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Research article

Qualitative analysis on the electrohydrodynamic flow equation

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Abstract: In this paper, we present a comprehensive analysis of the lower and upper bounds of solutions for a nonlinear second-order ordinary differential equation governing the electrohydrodynamic flow of a conducting fluid in cylindrical conduits. The equation describes the radial distribution of the flow velocity in an "ion drag" configuration, which is affected by an externally applied electric field. Our study involves the establishment of rigorous analytical bounds on the radial distribution, taking into account the Hartmann number H and a parameter α. An analytic approximate solution is obtained as an improvement of the a priori estimates and it is found to exhibit strong agreement with numerical solutions, particularly when considering small Hartmann numbers. Further, estimates for the central velocity $w(0)$ of the fluid occurring at the center of the cylindrical conduit were also established, and some interesting relationships were found between H and α. These findings establish a framework that illuminates the potential range of values for the physical parameter within the conduit.

Keywords: differential equation of electrohydrodynamics; approximate solution; lower and upper bounds
Mathematics Subject Classification: 35R35, 34L15, 34G20

1. Introduction

The nonlinear differential equations of electrohydrodynamics (EHD) describe the behavior of electrically charged fluids or particles in the presence of electric and magnetic fields. These equations are used to model a wide range of phenomena, including the motion of charged droplets or particles in electric fields, the behavior of ionic solutions, and various electrokinetic effects. The governing equations of EHD are typically derived from fundamental principles, such as the conservation of mass, momentum, and charge, along with the equations of electrostatics [1–3].

In EHD, the interaction between an applied electric field and a conducting fluid gives rise to various flow phenomena, and these phenomena can be described by partial differential equations (PDEs) or, as in this case, a nonlinear second-order ordinary differential equation (ODE). The (EHD) flow of a fluid in an ion drag configuration in a circular cylindrical conduit is governed by

$$\begin{cases} w''(r) + \frac{w'(r)}{r} + H^2\left(1 - \frac{w(r)}{1-\alpha w(r)}\right) = 0, \ 0 < r < 1, \\ w'(0) = 0, \ w(1) = 0. \end{cases} \tag{1.1}$$

Here, $w(r)$ represents the fluid velocity, r is the radial distance to the center of the conduit, and the parameter α measures the degree of nonlinearity. The equation includes second-order and first-order derivative terms concerning r. The choice of $w(r)$ depends on the specific problem being analyzed. As for the derivation of this boundary value problem (1.1), one can refer to the details provided in [4]. The equation of EHD, along with the boundary conditions, must be interpreted in the context of the physical system under consideration to determine the appropriate physical quantity that $w(r)$ represents, and it may represent the electric potential or voltage distribution within the conduit, the flow rate of the fluid, or the electrokinetic mobility, depending on the specific problem under study.

The above equation is an example of an ODE that can describe the radial distribution of velocity in a circular cylindrical conduit subjected to both electric and magnetic fields. The constant H is the Hartmann number, which is a parameter related to the conductivity or properties of the fluid, and characterizes the strength of the applied magnetic field. The coefficient H^2 indicates the influence of a magnetic field.

Due to their nonlinearity nature, solving these equations can be very challenging and often require numerical or semi-analytic methods. It is worth noting here that the authors in [4] obtained numerical solutions with perturbation expansions for small and large values of α. Paullet [5] claimed an error in the solutions given in [4] in the case of large α by finding a different solution in that case and justifying the error in [4]. In [5], the author proved an interesting result of the existence and uniqueness of the solution to the boundary value problem (BVP) (1.1):

Theorem 1.1. *For any* $\alpha > 0$ *and any* $H^2 \neq 0$, *there exists a solution to the BVP (1.1). Furthermore, this solution is monotonic and satisfies* $0 < w(r) < \frac{1}{\alpha+1}$ *for all* $r \in (0, 1)$.

The theorem has motivated the exploration of approximate solutions to the problem as noted in [6–9] using different perturbation and homotopy methods. The authors in [7] and [9] utilized homotopy and least square methods, respectively, to obtain semi-analytic solutions that are similar to the solution found in [5] for large α.

