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Research article

Advanced Hardy-type inequalities with negative parameters involving monotone functions in delta calculus on time scales

Ahmed M. Ahmed[1], **Ahmed I. Saied**[2], **Mohammed Zakarya**[3], **Amirah Ayidh I Al-Thaqfan**[4], **Maha Ali**[4] **and Haytham M. Rezk**[1,*]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Egypt

[2] Department of Mathematics, Faculty of Science, Benha University, Benha 13511, Egypt

[3] Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia

[4] Department of Mathematics, College of Arts and Sciences, King Khalid University, P.O. Box 64512, Abha 62529, Sarat Ubaidah, Saudi Arabia

* **Correspondence:** Email: haythamrezk@azhar.edu.eg.

Abstract: In this study, we introduced several novel Hardy-type inequalities with negative parameters for monotone functions within the framework of delta calculus on time scales $\mathbb{T}$. As an application, when $\mathbb{T} = \mathbb{N}_0$, we derived discrete inequalities with negative parameters for monotone sequences, offering fundamentally new results. When $\mathbb{T} = \mathbb{R}$, we established continuous analogues of inequalities that have appeared in previous literature. Additionally, we presented inequalities for other time scales, such as $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, which, to the best of the authors' knowledge, represented largely novel contributions.

Keywords: Hardy's inequality; negative parameter; monotone functions; Hölder's inequality; weighted inequalities; delta calculus; time scales
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1. Introduction

In [1], Hardy established a foundational result in the theory of inequalities with positive parameters, demonstrating the discrete inequality:

$$\sum_{s=1}^{\infty}\left(\frac{1}{s}\sum_{\kappa=1}^{s}\mathcal{E}(\kappa)\right)^{\gamma} \leq \left(\frac{\gamma}{\gamma-1}\right)^{\gamma}\sum_{s=1}^{\infty}\mathcal{E}^{\gamma}(s), \tag{1.1}$$

where $\gamma > 1$, $\mathcal{E}(s) \geq 0$ for $s \geq 1$, and $0 < \sum_{s=1}^{\infty} \mathcal{E}^{\gamma}(s) < \infty$. Hardy [2, Theorem A] also derived the corresponding integral inequality of (1.1):

$$\int_0^{\infty} \left(\frac{1}{\mathfrak{r}} \int_0^{\mathfrak{r}} \mho(\mathfrak{z}) d\mathfrak{z} \right)^{\gamma} d\mathfrak{r} \leq \left(\frac{\gamma}{\gamma - 1} \right)^{\gamma} \int_0^{\infty} \mho^{\gamma}(\mathfrak{r}) d\mathfrak{r}, \tag{1.2}$$

where $\gamma > 1$, and $\mho(\mathfrak{r}) \geq 0$ such that $0 < \int_0^{\infty} \mho^{\gamma}(\mathfrak{r}) d\mathfrak{r} < \infty$. The constant $(\gamma/(\gamma-1))^{\gamma}$ is optimal in both inequalities.

Since the emergence of these two inequalities (1.1) and (1.2), they have garnered significant attention from scientists and researchers, with many of them working on improving and generalizing them using various methods (see [3–5]). In parallel to advancements in positive parameter inequalities, there has been a growing interest in Hardy-type inequalities with negative parameters. For example, Bicheng [6] demonstrated that if $\gamma < 0$, $\varrho \in \mathbb{R}$, $\varrho \neq 1$, $\mho(\mathfrak{r}) \geq 0$, and $0 < \int_0^{\infty} \mathfrak{r}^{-\varrho} (\mathfrak{r}\mho(\mathfrak{r}))^{\gamma} d\mathfrak{r} < \infty$, then

$$\int_0^{\infty} \mathfrak{r}^{-\varrho} \left(\int_{\mathfrak{r}}^{\infty} \mho(\mathfrak{z}) d\mathfrak{z} \right)^{\gamma} d\mathfrak{r} \leq \left(\frac{\gamma}{1 - \varrho} \right)^{\gamma} \int_0^{\infty} \mathfrak{r}^{-\varrho} (\mathfrak{r}\mho(\mathfrak{r}))^{\gamma} d\mathfrak{r}; \quad \varrho > 1. \tag{1.3}$$

He also established that if $\varrho < 1$, then

$$\int_0^{\infty} \mathfrak{r}^{-\varrho} \left(\int_0^{\mathfrak{r}} \mho(\mathfrak{z}) d\mathfrak{z} \right)^{\gamma} d\mathfrak{r} \leq \left(\frac{\gamma}{\varrho - 1} \right)^{\gamma} \int_0^{\infty} \mathfrak{r}^{-\varrho} (\mathfrak{r}\mho(\mathfrak{r}))^{\gamma} d\mathfrak{r}, \tag{1.4}$$

where $(\gamma/(1-\varrho))^{\gamma}$ and $(\gamma/(\varrho-1))^{\gamma}$ is optimal in both inequalities (1.3) and (1.4).

Further advancements are presented in [7], where the authors extended these results. They showed that if $\gamma < 0$, $\varrho > 1$, and $\mho(\mathfrak{r}), \mathfrak{I}(\mathfrak{r}) > 0$ such that $\mathfrak{r}/\mathfrak{I}(\mathfrak{r})$ is a nondecreasing function, then

$$\int_0^{\infty} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_{\mathfrak{r}}^{\infty} \mho(\mathfrak{z}) d\mathfrak{z} \right)^{\gamma} d\mathfrak{r} \leq \left(\frac{\gamma}{1 - \varrho} \right)^{\gamma} \int_0^{\infty} (\mathfrak{r}\mho(\mathfrak{r}))^{\gamma} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} d\mathfrak{r}. \tag{1.5}$$

Additionally, if $0 \leq \varrho < 1$, and $\mho(\mathfrak{r}), \mathfrak{I}(\mathfrak{r}) > 0$ such that $\mathfrak{r}/\mathfrak{I}(\mathfrak{r})$ is a nonincreasing function, then

$$\int_0^{\infty} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_0^{\mathfrak{r}} \mho(\mathfrak{z}) d\mathfrak{z} \right)^{\gamma} d\mathfrak{r} \leq \left(\frac{\gamma}{\varrho - 1} \right)^{\gamma} \int_0^{\infty} (\mathfrak{r}\mho(\mathfrak{r}))^{\gamma} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} d\mathfrak{r}. \tag{1.6}$$

Moreover, if $\gamma < 0$, $\varrho < 0$, and $\mho(\mathfrak{r}), \mathfrak{I}(\mathfrak{r}) > 0$ such that $\mathfrak{r}/\mathfrak{I}(\mathfrak{r})$ is a nondecreasing function, then

$$\int_0^{\infty} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_0^{\mathfrak{r}} \mho(\mathfrak{z}) d\mathfrak{z} \right)^{\gamma} d\mathfrak{r} \leq \left(\frac{\gamma}{\varrho - 1} \right)^{\gamma} \int_0^{\infty} (\mathfrak{r}\mho(\mathfrak{r}))^{\gamma} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} d\mathfrak{r}. \tag{1.7}$$

The transition from Hardy's inequalities with positive parameters to those involving negative parameters illustrates a rich field of study, revealing a deeper structure and broader applicability of these mathematical tools. These developments highlight the ongoing evolution in the theory of Hardy-type inequalities, encompassing both positive and negative parameter cases and their various generalizations.

More recently, many scientists have used the famous theory known as time scale theory to study various classical inequalities, especially the famous Hardy inequality.

Among these scientists was P. Řehák [8], who was able to obtain the time scale form of Hardy's inequality, making discrete inequality (1.1) and integral version (1.2) a special case of it. He proved that if $\mathbb{T}$ is a time scale, $\varrho > 1$, and $\Omega(z) = \int_d^z \xi(\delta)\Delta\delta$, for $z \in [d, \infty)_{\mathbb{T}}$, then

$$\int_d^\infty \left(\frac{\Omega^\sigma(z)}{\sigma(z) - d}\right)^\varrho \Delta z < \left(\frac{\varrho}{\varrho - 1}\right)^\varrho \int_d^\infty \xi^\varrho(z)\Delta z, \tag{1.8}$$

unless $\xi \equiv 0$. If, in addition, $\mu(z)/z \to 0$ as $z \to \infty$, then $\left(\frac{\varrho}{\varrho-1}\right)^\varrho$ is sharp. Refer to Section 2 for the notations used here and for the calculus applied in proving the main results of this paper.

In [9], the authors presented a time scale form of (1.5)–(1.7) by using nabla calculus, respectively, as follows: Let $b \in \mathbb{T}$, $\varepsilon < 0$, $\varepsilon^* = \varepsilon/(\varepsilon - 1)$, $\varrho > 1$ and $\mho, \Im \in C_{ld}([b, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ with $(\zeta - b)/\Im(\zeta)$ is nondecreasing. If

$$\frac{\rho(\zeta) - b}{\zeta - b} \geq \frac{1}{k}, \text{ such that } \rho(\zeta) > b \text{ and } k \text{ is a positive constant}, \tag{1.9}$$

then

$$\int_b^\infty [\Im(\zeta)]^{-\varrho} [\mathcal{G}(\zeta)]^\varepsilon \nabla\zeta \leq \mathcal{Q} \int_b^\infty (\rho(\zeta) - b)^\varepsilon [\mho(\zeta)]^\varepsilon [\Im(\zeta)]^{-\varrho} \nabla\zeta, \tag{1.10}$$

where $\mathcal{G}(\zeta) = \int_\zeta^\infty \mho(\mathfrak{z})\nabla\mathfrak{z}$ and

$$\mathcal{Q} = \begin{cases} \left(\frac{\varepsilon}{1-\varrho}\right)^\varepsilon k^{\frac{\varrho-1}{\varepsilon^*}}, & 1 - \varrho \leq \varepsilon; \\ \left(\frac{\varepsilon}{1-\varrho}\right)^\varepsilon k^\varrho, & 1 - \varrho \geq \varepsilon. \end{cases}$$

