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Research article

Managing incomplete general hesitant linguistic preference relations and their application

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Abstract: Hesitant linguistic preference relations (HLPRs) are useful tools for decision makers (DMs) to express their qualitative judgements. However, the traditional HLPRs have one prominent drawback, which is to sort the linguistic values in a hesitant linguistic set. This will distort the DMs' initial judgements. In the present paper, a revised definition of HLPR, called general HLPR (GHLPR), was proposed. A characterization was explored for LPRs. Then, the characterization was extended to GHLPRs. Based on the characterization, the estimation of unknown entries in incomplete GHLPRs were carried out by two algorithms. The group decision-making problems with incomplete GHLPRs were settled by another algorithm. Finally, a case study was illustrated, and comparisons showed that our methods were more reasonable than the existent methods.

Keywords: incomplete general hesitant linguistic preference relation; additive consistency; characterization; missing value estimation; group decision-making
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1. Introduction

Pairwise comparisons are useful tools to convey the decision makers' (DMs) judgements in group decision-making (GDM). The most frequently used preference relations (PRs) are multiplicative PR [1], fuzzy PR [2–6], and linguistic PR (LPR) [7].

In the above traditional PRs, each entry only has one value, and this fails to express DMs' hesitant information when he uses a few values to articulate his preferences. In order to accommodate these cases, the hesitant fuzzy set (HFS) emerges ([8]). It also has many extensions, such as probabilistic picture hesitant fuzzy set [9], general hesitant intuitionistic fuzzy N-soft set [10], weighted hesitant fuzzy sets [11] and hesitant multi-fuzzy soft set [12]. After this, the hesitant linguistic term set (HLTS) is extended in [13] to deal with the linguistic hesitant. Some hesitant PRs are proposed, such as hesitant fuzzy PR [14,15] and hesitant multiplicative PR [14,16]. Based on the LPR and HLTS, the hesitant linguistic preference relations (HLPRs) is naturally extended ([17]).

At present, HLPRs have been widely investigated [18–21]. As different HLTSs have different numbers of values, one nature way is to make all the HLTSs with the same length. Therefore, some normalization methods are proposed, such as α-normalization, β-normalization [17], least common multiple expansion (LCME) [22], additive consistency (AC) based normalization for hesitant fuzzy preference relations [23], etc. Consistency and consensus are two important aspects of HLPRs in GDM. Zhu and Xu [17] first developed the HLPRs and studied the AC of HLPRs based on β-normalization. After this, several consistency problems were investigated for HLPRs from different points of view, such as multiplicative consistency (MC) [18], interval consistency index [24], consistency based on personalized individual semantics [25], worst consistency index [20], etc. For the consensus of HLPRs, generally, the consistency is also considered at the same time and diverse models are proposed, such as the expectation model [19], local adjustment strategy [26], optimization model [27], etc.

However, the DMs may lack necessary knowledge for a specific problem, limited expertise, be unwilling or unsure to give some of their pairwise preference values, or it is more convenient to omit some direct key comparisons. In this situation, missing information is emerged [28–33], i.e., the DM may develop the incomplete HLPRs. Several models have been developed to deal with the incomplete HLPRs, such as integer programming model [34], optimization models [35–37] and mathematical programming [38]. However, some drawbacks of the existent studies on incomplete HLPRs are:

(1) All the existing incomplete HLPRs are the extension of Zhu and Xu [17]'s HLPRs, in which the known values are sorted. As we can see in our example (see Example 1), the rearrangement of the values will misinterpret the DMs' judgment and also contradict with the consistency property of HLPRs. Thus, the definition of HLPRs is not reasonable.

(2) To estimate the unknown entries, the parameter of the β-normalization is randomly selected, which also will misinterpret the DMs' initial preferences.

All the existing work is very important for our study. However, as we have pointed out, there are some limitations for the existing work, such as the definition of the HLPR and the normalization process. The main work of the current paper is:

(1) A revised HLPR, called general HLPR (GHLPR), is proposed, in which the elements do not need to be sorted. As we can see, this will keep the DM's initial judgment. Furthermore, normalized GHLPR (NGHLPR) and incomplete NGHLPR are also similarly defined.

(2) A characterization is explored for LPRs, and the characterization is also extended to NGHLPRs.

(3) The incomplete GHLPR is studied and two algorithm procedures are proposed to estimate the unknown entries. The obtained NGHLPR will be more consistent than the existing ones.

The reminders are structured as follows. In Section 2, we first study the characterization of LPRs. Then, we illustrate an example to show the irrationality of the existing definition of HLPRs. We redefine the concept of HLPR, which is called GHLPR. In Section 3, we develop two algorithms to amend the incomplete NGHLPRs. A GDM with incomplete GHLPRs is developed. In Section 4, a case study is illustrated showing how to execute the algorithms. Section 5 provides the detailed comparisons with the previous studies to exhibit the performances of the current approach. Section 6 concludes the paper.

2. Preliminaries

2.1. LPRs

For sake of simplicity, let $N = \{1, 2, \cdots, n\}$.

Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set (LTS), where s_i represents a possible value for a linguistic variable, and $g + 1$ is the granularity in the LTS [39]. A typical LTS has 9 LTs, i.e., $S = \{s_0, s_1, \cdots, s_8\}$, in which s_0 denotes extremely poor, s_1 denotes very poor, s_2 denotes poor, s_3 denotes slightly poor and the median s_4 denotes fair. Inversely, s_5 to s_8 denotes the good corresponding to s_4 to s_0.

A continuous LTS $\overline{S} = \{s_\pi | s_0 \leq s_\pi \leq s_g, \pi \in [0, g]\}$ is an extension of the distinct LTS S in Xu [40], and the following holds:

(1) $s_\pi \oplus s_\rho = s_{\pi+\rho}$;

(2) $s_\pi \oplus s_\rho = s_\rho \oplus s_\pi$;

(3) $\delta s_\pi = s_{\delta\pi}$;

(4) $(\delta_1 + \delta_2) s_\pi = \delta_1 s_\pi \oplus \delta_2 s_\pi; \delta, \delta_1, \delta_2 \geq 0$;

(5) $\delta (s_\pi \oplus s_\rho) = \delta s_\pi \oplus \delta s_\rho$.

Here, $s_\pi, s_\rho \in \overline{S}$.

If $s = s_\pi$, we define the lower index of s as: $f(s_\pi) = \pi$. On the contrary, there exists the inverse function of f, such that: $f^{-1}(\pi) = s_\pi$.

Definition 1. [7] An LPR is depicted by a matrix $L = (l_{ij})_{n\times n}$, satisfying

$$l_{ij} \oplus l_{ji} = s_g,\ i, j \in N \tag{1}$$

Definition 2. [41] An LPR $L=(l_{ij})_{n\times n}$ is AC if

$$f(l_{ij}) = f(l_{ik}) + f(l_{kj}) - g/2,\ i, j, k \in N \tag{2}$$

Theorem 1. [41] Let $L=(l_{ij})_{n\times n}$ be an LPR, where $f(l_{ij}) = \frac{1}{n}\sum_{k=1}^{n}\left(f(l_{ik}) + f(l_{kj}) - g/2\right)$, then L is an AC LPR.

2.2. HLTS

HFS was initially proposed by Torra [8], which allows DM to give his preferences on alternatives in a set of fuzzy values. Motivated by this idea, HLTS is an extension of HFS in [13]. Let $A = \{a_1, a_2, \cdots, a_n\}$ be a set of alternatives.

Definition 3. [13] An HLTS H_S is a set of consecutive ordered LTs in S.

An HFLS can be simply specified as:

$$c_S = \{< a, c_S(a) > | a \in A\}$$

where $c_S(a)$ is a set of LTs in S, denoting the possible membership degrees of the element $a \in A$ to the set c. For simplicity, c_S is termed a hesitant linguistic element (HLE).

Remark 1. In [13], the LTs in an HLTS should be consecutive and ordered, and in this situation, the HLTS is given by one DM. However, Wang [42] deemed that HLTS may be the union of several LTs from different DMs, and in this situation, the HLTS may not be consecutive and called the extended HLTSs. In this paper, we still use HLTSs to denote the hesitant linguistic information provided by the DMs.

Definition 4. [43] For an HLE c, the score function of c is defined as

$$\zeta(c) = \frac{\sum_{s_\pi \in c} \pi}{\# c} \tag{3}$$

where $\#c$ is the cardinality of the LTs in c.

Comparison of two HLEs c_1 and c_2,

1) $c_1 > c_2$ if $\zeta(c_1) > \zeta(c_2)$;
2) $c_1 = c_2$ if $\zeta(c_1) = \zeta(c_2)$.

Definition 5. [13] For an HLE c,

(1) the upper bound c^+: $c^+ = \max(s_i)$;
(2) the lower bound c^-: $c^- = \min(s_i)$;

β-normalization is proposed to add some values in shorter HLEs in [17].

Definition 6. [17] For an HLE, $c = \{c^{\sigma(q)} | q = 1, 2, \cdots, \#c\}$, $\bar{c} = \xi c^+ + (1 - \xi) c^-$ $(0 \le \xi \le 1)$ is an added LT.

2.3. HLPRs

The concept of HLPR is introduced in Zhu and Xu [17], which is described as:

Definition 7. [17] Matrix $C = (c_{ij})_{n \times n}$ is an HLPR if

$$C = \begin{pmatrix} \{s_{g/2}\} & \{c_{12}^{(1)}, \cdots, c_{12}^{(\#c_{12})}\} & \cdots & \{c_{1n}^{(1)}, \cdots, c_{1n}^{(\#c_{1n})}\} \\ \{c_{21}^{(1)}, \cdots, c_{21}^{\#c_{21}}\} & \{s_{g/2}\} & \cdots & \{c_{2n}^{(1)}, \cdots, c_{2n}^{(\#c_{2n})}\} \\ \vdots & \ddots & \{s_{g/2}\} & \vdots \\ \{c_{n1}^{(1)}, \cdots, c_{n1}^{(\#c_{n1})}\} & \cdots & \cdots & \{s_{g/2}\} \end{pmatrix} \tag{4}$$

where $c_{ij} = \{c_{ij}^{\sigma(q)} | q = 1, 2, \cdots, \#c_{ij}\}$ ($\#c_{ij}$ is the cardinality of LTs in c_{ij}) is an HLE, and

$$\begin{cases} c_{ij}^{\sigma(q)} + c_{ji}^{\sigma(q)} = s_g, \\ c_{ii} = \{s_{g/2}\}, \\ \# c_{ij} = \# c_{ji}, \\ c_{ij}^{\sigma(q)} < c_{ij}^{\sigma(q+1)}, c_{ji}^{\sigma(q+1)} < c_{ji}^{\sigma(q)}, \forall i < j, i, j \in N \end{cases} \tag{5}$$

As the numbers in different entries c_{ij} are not identical, Zhu and Xu [17] used β-normalization to define a normalized HLPR (NHLPR).