A helpful technique involves identifying lower and upper solutions, from which one can obtain an approximate solution using the perturbation technique. These bounds could enhance our understanding of the system, provide insights into its behavior, and help perform a qualitative analysis. Furthermore, these bounds provide a foundation for assessing the reliability of the numerical simulations and whether a numerical solution falls within the established bounds, and can serve as a basis for model validation. The purpose of this paper is two-fold: The first goal is to find lower and upper bounds for the solution of (1.1). We will use these bounds to demonstrate the error in the solution obtained in [4] for large values of α, supporting the claims made by [5] and subsequent research. The second is to utilize the lower bound to establish estimates for the central velocity and an approximate explicit solution to the problem. To the best of our knowledge, there is no approximate explicit solution for (1.1) found in the literature. The proposed approximate solution shows significant efficiency in the case of small Hartmann numbers. The Hartmann number is given by $H = B\ell\sqrt{\frac{\sigma}{\mu\rho}}$, where σ is the electric conductivity, B is the magnetic field associated with the electric current, μ is the dynamic viscosity, ρ is the fluid density, and ℓ is the characteristic length scale. The Hartmann number represents the interaction between the electromagnetic and viscous forces. According to its definition, a larger Hartmann number indicates that electromagnetic forces outweigh the viscous forces, while a smaller Hartmann number refers to a relatively weaker electromagnetic field in comparison to the viscous forces effects on the fluid flow, which implies that viscous forces are the dominating force for the fluid flow, and this justifies why small Hartmann numbers are always preferred for the enhancement of heat transfer [10]. Some examples of substances with low Hartmann numbers include fluids with a relatively weak electric conductivity, such as distilled water, gases, and semiconductor materials. In these substances, the study and analysis of the behavior of the fluid flow is primarily based on viscous forces. In the study of the unsteady flow of an incompressible viscous and electrically conducting fluid, it was shown that ignoring the induced electromagnetic field is only permitted at small Hartmann [11–13]. Consideration of small Hartmann numbers finds practical applications in microfluidic devices, and microchannel cooling systems, in addition to electromagnetic and magnetohydrodynamic pumps and other chemical industries. Further, three-dimensional numerical simulations of inertial flows are limited to very low Hartmann numbers [14].

The present paper is structured as follows: Section two establishes the necessary lower and upper bounds of the solution to BVP (1.1), which will be used in section three to provide some estimates that will help understand the relationship between different parameters of the equation and give an estimate of the central radial velocity of the flow $\beta = w(0)$. In Section four, we establish an approximate solution for the problem via perturbation expansion. Section five provides a detailed analysis of the obtained results, in which the bound solutions are compared with numerical solutions for different parameter values associated with BVP (1.1). Section six summarizes the main findings and their conclusions.

2. A priori estimates for the solution $w(r)$

We first convert Pr. (1.1) to an equivalent BVP.

Lemma 2.1. *Pr.(1.1) can be converted to the nonlinear boundary value problem*

$$\begin{cases} y'' + \frac{y'}{r} + \frac{H^2}{y} - H^2(1+\alpha) = 0, \ 0 < r < 1, \\ y'(0) = 0, \ y(1) = 1, \end{cases} \tag{2.1}$$

where

$$y = 1 - \alpha w(r). \tag{2.2}$$

Proof. Rewrite the nonlinear equation of Pr.(1.1) in the form

$$-\frac{1}{\alpha}(1-\alpha w(r))'' - \frac{(1-\alpha w(r))'}{\alpha r} + H^2\left[1 + \frac{(1-\alpha w(r)) - 1}{\alpha(1-\alpha w(r))}\right] = 0 \tag{2.3}$$

or

$$-\frac{1}{\alpha}(1-\alpha w(r))'' - \frac{(1-\alpha w(r))'}{\alpha r} + H^2\left[1+\frac{1}{\alpha}-\frac{1}{\alpha(1-\alpha w(r))}\right] = 0. \tag{2.4}$$

A simple substitution of $y = 1-\alpha w(r)$ into (2.4) and taking into account that $w'(0) = 0$ and $w(1) = 0$, we obtain Pr. (2.1). □

