EL_{E_{44}} = 0.
\end{aligned}$$

(3)

$$\begin{aligned}
&r(B_{11}, A_{11}) = r(A_{11}), r\begin{pmatrix} E_{11} & C_{11}\\ B_{11}D_{11} & A_{11}\end{pmatrix} = r\begin{pmatrix} C_{11}\\ A_{11}\end{pmatrix}, r\begin{pmatrix} E_{11}\\ D_{11}\end{pmatrix} = r(D_{11}),\\
&r(B_{22}, A_{22}) = r(A_{22}), r\begin{pmatrix} E_{22} & C_{22}\\ B_4D_{22} & A_{22}\end{pmatrix} = r\begin{pmatrix} C_{22}\\ A_{22}\end{pmatrix}, r\begin{pmatrix} E_{22}\\ D_{22}\end{pmatrix} = r(D_{22}),\\
&r\begin{pmatrix} G_{11} & F_{11} & H_{11}\\ B_{11} & A_{11} & 0\\ B_{22}F_{22} & 0 & A_{22}\end{pmatrix} = r\begin{pmatrix} F_{11} & H_{11}\\ A_{11} & 0\\ 0 & A_{22}\end{pmatrix},\\
&r\begin{pmatrix} G_{11} & F_{11} & H_{11}\\ F_{22} & 0 & 0\\ B_{11} & A_{11} & 0\\ 0 & 0 & A_{22}\end{pmatrix} = r(F_{22}) + r\begin{pmatrix} F_{11} & H_{11}\\ A_{11} & 0\\ 0 & A_{22}\end{pmatrix},\\
&r\begin{pmatrix} H_{11} & F_{11} & G_{11}D_{11}\\ A_{22} & 0 & B_{22}F_{22}D_{11}\\ 0 & C_{11} & E_{11}\\ 0 & A_{11} & B_{11}D_{11}\end{pmatrix} = r\begin{pmatrix} H_{11} & F_{11}\\ 0 & C_{11}\\ 0 & A_{11}\\ A_{22} & 0\end{pmatrix},\\
&r\begin{pmatrix} H_{11} & F_{11} & G_{11}D_{11}\\ 0 & 0 & F_{22}D_{11}\\ 0 & C_{11} & E_{11}\\ 0 & A_{11} & B_{11}D_{11}\\ A_{22} & 0 & 0\end{pmatrix} = r\begin{pmatrix} H_{11} & F_{11}\\ 0 & C_{11}\\ 0 & A_{11}\\ A_{22} & 0\end{pmatrix} + r(F_{22}D_{11}),\\
&r\begin{pmatrix} G_{11} & F_{11} & H_{11} & 0\\ F_{22} & 0 & 0 & D_{22}\\ B_{11} & A_{11} & 0 & 0\\ 0 & 0 & C_{22} & -E_{22}\\ 0 & 0 & A_{22} & -B_{22}D_{22}\end{pmatrix} = r\begin{pmatrix} F_{11} & H_{11}\\ A_{11} & 0\\ 0 & C_{22}\\ 0 & A_{22}\end{pmatrix} + r(F_{22}, D_{22}),\\
&r\begin{pmatrix} G_{11} & F_{11}\\ F_{22} & 0\\ B_{11} & A_{11}\end{pmatrix} = r\begin{pmatrix} F_{11}\\ A_{11}\end{pmatrix} + r(F_{22}),
\end{aligned}$$

$$r\begin{pmatrix} H_{11} & F_{11} & 0 & G_{11}D_{11} \\ 0 & 0 & D_{22} & F_{22}D_{11} \\ C_{22} & 0 & -E_{22} & 0 \\ 0 & C_{11} & 0 & E_{11} \\ A_{22} & 0 & 0 & B_{22}F_{22}D_{11} \\ 0 & A_{11} & 0 & B_{11}D_{11} \end{pmatrix} = r\begin{pmatrix} H_{11} & F_{11} \\ C_{22} & 0 \\ 0 & C_{11} \\ A_{22} & 0 \\ 0 & A_{11} \end{pmatrix} + r(D_{22}, F_{22}D_{11}),$$

$$r\begin{pmatrix} F_{11} & G_{11}D_{11} \\ 0 & F_{22}D_{11} \\ C_{11} & E_{11} \\ A_{11} & B_{11}D_{11} \end{pmatrix} = r\begin{pmatrix} F_{11} \\ C_{11} \\ A_{11} \end{pmatrix} + r(F_{22}D_{11}),$$