Definition 8. [17] Matrix $\bar{C} = (\bar{c}_{ij})_{n \times n}$ is an NHLPR if

$$\overline{C}=\begin{pmatrix} \{s_{g/2}\} & \{\overline{c}_{12}^{(1)},\cdots,\overline{c}_{12}^{\#cN}\} & \cdots & \{\overline{c}_{1n}^{(1)},\cdots,\overline{c}_{1n}^{(\#cN)}\} \\ \{\overline{c}_{21}^{(1)},\cdots,\overline{c}_{21}^{(\#cN)}\} & \{s_{g/2}\} & \cdots & \{\overline{c}_{2n}^{(1)},\cdots,\overline{c}_{2n}^{(\#cN)}\} \\ \vdots & \ddots & \{s_{g/2}\} & \vdots \\ \{\overline{c}_{n1}^{(1)},\cdots,\overline{c}_{n1}^{(\#cN)}\} & \cdots & \cdots & \{s_{g/2}\} \end{pmatrix} \tag{6}$$

and

$$\begin{cases} \#cN=\max\{\#\overline{c}_{ij} \mid i,j\in N, i\neq j\} \\ \overline{c}_{ij}^{q}+\overline{c}_{ji}^{q}=s_g, \\ \overline{c}_{ii}=\{s_{g/2}\}, \\ \overline{c}_{ij}^{\sigma(q)}<\overline{c}_{ij}^{\sigma(q+1)}, \overline{c}_{ji}^{\sigma(q+1)}<\overline{c}_{ji}^{\sigma(q)}, \forall i<j, i,j\in N. \end{cases} \tag{7}$$

with some of the added values $\overline{c}_{ij}^{(q)}$ with ξ $(0\leq\xi\leq 1)$.

Definition 9. [17] Let $C=(c_{ij})_{n\times n}$ be an HLPR, and its NHLPR $\overline{C}=(\overline{c}_{ij})_{n\times n}$ with ξ, if

$$\overline{c}_{ij}^{\sigma(q)}\oplus s_{g/2}=\overline{c}_{ik}^{\sigma(q)}\oplus\overline{c}_{kj}^{\sigma(q)},\ \ i,k,j\in N. \tag{8}$$

Then, $C=(c_{ij})_{n\times n}$ is an AC HLPR with ξ.

However, in some time, the lower index of obtained entries in an HLPR may be out of the interval $[0, g]$, but in the interval $[-u, g+u]$, where $u > 0$. In such a case, we can utilize the next transformation function:

$$y\colon [-u, g+u]\to[0, g]$$

$$y(x)=g\frac{x+u}{g+2u}. \tag{9}$$

3. GHLPRs and incomplete GHLPRs

3.1. GHLPRs

In Definition 8, the values in each HLE are sorted. We show that this is problematic using the below example.

Example 1. Assume that two DMs $E=\{e_1, e_2\}$ participate to give their preferences on three alternatives in the LTS $S=\{s_0, s_1, \cdots, s_8\}$. Table 1 displays these judgments.

Table 1 shows that expert e_1 reckons that the alternative a_1 is very poor comparing to alternative a_2, and gives s_1. However, the second expert e_2 reckons that the two alternatives are indifference, gives s_4, and so on. Thus, the hesitant preferences of these two DMs are $\{s_1, s_4\}$, which constitutes an HLE. All these HLEs can be structured into the following HLPR C:

$$C=\begin{bmatrix} \{s_4\} & \{s_1,s_4\} & \{s_2,s_5\} & \{s_2,s_3\} \\ \{s_7,s_4\} & \{s_4\} & \{s_5,s_5\} & \{s_2,s_6\} \\ \{s_6,s_3\} & \{s_3,s_3\} & \{s_4\} & \{s_1,s_5\} \\ \{s_6,s_5\} & \{s_6,s_2\} & \{s_7,s_3\} & \{s_4\} \end{bmatrix}$$

Table 1. Experts' judgments.

Pairs of alternatives		Comparisons	
		Expert 1	Expert 2
a_1 versus	a_2	s_1	s_4
	a_3	s_2	s_5
	a_4	s_3	s_2
a_2 versus	a_1	s_7	s_4
	a_3	s_5	s_5
	a_4	s_6	s_2
a_3 versus	a_1	s_6	s_3
	a_2	s_3	s_3
	a_4	s_5	s_1
a_4 versus	a_1	s_5	s_6
	a_2	s_2	s_6
	a_3	s_3	s_7

To verify whether the HLPR C is AC, we have the following two LPRs:

$$\overline{C}^{\sigma(1)} = \begin{bmatrix} s_4 & s_1 & s_2 & s_2 \\ s_7 & s_4 & s_5 & s_2 \\ s_6 & s_3 & s_4 & s_1 \\ s_6 & s_6 & s_7 & s_4 \end{bmatrix}, \quad \overline{C}^{\sigma(2)} = \begin{bmatrix} s_4 & s_4 & s_5 & s_3 \\ s_4 & s_4 & s_5 & s_6 \\ s_3 & s_3 & s_4 & s_5 \\ s_5 & s_2 & s_3 & s_4 \end{bmatrix}.$$

As per Definition 9, it is not difficult to examine that $\overline{C}^{\sigma(1)}$ is inconsistent. According to Eq (8), $f(\overline{c}_{12}^{\sigma(1)}) + f(\overline{c}_{24}^{\sigma(1)}) - g/2 = 1 + 2 - 4 = -1 \neq f(\overline{c}_{14}^{\sigma(1)}) = 2$. Similarly, we can verify $f(\overline{c}_{12}^{\sigma(2)}) + f(\overline{c}_{24}^{\sigma(2)}) - g/2 = 4 + 6 - 4 = 6 \neq f(\overline{c}_{14}^{\sigma(2)}) = 3$. Therefore, C is not AC.

However, according to the preferences e_1 and e_2 given in Table 1, the two LPRs A_1 and A_2 of the two experts are:

$$L_1 = \begin{bmatrix} s_4 & s_1 & s_2 & s_3 \\ s_7 & s_4 & s_5 & s_6 \\ s_6 & s_3 & s_4 & s_5 \\ s_5 & s_2 & s_3 & s_4 \end{bmatrix}, \quad L_2 = \begin{bmatrix} s_4 & s_4 & s_5 & s_2 \\ s_4 & s_4 & s_5 & s_2 \\ s_3 & s_3 & s_4 & s_1 \\ s_6 & s_6 & s_7 & s_4 \end{bmatrix}.$$

As per Eq (2), both A_1 and A_2 are AC. Because $l_{ij} \oplus s_4 = l_{ik} \oplus l_{kj}$ holds for L_1 and L_2, if we combine L_1 and L_2 into an HLPR C, the consistency is destroyed as per Definition 9.

Remark 2. From the above Example 1, we conclude that the rearrangement of the values in an HLE distorts the DMs' initial preferences. What's more, the consistency may be inversely. To remedy the problems, in the following, we introduce the following definitions called GHLPR, and NGHLPR, respectively.

Definition 10. Matrix $C = (c_{ij})_{n\times n}$ is a GHLPR if

$$C=\begin{pmatrix} \{s_{g/2}\} & \{c_{12}^{1},\cdots,c_{12}^{\#c_{12}}\} & \cdots & \{c_{1n}^{1},\cdots,c_{1n}^{\#c_{1n}}\} \\ \{c_{21}^{1},\cdots,c_{21}^{\#c_{21}}\} & \{s_{g/2}\} & \cdots & \{c_{2n}^{1},\cdots,c_{2n}^{\#c_{2n}}\} \\ \vdots & \ddots & \{s_{g/2}\} & \vdots \\ \{c_{n1}^{1},\cdots,c_{n1}^{\#c_{n1}}\} & \{c_{n2}^{1},\cdots,c_{n2}^{\#c_{2n}}\} & \cdots & \{s_{g/2}\} \end{pmatrix} \tag{10}$$

where $c_{ij} = \{c_{ij}^{q}|\ q=1, 2, \cdots, \#c_{ij}\}$($\#c_{ij}$ is the number of LTs in c_{ij}) is an HLE, and for all $i, j \in N$

$$\begin{cases} c_{ij}^{q}+c_{ji}^{q}=s_g, \\ c_{ii}=\{s_{g/2}\}, \\ \#c_{ij}=\#c_{ji}. \end{cases} \tag{11}$$

Definition 11. Matrix $\overline{C} = (\overline{c}_{ij})_{n\times n}$ is an NGHLPR if

$$\overline{C}=\begin{pmatrix} \{s_{g/2}\} & \{\overline{c}_{12}^{1},\cdots,\overline{c}_{12}^{\#cN}\} & \cdots & \{\overline{c}_{1n}^{1},\cdots,\overline{c}_{1n}^{\#cN}\} \\ \{\overline{c}_{21}^{1},\cdots,\overline{c}_{21}^{\#cN}\} & \{s_{g/2}\} & \cdots & \{\overline{c}_{2n}^{1},\cdots,\overline{c}_{2n}^{\#cN}\} \\ \vdots & \ddots & \{s_{g/2}\} & \vdots \\ \{\overline{c}_{n1}^{1},\cdots,\overline{c}_{n1}^{\#cN}\} & \{\overline{c}_{n2}^{1},\cdots,\overline{c}_{n2}^{\#cN}\} & \cdots & \{s_{g/2}\} \end{pmatrix} \tag{12}$$

and

$$\begin{cases} \#cN=\max\{\#\overline{c}_{ij}\mid i,j\in N, i\neq j\} \\ \overline{c}_{ij}^{q}+\overline{c}_{ji}^{q}=s_g, \\ \overline{c}_{ii}=\{s_{g/2}\}. \end{cases} \tag{13}$$

with some added values $\overline{c}_{ij}^{q}$ with ξ.