Theorem 2.2. *For any $\alpha > 0$, the nonlinear boundary value problem (1.1) has lower and upper bounds solutions in $\mathbb{C}^2[0,1] \cap \mathbb{C}[0,1]$, such that*

$$w_1 \le w \le w_2 \text{ on } [0,1], \tag{2.5}$$

where

$$w_1(r) = \frac{H^2}{4}\left[\frac{\alpha+1}{\alpha} - \frac{1}{\alpha(1-\alpha w(0))}\right](1-r^2),\ 0 \le r \le 1 \tag{2.6}$$

and

$$w_2(r) = \frac{H^2}{4}(1-r^2),\ 0 \le r \le 1. \tag{2.7}$$

Proof. Rewrite the nonlinear equation of Pr.(1.1) in the following form

$$y'' + f(r,y,y') = 0, \tag{2.8}$$

where

$$f(r,y,y') = \frac{y'}{r} + \frac{H^2}{y} - H^2(1+\alpha). \tag{2.9}$$

Since w is monotonically decreasing, $w(0) \ge w(r) \ge w(1) = 0$, and in view of $y(r) = 1-\alpha w(r)$, we have

$$1-\alpha w(0) \le y \le 1. \tag{2.10}$$

By Theorem 1, we conclude by continuity that $w(0) \le \dfrac{1}{1+\alpha}$. That is, $w(0) < \frac{1}{\alpha}$, then $1 \le \frac{1}{y} \le \frac{1}{1-\alpha w(0)}$. This implies

$$f(r,y,y') > \frac{y'}{r} + H^2 - H^2(1+\alpha) = \frac{y'}{r} - \alpha H^2,\ 0 < r < 1. \tag{2.11}$$

So we have

$$-y'' > \frac{y'}{r} - \alpha H^2,\ 0 < r < 1 \tag{2.12}$$

or

$$-ry'' > y' - \alpha H^2 r,\ 0 < r < 1. \tag{2.13}$$

Integration from zero to r yields

$$-ry' \ge -\frac{\alpha H^2}{2}r^2,\ 0 < r < 1 \tag{2.14}$$

or

$$y' \le \frac{\alpha H^2}{2}r,\ 0 < r < 1. \tag{2.15}$$

Integrating again from r to one and taking into account that $y(1) = 1$, we obtain

$$1 - y \leq \frac{\alpha H^2}{4}(1 - r^2),\ 0 \leq r \leq 1, \tag{2.16}$$

or from (2.2),

$$\alpha w(r) \leq \frac{\alpha H^2}{4}(1 - r^2),\ 0 \leq r \leq 1. \tag{2.17}$$

Hence,

$$w(r) \leq \frac{H^2}{4}(1 - r^2),\ 0 \leq r \leq 1. \tag{2.18}$$

On the other hand, since $1 - \alpha w(0) > 0$ and $\frac{1}{y} \leq \frac{1}{1-\alpha w(0)}$, we get

$$f(r, y, y') < \frac{y'}{r} + \frac{H^2}{1 - \alpha w(0)} - H^2(\alpha + 1),\ 0 < r < 1. \tag{2.19}$$

Hence,

$$-y'' < \frac{y'}{r} + \frac{H^2}{1 - \alpha w(0)} - H^2(\alpha + 1),\ \ 0 \leq r \leq 1. \tag{2.20}$$

Proceeding as before, we obtain

$$w(r) \geq \frac{H^2}{4}\left[\frac{\alpha + 1}{\alpha} - \frac{1}{\alpha(1 - \alpha w(0))}\right](1 - r^2),\ 0 \leq r \leq 1. \tag{2.21}$$

Now, (2.5) follows from (2.18) and (2.21). □

Remark 2.3. *The lower bound (2.5) is particularly important in the sense that it satisfies the boundary conditions of BVP (1.1). Further, it can also be shown that it satisfies the properties of the solution as predicted by Theorem 1. Indeed,* (2.5) *implies that* $w_1(r) < \dfrac{1}{1+\alpha}$ *for all* $0 < r < 1$*. Since* $w(0) = \beta \leq \dfrac{1}{1+\alpha}$*, this implies that* $\beta \leq 1 - \alpha\beta$*. We rewrite* w_1 *as*

$$w_1(r) = \frac{H^2}{4}\left[1 - \frac{\beta}{1 - \alpha\beta}\right](1 - r^2), \tag{2.22}$$

from which we obtain $0 \leq w_1(r)$*. Finally, it is readily seen that* $w_1^{'}(r) < 0$ *for all* $0 < r < 1$*. The estimate* w_1 *is therefore in good position to serve as an approximate solution to (1.1), provided that additional enhancement to* w_1 *can be made using perturbation expansion.*

3. Estimates for the value $\beta = w(0)$

In this section, we are concerned with the value $\beta = w(0)$ of the radial velocity function. It was concluded earlier that $\beta \leq \dfrac{1}{1+\alpha}$. This allows us to obtain the following estimate