$$r\begin{pmatrix} G_{11} & F_{11} & 0 & 0 & 0 & H_{11} & 0 \\ F_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -G_{11}D_{11} & H_{11} & F_{11} & H_{11} & 0 \\ 0 & 0 & F_{22}D_{11} & 0 & 0 & 0 & B_{22} \\ B_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{22} & 0 & 0 & E_{22} \\ 0 & 0 & -E_{11} & 0 & C_{11} & 0 & 0 \\ 0 & 0 & -B_{22}F_{22}D_{11} & A_{22} & 0 & 0 & 0 \\ 0 & 0 & -B_{11}D_{11} & 0 & A_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{22} & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} F_{22} & 0 & 0 \\ 0 & D_{22} & F_{22}D_{11} \end{pmatrix} + r\begin{pmatrix} F_{11} & 0 & 0 & H_{11} \\ 0 & H_{11} & F_{11} & H_{11} \\ 0 & C_{22} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & C_{11} & 0 \\ 0 & 0 & A_{11} & 0 \\ A_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{22} \end{pmatrix}.$$

Under these conditions, the general solution of System (1.3) *is*

$$X_1 = A_{11}^{\dagger}B_{11} + L_{A_{11}}A_1^{\dagger}P_1D_{11}^{\dagger} + L_{A_{11}}L_{A_1}W_1 + L_{A_{11}}W_2R_{D_{11}},$$
$$X_2 = A_{22}^{\dagger}B_4 + L_{A_{22}}A_2^{\dagger}P_2D_{22}^{\dagger} + L_{A_{22}}L_{A_2}W_3 + L_{A_{22}}W_4R_{D_{22}},$$

where

$$\begin{aligned} W_1 &= \hat{A_1}^{\dagger}(B - \hat{A_2}W_1\hat{B_2} - \hat{A_{11}}W_3F_{22} - \hat{A_{22}}W_4\hat{B_4}) + L_{\hat{A_1}}U_1, \\ W_2 &= A_{12}^{\dagger}T\hat{B_2}^{\dagger} - A_{12}^{\dagger}A_{13}M_1^{\dagger}T\hat{B_2}^{\dagger} - A_{12}^{\dagger}S_1A_{13}^{\dagger}TN_1^{\dagger}F_{22}\hat{B_2}^{\dagger} \\ &\quad - A_{12}^{\dagger}S_1U_2R_{N_1}F_{22}\hat{B_2}^{\dagger} + L_{A_{12}}U_3 + U_4R_{\hat{B_2}}, \\ W_3 &= M_1^{\dagger}TF_{22}^{\dagger} + S_1^{\dagger}S_1A_{13}^{\dagger}TN_1^{\dagger} + L_{M_1}L_{S_1}U_5 + U_6R_{F_{22}} + L_{M_1}U_2R_{N_1}, \\ W_4 &= F_1 + L_{\hat{C_2}}V_1 + V_2R_{\hat{D_1}} + L_{\hat{C_1}}V_3R_{\hat{D_2}}, \ \textit{or} \ W_4 = F_2 - L_{\hat{C_{22}}}\hat{V}_1 - \hat{V}_2R_{\hat{D_{11}}} - L_{\hat{C_{11}}}\hat{V}_3R_{\hat{D_{22}}}, \end{aligned}$$

where $T = T_1 - \hat{A_{22}}W_4\hat{B_4}$, $U_i(i = \overline{1,6})$ *are arbitrary matrices with appropriate sizes over* $\mathbb{H}$,

$$
\begin{aligned}
V_1 &= (I_m,0)[C_{24}^{\dagger}(F - C_{21}V_3D_{12} - C_{33}\hat{V}_3D_{33}) - C_{24}^{\dagger}U_{11}D_{13} + L_{C_{24}}U_{12}],\\
\hat{V}_1 &= (0,I_m)[C_{24}^{\dagger}(F - C_{21}V_3D_{12} - C_{33}\hat{V}_3D_{33}) - C_{24}^{\dagger}U_{11}D_{13} + L_{C_{24}}U_{12}],\\
\hat{V}_2 &= [R_{C_{24}}(F - C_{21}V_3D_{12} - C_{33}\hat{V}_3D_{33})D_{13}^{\dagger} + C_{24}C_{24}^{\dagger}U_{11} + U_{21}R_{D_{13}}]\begin{pmatrix}0\\ I_n\end{pmatrix},\\
V_2 &= [R_{C_{24}}(F - C_{21}V_3D_{12} - C_{33}\hat{V}_3D_{33})D_{13}^{\dagger} + C_{24}C_{24}^{\dagger}U_{11} + U_{21}R_{D_{13}}]\begin{pmatrix}I_n\\ 0\end{pmatrix},\\
V_3 &= E_{11}^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}E_{22}M^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{44}E_{33}^{\dagger}\\
&\quad - E_{11}^{\dagger}SU_{31}R_NE_{44}E_{33}^{\dagger} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},\\
\hat{V}_3 &= M^{\dagger}FE_{44}^{\dagger} + S^{\dagger}SE_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{E_{44}},
\end{aligned}
$$

$U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$, and U_{42} are any matrices with appropriate sizes, and m and n denote the column number of H_{11} and the row number of D_{22}, respectively.