Remark 3. In Eq (12), some of the $\overline{c}_{ij}^{q}$ are the added values. Here, the added value $\overline{c}_{ij}^{q}$ may not be between the upper and lower bounds of c_{ij}, i.e., ξ may be not in the interval [0, 1], which is different from Definition 8. Furthermore, the added values are always appended after the original values.

Definition 12. Given an GHLPR $C = (c_{ij})_{n\times n}$ and its NGHLPR $\overline{C} = (\overline{c}_{ij})_{n\times n}$ with ξ, if

$$f(\overline{c}_{ij}^{q})=f(\overline{c}_{ik}^{q})+f(\overline{c}_{kj}^{q})-g/2,\ \ i,j,k\in N \tag{14}$$

then $C = (c_{ij})_{n\times n}$ is an AC GHLPR with ξ.

Example 2. For Example 1, we can obtain that GHLPR is:

$$C=\begin{bmatrix} \{s_4\} & \{s_1,s_4\} & \{s_2,s_5\} & \{s_3,s_2\} \\ \{s_7,s_4\} & \{s_4\} & \{s_5,s_5\} & \{s_6,s_2\} \\ \{s_6,s_3\} & \{s_3,s_3\} & \{s_4\} & \{s_5,s_1\} \\ \{s_5,s_6\} & \{s_2,s_6\} & \{s_3,s_7\} & \{s_4\} \end{bmatrix}.$$

As per Eq (14), it is easy to verify that C is AC, which shows that the DMs' original information is consistent, and conforms to the reality.

Herrera-Viedma, Herrera, and Chiclana et al. [44] explored the characterization of fuzzy PR, and Xu, Li, and Wang [45] extended the characterization to a more general case. In the following, we study the characterization of LPRs.

Proposition 1. An LPR $L = (l_{ij})_{n\times n}$ is AC and the following equations holds:

(1) $l_{ik} \oplus l_{kj} \oplus l_{ji} = \frac{3}{2} s_g$, $\forall i < k < j$.

(2) $l_{j(j+1)} \oplus l_{(j+1)(j+2)} \oplus \cdots \oplus l_{(k-1)k} \oplus a_{kj} = \frac{k-j+1}{2} s_g$, $\forall j < k$.

(3) $l_{jk_1} \oplus l_{k_1k_2} \oplus \cdots \oplus l_{k_{t-1}k_t} \oplus l_{k_tj} = \frac{t+1}{2} s_g$, $\forall j < k_1 < k_2 < \cdots < k_t$.

Based on Proposition 1, we have the following results:

Proposition 2. An NGHLPR $C = (c_{ij})_{n\times n}$ is AC, the following equations are identical:

(1) $\overline{c}_{ji}^{q} \oplus \overline{c}_{ik}^{q} \oplus \overline{c}_{kj}^{q} = \frac{3}{2} s_g$, $\forall i < k < j$; (15)

(2) $\overline{c}_{j(j+1)}^{q} \oplus \overline{c}_{(j+1)(j+2)}^{q} \oplus \cdots \oplus \overline{c}_{(k-1)k}^{q} \oplus \overline{c}_{kj}^{l} = \frac{k-j+1}{2} s_g$, $\forall j < k$; (16)

(3) $\overline{c}_{jk_1}^{q} \oplus \overline{c}_{k_1k_2}^{q} \oplus \cdots \oplus \overline{c}_{k_{t-1}k_t}^{q} \oplus \overline{c}_{k_tj}^{q} = \frac{t+1}{2} s_g$, $\forall j < k_1 < k_2 < \cdots < k_t$. (17)

Proof. (15)$\Rightarrow$(17). When $t = 1$, Eq (17) becomes $\overline{c}_{ij}^{l} \oplus \overline{c}_{ji}^{l} = s_g$. Next, we assume $t = n$ is true, i.e.,

$$f(\overline{c}_{jk_1}^{q}) + f(\overline{c}_{k_1k_2}^{q}) + \cdots + f(\overline{c}_{k_{n-1}k_n}^{q}) + f(\overline{c}_{k_nj}^{q}) = \frac{n+1}{2} g.$$

Then, when $t = n + 1$, we have:

$$\begin{aligned}
& f(\overline{c}_{jk_1}^{q}) + f(\overline{c}_{k_1k_2}^{q}) + \cdots + f(\overline{c}_{k_{n-1}k_n}^{q}) + f(\overline{c}_{k_nk_{n+1}}^{q}) + f(\overline{c}_{k_{n+1}j}^{q}) \\
&= \left(f(\overline{c}_{jk_1}^{q}) + f(\overline{c}_{k_1k_2}^{q}) + \cdots + f(\overline{c}_{k_{n-1}k_n}^{q}) \right) + f(\overline{c}_{k_nk_{n+1}}^{q}) + f(\overline{c}_{k_{n+1}j}^{q}) \\
&= \frac{n+1}{2} g - f(\overline{c}_{k_nj}^{q}) + f(\overline{c}_{k_nk_{n+1}}^{q}) + f(\overline{c}_{k_{n+1}j}^{q}) \\
&= \frac{n+1}{2} g - \left(g - f(\overline{c}_{jk_n}^{q}) \right) + f(\overline{c}_{k_nk_{n+1}}^{q}) + f(\overline{c}_{k_{n+1}j}^{q}) \\
&= \frac{n-1}{2} g + f(\overline{c}_{jk_n}^{q}) + f(\overline{c}_{k_nk_{n+1}}^{q}) + f(\overline{c}_{k_{n+1}j}^{q}) \\
&= \frac{n-1}{2} g + \frac{3}{2} g \\
&= \frac{(n+1)+1}{2} g.
\end{aligned}$$

So, the result is established.

(17)$\Rightarrow$(15)

As per Eq (17), one has:

$$f(\overline{c}_{jk_1}^{q}) + f(\overline{c}_{k_1k_2}^{q}) + \cdots + f(\overline{c}_{k_{t-1}i}^{q}) + f(\overline{c}_{ij}^{q}) = \frac{t+1}{2} g$$

$$f(\overline{c}_{ik_{1'}}^{q}) + f(\overline{c}_{k_{1'}k_{2'}}^{q}) + \cdots + f(\overline{c}_{k_{t'-1}k}^{q}) + f(\overline{c}_{ki}^{q}) = \frac{t+1}{2} g$$

$$f(\overline{c}_{kk_{1''}}^{q})+f(\overline{c}_{k_{1''}k_{2''}}^{q})+\cdots+f(\overline{c}_{k_{t''-1}i}^{q})+f(\overline{c}_{ik}^{q})=\frac{t+1}{2}g.$$

Adding the above three formulas, the left side of Eq (17) is:

$$\begin{aligned}
&f(\overline{c}_{jk_1}^{q})+f(\overline{c}_{k_1k_2}^{q})+\cdots+f(\overline{c}_{k_{t-1}i}^{q})+f(\overline{c}_{ij}^{q})+f(\overline{c}_{ik_{1'}}^{q})+f(\overline{c}_{k_{1'}k_{2'}}^{q})+\cdots\\
&\quad+f(\overline{c}_{k_{t'-1}k}^{q})+f(\overline{c}_{ki}^{q})+f(\overline{c}_{jk_{1''}}^{q})+f(\overline{c}_{k_{1''}k_{2''}}^{q})+\cdots+f(\overline{c}_{k_{t''-1}i}^{q})+f(\overline{c}_{ij}^{q})\\
&=\Big(f(\overline{c}_{jk_1}^{q})+f(\overline{c}_{k_1k_2}^{q})+\cdots+f(\overline{c}_{k_{t-1}i}^{q})+f(\overline{c}_{ik_{1'}}^{q})+f(\overline{c}_{k_{1'}k_{2'}}^{q})+\cdots\\
&\quad+f(\overline{c}_{k_{t'-1}k}^{q})+f(\overline{c}_{jk_{1''}}^{q})+f(\overline{c}_{k_{1''}k_{2''}}^{q})+\cdots+f(\overline{c}_{k_{t''-1}j}^{q})\Big)+f(\overline{c}_{ij}^{q})+f(\overline{c}_{ki}^{q})+f(\overline{c}_{jk}^{q})\\
&=\frac{3t}{2}g+f(\overline{c}_{ij}^{q})+f(\overline{c}_{ki}^{q})+f(\overline{c}_{jk}^{q})
\end{aligned}.$$

The right side of Eq (17) is: $\frac{3t+3}{2}g$.

Thus,

$$\frac{3t}{2}g+f(\overline{c}_{ij}^{q})+f(\overline{c}_{ki}^{q})+f(\overline{c}_{jk}^{q})=\frac{3t+3}{2}g$$

i.e.,

$$f(\overline{c}_{ij}^{q})+f(\overline{c}_{ki}^{q})+f(\overline{c}_{jk}^{q})=\frac{3}{2}g$$

$$\left(g-f(\overline{c}_{ji}^{q})\right)+\left(g-f(\overline{c}_{ik}^{q})\right)+\left(g-f(\overline{c}_{kj}^{q})\right)=\frac{3}{2}g$$

i.e.,

$$f(\overline{c}_{ji}^{q})+f(\overline{c}_{ik}^{q})+f(\overline{c}_{kj}^{q})=\frac{3}{2}g.$$

The results are proved.

Theorem 2. Let $C=(c_{ij})_{n\times n}$ be GHLPR, and its associated NGHLPR is $\overline{C}=(\overline{c}_{ij})_{n\times n}$ with ξ. Let

$$bc_{ij}^{q}=f^{-1}\left(\frac{1}{n}\sum_{k=1}^{n}\left(f(\overline{c}_{ik}^{q})+f(\overline{c}_{kj}^{q})-g/2\right)\right). \tag{18}$$

Then, $bC=(bc_{ij})_{n\times n}$ is called AC GHLPR with ξ.