Theorem 3.1. *Under the hypotheses of Theorem 3, we have*

$$\frac{H^2}{4} \leq \frac{\alpha}{(\alpha+1)\left[\alpha+1-\dfrac{1}{1-\alpha\beta}\right]}. \tag{3.1}$$

Proof. The first estimate follows by substituting with $r = 0$ in $w_1(r)$ and taking into account that

$$w_1(0) \leq \beta \leq \frac{1}{1+\alpha}, \tag{3.2}$$

and this gives

$$\frac{H^2}{4}\left[\frac{\alpha+1}{\alpha} - \frac{1}{\alpha(1-\alpha\beta)}\right] \leq \frac{1}{1+\alpha}. \tag{3.3}$$

Now, the inequality (3.1) follows directly from (3.3). □

Given the inequality above, we notice the following observations:

1. From (3.1), as H^2 increases, the quantity $\left(\alpha+1-\dfrac{1}{1-\alpha\beta}\right)$ should decrease, i.e. $\beta \longrightarrow \dfrac{1}{1+\alpha}$, which implies that for sufficiently large $H, \beta \cong \dfrac{1}{1+\alpha}$. Figure 1 presents this fact numerically for the case $H^2 = 25$. This conclusion agrees with the findings in [3].
2. (3.1) also suggests that if α increases without bounds, then the righthand side of (3.1) approaches to zero, from which we deduce that $H \to 0$. This suggests that for very high degrees of nonlinearity, the Hartmann number annihilates, which reflects the fact that the flow fluid is no longer influenced by the electromagnetic field.

Theorem 3.2. *Under the hypotheses of the previous Theorem, we have.*

1. If $\dfrac{\alpha H^2}{4} < 1$, then

$$\frac{H^2}{4}\left[\frac{\alpha+1}{\alpha} - \frac{1}{\alpha(1-\alpha\frac{H^2}{4})}\right] \leq \beta \leq \frac{H^2}{4}. \tag{3.4}$$

2. If $\dfrac{\alpha H^2}{4} \geq 1$, then

$$\frac{(\alpha+1)H^2-4}{(\alpha+1)H^2+\alpha(\alpha+1)H^2-4\alpha} \leq \beta \leq \frac{H^2}{4}. \tag{3.5}$$

Remark 3.3. *The condition $\dfrac{\alpha H^2}{4} < 1$ can be naturally obtained if we make the restriction that $w_2(0)$ doesn't exceed $\dfrac{1}{1+\alpha}$, the upper bound of the velocity $w(r)$. This constraint enhances our upper bound w_2 obtained in Theorem 4. Indeed,*

$$\frac{H^2}{4} = w_2(0) \leq \frac{1}{1+\alpha} < \frac{1}{\alpha}. \tag{3.6}$$

Proof. To prove (1), note that from Theorem 3, we have $w_1(0) \le w(0) \le w_2(0)$. This gives

$$\frac{H^2}{4}\left[\frac{\alpha+1}{\alpha} - \frac{1}{\alpha(1-\alpha\beta)}\right] \le \beta \le \frac{H^2}{4}. \tag{3.7}$$

From the condition $\frac{\alpha H^2}{4} < 1$, we see that $0 < 1 - \alpha\frac{H^2}{4} \le 1 - \alpha\beta$, which implies that

$$\frac{1}{\alpha(1-\alpha\beta)} \le \frac{1}{\alpha(1-\alpha H^2/4)}. \tag{3.8}$$

From (3.6)-(3.7) we obtain (3.4).
To prove (2), we use (3.1) to get

$$\frac{H^2}{4} \le \frac{\alpha}{(\alpha+1)^2 - \dfrac{\alpha+1}{1-\alpha\beta}} = \frac{1-\alpha\beta}{\alpha+1-\alpha^2\beta-2\alpha\beta-\beta} = \frac{1-\alpha\beta}{(1+\alpha)(1-\beta(\alpha+1))}. \tag{3.9}$$

This implies that

$$\frac{1-\beta(\alpha+1)}{1-\alpha\beta} \le \frac{4}{H^2(\alpha+1)} \tag{3.10}$$

or

$$1 - \frac{\beta}{1-\alpha\beta} \le \frac{4}{H^2(\alpha+1)}, \tag{3.11}$$

from which we get

$$\frac{\beta}{1-\alpha\beta} \ge 1 - \frac{4}{H^2(\alpha+1)} = \frac{H^2(\alpha+1)-4}{H^2(\alpha+1)}. \tag{3.12}$$