Additionally, if $b \in \mathbb{T}$, $\varepsilon < 0$, $0 \leq \varrho < 1$, and $\mho, \Im \in C_{ld}([b, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that $(\zeta - b)/\Im(\zeta)$ is a nonincreasing function. If (1.9) holds, then

$$\int_b^\infty [\Im(\zeta)]^{-\varrho} [\mathcal{M}(\zeta)]^\varepsilon \nabla\zeta \leq \mathcal{D} \int_b^\infty (\rho(\zeta) - b)^\varepsilon [\mho(\zeta)]^\varepsilon [\Im(\zeta)]^{-\varrho} \nabla\zeta, \tag{1.11}$$

where $\mathcal{M}(\zeta) = \int_b^\zeta \mho(\mathfrak{z})\nabla\mathfrak{z}$ and

$$\mathcal{D} = \begin{cases} \left(\frac{\varepsilon}{\varrho-1}\right)^\varepsilon k^{\frac{\varrho-1}{\varepsilon}}, & (\varrho - 1)/\varepsilon \geq 1; \\ \left(\frac{\varepsilon}{\varrho-1}\right)^\varepsilon k^\varrho, & (\varrho - 1)/\varepsilon \leq 1. \end{cases}$$

Moreover, if $b \in \mathbb{T}$, $\varepsilon < 0$, $\varrho < 0$, and $\mho, \Im \in C_{ld}([b, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that $(\zeta - b)/\Im(\zeta)$ is a nondecreasing function. If (1.9) holds, then

$$\int_b^\infty [\Im(\zeta)]^{-\varrho} [\mathcal{M}(\zeta)]^\varepsilon \nabla\zeta \leq \mathcal{A} \int_b^\infty (\rho(\zeta) - b)^\varepsilon [\mho(\zeta)]^\varepsilon [\Im(\zeta)]^{-\varrho} \nabla\zeta, \tag{1.12}$$

where $\mathcal{M}(\zeta) = \int_b^\zeta \mho(\mathfrak{z})\nabla\mathfrak{z}$ and

$$\mathcal{A} = \begin{cases} \left(\frac{\varepsilon}{\varrho-1}\right)^\varepsilon, & (\varrho - 1)/\varepsilon \leq 1; \\ \left(\frac{\varepsilon}{\varrho-1}\right)^\varepsilon k^{\frac{\varrho-1}{\varepsilon}}, & (\varrho - 1)/\varepsilon \geq 1. \end{cases}$$

Recently, new results have emerged regarding the Hardy inequality through various types of time scale calculus, such as the time scale delta integral (see [10–13]), which broadens the applications of dynamic inequalities in studying the qualitative behavior of dynamic equations, as referenced in [14–16].

In fact, the study of Hardy's inequality with a negative parameter using the idea of time scale $\mathbb{T}$ has not been exposed to many researchers. Therefore, in this paper, we will attempt to obtain some new results in this area through time scale calculus. Specifically, we will prove a time scale version of (1.5)–(1.7) and also obtain the discrete analogues of these inequalities.

The organization of the paper is as follows. In Section 2, we present some lemmas on time scales. In Section 3, we state and prove our results.

2. Definitions and basic lemmas

In 2001, Martin and Allan [17, 18] introduced the concept of a time scale $\mathbb{T}$, which is defined as a nonempty closed subset of $\mathbb{R}$. For any $\varkappa, \varrho \in \mathbb{T}$, the forward jump operator is defined by $\sigma(\varkappa) := \inf\{s \in \mathbb{T} : s > \varkappa\}$ and the backward jump operator by $\rho(\varrho) := \sup\{s \in \mathbb{T} : s > \varrho\}$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(\tau) := \sigma(\tau) - \tau \geq 0$. A point $\zeta \in \mathbb{T}$ is called:

- **Right-dense** if $\sigma(\zeta) = \zeta$;
- **Left-dense** if $\rho(\zeta) = \zeta$;
- **Right-scattered** if $\sigma(\zeta) > \zeta$;
- **Left-scattered** if $\rho(\zeta) < \zeta$.

If $\mathbb{T}$ has a left-scattered maximum η, then $\mathbb{T}^k = \mathbb{T} - \{\eta\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

In the following, for a function $\mho : \mathbb{T} \to \mathbb{R}$, we denote $\mho(\sigma(\tau))$ as $\mho^\sigma(\tau)$. The notation $[\tau, \varrho] \cap \mathbb{T}$ is denoted as $[\tau, \varrho]_\mathbb{T}$.

Definitions:

- **Rd-continuous function [17]:** A function $\mho : \mathbb{T} \to \mathbb{R}$ is *rd*-continuous if it is continuous at right-dense points and has finite left-sided limits at left-dense points. The set of rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.
- **Delta derivative [17]:** For $\mho : \mathbb{T} \to \mathbb{R}$ and $\mathfrak{z} \in \mathbb{T}$, the delta derivative $\mho^\Delta(\mathfrak{z})$ exists if, for any $\varepsilon > 0$, there is a neighborhood $W = (\mathfrak{z} - \delta, \mathfrak{z} + \delta) \cap \mathbb{T}$ of $\mathfrak{z}$ for some $\delta > 0$, such that

$$|\mho^\sigma(\mathfrak{z}) - \mho(s) - \mho^\Delta(\mathfrak{z})(\sigma(\mathfrak{z}) - s)| \leq \varepsilon|\sigma(\mathfrak{z}) - s|, \quad \forall s \in U, \; s \neq \sigma(\mathfrak{z}).$$

- **Antiderivative and delta integral [17]:** A function $\mathcal{G} : \mathbb{T} \to \mathbb{R}$ is a delta antiderivative of $\mho$ if $\mathcal{G}^\Delta(\mathfrak{z}) = \mho(\mathfrak{z})$, $\forall \mathfrak{z} \in \mathbb{T}^k$. The delta integral of $\mho$ is given by

$$\int_\varrho^\tau \mho(\mathfrak{z})\Delta\mathfrak{z} = \mathcal{G}(\tau) - \mathcal{G}(\varrho), \quad \forall \varrho, \tau \in \mathbb{T}.$$

It is noted that every rd-continuous function $\mho$ has an antiderivative. In particular, if $\mathfrak{z}_0 \in \mathbb{T}$, then

$$\left(\int_{\mathfrak{z}_0}^{\mathfrak{z}} \mho(\tau)\Delta\tau\right)^\Delta = \mho(\mathfrak{z}), \quad \mathfrak{z} \in \mathbb{T}.$$

We now present the main lemmas on $\mathbb{T}$ that will be utilized to support our conclusions.

Main lemmas:

• **Chain rule [17, Theorem 1.90]**: If $\mathfrak{I}:\mathbb{R}\to\mathbb{R}$ is continuous, $\mathfrak{I}:\mathbb{T}\to\mathbb{R}$ is $\Delta-$differentiable, and $\mho:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, then

$$(\mho\circ\mathfrak{I})^{\Delta}(\mathfrak{z}) = \mho^{'}(\mathfrak{I}(d))\,\mathfrak{I}^{\Delta}(\mathfrak{z}),\quad d\in[\mathfrak{z},\sigma(\mathfrak{z})]. \tag{2.1}$$

• **Integration by Parts [19]:** For $\varrho,\tau\in\mathbb{T}$ and $\phi,\varphi\in C_{rd}([\varrho,\tau]_{\mathbb{T}},\mathbb{R})$,

$$\int_{\varrho}^{\tau}\phi(\mathfrak{z})\varphi^{\Delta}(\mathfrak{z})\Delta\mathfrak{z} = [\phi(\mathfrak{z})\varphi(\mathfrak{z})]_{\varrho}^{\tau} - \int_{\varrho}^{\tau}\phi^{\Delta}(\mathfrak{z})\varphi^{\sigma}(\mathfrak{z})\Delta\mathfrak{z}. \tag{2.2}$$

• **Reversed Hölder's inequality [19]:** For $\varrho,\tau\in\mathbb{T}$ and $\phi,\omega\in C_{rd}([\varrho,\tau]_{\mathbb{T}},\mathbb{R}^{+})$,

$$\int_{\varrho}^{\tau}\phi(\mathfrak{z})\omega(\mathfrak{z})\Delta\mathfrak{z} \geq \left[\int_{\varrho}^{\tau}\phi^{\gamma}(\mathfrak{z})\Delta\mathfrak{z}\right]^{\frac{1}{\gamma}}\left[\int_{\varrho}^{\tau}\omega^{\nu}(\mathfrak{z})\Delta\mathfrak{z}\right]^{\frac{1}{\nu}}, \tag{2.3}$$

where $\gamma<0$, and $1/\gamma+1/\nu=1$.

3. Main results

In this section, we present our key results. Prior to stating the upcoming theorem, we establish a few preliminary assumptions: all integrals considered throughout the paper are assumed to exist. Additionally, we assume the presence of a positive constant $\aleph\geq 1$, such that

$$\frac{\mathfrak{r}-b}{\sigma(\mathfrak{r})-b}\geq\frac{1}{\aleph},\quad \mathfrak{r}\in(b,\infty)_{\mathbb{T}}. \tag{3.1}$$

In the following theorem, we will present the time scale version of inequality (1.5).