Theorem 4.3. *Consider the matrix equation* (1.4) *over $\mathbb{H}$, where $A_{ii}, B_{ii}, C_{ii}, D_{ii}, E_{ii}, F_{ii}(i = \overline{1,2})$, and G_{11} are given. Put*

$$
\begin{aligned}
&\hat{A}_1 = C_{11}L_{A_{11}}, P_1 = E_{11} - C_{11}A_{11}^{\dagger}B_{11}D_{11}, \hat{A}_2 = C_{22}L_{A_{22}}, P_2 = E_{22} - C_{22}A_{22}^{\dagger}B_{22}D_{22},\\
&A_5 = F_{11}L_{A_1}L_{\hat{A}_1}, A_6 = F_{11}L_{A_{11}}, A_7 = L_{A_{22}}L_{\hat{A}_2}, A_8 = L_{A_{22}}, B_5 = R_{D_{11}}, B_7 = R_{D_{22}}F_{22},\\
&B = G_{11} - F_{11}A_{11}^{\dagger}B_{11} - F_{11}L_{A_1}\hat{A}_1{}^{\dagger}P_1D_{11}^{\dagger} - A_{22}^{\dagger}B_{22}F_{22} - L_{A_{22}}\hat{A}_2{}^{\dagger}P_2D_{22}^{\dagger}F_{22},\\
&R_{A_5}A_6 = A_{11}, R_{A_5}A_7 = A_2, R_{A_5}A_8 = A_{33}, F_{22}L_{B_5} = N_1, R_{A_{11}}A_2 = M_1, S_1 = A_2L_{M_1},\\
&R_{A_5}B = T_1, C = R_{M_1}R_{A_{11}}, \hat{C}_1 = CA_{33}, \hat{C}_2 = R_{A_{11}}A_{33}, \hat{C_{11}} = R_{A_2}A_{33}, \hat{C}_4 = A_{33},\\
&D = L_{B_5}L_{N_1}, \hat{D}_1 = B_7, \hat{D}_2 = B_7L_{F_{22}}, \hat{D}_3 = B_7L_{B_5}, \hat{D}_4 = B_7D, \hat{E}_1 = CT_1, \hat{E}_2 = R_{A_{11}}T_1L_{F_{22}},\\
&\hat{E_{11}} = R_{A_2}T_1L_{B_5}, \hat{E}_4 = T_1D, C_1 = (L_{\hat{C}_2}, L_{\hat{C}_4}), D_{13} = \begin{pmatrix}R_{\hat{D}_1}\\ R_{\hat{D}_3}\end{pmatrix}, D_1 = L_{\hat{C}_1}, D_2 = R_{\hat{D}_2},\\
&C_{33} = L_{\hat{C_{11}}}, D_{33} = R_{\hat{D}_4}, E_{11} = R_{C_1}D_1, E_2 = R_{C_1}C_{33}, E_{33} = D_2L_{D_{13}}, E_{44} = D_{33}L_{D_{13}},\\
&M = R_{E_{11}}E_2, N = E_{44}L_{E_{33}}, F = \hat{F}_2 - \hat{F}_1, E = R_{C_1}FL_{D_{13}}, S = E_2L_M, F_{11} = \hat{C}_2L_{\hat{C}_1},\\
&\hat{G}_1 = \hat{E}_2 - \hat{C}_2\hat{C}_1{}^{\dagger}\hat{E}_1\hat{D}_1{}^{\dagger}\hat{D}_2, F_{33} = \hat{C}_4L_{\hat{C_{11}}}, \hat{G}_2 = \hat{E}_4 - \hat{C}_4\hat{C_{11}}{}^{\dagger}\hat{E_{11}}\hat{D}_3{}^{\dagger}\hat{D}_4,\\
&\hat{F}_1 = \hat{C}_1{}^{\dagger}\hat{E}_1\hat{D}_1{}^{\dagger} + L_{\hat{C}_1}\hat{C}_2{}^{\dagger}\hat{E}_2\hat{D}_2{}^{\dagger}, \hat{F}_2 = \hat{C_{11}}{}^{\dagger}\hat{E_{11}}\hat{D}_3{}^{\dagger} + L_{\hat{C_{11}}}\hat{C}_4{}^{\dagger}\hat{E}_4\hat{D}_4{}^{\dagger}.
\end{aligned}
$$

Then, the following statements are equivalent:

(1) *Equation* (1.4) *is consistent.*

(2)

$$
\begin{aligned}
&R_{A_{11}}B_{11} = 0, R_{\hat{A}_1}P_1 = 0, P_1L_{D_{11}} = 0, R_{A_{22}}B_{22} = 0,\\
&R_{\hat{A}_2}P_2 = 0, P_2L_{D_{22}} = 0,\ R_{\hat{C}_i}\hat{E}_i = 0, \hat{E}_iL_{\hat{D}_i} = 0(i = \overline{1,2}),\\
&R_{\hat{C_{11}}}\hat{E_{11}} = 0, R_{\hat{C}_4}\hat{E}_4 = 0, \hat{E_{11}}L_{\hat{D}_3} = 0, \hat{E}_4L_{\hat{D}_4} = 0, R_{E_{11}}EL_{E_{44}} = 0.
\end{aligned}
$$

(3)

$$r(B_{11},A_{11})=r(A_{11}),\ r\begin{pmatrix}E_{11} & C_{11}\\ B_{11}D_{11} & A_{11}\end{pmatrix}=r\begin{pmatrix}C_{11}\\ A_{11}\end{pmatrix},$$

$$r\begin{pmatrix}E_{11}\\ D_{11}\end{pmatrix}=r(D_{11}),\ \ r(B_{22},A_{22})=r(A_{22}),$$

$$r\begin{pmatrix}E_{22} & C_{22}\\ B_{22}D_{22} & A_{22}\end{pmatrix}=r\begin{pmatrix}C_{22}\\ A_{22}\end{pmatrix},\ r\begin{pmatrix}E_{22}\\ D_{22}\end{pmatrix}=r(D_{22}),$$

$$r\begin{pmatrix}B_{11} & A_{11}\\ A_{22}G_{11}-B_{22}F_{22} & A_{22}F_{11}\end{pmatrix}=r\begin{pmatrix}A_{11}\\ A_{22}F_{11}\end{pmatrix},$$

$$r\begin{pmatrix}F_{22} & 0\\ B_{11} & A_{11}\\ A_{22}G_{11} & A_{22}F_{11}\end{pmatrix}=r(F_{22})+r\begin{pmatrix}A_{11}\\ A_{22}F_{11}\end{pmatrix},$$

$$r\begin{pmatrix}C_{11} & E_{11}\\ A_{11} & B_{11}D_{11}\\ -A_{22}F_{11} & B_{22}F_{22}D_{11}-A_{22}G_{11}D_{11}\end{pmatrix}=r\begin{pmatrix}C_{11}\\ A_{11}\\ A_{22}F_{11}\end{pmatrix},$$

$$r\begin{pmatrix}0 & F_{22}D_{11}\\ C_{11} & E_{11}\\ A_{11} & B_{11}D_{11}\\ A_{22}F_{11} & A_{22}G_{11}D_{11}\end{pmatrix}=r\begin{pmatrix}C_{11}\\ A_{11}\\ A_{22}F_{11}\end{pmatrix}+r(F_{22}D_{11}),$$

$$r\begin{pmatrix}F_{22} & 0 & D_{22}\\ C_{22}G_{11} & C_{22}F_{11} & E_{22}\\ B_{11} & A_{11} & 0\\ A_{22}G_{11} & A_{22}F_{11} & B_{22}D_{22}\end{pmatrix}=r(F_{22},D_{22})+r\begin{pmatrix}C_{22}F_{11}\\ A_{22}F_{11}\\ A_{11}\end{pmatrix},$$

$$r\begin{pmatrix}G_{11} & F_{11}\\ F_{22} & 0\\ B_{11} & A_{11}\end{pmatrix}=r\begin{pmatrix}F_{11}\\ A_{11}\end{pmatrix}+r(F_{22}),$$

$$r\begin{pmatrix}0 & D_{22} & F_{22}D_{11}\\ C_{22}F_{11} & E_{22} & C_{22}G_{11}D_{11}\\ C_{11} & 0 & E_{22}\\ A_{22}F_{11} & 0 & A_{22}G_{11}D_{11}-B_{22}F_{22}D_{11}\\ A_{11} & 0 & B_{11}D_{11}\end{pmatrix}$$

$$=r\begin{pmatrix}C_{22}F_{11}\\ C_{11}\\ A_{22}F_{11}\\ A_{11}\end{pmatrix}+r(D_{22},F_{22}D_{11}),$$

$$r\begin{pmatrix}F_{11} & G_{11}D_{11}\\ 0 & F_{22}D_{11}\\ C_{11} & E_{11}\\ A_{11} & B_{11}D_{11}\end{pmatrix}=r\begin{pmatrix}F_{11}\\ C_{11}\\ A_{11}\end{pmatrix}+r(F_{22}D_{11}),$$