Now, the condition $\frac{\alpha H^2}{4} \ge 1$ allows us to write

$$\frac{H^2(\alpha+1)}{H^2(\alpha+1)-4} \ge \frac{1-\alpha\beta}{\beta} = \frac{1}{\beta} - \alpha. \tag{3.13}$$

Moving α to the other side and unifying denominators and taking the reciprocal again gives (3.5). □

4. Approximate solutions

In the following, we use a perturbation expansion to obtain approximate analytic solutions by applying our lower and upper bounds to the solution. Indeed, we write Pr. (1.1) as follows

$$w''(r) + \frac{w'(r)}{r} = -H^2\left(1 - \frac{w(r)}{1-\alpha w(r)}\right),\ 0 < r < 1. \tag{4.1}$$

Inserting the lower bound $w_2(r)$ of the solution of Pr. (1.1) into the righthand side of (4.1), we obtain

$$w''(r) + \frac{w'(r)}{r} = -H^2\left(1 - \frac{w_1(r)}{1-\alpha w_1(r)}\right),\ 0 < r < 1 \tag{4.2}$$

or

$$(rw'(r))' = -H^2\left(r - \frac{rw_1(r)}{1-\alpha w_1(r)}\right),\ 0 < r < 1. \tag{4.3}$$

Integrating (4.3) from zero to r and taking into account that $w'(0) = 0$, we obtain

$$w'(r) = \frac{-H^2}{2}r + H^2\frac{1}{r}\int_0^r \frac{tw_1(t)}{1-\alpha w_1(t)}dt. \tag{4.4}$$

Integrating again this equation from r to one and taking into account that $w(1) = 0$, we obtain

$$w_1(r,\alpha) = \frac{H^2}{4}(1-r^2) - H^2\int_r^1\left[\frac{1}{s}\int_0^s \frac{tw_1(t)}{1-\alpha w_1(t)}dt\right]ds, \tag{4.5}$$

which can be considered as an improved approximate solution with the given condition $\alpha < \frac{4}{H^2}$. Similarly, inserting the upper bound $w_2(r)$ of the solution into the righthand side of (4.1), we get

$$w_2(r,\alpha) = \frac{H^2}{4}(1-r^2) - H^2\int_r^1\left[\frac{1}{s}\int_0^s \frac{tw_2(t)}{1-\alpha w_2(t)}dt\right]ds. \tag{4.6}$$

This is the approximate solution of Pr. (1.1) with $\alpha < \frac{4}{H^2}$.

5. Discussions

We utilized the robust Maple software for numerical validation and developed a program that is effortlessly navigable with straightforward statements to tackle BVPs. The program can identify the type of issue and choose the most suitable algorithm to solve it. We employed the mid-defer method, which is a midpoint method that incorporates enhancement schemes. For enhancement schemes, the Richardson extrapolation method is the faster option, and deferred corrections are better suited for challenging problems due to their lower memory usage. Furthermore, this method can handle endpoint singularities that the trapezoidal scheme cannot address. Utilizing the continuation method is crucial in reducing global error when selecting an appropriate number of max mesh. This numerical technique modifies the coefficient of the second-order derivative and has been proven effective [15–19]. The numerical method utilizes carefully selected parameters. Specifically, the absolute error criterion is set to 10^{-4}, while the maximum mesh size is determined as 256.

Let's start by examining the numerical outcomes of the problem at hand with varying values of the parameter α, which pertains to the fluid's conductivity or characteristics, and for specific Hartmann numbers H, which indicates the intensity of the magnetic field applied.