Theorem 3.1. *Consider $b\in\mathbb{T}$, $\gamma<0$, $\gamma^{*}=\gamma/(\gamma-1)$, $\varrho>1$ and $\mho,\mathfrak{I}\in C_{rd}([b,\infty)_{\mathbb{T}},\mathbb{R}^{+})$ such that $(\mathfrak{r}-b)/\mathfrak{I}(\mathfrak{r})$ is nondecreasing. If (3.1) is satisfied, then*

$$\int_{b}^{\infty}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}[\mathcal{G}^{\sigma}(\mathfrak{r})]^{\gamma}\Delta\mathfrak{r}\leq\mathcal{Q}\int_{b}^{\infty}(\mathfrak{r}-b)^{\gamma}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\Delta\mathfrak{r}, \tag{3.2}$$

where $\mathcal{G}(\mathfrak{r})=\int_{\mathfrak{r}}^{\infty}\mho(\mathfrak{z})\Delta\mathfrak{z}$ and

$$\mathcal{Q}=\begin{cases}\left(\frac{\gamma}{1-\varrho}\right)^{\gamma}\aleph^{\frac{\varrho-1}{\gamma^{*}}}, & 1-\varrho\leq\gamma;\\ \left(\frac{\gamma}{1-\varrho}\right)^{\gamma}\aleph^{\varrho}, & 1-\varrho\geq\gamma.\end{cases}$$

Proof. Start with

$$\mathcal{G}^{\sigma}(\mathfrak{r})=\int_{\sigma(\mathfrak{r})}^{\infty}\mho(\mathfrak{z})\Delta\mathfrak{z}=\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{-\frac{1+\gamma-\varrho}{\gamma\gamma^{*}}}\left[(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^{*}}}\mho(\mathfrak{z})\right]\Delta\mathfrak{z}. \tag{3.3}$$

Applying reversed Hölder's inequality (2.3), we obtain

$$\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{-\frac{1+\gamma-\varrho}{\gamma\gamma^{*}}}\left[(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^{*}}}\mho(\mathfrak{z})\right]\Delta\mathfrak{z}\geq\left(\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{-\frac{1+\gamma-\varrho}{\gamma}}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma^{*}}}\left(\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^{*}}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}. \tag{3.4}$$

Applying (2.1) on $(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}$, we have

$$\frac{\gamma}{\varrho-1}\left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^{\Delta}=(d-b)^{\frac{\varrho-1-\gamma}{\gamma}},\ d\in[\mathfrak{z},\sigma(\mathfrak{z})]. \tag{3.5}$$

Since $\varrho>1$, $\gamma<0$, and $d\geq\mathfrak{z}$, then $(\varrho-1-\gamma)/\gamma<0$, so

$$(d-b)^{\frac{\varrho-1-\gamma}{\gamma}}\leq(\mathfrak{z}-b)^{\frac{\varrho-1-\gamma}{\gamma}}. \tag{3.6}$$

Substituting (3.6) into (3.5), we find

$$\frac{\gamma}{\varrho-1}\left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^{\Delta}\leq(\mathfrak{z}-b)^{\frac{\varrho-1-\gamma}{\gamma}}. \tag{3.7}$$

Integrating (3.7) over $\mathfrak{z}$ from $\sigma(\mathfrak{r})$ to ∞, we observe

$$\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{\frac{\varrho-1-\gamma}{\gamma}}\Delta\mathfrak{z}\geq\frac{\gamma}{\varrho-1}\int_{\sigma(\mathfrak{r})}^{\infty}\left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^{\Delta}\Delta\mathfrak{z}=\frac{\gamma}{1-\varrho}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma}}.$$

Since $\gamma^*>0$, we obtain

$$\left(\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{\frac{\varrho-1-\gamma}{\gamma}}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma^*}}\geq\left(\frac{\gamma}{1-\varrho}\right)^{\frac{1}{\gamma^*}}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma\gamma^*}}. \tag{3.8}$$

Substituting (3.8) into (3.4), we get

$$\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{-\frac{1+\gamma-\varrho}{\gamma\gamma^*}}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^*}}\mho(\mathfrak{z})\Delta\mathfrak{z}\geq\left(\frac{\gamma}{1-\varrho}\right)^{\frac{1}{\gamma^*}}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma\gamma^*}}\left(\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}. \tag{3.9}$$

From (3.3) and (3.9), we have for $\gamma<0$ that

$$[\mathcal{G}^{\sigma}(\mathfrak{r})]^{\gamma}\leq\left(\frac{\gamma}{1-\varrho}\right)^{\gamma-1}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}. \tag{3.10}$$

Multiplying (3.10) by $[\Im(\mathfrak{r})]^{-\varrho}$ and then integrating over $\mathfrak{r}$ from b to ∞, we get

$$\begin{aligned}&\int_{b}^{\infty}[\Im(\mathfrak{r})]^{-\varrho}[\mathcal{G}^{\sigma}(\mathfrak{r})]^{\gamma}\Delta\mathfrak{r}\\&\leq\left(\frac{\gamma}{1-\varrho}\right)^{\gamma-1}\int_{b}^{\infty}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\Im(\mathfrak{r})]^{-\varrho}\left(\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)\Delta\mathfrak{r}.\end{aligned} \tag{3.11}$$

Applying (2.2) on $\int_{b}^{\infty}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\Im(\mathfrak{r})]^{-\varrho}\left(\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)\Delta\mathfrak{r}$, we conclude

$$\begin{aligned}&\int_{b}^{\infty}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\Im(\mathfrak{r})]^{-\varrho}\left(\int_{\sigma(\mathfrak{r})}^{\infty}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)\Delta\mathfrak{r}\\&=\ u_1(\mathfrak{r})\left(\int_{\mathfrak{r}}^{\infty}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)\Big|_{b}^{\infty}+\int_{b}^{\infty}(\mathfrak{r}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{r})]^{\gamma}u_1(\mathfrak{r})\Delta\mathfrak{r},\end{aligned}$$

where $u_1(\mathfrak{r}) = \int_b^{\mathfrak{r}} (\sigma(\mathfrak{z}) - b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{z})]^{-\varrho} \Delta\mathfrak{z}$. Using (3.1), we have

$$\begin{aligned} &\int_b^\infty (\sigma(\varrho) - b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_{\sigma(\mathfrak{r})}^\infty (\mathfrak{z} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z} \right) \Delta\mathfrak{r} \\ = &\int_b^\infty (\mathfrak{r} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma \left(\int_b^{\mathfrak{r}} (\sigma(\mathfrak{z}) - b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \left(\frac{\sigma(\mathfrak{z}) - b}{\mathfrak{I}(\mathfrak{z})} \right)^\varrho \Delta\mathfrak{z} \right) \Delta\mathfrak{r} \\ \leq &\aleph^\varrho \int_b^\infty (\mathfrak{r} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma \left(\int_b^{\mathfrak{r}} (\sigma(\mathfrak{z}) - b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \left(\frac{\mathfrak{z} - b}{\mathfrak{I}(\mathfrak{z})} \right)^\varrho \Delta\mathfrak{z} \right) \Delta\mathfrak{r}. \end{aligned} \tag{3.12}$$

Since $(\mathfrak{z} - b) / \mathfrak{I}(\mathfrak{z})$ is nondecreasing and $\varrho > 1$, we have for $\mathfrak{z} \leq \mathfrak{r}$ that

$$\begin{aligned} &\int_b^{\mathfrak{r}} (\sigma(\mathfrak{z}) - b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \left(\frac{\mathfrak{z} - b}{\mathfrak{I}(\mathfrak{z})} \right)^\varrho \Delta\mathfrak{z} \\ \leq &\left(\frac{\mathfrak{r} - b}{\mathfrak{I}(\mathfrak{r})} \right)^\varrho \int_b^{\mathfrak{r}} (\sigma(\mathfrak{z}) - b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \Delta\mathfrak{z} = \left(\frac{\mathfrak{r} - b}{\mathfrak{I}(\mathfrak{r})} \right)^\varrho \int_b^{\mathfrak{r}} (\sigma(\mathfrak{z}) - b)^{\frac{1-\varrho}{\gamma}-1} \Delta\mathfrak{z}. \end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.12), we observe that

$$\begin{aligned} &\int_b^\infty (\sigma(\mathfrak{r}) - b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_{\sigma(\mathfrak{r})}^\infty (\mathfrak{z} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z} \right) \Delta\mathfrak{r} \\ \leq &\aleph^\varrho \int_b^\infty (\mathfrak{r} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}+\varrho} [\mho(\mathfrak{r})]^\gamma [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_b^{\mathfrak{r}} (\sigma(\mathfrak{z}) - b)^{\frac{1-\varrho}{\gamma}-1} \Delta\mathfrak{z} \right) \Delta\mathfrak{r}. \end{aligned} \tag{3.14}$$

From (3.11) and (3.14), we get

$$\begin{aligned} &\int_b^\infty [\mathfrak{I}(\mathfrak{r})]^{-\varrho} [\mathcal{G}^\sigma(\mathfrak{r})]^\gamma \Delta\mathfrak{r} \\ \leq &\left(\frac{\gamma}{1 - \varrho} \right)^{\gamma-1} \aleph^\varrho \int_b^\infty (\mathfrak{r} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}+\varrho} [\mho(\mathfrak{r})]^\gamma [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_b^{\mathfrak{r}} (\sigma(\mathfrak{z}) - b)^{\frac{1-\varrho}{\gamma}-1} \Delta\mathfrak{z} \right) \Delta\mathfrak{r}. \end{aligned} \tag{3.15}$$

Applying (2.1) on $(\mathfrak{z} - b)^{\frac{1-\varrho}{\gamma}}$, we observe

$$\frac{\gamma}{1 - \varrho} \left((\mathfrak{z} - b)^{\frac{1-\varrho}{\gamma}} \right)^\Delta = (d - b)^{\frac{1-\varrho}{\gamma}-1}, \ d \in [\mathfrak{z}, \sigma(\mathfrak{z})]. \tag{3.16}$$

Now, we consider two cases:

Case 1: For $1 - \varrho \leq \gamma$, we have $\frac{1-\varrho}{\gamma} - 1 \geq 0$. From (3.16), we have $(d - b)^{\frac{1-\varrho}{\gamma}-1} \geq (\mathfrak{z} - b)^{\frac{1-\varrho}{\gamma}-1}$, and

$$\frac{\gamma}{1 - \varrho} \left((\mathfrak{z} - b)^{\frac{1-\varrho}{\gamma}} \right)^\Delta \geq (\mathfrak{z} - b)^{\frac{1-\varrho}{\gamma}-1}. \tag{3.17}$$

Integrating (3.17) over $\mathfrak{z}$ from b to $\mathfrak{r}$, we get

$$\int_b^{\mathfrak{r}} (\mathfrak{z} - b)^{\frac{1-\varrho}{\gamma}-1} \Delta\mathfrak{z} \leq \frac{\gamma}{1 - \varrho} \int_b^{\mathfrak{r}} \left((\mathfrak{z} - b)^{\frac{1-\varrho}{\gamma}} \right)^\Delta \Delta\mathfrak{z} = \frac{\gamma}{1 - \varrho} (\mathfrak{r} - b)^{\frac{1-\varrho}{\gamma}}. \tag{3.18}$$

Using (3.1), (3.15), and (3.18), we have

$$\begin{aligned}&\int_b^{\infty}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\left[\mathcal{G}^{\sigma}(\mathfrak{r})\right]^{\gamma}\Delta\mathfrak{r}\\ \leq&\left(\frac{\gamma}{1-\varrho}\right)^{\gamma-1}\aleph^{\frac{1-\varrho}{\gamma}-1+\varrho}\int_b^{\infty}(\mathfrak{r}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}+\varrho}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\left(\int_b^{\mathfrak{r}}(\mathfrak{z}-b)^{\frac{1-\varrho}{\gamma}-1}\Delta\mathfrak{z}\right)\Delta\mathfrak{r}\\ \leq&\left(\frac{\gamma}{1-\varrho}\right)^{\gamma}\aleph^{\frac{\varrho-1}{\gamma^*}}\int_b^{\infty}(\mathfrak{r}-b)^{\gamma}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\Delta\mathfrak{r},\end{aligned}$$

which matches (3.2) with $Q=(\gamma/(1-\varrho))^{\gamma}\aleph^{\frac{\varrho-1}{\gamma^*}}$.