$$
r\begin{pmatrix}
F_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & F_{22}D_{11} & 0 & B_{22} \\
B_{11} & A_{11} & 0 & 0 & 0 \\
C_{22}G_{11} & C_{22}F_{11} & C_{22}G_{11}D_{11} & -C_{22}F_{11} & E_{22} \\
0 & 0 & -E_{11} & C_{11} & 0 \\
A_{22}G_{11} & A_{22}F_{11} & A_{22}G_{11}D_{11}-B_{22}F_{22}D_{11} & -A_{22}F_{11} & 0 \\
0 & 0 & -B_{11}D_{11} & A_{11} & 0 \\
A_{22}G_{11} & A_{22}F_{11} & 0 & 0 & 0
\end{pmatrix}
$$

$$
= r\begin{pmatrix}
F_{22} & 0 & 0 \\
0 & F_{22}D_{11} & D_{22}
\end{pmatrix} + r\begin{pmatrix}
-C_{22}F_{11} & C_{22}F_{11} \\
-A_{22}F_{11} & A_{22}F_{11} \\
0 & C_{11} \\
0 & A_{11} \\
A_{11} & 0 \\
A_{11} & 0 \\
-A_{22}F_{11} & 0
\end{pmatrix}.
$$

Under these conditions, the general solution of System (1.4) *is*

$$
X_1 = A_{11}^{\dagger}B_{11} + L_{A_1}\hat{A_1}^{\dagger}P_1\hat{B_1}^{\dagger} + L_{A_1}L_{\hat{A_1}}W_1 + L_{A_1}W_2R_{\hat{B_1}},
$$
$$
X_2 = A_2^{\dagger}B_{22} + L_{A_2}\hat{A_2}^{\dagger}P_2\hat{B_2}^{\dagger} + L_{A_2}L_{\hat{A_2}}W_3 + L_{A_3}W_4R_{\hat{B_2}},
$$

where

$$
\begin{aligned}
W_1 &= A_5^{\dagger}(B - A_6W_1B_5 - A_7W_3F_{22} - A_8W_4B_7) + L_{A_5}U_1,\\
W_2 &= A_1^{\dagger}TB_5^{\dagger} - A_1^{\dagger}A_2M_1^{\dagger}TB_5^{\dagger} - A_1^{\dagger}S_1A_2^{\dagger}TN_1^{\dagger}F_{22}B_5^{\dagger}\\
&\quad - A_1^{\dagger}S_1U_2R_{N_1}F_{22}B_5^{\dagger} + L_{A_1}U_3 + U_4R_{B_5},\\
W_3 &= M_1^{\dagger}TF_{22}^{\dagger} + S_1^{\dagger}S_1A_2^{\dagger}TN_1^{\dagger} + L_{M_1}L_{S_1}U_5 + U_6R_{F_{22}} + L_{M_1}U_2R_{N_1},\\
W_4 &= \hat{F}_1 + L_{\hat{C}_2}V_1 + V_2R_{\hat{D}_1} + L_{\hat{C}_1}V_3R_{\hat{D}_2},\ or\ W_4 = \hat{F}_2 - L_{\hat{C}_4}\hat{V}_1 - \hat{V}_2R_{\hat{D}_3} - L_{\hat{C_{11}}}\hat{V}_3R_{\hat{D}_4},
\end{aligned}
$$

where $T = T_1 - A_8W_4B_7, U_i(i = \overline{1,6})$ *are arbitrary matrices with appropriate sizes over* $\mathbb{H}$*,*

$$
\begin{aligned}
V_1 &= (I_m, 0)[C_1^{\dagger}(F - D_1V_3D_2 - C_{33}\hat{V}_3D_{33}) - C_1^{\dagger}U_{11}D_1 + L_{C_1}U_{12}],\\
\hat{V}_1 &= (0, I_m)[C_1^{\dagger}(F - D_1V_3D_2 - C_{33}\hat{V}_3D_{33}) - C_1^{\dagger}U_{11}D_1 + L_{C_1}U_{12}],\\
\hat{V}_2 &= [R_{C_1}(F - D_1V_3D_2 - C_{33}\hat{V}_3D_{33})D_1^{\dagger} + C_1C_1^{\dagger}U_{11} + U_{21}R_{D_1}]\begin{pmatrix} 0 \\ I_n \end{pmatrix},\\
V_2 &= [R_{C_1}(F - C_2V_3D_2 - C_{33}\hat{V}_3D_{33})D_1^{\dagger} + C_1C_1^{\dagger}U_{11} + U_{21}R_{D_1}]\begin{pmatrix} I_n \\ 0 \end{pmatrix},\\
V_3 &= E_{11}^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}E_2M^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}SE_2^{\dagger}FN^{\dagger}E_{44}E_{33}^{\dagger}\\
&\quad - E_{11}^{\dagger}SU_{31}R_NE_{44}E_{33}^{\dagger} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},\\
\hat{V}_3 &= M^{\dagger}FE_{44}^{\dagger} + S^{\dagger}SE_2^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{E_{44}},
\end{aligned}
$$

$U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$, *and* U_{42} *are any quaternion matrices with appropriate sizes, and* m *and* n *denote the column number of* A_{22} *and the row number of* D_{22}*, respectively.*

5. An application of the system (1.1)

In this section, we use the Lemma 2.2 and the Theorem 3.1 to study matrix equation (1.9) involving η-Hermicity matrices.