In Figure 1, we present the numerical solutions for three cases ($H = 1, 2, 5$), depicting the plotted values of ($\alpha = 0, 2, 4, 6$) against the independent variable r. Notably, the numerical values significantly increase with an increase in the Hartmann number, while a decrease in the α parameter results in a

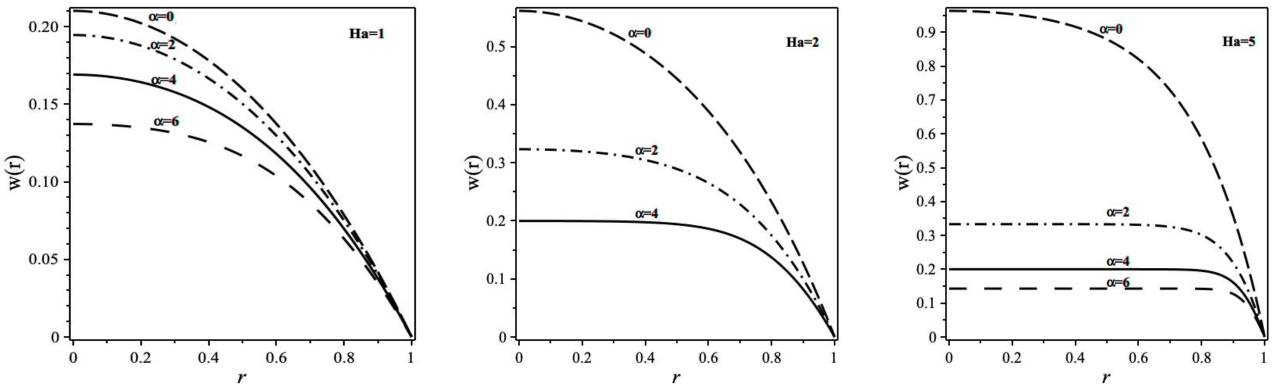

Figure 1. The numerical solution for the boundary problem (1.1) for different values of H and the parameter α.

decrease in the numerical solution. Additionally, it is evident that as the Hartmann number and the parameter α both increase, the solution tends to become flatter and is bounded by $1/(1+\alpha)$. This observation aligns closely with the findings presented by Paullet in reference [5].

However, it is evident that the lower bound is contingent upon $\beta = w(0)$, of the solution. Our task is to elucidate the approach for estimating this value. Presented in the following Table 1 is a comparison between the numerical central value and the initial lower and upper bounds about the condition $\alpha\frac{H^2}{4} \leq 1$ for different values of the Hartmann's number and the parameter α. Notably, the value $w(0)$ can be approximated as the midpoint between these lower and upper bounds.

Figure 2 displays a comparison between the numerical solution, lower and upper approximations for $\alpha = 2$ and $H = 0.5, 1., 1.4$ against r. The accuracy of the lower solution in approximating numerical solutions is undeniable, regardless of the Hartmann constant's value. It is particularly noteworthy that the lower solution shows excellent alignment with the numerical solution when the Hartmann number is small.

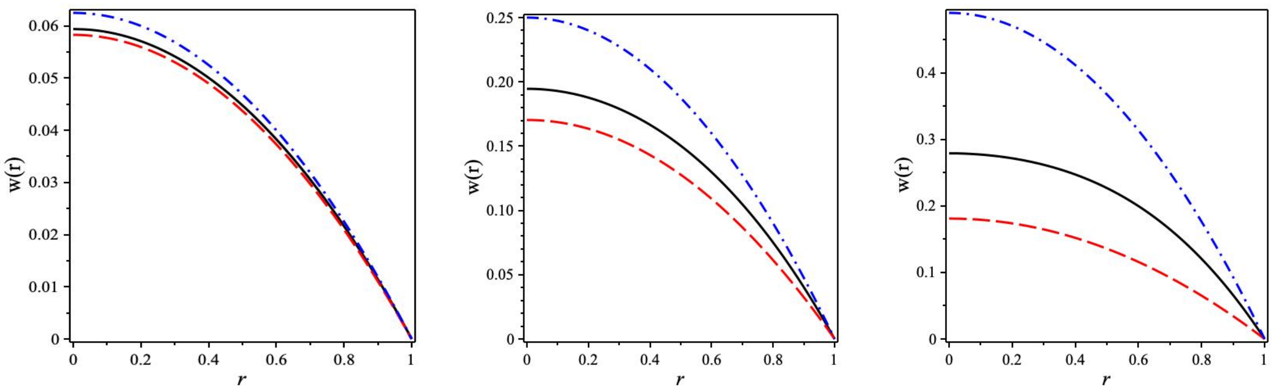

Figure 2. The numerical solution, lower and upper approximations for the boundary problem (1.1) for different values of $H = 0.5, 1, 1.4$ (from the left to the right panel, respectively) and the parameter $\alpha = 2$. Black solid line (Numerical solution). Red dashed line (Lower approximation. Blue dash-dotted line (Upper approximation)

Table 1. An estimation of $w(0)$ by examining the lower and upper bounds of BVP (1.1) for varying values of the Hartmann number, specifically, $H = 0.5$ and 1, in conjunction with different values of the parameter α. It's important to note that this analysis is valid when the condition $1 - \alpha w(0) > 0$ is satisfied, which conforms with 3.4.