Proposition 3. Let $\overline{C}_1$, $\overline{C}_2$, ⋯, $\overline{C}_m$, be m NGHLPRs, then their weighted average

$$\overline{C}=\lambda_1\overline{C}_1\oplus\lambda_2\overline{C}_2\oplus\cdots\oplus\lambda_m\overline{C}_m,\ \ \lambda_i\in[0,1],\ \ \sum_{i=1}^{m}\lambda_i=1 \tag{19}$$

is also an NGHLPR, which satisfies $\overline{c}_{ij}^{q}\oplus\overline{c}_{ji}^{q}=s_g$, $\forall i,j=1,2,\cdots,n$.

Proof. Since $\overline{C}_1$, $\overline{C}_2$, ⋯, $\overline{C}_m$ are NGHLPRs, it follows that

$$\overline{c}_{ij}^{q}\oplus\overline{c}_{ji}^{q}=s_g.$$

Then, by Eq (19), we have

$$\overline{c}_{ij}^{q} = \lambda_1 \overline{c}_{ij,1}^{q} \oplus \lambda_2 \overline{c}_{ij,2}^{q} \oplus \cdots \oplus \lambda_m \overline{c}_{ij,m}^{q},$$

$$\overline{c}_{ji}^{q} = \lambda_1 \overline{c}_{ji,1}^{q} \oplus \lambda_2 \overline{c}_{ji,2}^{q} \oplus \cdots \oplus \lambda_m \overline{c}_{ji,m}^{q},$$

$$\begin{aligned}
\overline{c}_{ij}^{q} \oplus \overline{c}_{ji}^{q} &= \left(\lambda_1 \overline{c}_{ij,1}^{q} \oplus \lambda_2 \overline{c}_{ij,2}^{q} \oplus \cdots \oplus \lambda_m \overline{c}_{ij,m}^{q}\right) \oplus \left(\lambda_1 \overline{c}_{ji,1}^{q} \oplus \lambda_2 \overline{c}_{ji,2}^{q} \oplus \cdots \oplus \lambda_m \overline{c}_{ji,m}^{q}\right) \\
&= \lambda_1 (\overline{c}_{ij,1}^{q} \oplus \overline{c}_{ji,1}^{q}) \oplus \lambda_2 (\overline{c}_{ij,2}^{q} \oplus \overline{c}_{ji,2}^{q}) \oplus \cdots \oplus \lambda_m (\overline{c}_{ij,m}^{q} \oplus \overline{c}_{ji,m}^{q}) \\
&= \lambda_1 s_g \oplus \lambda_2 s_g \oplus \cdots \oplus \lambda_m s_g \\
&= \mathop{\oplus}_{i=1}^{m} \lambda_i s_g \\
&= s_g
\end{aligned}.$$

The proof is thus completed.

3.2. Incomplete GHLPRs

In this part, we introduce the incomplete GHLPRs.

Tang, Liao, and Li et al. [46] extended the HLPRs into incomplete environment and defined the incomplete HLPRs. It also needs the known values in an HFLE to be in ascending or descending order. They further developed some produces to estimate the missing hesitant LTs. For the same reason presented, there exist some problems for the estimated complete HLPRs. This also can be verified from their Example 4, the estimated value $\overline{c}_{45}' = \{s_{-5}, s_{-5}, s_{-6}, s_{-7}\}$ which contradicts with their definition for HLPRs and requires $c_{ij}^{\sigma(q)} < c_{ij}^{\sigma(q+1)}$ $(i < j)$. Similarly, we define incomplete GHLPRs.

Definition 13. An incomplete GHLPR is matrix $C = (c_{ij})_{n\times n}$ if some of the entries are not provided, and the given entries satisfy Eq (11).

Generally, the incomplete GHLPR can be expressed in the form of matrix as:

$$C = \begin{pmatrix} \{s_{g/2}\} & \{c_{12}^{1}, \cdots, c_{12}^{\#c_{12}}\} & \cdots & \cdots & \{c_{1n}^{1}, \cdots, c_{1n}^{\#c_{1n}}\} \\ \{c_{21}^{1}, \cdots, c_{21}^{\#c_{21}}\} & \{s_{g/2}\} & \cdots & \cdots & \{c_{2n}^{1}, \cdots, c_{2n}^{\#c_{2n}}\} \\ \vdots & \vdots & \ddots & x & \vdots \\ \vdots & \vdots & x & \ddots & \vdots \\ \{c_{n1}^{1}, \cdots, c_{n1}^{\#c_{n1}}\} & \{c_{n2}^{1}, \cdots, c_{n2}^{\#c_{n2}}\} & \cdots & \cdots & \{s_{g/2}\} \end{pmatrix}. \tag{20}$$

Definition 14. An incomplete NGHLPR is $\overline{C} = (\overline{c}_{ij})_{n\times n}$ if some of the entries are not provided, and the given entries satisfy Eq (13) where some of the added values are $\overline{c}_{ij}^{q}$ with ξ.

Similarly, the incomplete NGHLPR can be expressed in the form of matrix as:

$$\overline{C} = \begin{pmatrix} \{s_{g/2}\} & \{\overline{c}_{12}^{1}, \cdots, \overline{c}_{12}^{\#cN}\} & \cdots & \cdots & \{\overline{c}_{1n}^{1}, \cdots, \overline{c}_{1n}^{\#cN}\} \\ \{\overline{c}_{21}^{1}, \cdots, \overline{c}_{21}^{\#cN}\} & \{s_{g/2}\} & \cdots & \cdots & \{\overline{c}_{2n}^{1}, \cdots, \overline{c}_{2n}^{\#cN}\} \\ \vdots & \vdots & \ddots & x & \vdots \\ \vdots & \vdots & x & \ddots & \vdots \\ \{\overline{c}_{n1}^{1}, \cdots, \overline{c}_{n1}^{\#cN}\} & \{\overline{c}_{n2}^{1}, \cdots, \overline{c}_{n2}^{\#cN}\} & \cdots & \cdots & \{s_{g/2}\} \end{pmatrix}. \tag{21}$$

Remark 4. In the incomplete (N)GHLPR, we use "x" to denote the entries which are not provided by the DM. This means that the DM does not provide his hesitant preferences due to various reasons, such as do not have the knowledge on the problem, do not have time, or are reluctant to provide his judgements for some sensitive issues. For the incomplete preference relations, the number of unknown entries "*x*" should not be very large. Each alternative should be compared with the other alternatives at least once. Otherwise, some of the unknown "*x*" could not be estimated. In this case, it is unacceptable. In this paper, we assume the incomplete GHLPR is acceptable. That is, all the unknown entries "*x*" could be finally inferred by other entries directly or indirectly.

4. Missing elements estimation for incomplete GHLPRs based on AC

4.1. Estimation of unknown entries in incomplete GHLPRs

As the GHLPRs may be incomplete, the main purpose is to estimate the unknown entries, and this is mainly to fill the unknown elements to get a complete NGHLPRs. Thus, we could first get an incomplete NGHLPR. We have two methods. One is the parameter ξ predetermined in Definition 6, and the other is the parameter ξ unknown. Thus, we design these two methods separately.

(1) The parameter ξ predetermined based method

As the parameter ξ is predetermined, we can obtain an incomplete NGLPR as per Definition 14. In this case, to estimate an unknown entry, we should find a chain $\overline{c}_{jk_1}$, $\overline{c}_{k_1k_2}$, $\cdots, \overline{c}_{k_{t-1}k_t}, \overline{c}_{k_tj}$ ($j < k_1 < k_2 < \cdots < k_t$) which only includes one unknown entry, and assume the only unknown entry is $\overline{c}_{k_{i-1}k_i}$. If $k_{i-1} < k_i$, then $\overline{c}_{k_{i-1}k_i}$ is the intermediate entry in the chain, which can be measured by Eq (17), that is:

$$\overline{c}^q_{k_{i-1}k_i} = I^{-1}\left(\frac{1}{\#Q}\sum_{j<k_1<k_2<\ldots<k_t}\left(\frac{t+1}{2}g - f(\overline{c}^q_{jk_1}) - \cdots - f(\overline{c}^q_{k_{i-2}k_{i-1}}) - f(\overline{c}^q_{k_ik_{i+1}}) - \cdots - f(\overline{c}^q_{k_tj})\right)\right). \tag{22}$$

If $k_{i-1} > k_i$, $\overline{c}_{k_{i-1}k_i}$ is the final entry in the chain, i.e., $\overline{c}_{k_{i-1}k_i} = \overline{c}_{k_tj}$, $\overline{c}^q_{k_{i-1}k_i}$ equals:

$$\overline{c}^q_{k_tj} = f^{-1}\left(\frac{1}{\#Q}\sum_{j<k_1<k_2<\ldots<k_t}\left(\frac{t+1}{2}g - f(\overline{c}^q_{jk_1}) - f(\overline{c}^q_{k_1k_2}) - \cdots - f(\overline{c}^q_{k_{t-1}k_t})\right)\right) \tag{23}$$

where $\#Q$ denotes the number of chains ($\overline{c}_{jk_1}$, $\overline{c}_{k_1k_2}$, $\cdots$, $\overline{c}_{k_{t-1}k_t}$, $\overline{c}_{k_tj}$), which includes $\overline{c}_{k_{i-1}k_i}$.