Case 2: For $1-\varrho\geq\gamma$, we have $\frac{1-\varrho}{\gamma}-1\leq 0$. From (3.16), we see that $(d-b)^{\frac{1-\varrho}{\gamma}-1}\geq(\sigma(\mathfrak{z})-b)^{\frac{1-\varrho}{\gamma}-1}$, and

$$\frac{\gamma}{1-\varrho}\left((\mathfrak{z}-b)^{\frac{1-\varrho}{\gamma}}\right)^{\Delta}\geq(\sigma(\mathfrak{z})-b)^{\frac{1-\varrho}{\gamma}-1}. \tag{3.19}$$

Integrating (3.19) over $\mathfrak{z}$ from b to $\mathfrak{r}$, we have

$$\int_b^{\mathfrak{r}}(\sigma(\mathfrak{z})-b)^{\frac{1-\varrho}{\gamma}-1}\Delta\mathfrak{z}\leq\frac{\gamma}{1-\varrho}\int_b^{\mathfrak{r}}\left((\mathfrak{z}-b)^{\frac{1-\varrho}{\gamma}}\right)^{\Delta}\Delta\mathfrak{z}=\frac{\gamma}{1-\varrho}(\mathfrak{r}-b)^{\frac{1-\varrho}{\gamma}}. \tag{3.20}$$

Substituting (3.20) into (3.15), we observe

$$\int_b^{\infty}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\left[\mathcal{G}^{\sigma}(\mathfrak{r})\right]^{\gamma}\Delta\mathfrak{r}\leq\left(\frac{\gamma}{1-\varrho}\right)^{\gamma}\aleph^{\varrho}\int_b^{\infty}(\mathfrak{r}-b)^{\gamma}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\Delta\mathfrak{r},$$

which matches (3.2) with $Q=(\gamma/(1-\varrho))^{\gamma}\aleph^{\varrho}$. □

Remark 3.1. *In Theorem 3.1, when* $\mathbb{T}=\mathbb{R}$ *and* $b=0$, *we have* $\sigma(\mathfrak{r})=\mathfrak{r}$. *Consequently, we see that (3.1) holds with* $\aleph=1$. *As a result, (3.2) simplifies to (1.5), and for* $\mathfrak{I}(\mathfrak{r})=\mathfrak{r}$, *we obtain (1.3).*

Corollary 3.1. *If* $\mathbb{T}=\mathbb{N}_0$, $b=0$, $\varrho>1$, $\gamma<0$, *and* $\{s_n\}_{n=0}^{\infty}$, $\{t_n\}_{n=0}^{\infty}$ *are positive sequences such that* n/t_n *is nondecreasing, then*

$$\sum_{n=0}^{\infty}[t_n]^{-\varrho}\left(\sum_{k=n+1}^{\infty}s_k\right)^{\gamma}\leq Q\sum_{n=0}^{\infty}n^{\gamma}[s_n]^{\gamma}[t_n]^{-\varrho},$$

where

$$Q=\begin{cases}2^{\frac{\varrho-1}{\gamma^*}}\left(\frac{\gamma}{1-\varrho}\right)^{\gamma}, & 1-\varrho\leq\gamma;\\ 2^{\varrho}\left(\frac{\gamma}{1-\varrho}\right)^{\gamma}, & 1-\varrho\geq\gamma.\end{cases}$$

Here,

$$\frac{n-b}{\sigma(n)-b}=\frac{n}{n+1}=1-\frac{1}{n+1},\ n\geq 1.$$

Since $-1/(n+1)\geq-1/2$, *then* $(n-b)/(\sigma(n)-b)\geq 1/2$, *and (3.1) holds with* $\aleph=2$.

Corollary 3.2. *Let* $\mathbb{T} = q^{\mathbb{N}_0}$ *for* $q > 1$, $b \in \mathbb{T}$, $\gamma < 0$, $\gamma^* = \gamma/(\gamma-1)$, $\varrho > 1$, *and* $\mho, \Im$ *be posituve sequences on* $[b, \infty)$ *such that* $(\mathfrak{r}-b)/\Im(\mathfrak{r})$ *is nondecreasing. If*

$$\frac{\mathfrak{r}-b}{q\mathfrak{r}-b} \geq \frac{1}{\aleph}, \quad \mathfrak{r} \in (b,\infty)_{\mathbb{T}},$$

then

$$\sum_{\mathfrak{r}=b}^{\infty} \mathfrak{r}\left[\Im(\mathfrak{r})\right]^{-\varrho}\left[\mathcal{G}(q\mathfrak{r})\right]^{\gamma} \leq \mathcal{Q}\sum_{\mathfrak{r}=b}^{\infty} \mathfrak{r}\left(\mathfrak{r}-b\right)^{\gamma}\left[\mho(\mathfrak{r})\right]^{\gamma}\left[\Im(\mathfrak{r})\right]^{-\varrho},$$

where $\mathcal{G}(\mathfrak{r}) = \sum_{\mathfrak{z}=\mathfrak{r}}^{\infty}(q-1)\mathfrak{z}\mho(\mathfrak{z})$ *and*

$$\mathcal{Q} = \begin{cases} \left(\frac{\gamma}{1-\varrho}\right)^{\gamma}\aleph^{\frac{\varrho-1}{\gamma^*}}, & 1-\varrho \leq \gamma; \\ \left(\frac{\gamma}{1-\varrho}\right)^{\gamma}\aleph^{\varrho}, & 1-\varrho \geq \gamma. \end{cases}$$

In the following theorem, we will present the time scale version of inequality (1.6).

Theorem 3.2. *Assume* $b \in \mathbb{T}$, $\gamma < 0$, $\gamma^* = \gamma/(\gamma-1)$, $0 \leq \varrho < 1$, *and* $\mho, \Im \in C_{rd}([b,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ *such that* $(\mathfrak{r}-b)/\Im(\mathfrak{r})$ *is nonincreasing. If (3.1) holds, then*

$$\int_b^{\infty}\left[\Im(\mathfrak{r})\right]^{-\varrho}\left[\Omega^{\sigma}(\mathfrak{r})\right]^{\gamma}\Delta\mathfrak{r} \leq \mathcal{J}\int_b^{\infty}(\mathfrak{r}-b)^{\gamma}\left[\mho(\mathfrak{r})\right]^{\gamma}\left[\Im(\mathfrak{r})\right]^{-\varrho}\Delta\mathfrak{r}, \tag{3.21}$$

where $\Omega(\mathfrak{r}) = \int_b^{\mathfrak{r}}\mho(\mathfrak{z})\Delta\mathfrak{z}$ *and*

$$\mathcal{J} = \begin{cases} \aleph^{\frac{\varrho-1}{\gamma}}\left(\frac{\gamma}{\varrho-1}\right)^{\gamma}, & (\varrho-1)/\gamma \geq 1; \\ \aleph^{\varrho}\left(\frac{\gamma}{\varrho-1}\right)^{\gamma}, & (\varrho-1)/\gamma \leq 1. \end{cases}$$

Proof. To prove this theorem, we consider two cases:

Case 1: For $(\varrho-1)/\gamma \geq 1$. Start with

$$\Omega^{\sigma}(\mathfrak{r}) = \int_b^{\sigma(\mathfrak{r})}\mho(\mathfrak{z})\Delta\mathfrak{z} = \int_b^{\sigma(\mathfrak{r})}\left[(\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma\gamma^*}}\right]\left[(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^*}}\mho(\mathfrak{z})\right]\Delta\mathfrak{z}. \tag{3.22}$$

Applying (2.3) on (3.22), we get

$$\begin{aligned} &\int_b^{\sigma(\mathfrak{r})}\left[(\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma\gamma^*}}\right]\left[(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^*}}\mho(\mathfrak{z})\right]\Delta\mathfrak{z} \\ \geq\ &\left(\int_b^{\sigma(\mathfrak{r})}(\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma}}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma^*}}\left(\int_b^{\sigma(\mathfrak{r})}(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}\left[\mho(\mathfrak{z})\right]^{\gamma}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}. \end{aligned}$$

From this and the previous inequality, we have

$$\Omega^{\sigma}(\mathfrak{r}) \geq \left(\int_b^{\sigma(\mathfrak{r})}(\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma}}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma^*}}\left(\int_b^{\sigma(\mathfrak{r})}(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}\left[\mho(\mathfrak{z})\right]^{\gamma}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}. \tag{3.23}$$

Applying (2.1) on $(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}$, we observe that

$$\frac{\gamma}{\varrho-1}\left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^{\Delta}=(d-b)^{\frac{\varrho-\gamma-1}{\gamma}},\ d\in[\mathfrak{z},\sigma(\mathfrak{z})]. \tag{3.24}$$

Since $(\varrho-1)/\gamma\geq 1$, then $(d-b)^{\frac{\varrho-\gamma-1}{\gamma}}\leq(\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma}}$, and (3.24) becomes

$$\frac{\gamma}{\varrho-1}\left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^{\Delta}\leq(\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma}}. \tag{3.25}$$

By integrating (3.25) over $\mathfrak{z}$ from b to $\sigma(\mathfrak{r})$, we get

$$\begin{aligned}&\int_b^{\sigma(\mathfrak{r})}(\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma}}\Delta\mathfrak{z}\\ \geq\ &\frac{\gamma}{\varrho-1}\int_b^{\sigma(\mathfrak{r})}\left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^{\Delta}\Delta\mathfrak{z}=\frac{\gamma}{\varrho-1}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma}}.\end{aligned} \tag{3.26}$$