H = 0.5

α	initial value of lower bound	$w_N(0)$	initial value upper bound
.5	.058468	0.059634	0.0625
1.	.0583334	0.059563	0.0625
2.	.0580358	0.059411	0.0625
3.	.0576922	0.059243	0.0625
4.	.0572916	0.059055	0.0625
5.	.0568181	0.058845	0.0625
6.	.0562500	0.058608	0.0625
8.	.0546875	0.058020	0.0625

H = 1

α	initial value of lower bound	$w_N(0)$	initial value upper bound
.1	.18590	0.209556	.25
.2	.18421	0.208944	.25
.3	.18243	0.208316	.25
.4	.18056	0.207670	.25
.5	.17857	0.207010	.25
1.	.16667	0.203415	.25

In Figure 3, we explored the results obtained by choosing a small Hartmann number and varying the parameter α between 4, 6, and 10. The lower solution aligns well with the numerical solution when the parameter α has modest values. On the other hand, the upper approximation is not affected by the parameter α and is in accordance with the numerical solution when α has small values, but diverges when α is large.

To explore the range of errors in lower and upper approximations compared to numerical solutions, we have provided the Tables 2 and 3 displaying numerical and approximate values and absolute errors for various Hartmann numbers and α parameters. In all instances, the lower approximation effectively aligns with the numerical solutions. It is imperative that we thoroughly scrutinize the enhanced approximation solutions we have acquired, specifically $w_i(r, \alpha)$, to determine their level of accuracy. In Table 4, we compare the numerical solution to the improved approximation solutions obtained using Eqs (4.5) and (4.6), as well as the corresponding absolute errors. It is imperative to note that, without exception, for all values of α and $H = 1$, the second improved solution is indisputably more accurate and displays impeccable agreement with the numerical solution for the given problem. The condition $1 - \alpha w(0) > 0$ is also taken into consideration. On the other hand, we examine situations where the Hartmann number, H, has a high value.

Finally, we compare our results with those obtained from the pure perturbation theory given by

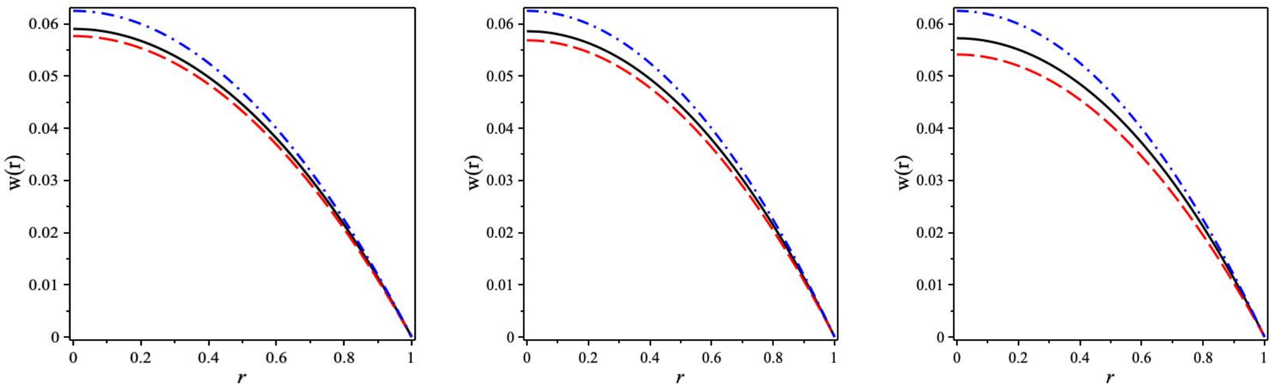

Figure 3. The numerical solution, lower and upper approximations for the boundary problem (1.1) for different values of $\alpha = 4, 6, 10$ (from the left to the right panel, respectively) and the Hartmann number $H = 0.5$. Black solid line (Numerical solution). Red dashed line (Lower approximation. Blue dash-dotted line (Upper approximation)