Theorem 5.1. *Let* $A_{11}, B_{11}, C_{11}, E_{11}, F_{11}, F_{22}$, *and* $G_{11}(G_{11} = G_{11}^{\eta^*})$ *be given matrices. Put*

$$\begin{aligned}
&A_1 = C_{11}L_{A_{11}}, P_1 = E_{11} - C_{11}A_{11}^{\dagger}B_{11}C_{11}^{\eta^*}, B_2 = A_1^{\eta^*}, P_2 = P_1^{\eta^*}, \hat{B}_1 = R_{B_2}\left(F_{22}L_{A_{11}}\right)^{\eta^*},\\
&\hat{A}_3 = F_{11}L_{A_{11}}, \hat{A}_2 = \hat{A}_3L_{A_1}, \hat{A}_4 = L_{C_{11}}, \hat{B}_3 = \left(F_{11}\hat{A}_4\right)^{\eta^*}, \hat{B}_4 = \left(F_{22}L_{A_{11}}\right)^{\eta^*}, F_{11}^{\eta^*}L_{\hat{B}_1} = \hat{B}_{11},\\
&P = G_{11} - F_{11}A_{11}^{\dagger}B_{11}F_{11}^{\eta^*} - \hat{A}_3A_1^{\dagger}P_1\left(F_{11}C_{11}^{\dagger}\right)^{\eta^*} - \left(F_{22}A_{11}^{\dagger}B_{11}\right)^{\eta^*} - C_{11}^{\dagger}P_2B_2^{\dagger}\hat{B}_4, \hat{B}_{22}L_{B_{11}} = N_1,\\
&\hat{B}_3L_{\hat{B}_1} = \hat{B}_{22}, \hat{B}_4L_{\hat{B}_1} = \hat{B}_{33}, R_{\hat{A}_2}\hat{A}_3 = \hat{M}_1, S_1 = \hat{A}_3L_{M_1}, T_1 = PL_{\hat{B}_1}, C = R_{M_1}R_{\hat{A}_2}, C_1 = C\hat{A}_4,\\
&C_2 = R_{\hat{A}_2}\hat{A}_4, C_3 = R_{\hat{A}_3}\hat{A}_4, C_4 = \hat{A}_4, D = L_{\hat{B}_{11}}L_{N_1}, D_1 = \hat{B}_{33}, D_2 = \hat{B}_{33}L_{\hat{B}_{22}}, D_4 = \hat{B}_{33}D,\\
&E_1 = CT_1, E_2 = R_{\hat{A}_2}T_1L_{\hat{B}_{11}}, E_4 = T_1D, \hat{C}_{11} = \left(L_{C_2}, L_{C_4}\right), D_3 = \hat{B}_{33}L_{\hat{B}_{11}}, \hat{D}_{11} = \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix},\\
&\hat{C}_{22} = L_{C_1}, \hat{D}_{22} = R_{D_2},\ \hat{C}_{33} = L_{C_3}, \hat{D}_{33} = R_{D_4}, \hat{E}_{11} = R_{\hat{C}_{11}}\hat{C}_{22}, \hat{E}_{22} = R_{\hat{C}_{11}}\hat{C}_{33},\\
&\hat{E}_{33} = \hat{D}_{22}L_{\hat{D}_{11}}, \hat{E}_{44} = \hat{D}_{33}L_{\hat{D}_{11}}, M = R_{\hat{E}_{11}}\hat{E}_{22}, N = \hat{E}_{44}L_{\hat{E}_{33}},\ F = F_2 - F_1, E = R_{\hat{C}_{11}}FL_{\hat{D}_{11}},\\
&S = \hat{E}_{22}L_M, \hat{F_{11}} = C_2L_{C_1}, G_1 = E_2 - C_2C_1^{\dagger}E_1D_1^{\dagger}D_2, \hat{F_{22}} = C_4L_{C_3}, G_2 = E_4 - C_4C_3^{\dagger}E_3D_3^{\dagger}D_4,\\
&F_1 = C_1^{\dagger}E_1D_1^{\dagger} + L_{C_1}^{\dagger}C_2^{\dagger}E_2D_2^{\dagger}, F_2 = C_3^{\dagger}E_3D_3^{\dagger} + L_{C_3}C_4^{\dagger}E_4D_4^{\dagger}.
\end{aligned}$$