Substituting (3.26) into (3.23), since $\gamma^*>0$, we observe

$$\Omega^{\sigma}(\mathfrak{r})\geq\left(\frac{\gamma}{\varrho-1}\right)^{\frac{1}{\gamma^*}}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma\gamma^*}}\left(\int_b^{\sigma(\mathfrak{r})}(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}.$$

For $\gamma<0$, this yields

$$[\Omega^{\sigma}(\mathfrak{r})]^{\gamma}\leq\left(\frac{\gamma}{\varrho-1}\right)^{\gamma-1}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}\int_b^{\sigma(\mathfrak{r})}(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}. \tag{3.27}$$

Multiplying (3.27) by $[\mathfrak{I}(\mathfrak{r})]^{-\varrho}$ and then integrating over $\mathfrak{r}$ from b to ∞, we find

$$\begin{aligned}&\int_b^{\infty}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}[\Omega^{\sigma}(\mathfrak{r})]^{\gamma}\Delta\mathfrak{r}\\ \leq\ &\left(\frac{\gamma}{\varrho-1}\right)^{\gamma-1}\int_b^{\infty}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\left(\int_b^{\sigma(\mathfrak{r})}(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)\Delta\mathfrak{r}.\end{aligned} \tag{3.28}$$

Applying (2.2) on

$$\int_b^{\infty}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\left(\int_b^{\sigma(\mathfrak{r})}(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)\Delta\mathfrak{r},$$

we obtain

$$\begin{aligned}&\int_b^{\infty}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\left(\int_b^{\sigma(\mathfrak{r})}(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)\Delta\mathfrak{r}\\ =\ &u_3(\mathfrak{r})\left(\int_b^{\mathfrak{r}}(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)\Bigg|_b^{\infty}-\int_b^{\infty}u_3(\mathfrak{r})(\sigma(\mathfrak{r})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{r})]^{\gamma}\Delta\mathfrak{r},\end{aligned}$$

where

$$u_3(\mathfrak{r})=-\int_{\mathfrak{r}}^{\infty}(\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}}[\mathfrak{I}(\mathfrak{z})]^{-\varrho}\Delta\mathfrak{z}.$$

Using (3.1), we have

$$\begin{aligned}
&\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)\Delta\mathfrak{r} \\
=\ & \int_b^\infty \left[\int_\mathfrak{r}^\infty (\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{z})]^{-\varrho} \Delta\mathfrak{z}\right] (\sigma(\mathfrak{r})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma \Delta\mathfrak{r} \\
=\ & \int_b^\infty \left[\int_\mathfrak{r}^\infty (\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \left(\frac{\sigma(\mathfrak{z})-b}{\mathfrak{I}(\mathfrak{z})}\right)^\varrho \Delta\mathfrak{z}\right] (\sigma(\mathfrak{r})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma \Delta\mathfrak{r} \\
\leq\ & \aleph^\varrho \int_b^\infty \left[\int_\mathfrak{r}^\infty (\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \left(\frac{\mathfrak{z}-b}{\mathfrak{I}(\mathfrak{z})}\right)^\varrho \Delta\mathfrak{z}\right] (\sigma(\mathfrak{r})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma \Delta\mathfrak{r}. \qquad (3.29)
\end{aligned}$$

Since $(\mathfrak{z}-b)/\mathfrak{I}(\mathfrak{z})$ is nonincreasing and $0 \leq \varrho < 1$, we have for $\mathfrak{z} \geq \mathfrak{r}$ that $((\mathfrak{z}-b)/\mathfrak{I}(\mathfrak{z}))^\varrho \leq ((\mathfrak{r}-b)/\mathfrak{I}(\mathfrak{r}))^\varrho$, and then (3.29) becomes

$$\begin{aligned}
&\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)\Delta\mathfrak{r} \\
\leq\ & \aleph^\varrho \int_b^\infty \left[\int_\mathfrak{r}^\infty (\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \Delta\mathfrak{z}\right] (\sigma(\mathfrak{r})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} (\mathfrak{r}-b)^\varrho [\mho(\mathfrak{r})]^\gamma [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \Delta\mathfrak{r}. \qquad (3.30)
\end{aligned}$$

From (3.16), since $(1-\varrho)/\gamma < 0$, we have $(d-b)^{\frac{1-\varrho}{\gamma}-1} \geq (\sigma(\mathfrak{z})-b)^{\frac{1-\varrho}{\gamma}-1}$ and

$$\frac{\gamma}{1-\varrho}\left[(\mathfrak{z}-b)^{\frac{1-\varrho}{\gamma}}\right]^\Delta \geq (\sigma(\mathfrak{z})-b)^{\frac{1-\varrho}{\gamma}-1} = (\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}-\varrho},$$

thus,

$$\int_\mathfrak{r}^\infty (\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \Delta\mathfrak{z} \leq \frac{\gamma}{1-\varrho}\int_\mathfrak{r}^\infty \left[(\mathfrak{z}-b)^{\frac{1-\varrho}{\gamma}}\right]^\Delta \Delta\mathfrak{z} = \frac{\gamma}{\varrho-1}(\mathfrak{r}-b)^{\frac{1-\varrho}{\gamma}}. \qquad (3.31)$$

Substituting (3.31) into (3.30) and using (3.1), since $(1-\varrho)/\gamma^* - 1 > 0$, we observe

$$\begin{aligned}
&\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)\Delta\mathfrak{r} \\
\leq\ & \aleph^\varrho \left(\frac{\gamma}{\varrho-1}\right) \int_b^\infty (\sigma(\mathfrak{r})-b)^\gamma (\sigma(\mathfrak{r})-b)^{\frac{1-\varrho}{\gamma^*}-1} (\mathfrak{r}-b)^{\varrho+\frac{1-\varrho}{\gamma}} [\mho(\mathfrak{r})]^\gamma [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \Delta\mathfrak{r} \\
\leq\ & \aleph^\varrho \left(\frac{\gamma}{\varrho-1}\right) \int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{1-\varrho}{\gamma^*}-1} (\mathfrak{r}-b)^{\gamma+\varrho+\frac{1-\varrho}{\gamma}} [\mho(\mathfrak{r})]^\gamma [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \Delta\mathfrak{r} \\
\leq\ & \aleph^{\frac{\varrho-1}{\gamma}} \left(\frac{\gamma}{\varrho-1}\right) \int_b^\infty (\mathfrak{r}-b)^\gamma [\mho(\mathfrak{r})]^\gamma [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \Delta\mathfrak{r}. \qquad (3.32)
\end{aligned}$$

Substituting (3.32) into (3.28), we have

$$\int_b^\infty [\mathfrak{I}(\mathfrak{r})]^{-\varrho} [\Omega^\sigma(\mathfrak{r})]^\gamma \Delta\mathfrak{r} \leq \aleph^{\frac{\varrho-1}{\gamma}} \left(\frac{\gamma}{\varrho-1}\right)^\gamma \int_b^\infty (\mathfrak{r}-b)^\gamma [\mho(\mathfrak{r})]^\gamma [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \Delta\mathfrak{r},$$

which matches (3.21) with $\mathcal{J} = \aleph^{\frac{\varrho-1}{\gamma}} (\gamma/(\varrho-1))^\gamma$.

Case 2: For $(\varrho - 1)/\gamma \leq 1$. We have

$$\Omega^{\sigma}(\mathfrak{r}) = \int_b^{\sigma(\mathfrak{r})} \mho(\mathfrak{z})\Delta\mathfrak{z} = \int_b^{\sigma(\mathfrak{r})} (\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma\gamma^*}} \left[(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^*}} \mho(\mathfrak{z})\right] \Delta\mathfrak{z}. \tag{3.33}$$

Applying (2.3) on $\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma\gamma^*}} \left[(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^*}} \mho(\mathfrak{z})\right] \Delta\mathfrak{z}$, we find

$$\begin{aligned}
&\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma\gamma^*}} \left[(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^*}} \mho(\mathfrak{z})\right] \Delta\mathfrak{z} \\
\geq\ & \left(\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma}} \Delta\mathfrak{z}\right)^{\frac{1}{\gamma^*}} \left(\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^{\gamma} \Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}.
\end{aligned} \tag{3.34}$$

From (3.33) and (3.34), we have

$$\Omega^{\sigma}(\mathfrak{r}) \geq \left(\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma}} \Delta\mathfrak{z}\right)^{\frac{1}{\gamma^*}} \left(\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^{\gamma} \Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}. \tag{3.35}$$

Using (3.5), since $0 < (\varrho - 1)/\gamma \leq 1$, we have $(d-b)^{\frac{\varrho-\gamma-1}{\gamma}} \leq (\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma}}$ and

$$\frac{\gamma}{\varrho-1}\left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^{\Delta} \leq (\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma}},$$

and then

$$\begin{aligned}
\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma}} \Delta\mathfrak{z} &\geq \frac{\gamma}{\varrho-1}\int_b^{\sigma(\mathfrak{r})} \left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^{\Delta} \Delta\mathfrak{z} \\
&= \frac{\gamma}{\varrho-1}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma}}.
\end{aligned} \tag{3.36}$$

Substituting (3.36) into (3.35), since $\gamma^* > 0$, we conclude

$$\Omega^{\sigma}(\mathfrak{r}) \geq \left(\frac{\gamma}{\varrho-1}\right)^{\frac{1}{\gamma^*}} (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma\gamma^*}} \left(\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^{\gamma} \Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}.$$

For $\gamma < 0$, this yields

$$[\Omega^{\sigma}(\mathfrak{r})]^{\gamma} \leq \left(\frac{\gamma}{\varrho-1}\right)^{\gamma-1} (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}} \int_b^{\sigma(\mathfrak{r})} (\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^{\gamma} \Delta\mathfrak{z}.$$

Multiplying the last inequality by $[\mathfrak{I}(\mathfrak{r})]^{-\varrho}$ and then integrating over $\mathfrak{r}$ from b to ∞, we observe

$$\begin{aligned}
&\int_b^{\infty} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} [\Omega^{\sigma}(\mathfrak{r})]^{\gamma} \Delta\mathfrak{r} \\
\leq\ & \left(\frac{\gamma}{\varrho-1}\right)^{\gamma-1} \int_b^{\infty} (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^{\gamma} \Delta\mathfrak{z}\right) \Delta\mathfrak{r}.
\end{aligned} \tag{3.37}$$