Example 2. Let $S = \{s_\pi \mid \pi = 0, 1, \cdots, 8\}$, $C = (c_{ij})_{4\times4}$ be an incomplete GHLPR:

$$C = \begin{bmatrix} \{s_4\} & \{s_3, s_5\} & \{s_2\} & \{s_2, s_3\} \\ \{s_5, s_3\} & \{s_4\} & \{s_6, s_8\} & x \\ \{s_6\} & \{s_2, s_0\} & \{s_4\} & x \\ \{s_6, s_5\} & x & x & \{s_4\} \end{bmatrix}.$$

Step 1. We use Definition 14 to convert C to its incomplete normalized NGHLPR with $\xi = 1$, and we have:

$$\overline{C}=\begin{bmatrix}\{s_4\} & \{s_3,s_5\} & \{s_2,s_2\} & \{s_2,s_3\}\\ \{s_5,s_3\} & \{s_4\} & \{s_6,s_8\} & x\\ \{s_6,s_6\} & \{s_2,s_0\} & \{s_4\} & x\\ \{s_6,s_5\} & x & x & \{s_4\}\end{bmatrix}.$$

Step 2. By Eqs (22) or (23), we estimate the unknown entry $\overline{c}_{24}$ as follows:

There exists only one eligible sequence $\{\overline{c}_{12}, \overline{c}_{24}, \overline{c}_{41}\}$, then we have:

$$\overline{c}_{24}^1=f^{-1}\left(\frac{3}{2}\times 8-f(\overline{c}_{12}^1)-f(\overline{c}_{41}^1)\right)=s_3.$$

$$\overline{c}_{24}^2=f^{-1}\left(\frac{3}{2}\times 8-f(\overline{c}_{12}^2)-f(\overline{c}_{41}^2)\right)=s_2.$$

Therefore,

$$\overline{c}_{24}=\{s_3,s_2\}.$$

Similarly, the other missing HFLEs can be obtained:

$$\overline{c}_{34}=\{s_4,s_5\}.$$

Step 3. The complete NGHLPR of *C* is:

$$\overline{C}=\begin{bmatrix}\{s_4\} & \{s_3,s_5\} & \{s_2,s_2\} & \{s_2,s_3\}\\ \{s_5,s_3\} & \{s_4\} & \{s_6,s_8\} & \{s_3,s_2\}\\ \{s_7\} & \{s_2,s_0\} & \{s_4\} & \{s_4,s_5\}\\ \{s_6,s_5\} & \{s_5,s_7\} & \{s_4,s_3\} & \{s_4\}\end{bmatrix}.$$

We summarize the parameter ξ predetermined based method to estimate the unknown entries in the following Algorithm 1.

Algorithm 1.

Input: The incomplete GHLPR, $C=(c_{ij})_{n\times n}$.

Output: The complete NGHLPR.

Step 1. Set a predetermined parameter ξ, and we obtained an incomplete NGHLPR.

Step 2. By Eqs (22) and (23), we estimate the unknown entries, and then obtain a complete NGHLPR.

Step 3. If there are some of the values in the NGHLPR that are out of the scope $[0, g]$ but within $[-u, g+u]$, we use Eq (9) to transform it.

Step 4. End.

(2) The parameter ξ unknown based method

The above parameter ξ predetermined based method is how to determine the parameter. In the practice, there is no principle to determine the value. For instance, suppose an HLE $c_{12}=\{s_4, s_7\}$ and $\#\overline{c}_{ij}=3$. Let $\xi=0.7$. The added LT is $\overline{c}_{12}^3=0.7\times s_7\oplus 0.3\times s_4=s_{6.1}$, then $\overline{c}_{12}=\{s_4, s_7, s_{6.1}\}$. How to set ξ is randomly. In the following, we deem that the parameter ξ is unknown. In the initial normalization process, we also look at the added LT as unknown values, which also need to be estimated by the existing values, and can be estimated by Eqs (22) and (23). We reckon this can use the DM's information sufficiently. Below, we still use Example 2 to explain this method.

Example 3. Let $C = (c_{ij})_{4\times4}$ be an incomplete GHLPR in Example 2.

To acquire incomplete NGHLPR of C, added LTs are also looked as unknown values.

Step 1. C is transformed into the following two LPRs:

$$\overline{C}^1 = \begin{bmatrix} s_4 & s_3 & s_2 & s_2 \\ s_5 & s_4 & s_6 & x \\ s_6 & s_2 & s_4 & x \\ s_6 & x & x & s_4 \end{bmatrix}, \quad \overline{C}^2 = \begin{bmatrix} s_4 & s_5 & x & s_3 \\ s_3 & s_4 & s_8 & x \\ x & s_0 & s_4 & x \\ s_5 & x & x & s_4 \end{bmatrix}.$$

Step 2. Both $\overline{C}^1$ and $\overline{C}^2$ are incomplete LPRs. To estimate the unknown values "x", the computation is given below:

For $\overline{c}_{24}^1$ in $\overline{C}^1$, there exists only one chain $\{\overline{c}_{12}^1, \overline{c}_{24}^1, \overline{c}_{41}^1\}$ which includes $\overline{c}_{24}^1$, then we have

$$\overline{c}_{24}^1 = f^{-1}\left(\frac{3}{2}\times 8 - f(\overline{c}_{12}^1) - f(\overline{c}_{41}^1)\right) = s_3 .$$

Similarly, the other missing LTs can be obtained:

$$\overline{c}_{34}^1 = s_4 , \ \overline{c}_{42}^1 = s_5 , \ \overline{c}_{43}^1 = s_4 .$$

$\overline{C}^1$ is estimated as:

$$\overline{C}^1 = \begin{bmatrix} s_4 & s_3 & s_2 & s_2 \\ s_5 & s_4 & s_6 & s_3 \\ s_6 & s_2 & s_4 & s_4 \\ s_6 & s_5 & s_4 & s_4 \end{bmatrix}.$$

For $\overline{C}^2$, we first estimate $\overline{c}_{13}^2$. There exists only one chain $\{\overline{c}_{13}^2, \overline{c}_{32}^2, \overline{c}_{21}^2\}$, and we have:

$$\overline{c}_{13}^2 = f^{-1}\left(\frac{3}{2}\times 8 - f(\overline{c}_{32}^2) - f(\overline{c}_{21}^2)\right) = s_9 \text{, and thus } \overline{c}_{31}^2 = s_{-1} .$$

For $\overline{c}_{24}^1$, there exists only chain $\{\overline{c}_{12}^2, \overline{c}_{24}^2, \overline{c}_{41}^2\}$, then we have:

$$\overline{c}_{24}^2 = f^{-1}\left(\frac{3}{2}\times 8 - f(\overline{c}_{12}^2) - f(\overline{c}_{41}^2)\right) = s_2 \text{, and thus } \overline{c}_{42}^2 = s_6 .$$

for $\overline{c}_{34}^2$ and $\overline{c}_{43}^2$ cannot be estimated in the first round. Then, $\overline{C}^2$ becomes:

$$\overline{C}^2 = \begin{bmatrix} s_4 & s_5 & s_9 & s_3 \\ s_3 & s_4 & s_8 & s_2 \\ s_{-1} & s_0 & s_4 & x \\ s_5 & s_6 & x & s_4 \end{bmatrix}.$$

Then, we continue to estimate $\overline{c}_{34}^2$ and $\overline{c}_{43}^2$. For $\overline{c}_{34}^2$, there are two chains $\{\overline{c}_{13}^2, \overline{c}_{34}^2, \overline{c}_{41}^2\}$, $\{\overline{c}_{23}^2, \overline{c}_{34}^2, \overline{c}_{42}^2\}$ which includes $\overline{c}_{34}^2$, and we have:

$$\overline{c}_{34}^{2,1} = f^{-1}\left(\frac{3}{2}\times 8 - f(\overline{c}_{13}^{2}) - f(\overline{c}_{41}^{2})\right) = s_{-2}, \quad \overline{c}_{34}^{2,2} = f^{-1}\left(\frac{3}{2}\times 8 - f(\overline{c}_{23}^{2}) - f(\overline{c}_{42}^{2})\right) = s_{-2}.$$

Then,

$$\overline{c}_{34}^{2} = \frac{1}{2}(\overline{c}_{34}^{2,1} + \overline{c}_{34}^{2,2}) = s_{-2}.$$

Thus,

$$\overline{c}_{43}^{2} = s_{10}.$$

$\overline{C}^2$ is estimated as:

$$\overline{C}^2 = \begin{bmatrix} s_4 & s_5 & s_9 & s_3 \\ s_3 & s_4 & s_8 & s_2 \\ s_{-1} & s_0 & s_4 & s_{-2} \\ s_5 & s_6 & s_{10} & s_4 \end{bmatrix}.$$

$\overline{C}$ is estimated as:

$$\overline{C} = \begin{bmatrix} \{s_4\} & \{s_3, s_5\} & \{s_2, s_9\} & \{s_2, s_3\} \\ \{s_5, s_3\} & \{s_4\} & \{s_6, s_8\} & \{s_3, s_2\} \\ \{s_6, s_{-1}\} & \{s_2, s_0\} & \{s_4\} & \{s_4, s_{-2}\} \\ \{s_6, s_5\} & \{s_5, s_6\} & \{s_4, s_{10}\} & \{s_4\} \end{bmatrix}.$$

Step 3. As the lower index of $\overline{c}_{43}^{2} = s_{10}$ is 10, which falls outside the interval [0, 8], and $u = 2$, a transformation function $y(x) = \frac{2}{3}(x + 2)$ is applied to the values in $\overline{C}$, and, eventually, $\overline{C}$ is:

$$\overline{C} = \begin{bmatrix} \{s_4\} & \{s_{3.333}, s_{4.667}\} & \{s_{2.667}, s_{7.333}\} & \{s_{2.667}, s_{3.333}\} \\ \{s_{4.667}, s_{3.333}\} & \{s_4\} & \{s_{5.333}, s_{6.667}\} & \{s_{3.333}, s_{2.667}\} \\ \{s_{5.333}, s_{0.667}\} & \{s_{2.667}, s_{1.333}\} & \{s_4\} & \{s_4, s_0\} \\ \{s_{5.333}, s_{4.667}\} & \{s_{4.667}, s_{5.333}\} & \{s_4, s_8\} & \{s_4\} \end{bmatrix}.$$

Remark 5. Sometimes, the separated LPR may be unacceptable [47]. In such cases, we randomly set one added LT known by the first method.

In summary, we design the following Algorithm 2.

Algorithm 2.

Input: The incomplete GHLPR, $C = (c_{ij})_{n\times n}$.

Output: The complete NGHLPR.

Step 1. Let $\#cN = \max\{\#c_{ij} \mid i, j \in N, i \neq j\}$. If the number of values in each entry is not $\#cN$, we fill "x" to the entry to get an incomplete NGHLPR $\overline{C} = (\overline{c}_{ij})_{n\times n}$.