Then, the following statements are equivalent:
(1) *System* (1.9) *is solvable.*
(2)

$$R_{A_{11}}B_{11} = 0, R_{A_1}P_1 = 0, P_1\left(R_{C_{11}}\right)^{\eta^*} = 0, R_{C_i}E_i = 0, E_iL_{D_i} = 0 (i = \overline{1,4}), R_{\hat{E}_{11}}EL_{\hat{E}_{44}} = 0.$$

(3)

$$\begin{aligned}
&r(B_{11}, A_{11}) = r(A_{11}), r\begin{pmatrix} E_{11} & C_{11} \\ B_{11}C_{11}^{\eta^*} & A_{11} \end{pmatrix} = r\begin{pmatrix} C_{11} \\ A_{11} \end{pmatrix},\ r\begin{pmatrix} E_{11} \\ C_{11}^{\eta^*} \end{pmatrix} = r(C_{11}),\\
&r\begin{pmatrix} F_{22}^{\eta^*} & 0 & C_{11}^{\eta^*} & A_{11}^{\eta^{\eta^*}} \\ B_{11}F_{11}^{\eta^*} & A_{11} & 0 & 0 \\ C_{11}G_{11} & C_{11}F_{11} & E_{11}^{\eta^*} & C_{11}B_{11}^{\eta^*} \end{pmatrix} = r\left(F_{22}^{\eta^*}, C_{11}^{\eta^*}, A_{11}^{\eta^*}\right) + r\begin{pmatrix} A_{11} \\ C_{11}F_{11} \end{pmatrix},\\
&r\begin{pmatrix} F_{11}^{\eta^*} & 0 & -C_{11}^{\eta^*} & 0 & 0 \\ F_{22}^{\eta^*} & 0 & 0 & C_{11}^{\eta^*} & A_{11}^{\eta^*} \\ 0 & C_{11} & E_{11} & 0 & 0 \\ 0 & A_{11} & B_{11}C_{11}^{\eta^*} & 0 & 0 \\ C_{11}G_{11} & C_{11}F_{11} & 0 & E_{11}^{\eta^*} & C_{11}B_{11}^{\eta^*} \end{pmatrix} =\\
&r\begin{pmatrix} C_{11} \\ A_{11} \\ 0 \end{pmatrix} + r\begin{pmatrix} F_{11}^{\eta^*} & C_{11}^{\eta^*} & 0 & 0 \\ F_{22}^{\eta^*} & 0 & C_{11}^{\eta^*} & A_{11}^{\eta^*} \end{pmatrix},
\end{aligned}$$

$$
r\begin{pmatrix} F_{11}^{\eta^*} & 0 & 0 & 0 \\ F_{22}^{\eta^*} & 0 & C_{11}^{\eta^*} & A_{11}^{\eta^*} \\ 0 & A_{11} & 0 & 0 \\ C_{11}G_{11} & C_{11}F_{11} & E_{11}^{\eta^*} & C_{11}B_{11}^{\eta^*} \end{pmatrix} = r\begin{pmatrix} F_{11}^{\eta^*} & 0 & 0 \\ F_{22}^{\eta^*} & C_{11}^{\eta^*} & A_{11}^{\eta^*} \end{pmatrix} + r\begin{pmatrix} A_{11} \\ C_{11}F_{11} \end{pmatrix},
$$

$$
r\begin{pmatrix} F_{11}^{\eta^*} & 0 & 0 \\ F_{22}^{\eta^*} & C_{11}^{\eta^*} & A_{11}^{\eta^*} \\ C_{11}G_{11} & E_{11}^{\eta^*} & C_{11}B_{11}^{\eta^*} \end{pmatrix} = r\begin{pmatrix} F_{11}^{\eta^*} & 0 & 0 \\ F_{22}^{\eta^*} & C_{11}^{\eta^*} & A_{11}^{\eta^*}, \end{pmatrix},
$$

$$
r\begin{pmatrix} G_{11} & F_{11} & B_{11}^{\eta^*} \\ F_{22}^{\eta^*} & 0 & A_{11}^{\eta^*} \\ B_{11}F_{11}^{\eta^*} & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} F_{11} \\ A_{11} \end{pmatrix} + r\left(F_{22}^{\eta^*}, A_{11}^{\eta^*}\right),
$$

$$
r\begin{pmatrix} G_{11} & F_{11} & 0 & B_{11}^{\eta^*} \\ F_{11}^{\eta^*} & 0 & -C_{11}^{\eta^*} & 0 \\ F_{22}^{\eta^*} & 0 & 0 & A_{11}^{\eta^*} \\ 0 & C_{11} & E_{11} & 0 \\ 0 & A_{11} & B_{11}C_{11}^{\eta^*} & 0 \end{pmatrix} = r\begin{pmatrix} F_{11}^{\eta^*} & C_{11}^{\eta^*} & 0 \\ F_{22}^{\eta^*} & 0 & A_{11}^{\eta^*} \end{pmatrix} + r\begin{pmatrix} F_{11} \\ C_{11} \\ A_{11} \end{pmatrix},
$$