Applying (2.2) on $\int_b^\infty (\sigma(\mathfrak{r}) - b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right) \Delta\mathfrak{r}$, we obtain

$$\begin{aligned} &\int_b^\infty (\sigma(\mathfrak{r}) - b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right) \Delta\mathfrak{r} \\ = \;& u_4(\mathfrak{r}) \left(\int_b^{\mathfrak{r}} (\mathfrak{z} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)\Bigg|_b^\infty - \int_b^\infty u_4(\mathfrak{r}) (\mathfrak{r} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma \Delta\mathfrak{r}, \end{aligned}$$

where

$$u_4(\mathfrak{r}) = -\int_{\mathfrak{r}}^\infty (\sigma(\mathfrak{z}) - b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{z})]^{-\varrho} \Delta\mathfrak{z}.$$

With (3.1), $(\mathfrak{z} - b)/\mathfrak{I}(\mathfrak{z})$ is nonincreasing, and $\varrho > 0$, we have for $\mathfrak{z} \geq \mathfrak{r}$ that

$$\begin{aligned} &\int_b^\infty (\sigma(\mathfrak{r}) - b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right) \Delta\mathfrak{r} \\ = \;& \int_b^\infty (\mathfrak{r} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma \left[\int_{\mathfrak{r}}^\infty (\sigma(\mathfrak{z}) - b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{z})]^{-\varrho} \Delta\mathfrak{z}\right] \Delta\mathfrak{r} \\ = \;& \int_b^\infty (\mathfrak{r} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma \left[\int_{\mathfrak{r}}^\infty \left[\frac{\sigma(\mathfrak{z}) - b}{\mathfrak{I}(\mathfrak{z})}\right]^\varrho (\sigma(\mathfrak{z}) - b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \Delta\mathfrak{z}\right] \Delta\mathfrak{r} \\ \leq \;& \aleph^\varrho \int_b^\infty (\mathfrak{r} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma \left[\int_{\mathfrak{r}}^\infty \left[\frac{\mathfrak{z} - b}{\mathfrak{I}(\mathfrak{z})}\right]^\varrho (\sigma(\mathfrak{z}) - b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \Delta\mathfrak{z}\right] \Delta\mathfrak{r} \\ \leq \;& \aleph^\varrho \int_b^\infty (\mathfrak{r} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}+\varrho} [\mho(\mathfrak{r})]^\gamma [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left[\int_{\mathfrak{r}}^\infty (\sigma(\mathfrak{z}) - b)^{\frac{1-\varrho}{\gamma}-1} \Delta\mathfrak{z}\right] \Delta\mathfrak{r}. \end{aligned} \tag{3.38}$$

Since $(1 - \varrho)/\gamma < 0$, then by using (3.16), we have

$$\frac{\gamma}{1-\varrho} \left[(\mathfrak{z} - b)^{\frac{1-\varrho}{\gamma}}\right]^\Delta \geq (\sigma(\mathfrak{z}) - b)^{\frac{1-\varrho}{\gamma}-1},$$

therefore,

$$\int_{\mathfrak{r}}^\infty (\sigma(\mathfrak{z}) - b)^{\frac{1-\varrho}{\gamma}-1} \Delta\mathfrak{z} \leq \frac{\gamma}{1-\varrho} \int_{\mathfrak{r}}^\infty \left[(\mathfrak{z} - b)^{\frac{1-\varrho}{\gamma}}\right]^\Delta \Delta\mathfrak{z} = \frac{\gamma}{\varrho - 1} (\mathfrak{r} - b)^{\frac{1-\varrho}{\gamma}}. \tag{3.39}$$

Substituting (3.39) into (3.38), we obtain

$$\begin{aligned} &\int_b^\infty (\sigma(\mathfrak{r}) - b)^{\frac{\varrho-1}{\gamma^*}} [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\mathfrak{z} - b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right) \Delta\mathfrak{r} \\ \leq \;& \aleph^\varrho \left(\frac{\gamma}{\varrho - 1}\right) \int_b^\infty (\mathfrak{r} - b)^\gamma [\mho(\mathfrak{r})]^\gamma [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \Delta\mathfrak{r}. \end{aligned} \tag{3.40}$$

Substituting (3.40) into (3.37), we get

$$\int_b^\infty [\mathfrak{I}(\mathfrak{r})]^{-\varrho} [\Omega^\sigma(\mathfrak{r})]^\gamma \Delta\mathfrak{r} \leq \aleph^\varrho \left(\frac{\gamma}{\varrho - 1}\right)^\gamma \int_b^\infty (\mathfrak{r} - b)^\gamma [\mho(\mathfrak{r})]^\gamma [\mathfrak{I}(\mathfrak{r})]^{-\varrho} \Delta\mathfrak{r},$$

which matches (3.21) with $\mathcal{J} = \aleph^\varrho (\gamma/(\varrho - 1))^\gamma$. □

Remark 3.2. *In Theorem 3.2, when* $\mathbb{T}=\mathbb{R}$ *and* $b=0$, *we have* $\sigma(\mathfrak{r})=\mathfrak{r}$. *Consequently, we see that (3.1) holds with* $\aleph=1$. *As a result, (3.21) simplifies to (1.6), and for* $\mathfrak{I}(\mathfrak{r})=\mathfrak{r}$, *we get (1.4).*

Corollary 3.3. *If* $\mathbb{T}=\mathbb{N}_0$, $b=0$ *and* $\{s_n\}_{n=0}^{\infty}$, $\{t_n\}_{n=0}^{\infty}$ *are positive sequences with the property that* n/t_n *is nonincreasing, then*

$$\sum_{n=0}^{\infty}[t_n]^{-\varrho}\left[\sum_{k=0}^{n}s_k\right]^{\gamma}\leq\mathcal{J}\sum_{n=0}^{\infty}n^{\gamma}[s_n]^{\gamma}[t_n]^{-\varrho}, \tag{3.41}$$

where

$$\mathcal{J}=\begin{cases}2^{\frac{\varrho-1}{\gamma}}\left(\frac{\gamma}{\varrho-1}\right)^{\gamma}, & (\varrho-1)/\gamma\geq 1;\\ 2^{\varrho}\left(\frac{\gamma}{\varrho-1}\right)^{\gamma}, & (\varrho-1)/\gamma\leq 1.\end{cases}$$

Here, the inequality (3.1) holds with $\aleph=2$.

Corollary 3.4. *Let* $\mathbb{T}=q^{\mathbb{N}_0}$ *for* $q>1$, $b\in\mathbb{T}$, $\gamma<0$, $\gamma^*=\gamma/(\gamma-1)$, $0\leq\varrho<1$, *and* $\mho,\mathfrak{I}$ *be posituve sequences on* $[b,\infty)$ *such that* $(\mathfrak{r}-b)/\mathfrak{I}(\mathfrak{r})$ *is nonincreasing. If*

$$\frac{\mathfrak{r}-b}{q\mathfrak{r}-b}\geq\frac{1}{\aleph},\quad \mathfrak{r}\in(b,\infty)$$

holds, then

$$\sum_{\mathfrak{r}=b}^{\infty}\mathfrak{r}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}[\Omega(q\mathfrak{r})]^{\gamma}\leq\mathcal{J}\sum_{\mathfrak{r}=b}^{\infty}\mathfrak{r}(\mathfrak{r}-b)^{\gamma}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho},$$

where $\Omega(\mathfrak{r})=\sum_{\mathfrak{z}=b}^{\mathfrak{r}/q}(q-1)\mathfrak{z}\mho(\mathfrak{z})$ *and*

$$\mathcal{J}=\begin{cases}\aleph^{\frac{\varrho-1}{\gamma}}\left(\frac{\gamma}{\varrho-1}\right)^{\gamma}, & (\varrho-1)/\gamma\geq 1;\\ \aleph^{\varrho}\left(\frac{\gamma}{\varrho-1}\right)^{\gamma}, & (\varrho-1)/\gamma\leq 1.\end{cases}$$

In the following theorem, we will present the time scale version of inequality (1.7).

Theorem 3.3. *Assume* $b\in\mathbb{T}$, $\gamma<0$, $\gamma^*=\gamma/(\gamma-1)$, $\varrho<0$, *and* $\mho,\mathfrak{I}\in C_{rd}([b,\infty)_{\mathbb{T}},\mathbb{R}^+)$ *such that* $(\mathfrak{r}-b)/\mathfrak{I}(\mathfrak{r})$ *is nondecreasing. If (3.1) holds, then*

$$\int_b^{\infty}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}[\Omega^{\sigma}(\mathfrak{r})]^{\gamma}\Delta\mathfrak{r}\leq\mathcal{M}\int_b^{\infty}(\mathfrak{r}-b)^{\gamma}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\Delta\mathfrak{r}, \tag{3.42}$$

where $\Omega(\mathfrak{r})=\int_b^{\mathfrak{r}}\mho(\mathfrak{z})\Delta\mathfrak{z}$ *and*

$$\mathcal{M}=\begin{cases}\left(\frac{\gamma}{\varrho-1}\right)^{\gamma}, & (\varrho-1)/\gamma\leq 1;\\ \left(\frac{\gamma}{\varrho-1}\right)^{\gamma}\aleph^{\frac{\varrho-1}{\gamma}-\varrho}, & (\varrho-1)/\gamma\geq 1.\end{cases}$$

Proof. We consider the following two cases to prove this theorem.