Step 2. We separate $\overline{C} = (\overline{c}_{ij})_{n\times n}$ into $\#cN$ number of LPRs. If some of the LPRs are unacceptable, we randomly choose one unknown entry in the LPR to duplicate its value in in its original HLE until it is acceptable; otherwise, go to the next step.

Step 3. As per Eqs (22) and (23), we estimate the unknown values in these LPRs. The estimation process may be iterative. Then, an NGHLPR is obtained.

Step 4. If there are some of the values in the NGHLPR are out of the scope $[0, g]$, but within $[-u, g+u]$, we use Eq (9) to transform it.

Step 5. End.

4.2. GDM with incomplete GHLPRs

In the real situation, more and more problems are solved by a group of experts. Thus, to deal with a group of incomplete GHLPRs is a challenging work. In the following, we devise Algorithm 3 to manage GDM problems with incomplete GHLPRs.

Algorithm 3.

For a GDM problem, let $A = \{a_1, a_2, \cdots, a_n\}$, $E = \{e_1, e_2, \cdots, e_m\}$ be sets of alternatives and DMs, respectively. $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T$ is the normalized weight vector of DMs. The decision procedure with acceptable incomplete GHLPR is detailed below:

Step 1. Each DM e_t (t=1, 2, $\cdots$, m) gives his/her evaluations on the alternatives by an incomplete GHLPR $C_t = (c_{ij,t})_{n\times n}$.

Step 2. By Algorithm 1 or 2, we obtain the complete group NGHLPR.

Step 3. Use the hesitant linguistic average (HLA) operator to obtain the group preference LTs for everyone alternative.

$$c_i = HLA(c_{i1}, c_{i2}, \cdots, c_{in}) = \frac{1}{n} \mathop{\oplus}_{j=1}^{n} c_{ij}, \quad i \in N. \tag{24}$$

Step 4. Use Eq (3) to get the scores $\zeta(c_i)$ for every one alternative.

Step 5. The alternatives' rankings are obtained by the values $\zeta(c_i)$.

Step 6. End.

5. A case study

Now, there is a GDM problem which is to evaluate and select cloud server suppliers for the computer and software school. The school has to choose five potential suppliers, including Alibaba, Tencent, Huawei, Baidu, and Jinshan. Four aspects should be considered: usability, performance, security, and costs.

Three leaders of the school e_t (t =1, 2, 3) (the importance vector is $\lambda = (1/3, 1/3, 1/3)^T$) are organized to furnish their evaluations on the five alternatives a_i (i =1, 2, $\cdots$, 5), and these DMs give the following incomplete GHLPRs $C_t = (c_{ij,t})_{n\times n}$ (t =1, 2, 3):

$$C_1 = \begin{bmatrix} \{s_4\} & x & x & \{s_1, s_5, s_6\} & \{s_3, s_6, s_8\} \\ x & \{s_4\} & \{s_6, s_7\} & \{s_5, s_7, s_8\} & x \\ x & \{s_2, s_1\} & \{s_4\} & \{s_3, s_5, s_7\} & x \\ \{s_7, s_3, s_2\} & \{s_3, s_1, s_0\} & \{s_5, s_3, s_1\} & \{s_4\} & \{s_4, s_3, s_2\} \\ \{s_5, s_2, s_0\} & x & x & \{s_4, s_5, s_6\} & \{s_4\} \end{bmatrix},$$

$$C_2=\begin{bmatrix} \{s_4\} & x & \{s_1,s_4,s_5\} & x & x \\ x & \{s_4\} & \{s_2,s_4,s_5\} & x & x \\ \{s_7,s_4,s_3\} & \{s_6,s_4,s_3\} & \{s_4\} & \{s_3,s_5,s_7\} & \{s_2,s_3,s_4\} \\ x & x & \{s_5,s_3,s_1\} & \{s_4\} & x \\ x & x & \{s_6,s_5,s_4\} & x & \{s_4\} \end{bmatrix},$$

$$C_3=\begin{bmatrix} \{s_4\} & \{s_2,s_3,s_4\} & \{s_3,s_5,s_8\} & \{s_0,s_1,s_2\} & x \\ \{s_6,s_5,s_4\} & \{s_4\} & x & \{s_3,s_4,s_5\} & \{s_3,s_6,s_7\} \\ \{s_5,s_3,s_0\} & x & \{s_4\} & \{s_1,s_2,s_4\} & x \\ \{s_8,s_7,s_6\} & \{s_5,s_4,s_3\} & \{s_7,s_6,s_4\} & \{s_4\} & \{s_6,s_7,s_8\} \\ x & \{s_5,s_2,s_1\} & x & \{s_2,s_1,s_0\} & \{s_4\} \end{bmatrix}.$$

The following procedures are involved:

Step 1. As per Algorithm 2, we fill the missing values for these incomplete GHLPRs. For C_1, we first obtain its incomplete NHLPR $\overline{C}_1$:

$$\overline{C}_1=\begin{bmatrix} \{s_4\} & x & x & \{s_1,s_5,s_6\} & \{s_3,s_6,s_8\} \\ x & \{s_4\} & \{s_6,s_7,x\} & \{s_5,s_7,s_8\} & x \\ x & \{s_2,s_1,x\} & \{s_4\} & \{s_3,s_5,s_7\} & x \\ \{s_7,s_3,s_2\} & \{s_3,s_1,s_0\} & \{s_5,s_3,s_1\} & \{s_4\} & \{s_4,s_3,s_2\} \\ \{s_5,s_2,s_0\} & x & x & \{s_4,s_5,s_6\} & \{s_4\} \end{bmatrix}..$$

Then, we have:

$$\begin{aligned}\overline{c}_{12,1}^{1}&=\frac{1}{4}f^{-1}\left(\left(\frac{5}{2}\times 8-f(\overline{c}_{23,1}^{1})-f(\overline{c}_{34,1}^{1})-f(\overline{c}_{45,1}^{1})-f(\overline{c}_{51,1}^{1})\right)+\right.\\&\qquad\left(2\times 8-f(\overline{c}_{24,1}^{1})-f(\overline{c}_{45,1}^{1})-f(\overline{c}_{51,1}^{1})\right)+\left(2\times 8-f(\overline{c}_{23,1}^{1})-f(\overline{c}_{34,1}^{1})-f(\overline{c}_{41,1}^{1})\right)+\\&\qquad\left.\left(\frac{3}{2}\times 8-f(\overline{c}_{24,1}^{1})-f(\overline{c}_{41,1}^{1})\right)\right)\\&=\frac{1}{4}f^{-1}\left((20-6-3-4-5)+(16-5-4-5)+(16-6-3-7)+(12-5-7)\right)\\&=s_1\end{aligned}$$

$$\begin{aligned}\overline{c}_{12,1}^{2}&=\frac{1}{4}f^{-1}\left(\left(\frac{5}{2}\times 8-f(\overline{c}_{23,1}^{2})-f(\overline{c}_{34,1}^{2})-f(\overline{c}_{45,1}^{2})-f(\overline{c}_{51,1}^{2})\right)+\right.\\&\qquad\left(2\times 8-f(\overline{c}_{24,1}^{2})-f(\overline{c}_{45,1}^{2})-f(\overline{c}_{51,1}^{2})\right)+\\&\qquad\left.\left(2\times 8-f(\overline{c}_{23,1}^{2})-f(\overline{c}_{34,1}^{2})-f(\overline{c}_{41,1}^{2})\right)+\left(\frac{3}{2}\times 8-f(\overline{c}_{24,1}^{2})-f(\overline{c}_{41,1}^{2})\right)\right)\\&=\frac{1}{4}f^{-1}\left((20-7-5-3-2)+(16-7-3-2)+(16-7-5-3)+(12-7-3)\right)\\&=s_{2.5}\end{aligned}$$

$$\overline{c}_{12,1}^{3}=\frac{1}{2}f^{-1}\left(\left(2\times 8-f(\overline{c}_{24,1}^{3})-f(\overline{c}_{45,1}^{3})-f(\overline{c}_{51,1}^{3})\right)+\left(\frac{3}{2}\times 8-f(\overline{c}_{24,1}^{3})-f(\overline{c}_{41,1}^{3})\right)\right)$$

$$=\frac{1}{2}f^{-1}\left((16-8-2-0)+(12-8-2)\right)=s_4$$

$$\overline{c}_{13,1}^{1}=\frac{1}{2}f^{-1}\left(\left(2\times 8-f(\overline{c}_{34,1}^{1})-f(\overline{c}_{45,1}^{1})-f(\overline{c}_{51,1}^{1})\right)+\left(\frac{3}{2}\times 8-f(\overline{c}_{34,1}^{1})-f(\overline{c}_{41,1}^{1})\right)\right)$$

$$=\frac{1}{2}f^{-1}\left((16-3-4-5)+(12-3-7)\right)=s_3$$

$$\overline{c}_{13,1}^{2}=\frac{1}{2}f^{-1}\left(\left(2\times 8-f(\overline{c}_{34,1}^{2})-f(\overline{c}_{45,1}^{2})-f(\overline{c}_{51,1}^{2})\right)+\left(\frac{3}{2}\times 8-f(\overline{c}_{34,1}^{2})-f(\overline{c}_{41,1}^{2})\right)\right)$$

$$=\frac{1}{2}f^{-1}\left((16-5-3-2)+(12-5-3)\right)=s_5$$

$$\overline{c}_{13,1}^{3}=\frac{1}{2}f^{-1}\left(\left(2\times 8-f(\overline{c}_{34,1}^{3})-f(\overline{c}_{45,1}^{3})-f(\overline{c}_{51,1}^{3})\right)+\left(\frac{3}{2}\times 8-f(\overline{c}_{34,1}^{3})-f(\overline{c}_{41,1}^{3})\right)\right)$$

$$=\frac{1}{2}f^{-1}\left((16-7-2-0)+(12-7-2)\right)=s_5$$

$$\overline{c}_{52,1}^{1}=\frac{1}{2}f^{-1}\left(\left(2\times 8-f(\overline{c}_{23,1}^{1})-f(\overline{c}_{34,1}^{1})-f(\overline{c}_{45,1}^{1})\right)+\left(\frac{3}{2}\times 8-f(\overline{c}_{24,1}^{1})-f(\overline{c}_{45,1}^{1})\right)\right)$$