$$
r\begin{pmatrix} G_{11} & F_{11} & B_{11}^{\eta^*} \\ F_{11}^{\eta^*} & 0 & 0 \\ F_{22}^{\eta^*} & 0 & A_{11}^{\eta^*} \\ 0 & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} F_{11}^{\eta^*} & 0 \\ F_{22}^{\eta^*} & A_{11}^{\eta^*} \end{pmatrix} + r\begin{pmatrix} F_{11} \\ A_{11} \end{pmatrix},
$$

$$
r\begin{pmatrix} G_{11} & B_{11}^{\eta^*} \\ F_{11}^{\eta^*} & 0 \\ F_{22}^{\eta^*} & A_{11}^{\eta^*} \end{pmatrix} = r\begin{pmatrix} F_{11}^{\eta^*} & 0 \\ F_{22}^{\eta^*} & A_{11}^{\eta^*} \end{pmatrix},
$$

$$
r\begin{pmatrix} F_{11}^{\eta^*} & 0 & 0 & 0 & 0 & 0 & 0 & C_{11}^{\eta^*} & 0 \\ F_{22}^{\eta^*} & 0 & 0 & 0 & 0 & C_{11}^{\eta^*} & A_{11}^{\eta^*} & 0 & 0 \\ 0 & 0 & F_{11}^{\eta^*} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F_{22}^{\eta^*} & C_{11}^{\eta^*} & A_{11}^{\eta^*} & 0 & 0 & 0 & 0 \\ F_{22}^{\eta^*} & 0 & F_{22}^{\eta^*} & 0 & 0 & 0 & 0 & 0 & A_{11}^{\eta^*} \\ 0 & C_{11} & 0 & 0 & 0 & 0 & 0 & -E_{11} & 0 \\ 0 & A_{11} & 0 & 0 & 0 & 0 & 0 & -B_{11}C_{11}^{\eta^*} & 0 \\ C_{11}G_{11} & C_{11}F_{11} & 0 & 0 & 0 & E_{11}^{\eta^*} & C_{11}B_{11}^{\eta^*} & 0 & 0 \end{pmatrix}
$$

$$
= r\begin{pmatrix} F_{11}^{\eta^*} & 0 & 0 & 0 & 0 & 0 & C_{11}^{\eta^*} & 0 \\ F_{22}^{\eta^*} & 0 & 0 & 0 & C_{11}^{\eta^*} & A_{11}^{\eta^*} & 0 & 0 \\ 0 & F_{11}^{\eta^*} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{22}^{\eta^*} & C_{11}^{\eta^*} & A_{11}^{\eta^*} & 0 & 0 & 0 & 0 \\ F_{22}^{\eta^*} & F_{22}^{\eta^*} & 0 & 0 & 0 & 0 & 0 & A_{11}^{\eta^*} \end{pmatrix} + r\begin{pmatrix} C_{11} \\ A_{11} \\ C_{11}F_{11} \end{pmatrix}.
$$

Proof. Evidently, the system of Eq (1.9) has a solution if and only if the following matrix equation has a solution:

$$\begin{aligned}
&A_{11}\hat{X}_1 = B_{11}, C_{11}\hat{X}_1C_{11}^{\eta^*} = E_{11},\\
&\hat{X}_2A_{11}^{\eta^*} = B_{11}^{\eta^*}, C_{11}\hat{X}_2C_{11}^{\eta^*} = E_{11}^{\eta^*},\\
&F_{11}X_1F_{11}^{\eta^*} + \hat{X}_2^{\eta^*}F_{22}^{\eta^*} = G_{11}.
\end{aligned} \tag{5.1}$$

If (1.9) has a solution, say, X_1, then $(\hat{X}_1,\ \hat{X}_2) := (X_1,\ X_1^{\eta^*})$ is a solution of (5.1). Conversely, if (5.1) has a solution, say $(\hat{X}_1,\ \hat{X}_2)$, then it is easy to show that (1.5) has a solution

$$X_1 := \frac{\hat{X}_1 + X_2^{\eta^*}}{2}.$$

According to Theorem 3.1, we can deduce that this theorem holds. □

6. Conclusions

We have established the solvability conditions and the expression of the general solutions to some constrained systems (1.1)–(1.4). As an application, we have investigated some necessary and sufficient conditions for Eq (1.9) to be consistent. It should be noted that the results of this paper are valid for the real number field and the complex number field as special number fields.

Author contributions

Long-Sheng Liu, Shuo Zhang and Hai-Xia Chang: Conceptualization, formal analysis, investigation, methodology, software, validation, writing an original draft, writing a review, and editing. All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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