Case 1: For $(\varrho-1)/\gamma\leq 1$. Start with

$$\Omega^{\sigma}(\mathfrak{r})=\int_b^{\sigma(\mathfrak{r})}\mho(\mathfrak{z})\Delta\mathfrak{z}=\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma\gamma^*}}\left[(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^*}}\mho(\mathfrak{z})\right]\Delta\mathfrak{z}. \tag{3.43}$$

Applying (2.3) on $\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma\gamma^*}}\left[(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^*}}\mho(\mathfrak{z})\right]\Delta\mathfrak{z}$, we get

$$\begin{aligned}&\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma\gamma^*}}\left[(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^*}}\mho(\mathfrak{z})\right]\Delta\mathfrak{z}\\ \geq\;&\left(\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma}}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma^*}}\left(\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}.\end{aligned}\tag{3.44}$$

From (3.43) and (3.44), we have

$$\Omega^{\sigma}(\mathfrak{r})\geq\left(\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma}}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma^*}}\left(\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}.\tag{3.45}$$

Since $0<(\varrho-1)/\gamma\leq 1$, and by using (3.5), we find

$$\frac{\gamma}{\varrho-1}\left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^{\Delta}\leq(\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma}},$$

thus,

$$\begin{aligned}\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{\varrho-\gamma-1}{\gamma}}\Delta\mathfrak{z}\;&\geq\;\frac{\gamma}{\varrho-1}\int_b^{\sigma(\mathfrak{r})}\left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^{\Delta}\Delta\mathfrak{z}\\ &=\;\frac{\gamma}{\varrho-1}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma}}.\end{aligned}\tag{3.46}$$

Substituting (3.46) into (3.45), we conclude

$$\Omega^{\sigma}(\mathfrak{r})\geq\left(\frac{\gamma}{\varrho-1}\right)^{\frac{1}{\gamma^*}}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma\gamma^*}}\left(\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}.$$

For $\gamma<0$, we have

$$[\Omega^{\sigma}(\mathfrak{r})]^{\gamma}\leq\left(\frac{\gamma}{\varrho-1}\right)^{\gamma-1}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}.$$

Multiplying the last inequality by $[\Im(\mathfrak{r})]^{-\varrho}$ and then integrating over $\mathfrak{r}$ from b to ∞, we observe

$$\begin{aligned}&\int_b^{\infty}[\Im(\mathfrak{r})]^{-\varrho}[\Omega^{\sigma}(\mathfrak{r})]^{\gamma}\Delta\mathfrak{r}\\ \leq\;&\left(\frac{\gamma}{\varrho-1}\right)^{\gamma-1}\int_b^{\infty}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\Im(\mathfrak{r})]^{-\varrho}\left(\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)\Delta\mathfrak{r}.\end{aligned}\tag{3.47}$$

Applying (2.2) on

$$\int_b^{\infty}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\Im(\mathfrak{r})]^{-\varrho}\left(\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)\Delta\mathfrak{r},$$

we observe that

$$\begin{aligned} &\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\Im(\mathfrak{r})]^{-\varrho}\left(\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^\gamma\Delta\mathfrak{z}\right)\Delta\mathfrak{r} \\ = \;& u_5(\mathfrak{r})\left(\int_b^{\mathfrak{r}}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^\gamma\Delta\mathfrak{z}\right)\Bigg|_b^\infty-\int_b^\infty(\mathfrak{r}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}u_5(\mathfrak{r})[\mho(\mathfrak{r})]^\gamma\Delta\mathfrak{r}, \end{aligned}$$

where

$$u_5(\mathfrak{r})=-\int_{\mathfrak{r}}^\infty(\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}}[\Im(\mathfrak{z})]^{-\varrho}\Delta\mathfrak{z}.$$

Since $\sigma(\mathfrak{z})\geq\mathfrak{z}$ and $\varrho<0$, we get

$$\begin{aligned} &\int_b^\infty(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\Im(\mathfrak{r})]^{-\varrho}\left(\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^\gamma\Delta\mathfrak{z}\right)\Delta\mathfrak{r} \\ = \;& \int_b^\infty(\mathfrak{r}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{r})]^\gamma\left[\int_{\mathfrak{r}}^\infty(\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}}[\Im(\mathfrak{z})]^{-\varrho}\Delta\mathfrak{z}\right]\Delta\mathfrak{r} \\ = \;& \int_b^\infty(\mathfrak{r}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{r})]^\gamma\left[\int_{\mathfrak{r}}^\infty\left[\frac{\sigma(\mathfrak{z})-b}{\Im(\mathfrak{z})}\right]^\varrho(\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}-\varrho}\Delta\mathfrak{z}\right]\Delta\mathfrak{r} \\ \leq \;& \int_b^\infty(\mathfrak{r}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{r})]^\gamma\left[\int_{\mathfrak{r}}^\infty\left[\frac{\mathfrak{z}-b}{\Im(\mathfrak{z})}\right]^\varrho(\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}-\varrho}\Delta\mathfrak{z}\right]\Delta\mathfrak{r}. \end{aligned} \tag{3.48}$$

Since $(\mathfrak{z}-b)/\Im(\mathfrak{z})$ is nondecreasing and $\varrho<0$, (3.48) becomes

$$\begin{aligned} &\int_b^\infty(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\Im(\mathfrak{r})]^{-\varrho}\left(\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^\gamma\Delta\mathfrak{z}\right)\Delta\mathfrak{r} \\ \leq \;& \int_b^\infty(\mathfrak{r}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}+\varrho}[\mho(\mathfrak{r})]^\gamma[\Im(\mathfrak{r})]^{-\varrho}\left[\int_{\mathfrak{r}}^\infty(\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}-\varrho}\Delta\mathfrak{z}\right]\Delta\mathfrak{r} \\ = \;& \int_b^\infty(\mathfrak{r}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}+\varrho}[\mho(\mathfrak{r})]^\gamma[\Im(\mathfrak{r})]^{-\varrho}\left[\int_{\mathfrak{r}}^\infty(\sigma(\mathfrak{z})-b)^{\frac{1-\varrho}{\gamma}-1}\Delta\mathfrak{z}\right]\Delta\mathfrak{r}. \end{aligned} \tag{3.49}$$

Since $\varrho<0$ and $\gamma<0$, by using (3.16), we get

$$\frac{\gamma}{1-\varrho}\left[(\mathfrak{z}-b)^{\frac{1-\varrho}{\gamma}}\right]^\Delta\geq(\sigma(\mathfrak{z})-b)^{\frac{1-\varrho}{\gamma}-1},$$

and then

$$\int_{\mathfrak{r}}^\infty(\sigma(\mathfrak{z})-b)^{\frac{1-\varrho}{\gamma}-1}\Delta\mathfrak{z}\leq\frac{\gamma}{1-\varrho}\int_{\mathfrak{r}}^\infty\left[(\mathfrak{z}-b)^{\frac{1-\varrho}{\gamma}}\right]^\Delta\Delta\mathfrak{z}=\frac{\gamma}{\varrho-1}(\mathfrak{r}-b)^{\frac{1-\varrho}{\gamma}}. \tag{3.50}$$

Substituting (3.50) into (3.49), we obtain

$$\begin{aligned} &\int_b^\infty(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\Im(\mathfrak{r})]^{-\varrho}\left(\int_b^{\sigma(\mathfrak{r})}(\mathfrak{z}-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^\gamma\Delta\mathfrak{z}\right)\Delta\mathfrak{r} \\ \leq \;& \frac{\gamma}{\varrho-1}\int_b^\infty(\mathfrak{r}-b)^\gamma[\mho(\mathfrak{r})]^\gamma[\Im(\mathfrak{r})]^{-\varrho}\Delta\mathfrak{r}. \end{aligned} \tag{3.51}$$

Substituting (3.51) into (3.47), we observe

$$\int_b^\infty [\Im(\mathfrak{r})]^{-\varrho} [\Omega^\sigma(\mathfrak{r})]^\gamma \Delta\mathfrak{r} \le \left(\frac{\gamma}{\varrho-1}\right)^\gamma \int_b^\infty (\mathfrak{r}-b)^\gamma [\mho(\mathfrak{r})]^\gamma [\Im(\mathfrak{r})]^{-\varrho} \Delta\mathfrak{r},$$

which is (3.42) with $\mathcal{M} = (\gamma/(\varrho-1))^\gamma$.

Case 2: For $(\varrho-1)/\gamma \ge 1$. We have

$$\Omega^\sigma(\mathfrak{r}) = \int_b^{\sigma(\mathfrak{r})} \mho(\mathfrak{z})\Delta\mathfrak{z} = \int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma\gamma^*}} \left[(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^*}} \mho(\mathfrak{z})\right] \Delta\mathfrak{z}. \tag{3.52}$$

Applying (2.3), we get

$$\begin{aligned} &\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma\gamma^*}} \left[(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma\gamma^*}} \mho(\mathfrak{z})\right] \Delta\mathfrak{z} \\ \ge\ & \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma}} \Delta\mathfrak{z}\right)^{\frac{1}{\gamma^*}} \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}. \end{aligned} \tag{3.53}$$

Substituting (3.53) into (3.52), we obtain

$$\Omega^\sigma(\mathfrak{r}) \ge \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma}} \Delta\mathfrak{z}\right)^{\frac{1}{\gamma^*}} \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}. \tag{3.54}$$

Since $(\varrho-1)/\gamma \ge 1$, then by using (3.5), we have

$$\frac{\gamma}{\varrho-1}\left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^\Delta \le (\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma}}. \tag{3.55}$$

By integrating (3.55) over $\mathfrak{z}$ from b to $\sigma(\mathfrak{r})$, we get

$$\begin{aligned} \int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{\varrho-\gamma-1}{\gamma}} \Delta\mathfrak{z} &\ge \frac{\gamma}{\varrho-1}\int_b^{\sigma(\mathfrak{r})} \left[(\mathfrak{z}-b)^{\frac{\varrho-1}{\gamma}}\right]^\Delta \Delta\mathfrak{z} \\ &= \frac{\gamma}{\varrho-1}(\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma}}. \end{aligned} \tag{3.56}$$

Substituting (3.56) into (3.54), since $\gamma^* > 0$, we observe

$$\Omega^\sigma(\mathfrak{r}) \ge \left(\frac{\gamma}{\varrho-1}\right)^{\frac{1}{\gamma^*}} (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma\gamma^*}} \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)^{\frac{1}{\gamma}}.$$

For $\gamma < 0$, this yields

$$[\Omega^\sigma(\mathfrak{r})]^\gamma \le \left(\frac{\gamma}{\varrho-1}\right)^{\gamma-1} (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}} \int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}. \tag{3.57}$$

Multiplying (3.57) with $[\Im(\mathfrak{r})]^{-\varrho}$ and then integrating over $\mathfrak{r}$ from b to ∞, we see

$$
\begin{aligned}
&\int_b^\infty [\Im(\mathfrak{r})]^{-\varrho} [\Omega^\sigma(\mathfrak{r})]^\gamma \Delta\mathfrak{r} \\
\leq\ & \left(\frac{\gamma}{\varrho-1}\right)^{\gamma-1} \int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}} [\Im(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)\Delta\mathfrak{r}. \qquad (3.58)
\end{aligned}
$$