$$=\frac{1}{2}f^{-1}\left((16-6-3-4)+(12-5-4)\right)=s_3$$

$$\overline{c}_{52,1}^{2}=\frac{1}{2}f^{-1}\left(\left(2\times 8-f(\overline{c}_{23,1}^{2})-f(\overline{c}_{34,1}^{2})-f(\overline{c}_{45,1}^{2})\right)+\left(\frac{3}{2}\times 8-f(\overline{c}_{24,1}^{2})-f(\overline{c}_{45,1}^{2})\right)\right)$$

$$=\frac{1}{2}f^{-1}\left((16-7-5-3)+(12-7-3)\right)=s_{1.5}$$

$$\overline{c}_{52,1}^{3}=f^{-1}\left(\frac{3}{2}\times 8-f(\overline{c}_{24,1}^{3})-f(\overline{c}_{45,1}^{3})\right)=f^{-1}\left(12-8-2\right)=s_2$$

$$\overline{c}_{53,1}^{1}=f^{-1}\left(\frac{3}{2}\times 8-f(\overline{c}_{34,1}^{1})-f(\overline{c}_{45,1}^{1})\right)=f^{-1}\left(12-3-4\right)=s_5$$

$$\overline{c}_{53,1}^{2}=f^{-1}\left(\frac{3}{2}\times 8-f(\overline{c}_{34,1}^{2})-f(\overline{c}_{45,1}^{2})\right)=f^{-1}\left(12-5-3\right)=s_4$$

$$\overline{c}_{53,1}^{3}=f^{-1}\left(\frac{3}{2}\times 8-f(\overline{c}_{34,1}^{3})-f(\overline{c}_{45,1}^{3})\right)=f^{-1}\left(12-7-2\right)=s_3$$

$$\overline{c}_{23,1}^{3}=f^{-1}\left(\frac{3}{2}\times 8-f(\overline{c}_{34,1}^{3})-f(\overline{c}_{42,1}^{3})\right)=f^{-1}\left(12-7-0\right)=s_5\,.$$

Then, we have:

$$\overline{c}_{21,1}^{1}=s_7\,,\ \overline{c}_{21,1}^{2}=s_{5.5}\,,\ \overline{c}_{21,1}^{3}=s_4\,,\ \overline{c}_{31,1}^{1}=s_5\,,\ \overline{c}_{31,1}^{2}=s_3\,,\ \overline{c}_{31,1}^{3}=s_3\,,$$

$$\overline{c}_{25,1}^{1}=s_5\,,\ \overline{c}_{25,1}^{2}=s_{6.5}\,,\ \overline{c}_{25,1}^{3}=s_6\,,\ \overline{c}_{35,1}^{1}=s_3\,,\ \overline{c}_{35,1}^{2}=s_4\,,\ \overline{c}_{35,1}^{3}=s_5\,,\ \overline{c}_{32,1}^{3}=s_3\,.$$

Then, we obtain a complete NHLPR $\overline{C}_1$:

$$\bar{C}_1=\begin{bmatrix} \{s_4\} & \{s_1,s_{2.5},s_4\} & \{s_3,s_5,s_5\} & \{s_1,s_5,s_6\} & \{s_3,s_6,s_8\} \\ \{s_7,s_{5.5},s_4\} & \{s_4\} & \{s_6,s_7,s_3\} & \{s_5,s_7,s_8\} & \{s_5,s_{6.5},s_6\} \\ \{s_5,s_3,s_3\} & \{s_2,s_1,s_5\} & \{s_4\} & \{s_3,s_5,s_7\} & \{s_3,s_4,s_5\} \\ \{s_7,s_3,s_2\} & \{s_3,s_1,s_0\} & \{s_5,s_3,s_1\} & \{s_4\} & \{s_4,s_3,s_2\} \\ \{s_5,s_2,s_0\} & \{s_3,s_{1.5},s_2\} & \{s_5,s_4,s_3\} & \{s_4,s_5,s_6\} & \{s_4\} \end{bmatrix}.$$

Similarly, we obtain

$$\bar{C}_2=\begin{bmatrix} \{s_4\} & \{s_3,s_4,s_4\} & \{s_1,s_4,s_5\} & \{s_0,s_5,s_8\} & \{s_{-1},s_3,s_5\} \\ \{s_5,s_4,s_4\} & \{s_4\} & \{s_2,s_4,s_5\} & \{s_1,s_5,s_8\} & \{s_0,s_3,s_5\} \\ \{s_7,s_4,s_3\} & \{s_6,s_4,s_3\} & \{s_4\} & \{s_3,s_5,s_7\} & \{s_2,s_3,s_4\} \\ \{s_8,s_3,s_0\} & \{s_7,s_3,s_0\} & \{s_5,s_3,s_1\} & \{s_4\} & \{s_8,s_3,s_0\} \\ \{s_9,s_5,s_3\} & \{s_8,s_5,s_3\} & \{s_6,s_5,s_4\} & \{s_0,s_5,s_8\} & \{s_4\} \end{bmatrix}.$$

By $y(x)=\frac{8}{3}(x+1)$, we convert it to an NGHLPR as

$$\bar{C}_2=\begin{bmatrix} \{s_4\} & \{s_{3.200},s_{4.000},s_{4.000}\} & \{s_{1.600},s_{4.000},s_{4.800}\} & \{s_{0.800},s_{4.800},s_{7.200}\} & \{s_0,s_{3.200},s_{4.800}\} \\ \{s_{4.800},s_{4.000},s_{4.000}\} & \{s_4\} & \{s_{2.400},s_{4.000},s_{4.800}\} & \{s_{1.600},s_{4.800},s_{7.200}\} & \{s_{0.800},s_{3.200},s_{4.800}\} \\ \{s_{6.400},s_{4.000},s_{3.200}\} & \{s_{5.600},s_{4.000},s_{3.200}\} & \{s_4\} & \{s_{3.200},s_{4.800},s_{6.400}\} & \{s_{2.400},s_{3.200},s_{4.000}\} \\ \{s_{7.200},s_{3.200},s_{0.800}\} & \{s_{6.400},s_{3.200},s_{0.800}\} & \{s_{4.800},s_{3.200},s_{1.600}\} & \{s_4\} & \{s_{7.200},s_{3.200},s_{0.800}\} \\ \{s_{8.000},s_{4.800},s_{3.200}\} & \{s_{7.200},s_{4.800},s_{3.200}\} & \{s_{5.600},s_{4.800},s_{4.000}\} & \{s_{0.800},s_{4.800},s_{7.200}\} & \{s_4\} \end{bmatrix}.$$

For DM e_3, the complete NGHLPR is

$$\bar{C}_3=\begin{bmatrix} \{s_4\} & \{s_{2.2222},s_{3.1111},s_{4.0000}\} & \{s_{3.1111},s_{4.8889},s_{7.5556}\} & \{s_{0.4444},s_{1.3333},s_{2.2222}\} & \{s_{2.2222},s_{5.1111},s_{8.0000}\} \\ \{s_{5.7778},s_{4.8889},s_{4.0000}\} & \{s_4\} & \{s_{4.8889},s_{5.1111},s_{4.4444}\} & \{s_{3.1111},s_{4.0000},s_{4.8889}\} & \{s_{3.1111},s_{5.7778},s_{6.6667}\} \\ \{s_{4.8889},s_{3.1111},s_{0.4444}\} & \{s_{3.1111},s_{2.8889},s_{3.5556}\} & \{s_4\} & \{s_{1.3333},s_{2.2222},s_{4.0000}\} & \{s_{3.1111},s_{4.8889},s_{7.5556}\} \\ \{s_{7.5556},s_{6.6667},s_{5.7778}\} & \{s_{4.8889},s_{4.0000},s_{3.1111}\} & \{s_{6.6667},s_{5.7778},s_{4.0000}\} & \{s_4\} & \{s_{5.7778},s_{6.6667},s_{7.5556}\} \\ \{s_{5.7778},s_{2.8889},s_0\} & \{s_{4.8889},s_{2.2222},s_{1.3333}\} & \{s_{4.8889},s_{3.1111},s_{0.4444}\} & \{s_{2.2222},s_{1.3333},s_{0.4444}\} & \{s_4\} \end{bmatrix}.$$

Step 2. Proposition 3 is utilized to fuse all individual complete GHLPR $\bar{C}_t=(\bar{c}_{ij,t})_{n\times n}$ $(t=1,2,3)$

into an NGHLPR $C=(c_{ij})_{n\times n}$ as

$$C=\begin{bmatrix} \{s_4\} & \{s_{2.1407},s_{3.2037},s_{4.0000}\} & \{s_{2.5704},s_{4.6296},s_{5.7852}\} & \{s_{0.7481},s_{3.7111},s_{5.1407}\} & \{s_{1.7407},s_{4.7704},s_{6.9333}\} \\ \{s_{5.8593},s_{4.7963},s_{4.0000}\} & \{s_4\} & \{s_{4.4296},s_{5.3704},s_{4.0815}\} & \{s_{3.2370},s_{5.2667},s_{6.6963}\} & \{s_{2.9704},s_{5.1593},s_{5.8222}\} \\ \{s_{5.4296},s_{3.3704},s_{2.2148}\} & \{s_{3.5704},s_{2.6296},s_{3.9185}\} & \{s_4\} & \{s_{2.5111},s_{4.0074},s_{5.8000}\} & \{s_{2.8370},s_{4.0296},s_{5.5185}\} \\ \{s_{7.2519},s_{4.2889},s_{2.8593}\} & \{s_{4.7630},s_{2.7333},s_{1.3037}\} & \{s_{5.4889},s_{3.9926},s_{2.2000}\} & \{s_4\} & \{s_{5.6593},s_{4.2889},s_{3.4519}\} \\ \{s_{6.2593},s_{3.2296},s_{1.0667}\} & \{s_{5.0296},s_{2.8407},s_{2.1778}\} & \{s_{5.1630},s_{3.9704},s_{2.4815}\} & \{s_{2.3407},s_{3.7111},s_{4.5481}\} & \{s_4\} \end{bmatrix}.$$

Step 3. The preference LTs for each alternative a_i (i =1, 2, ···, 5) are:

$$c_1=\{s_{2.2400},s_{4.0630},s_{5.1718}\},\ c_2=\{s_{4.0993},s_{4.9185},s_{4.9200}\},\ c_3=\{s_{3.6696},s_{3.6074},s_{4.2904}\},$$

$$c_4=\{s_{5.4326},s_{3.8607},s_{2.7630}\},\ c_5=\{s_{4.5585},s_{3.5504},s_{2.8548}\}.$$

Step 4. As per Definition 4, the score of each alternative $\zeta(b_i)$ is:

$$\zeta(b_1)=3.8250,\ \zeta(b_2)=4.6459,\ \zeta(b_3)=3.8558,\ \zeta(b_4)=4.1877,\ \zeta(b_5)=3.6546.$$

Step 5. According to the $\zeta(b_i)$, we rank the alternatives:

$$a_2 \succ a_4 \succ a_3 \succ a_1 \succ a_5.$$

Therefore, a_2 is the best choice, i.e., the school should choose Tencent as its final decision.