Applying (2.2) on $\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}} [\Im(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)\Delta\mathfrak{r}$, we get

$$
\begin{aligned}
&\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}} [\Im(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)\Delta\mathfrak{r} \\
=\ & u_6(\mathfrak{r})\left(\int_b^{\mathfrak{r}} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)\Bigg|_b^\infty - \int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma u_6(\mathfrak{r})\Delta\mathfrak{r},
\end{aligned}
$$

where

$$
u_6(\mathfrak{r}) = -\int_{\mathfrak{r}}^\infty (\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}} [\Im(\mathfrak{z})]^{-\varrho} \Delta\mathfrak{z}.
$$

Since $\sigma(\mathfrak{z}) \geq \mathfrak{z}$, and $\varrho < 0$, we find

$$
\begin{aligned}
&\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}} [\Im(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)\Delta\mathfrak{r} \\
=\ & \int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma \left[\int_{\mathfrak{r}}^\infty (\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}} [\Im(\mathfrak{z})]^{-\varrho} \Delta\mathfrak{z}\right]\Delta\mathfrak{r} \\
\leq\ & \int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{r})]^\gamma \left[\int_{\mathfrak{r}}^\infty (\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \left(\frac{\mathfrak{z}-b}{\Im(\mathfrak{z})}\right)^\varrho \Delta\mathfrak{z}\right]\Delta\mathfrak{r}. \qquad (3.59)
\end{aligned}
$$

Since $(\mathfrak{z}-b)/\Im(\mathfrak{z})$ is nondecreasing, $\varrho < 0$ and $\mathfrak{z} \geq \mathfrak{r}$, (3.59) becomes

$$
\begin{aligned}
&\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}} [\Im(\mathfrak{r})]^{-\varrho} \left(\int_b^{\sigma(\mathfrak{r})} (\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} [\mho(\mathfrak{z})]^\gamma \Delta\mathfrak{z}\right)\Delta\mathfrak{r} \\
\leq\ & \int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} (\mathfrak{r}-b)^\varrho [\mho(\mathfrak{r})]^\gamma [\Im(\mathfrak{r})]^{-\varrho} \left[\int_{\mathfrak{r}}^\infty (\sigma(\mathfrak{z})-b)^{\frac{\varrho-1}{\gamma^*}-\varrho} \Delta\mathfrak{z}\right]\Delta\mathfrak{r} \\
=\ & \int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}} (\mathfrak{r}-b)^\varrho [\mho(\mathfrak{r})]^\gamma [\Im(\mathfrak{r})]^{-\varrho} \left[\int_{\mathfrak{r}}^\infty (\sigma(\mathfrak{z})-b)^{\frac{1-\varrho}{\gamma}-1} \Delta\mathfrak{z}\right]\Delta\mathfrak{r}. \qquad (3.60)
\end{aligned}
$$

Since $\varrho < 0$ and $\gamma < 0$, then by using (3.16), we have

$$
\frac{\gamma}{1-\varrho}\left[(\mathfrak{z}-b)^{\frac{1-\varrho}{\gamma}}\right]^\Delta \geq (\sigma(\mathfrak{z})-b)^{\frac{1-\varrho}{\gamma}-1},
$$

and then

$$
\begin{aligned}
\int_{\mathfrak{r}}^\infty (\sigma(\mathfrak{z})-b)^{\frac{1-\varrho}{\gamma}-1} \Delta\mathfrak{z} \ &\leq\ \frac{\gamma}{1-\varrho}\int_{\mathfrak{r}}^\infty \left[(\mathfrak{z}-b)^{\frac{1-\varrho}{\gamma}}\right]^\Delta \Delta\mathfrak{z} \\
&=\ \frac{\gamma}{\varrho-1}(\mathfrak{r}-b)^{\frac{1-\varrho}{\gamma}}. \qquad (3.61)
\end{aligned}
$$

Substituting (3.61) into (3.60) and using (3.1) (note that $\frac{\varrho-1}{\gamma}-\varrho\geq 0$), we observe that

$$\begin{aligned}
&\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma^*}}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\left(\int_b^{\sigma(\mathfrak{r})}(\sigma(\mathfrak{z})-b)^{\frac{1+\gamma-\varrho}{\gamma^*}}[\mho(\mathfrak{z})]^{\gamma}\Delta\mathfrak{z}\right)\Delta\mathfrak{r}\\
\leq\ & \frac{\gamma}{\varrho-1}\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{1-\varrho}{\gamma^*}-1+\gamma}(\mathfrak{r}-b)^{\varrho+\frac{1-\varrho}{\gamma}}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\Delta\mathfrak{r}\\
\leq\ & \frac{\gamma}{\varrho-1}\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{1-\varrho}{\gamma^*}-1}(\mathfrak{r}-b)^{\varrho+\gamma+\frac{1-\varrho}{\gamma}}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\Delta\mathfrak{r}\\
\leq\ & \frac{\gamma}{\varrho-1}\int_b^\infty (\sigma(\mathfrak{r})-b)^{\frac{\varrho-1}{\gamma}-\varrho}(\mathfrak{r}-b)^{\varrho+\gamma+\frac{1-\varrho}{\gamma}}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\Delta\mathfrak{r}\\
\leq\ & \frac{\gamma}{\varrho-1}\aleph^{\frac{\varrho-1}{\gamma}-\varrho}\int_b^\infty (\mathfrak{r}-b)^{\gamma}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\Delta\mathfrak{r}.
\end{aligned}\tag{3.62}$$

Substituting (3.62) into (3.58), we obtain

$$\int_b^\infty [\mathfrak{I}(\mathfrak{r})]^{-\varrho}[\Omega^{\sigma}(\mathfrak{r})]^{\gamma}\Delta\mathfrak{r}\leq\left(\frac{\gamma}{\varrho-1}\right)^{\gamma}\aleph^{\frac{\varrho-1}{\gamma}-\varrho}\int_b^\infty(\mathfrak{r}-b)^{\gamma}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}\Delta\mathfrak{r},$$

which is (3.42) with $\mathcal{M}=(\gamma/(\varrho-1))^{\gamma}\aleph^{\frac{\varrho-1}{\gamma}-\varrho}$. □

Remark 3.3. *In Theorem 3.3, if* $\mathbb{T}=\mathbb{R}$, *and* $b=0$, *then (3.1) holds with* $\aleph=1$, *and (3.42) reduces to (1.7). In addition, for* $\mathfrak{I}(\mathfrak{r})=\mathfrak{r}$, *we get (1.4).*

Corollary 3.5. *If* $\mathbb{T}=\mathbb{N}_0$, $b=0$, $\varrho,\gamma<0$, *and* $\{s_n\}_{n=0}^{\infty}$, $\{t_n\}_{n=0}^{\infty}$ *are positive sequences with the property* n/t_n *bing nondecreasing, then (3.1) holds with* $\aleph=2$. *Consequently, the following inequality holds:*

$$\sum_{n=0}^{\infty}[t_n]^{-\varrho}\left[\sum_{k=0}^{n}s_k\right]^{\gamma}\leq\mathcal{M}\sum_{n=0}^{\infty}n^{\gamma}[s_n]^{\gamma}[t_n]^{-\varrho},$$

where

$$\mathcal{M}=\begin{cases}\left(\frac{\gamma}{\varrho-1}\right)^{\gamma}, & (\varrho-1)/\gamma\leq 1;\\ 2^{\frac{\varrho-1}{\gamma}-\varrho}\left(\frac{\gamma}{\varrho-1}\right)^{\gamma}, & (\varrho-1)/\gamma\geq 1.\end{cases}$$

Corollary 3.6. *Let* $\mathbb{T}=q^{\mathbb{N}_0}$ *for* $q>1$, $b\in\mathbb{T}$, $\gamma,\gamma^*=\gamma/(\gamma-1)$, $\varrho<0$, *and* $\mho,\mathfrak{I}$ *be positive sequences on* $[b,\infty)$ *such that* $(\mathfrak{r}-b)/\mathfrak{I}(\mathfrak{r})$ *is nondecreasing. If*

$$\frac{\mathfrak{r}-b}{q\mathfrak{r}-b}\geq\frac{1}{\aleph},\quad \mathfrak{r}\in(b,\infty)$$

holds, then

$$\sum_{\mathfrak{r}=b}^{\infty}\mathfrak{r}[\mathfrak{I}(\mathfrak{r})]^{-\varrho}[\Omega(q\mathfrak{r})]^{\gamma}\leq\mathcal{M}\sum_{\mathfrak{r}=b}^{\infty}\mathfrak{r}(\mathfrak{r}-b)^{\gamma}[\mho(\mathfrak{r})]^{\gamma}[\mathfrak{I}(\mathfrak{r})]^{-\varrho},$$

where $\Omega(\mathfrak{r})=\sum_{\mathfrak{z}=b}^{\mathfrak{r}/q}(q-1)\mathfrak{z}\mho(\mathfrak{z})$ *and*

$$\mathcal{M}=\begin{cases}\left(\frac{\gamma}{\varrho-1}\right)^{\gamma}, & (\varrho-1)/\gamma\leq 1;\\ \left(\frac{\gamma}{\varrho-1}\right)^{\gamma}\aleph^{\frac{\varrho-1}{\gamma}-\varrho}, & (\varrho-1)/\gamma\geq 1.\end{cases}$$

4. Conclusions

This work extends Hardy's foundational inequalities by exploring their generalizations with negative parameters within the framework of time scale theory. We have derived new results by providing time scale versions of previously established inequalities, along with their discrete analogues. These contributions offer a more comprehensive perspective on Hardy-type inequalities, demonstrating their flexibility and potential for further research. Our findings underscore the importance of integrating time scale calculus into classical inequality theory, unveiling promising directions for future investigations.

Looking ahead, we plan to expand on these results by applying alpha-conformable fractional derivatives on time scales, facilitating a deeper exploration of fractional calculus in this setting. Additionally, we aim to broaden our findings by examining their application within the framework of diamond-alpha calculus, which we believe will offer fresh insights into this developing field.

Author contributions

A. M. Ahmed, A. I. Saied and H. M. Rezk: Investigation, Software, Supervision, Writing-original draft; M. Zakarya, A. A. I Al-Thaqfan and M. Ali: Writing-review editing, Funding. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that there are no conflict of interests in this paper.

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