6. Comparative analysis

To show the performances of the presented method, in the following, we do some comparisons with the published methods.

As per Eq (8), the consistency degree of GHLPR $C=(c_{ij})_{n\times n}$ can be calculated by the difference between $\overline{c}_{ij}^{\sigma(l)}\oplus s_{g/2}$ and $\overline{c}_{ik}^{\sigma(l)}\oplus\overline{c}_{kj}^{\sigma(l)}$. Thus, we propose the CI(C) as follows:

$$\mathrm{CI}(C)=\frac{2}{n(n-1)(n-2)g}\frac{1}{\#\overline{c}_{ij}}\sum_{i<j<k}^{n}\sum_{q=1}^{\#\overline{c}_{ij}}\varepsilon_{ijk}^{q} \tag{25}$$

where

$$\varepsilon_{ijk}^{q}=\left|f(\overline{c}_{ij}^{q})+g/2-f(\overline{c}_{ik}^{q})-f(\overline{c}_{kj}^{q})\right|. \tag{26}$$

Obviously, $0\le \mathrm{CI}(C)\le 1$. The smaller the CI(C), the more reliable the DM.

6.1. The performance of GHLPR

In Example 1, if we use Eq (25) to measure the CI of C, which is derived by Zhu and Xu [17]'s definition of HLPR, we obtain CI(C)=0.026. This shows that C is not AC. However, if we use the proposed definition of GHLPR in this paper, we have CI(C)=0, which denotes that C is perfectly AC. This also conforms to our analysis, that is, the information provided by DM is consistent. However, all the existing HLPRs use Eq (4), and as we have pointed out in Example 1, the main difference is that the existing HLPR should reorder the values in HLE. If we do not sort the values, the obtained HLPR will conflict with Eq (4). On the other hand, if we sort these values, the obtained HLPR will not be consistent, the DM's initial information is distorted. Wu, Zhou, and Chen et al. [22] also found the problem; please see their comments after Example 3. This also can be verified in so many other papers. Thus, the reordering of values is not reasonable. This shows the reasonableness of the GHLPR.

6.2. The performance of normalization

If we use Zhu and Xu [17]'s normalization method to obtain the NHLPR, and $\xi = 1$, we have:

$$\bar{C}_{Zhu} = \begin{bmatrix} \{s_4\} & \{s_5, s_5\} & \{s_2, s_3\} & \{s_6, s_6\} \\ \{s_3, s_3\} & \{s_4\} & \{s_3, s_3\} & \{s_4, s_4\} \\ \{s_6, s_5\} & \{s_5, s_5\} & \{s_4\} & \{s_4, s_7\} \\ \{s_2, s_2\} & \{s_4, s_4\} & \{s_4, s_1\} & \{s_4\} \end{bmatrix}.$$

By Eq (25), CI(C_{Zhu}) = 0.0365.

However, if we use the proposed method, we have:

$$\bar{C} = \begin{bmatrix} \{s_4\} & \{s_5, s_5\} & \{s_2, s_3\} & \{s_6, s_6\} \\ \{s_3, s_3\} & \{s_4\} & \{s_3, s_2\} & \{s_4, s_5\} \\ \{s_6, s_5\} & \{s_5, s_6\} & \{s_4\} & \{s_4, s_7\} \\ \{s_2, s_2\} & \{s_4, s_3\} & \{s_4, s_1\} & \{s_4\} \end{bmatrix}.$$

By Eq (25) CI(B) = 0.0234, which is smaller than 0.0365, denoting the obtained NHLPR based on AC is more consistent than the existing one. If we further investigate, this is because Zhu and Xu [17]'s normalization method randomly adds the value, while the proposed method uses the AC property to add the values. Furthermore, we can find that the added LPR is:

$$\bar{C}^2 = \begin{bmatrix} s_4 & s_5 & s_3 & s_6 \\ s_3 & s_4 & s_2 & s_5 \\ s_5 & s_6 & s_4 & s_7 \\ s_2 & s_3 & s_1 & s_4 \end{bmatrix}$$

which is perfectly additively consistent LPR. Meanwhile the obtained $\bar{C}_{Zhu}^2$ is:

$$\overline{C}_{Zhu}^{2}=\begin{bmatrix} s_4 & s_5 & s_3 & s_6 \\ s_3 & s_4 & s_3 & s_4 \\ s_5 & s_5 & s_4 & s_7 \\ s_2 & s_4 & s_1 & s_4 \end{bmatrix}$$

which is not AC LPR, and this is why the CI(C_{Zhu}) is larger than CI(C).

Additionally, in the obtained C by the proposed method, $c_{12} = \{s_3, s_2\}$, if we reorder the values, will be more inconsistent. This also shows that the proposed GHLPR is more reasonable.

6.3. Comparison with other methods

In the following, we do detailed comparisons with the other existing methods, which also study the incomplete HLPRs. The results are summarized in Table 2. From Table 2, we can see that all the other methods rearrange the values in HLTS, in which the consistency and the concept of HLPRs will conflict with each other. In the following, we describe the drawbacks of the other methods, respectively.

Table 2. Comparisons with the existent methods.

	Rearrange values	Add values	Consistency type	Methods to estimate missing elements
The proposed method	N	Y	AC	Estimated by AC property
Song and Li [38]	Y	N	MC	Mathematical programming
Liu, Ma, and Jiang [35]	Y	Y	AC	Optimization models
Wu, Li, Merigó et al. [34]	Y	Y	AC	Integer programming
Li, Zhang, and Yu [36]	Y	Y	AC	Optimization models

Song and Li [38] constructed the mathematical programming models to estimate the missing elements based on the MC. However, their method only can determine one MC LPR, and, thus, the estimated missing HLT only has one value. This may obey the real situation, as the determined elements only have one linguistic term, which is not the hesitant. Another is that, for the MC of HLPRs, the lowest and largest linguistic terms cannot be used, i.e., the evaluation values cannot be equal to s_0 or s_g. However, in the real application, the lowest evaluation (s_0= extremely poor) and the best evaluation (s_8 = extremely good) values are always offered by the DMs. This limits the application of Song and Li [38]'s methods.

Liu, Ma, and Jiang [35] used the worst consistency index (WCI) and best consistency index (BCI) to determine the lower and upper bounds of the missing elements of an HLPR. However, as we can see from their example (Example 1 in [35]), the estimated HLTs have a wide range, for example, $\{s_0, \cdots, s_6\}$. This may be far away from the real situation, as the provided HLTs are only one or two linguistic terms, but the estimated elements have 7 linguistic values.

The LCME principle for normalization HLTSs in [22]. As we can see, their method will add more elements to an HLTS, which will cause more inconsistency. Furthermore, Wu, Li, Merigó et al. [34] proposed an integer programming model to deal with incomplete HLPRs. As we can see, in their examples, the LCME method will generate some redundant LPRs, and the CI obtained by their method will be larger than ours.

Li, Zhang, and Yu [36] used the average consistency index (ACI) to impute the missing HLTSs. However, their method also only can determine one linguistic term, and needs to preset the threshold of ACI. Different thresholds may generate different missing values, and there is no rule how to set the thresholds. Tang, Liao, and Li et al. [46] also studied the incomplete HLPR and they used the existing definition of HLPR. In their Example 4, the estimated value $\overrightarrow{c}'_{45} = \{s_{-5}, s_{-5}, s_{-6}, s_{-7}\}$, which contradicts with the definition of HLPRs, and requires $c_{ij}^{\sigma(q)} < c_{ij}^{\sigma(q+1)} (i < j)$. This also shows the unreasonableness of Tang, Liao, and Li et al. [46]'s definition of incomplete HLPRs.

7. Conclusions

In the GDM, the DMs may use linguistic terms to articulate their pair-wise preferences over a given set of alternatives. Then, these DMs' preferences are organized into an HLPR, and each HLE has some linguistic values, and these different linguistic values are sorted in the existent studies. However, our illustrated example shows that this will obey or distort the DMs' initial preferences and also conflict with the AC property of HLPRs. In order to rectify the unreasonable arrangement of the sorting values, we redefined the HLPRs, which is named GHLPR in the present work. The NGHLPR and incomplete GHLPR (NGHLPRs) are defined accordingly. We also have investigated the characterization of additively consistent LPRs, and the characterization of LPRs is extended to accumulate the NGHLPRs. In the real application, the DMs may provide incomplete GHLPRs, and we have developed two algorithms to fill the unknown values in NGHLPRs. The first algorithm is based on the traditional normalization idea, which adds some linguistic values to the shorter HLEs until all the known HLEs in an GHLPR have the same length. The second algorithm is based on the AC property of GHLPRs. The shorter HLEs are also treated with unknown linguistic terms, and these unknown linguistic terms should be estimated by the known values. Both algorithms to estimate the missing values are based on the characterization of NGHLPRs. We also develop an algorithm to solve GDM with incomplete HLPRs. A case study is also furnished to illustrate how to perform the algorithms. A comprehensive comparative study is carried out to show the merits of the proposed methods.

Conflict of interest

The author declares no conflict of interest.

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