Mackee et al. [4]. The so-called large-α perturbation solution is given as

$$w_{Mackee}(r;\alpha_l) = \frac{H^2}{4}(1+\frac{1}{\alpha})(1-r^2) + \frac{1}{\alpha^2}\Big(2\int_0^r \frac{log(1-s^2)}{s}ds + \frac{\pi^2}{6}\Big). \tag{5.1}$$

In Table 5, we present a comparison of practical approaches for the solution of BVP (1.1). We evaluate the numerical solution $w_N(r)$ alongside analytical lower and upper approximations, denoted as $w_L(r)$ and $w_U(r)$, respectively. Additionally, we include the large-α perturbation solution $w_{Mackee}(r;\alpha_l)$, as obtained by Mckee et al. [4], specifically for the Hartmann number ($H = 0.5$) and parameter values ($\alpha = 4, 8$).

It is worth noting that our modified approximation solution, as presented in Eq (4.6), demonstrates excellent agreement with the numerical solutions. Furthermore, the large-α perturbation solution yields significantly higher values for the solution.

Further, it is interesting to notice that the numerical values of the Mackee large α perturbation solution (5.1) exceed the upper bound w_2 for $\alpha = 4, 8$. This in turn implies that the formula (5.1) cannot be considered a valid approximate solution of 1.1 according to (2.5). This supports the claims made by [3] of an error in the perturbation order of the leading term of the formula given by [2].

In summary, our comparative analysis provides actionable insights into the effectiveness of various solution approaches, highlighting the reliability of our modified approximation method when dealing with this specific problem.

6. Conclusions

This study delved into the nonlinear EHD differential equation (1.1). We investigated the upper and lower bounds for the solution and used the lower bound to establish an improved approximate solution that satisfies the boundary conditions and agrees with the result of Theorem 1. We also found that this approximate solution shows excellent alignment with the numerical solution, especially when the Hartmann number is small. As pointed out in the introduction, semiconductors materials, gases,

Table 2. Comparison between the numerical solution $w_N(r)$ of BVP (1.1), the analytical lower and upper approximations $w_L(r), w_U(r)$ and the correspondent absolute errors (Err_L, Err_U) for the Hartmann number $H = 0.5$ and the parameter $\alpha = 2, 6$, where the condition $1 - \alpha w(0) > 0$ is satisfied.

$\alpha = \mathbf{2}$

r	$w_N(r)$	$w_L(r)$	ErrL	$w_U(r)$	ErrU
0.	.594114e-1	.582860e-1	.11254e-2	.625000e-1	.30886e-2
.100000	.588281e-1	.577032e-1	.11249e-2	.618750e-1	.30469e-2
.200000	.570776e-1	.559545e-1	.11231e-2	.600000e-1	.29224e-2
.300000	.541554e-1	.530402e-1	.11152e-2	.568750e-1	.27196e-2
.400000	.500544e-1	.489602e-1	.10942e-2	.525000e-1	.24456e-2
.500000	.447654e-1	.437145e-1	.10509e-2	.468750e-1	.21096e-2
.600000	.382762e-1	.373030e-1	.9732e-3	.400000e-1	.17238e-2
.700000	.305723e-1	.297258e-1	.8465e-3	.318750e-1	.13027e-2
.800000	.216376e-1	.209830e-1	.6546e-3	.225000e-1	.8624e-3
.900000	.114537e-1	.110744e-1	.3793e-3	.118750e-1	.4213e-3
1.	0.	0.	0.	0.	0.

$\alpha = \mathbf{6}$

r	$w_N(r)$	$w_L(r)$	ErrL	$w_U(r)$	ErrU
0.	.590550e-1	.576675e-1	.13875e-2	.625000e-1	.34450e-2
.100000	.584777e-1	.570908e-1	.13869e-2	.618750e-1	.33973e-2
.200000	.567454e-1	.553608e-1	.13846e-2	.600000e-1	.32546e-2
.300000	.538517e-1	.524775e-1	.13742e-2	.568750e-1	.30233e-2
.400000	.497880e-1	.484408e-1	.13472e-2	.525000e-1	.27120e-2
.500000	.445421e-1	.432505e-1	.12916e-2	.468750e-1	.23329e-2
.600000	.380996e-1	.369072e-1	.11924e-2	.400000e-1	.19004e-2
.700000	.304437e-1	.294105e-1	.10332e-2	.318750e-1	.14313e-2
.800000	.215553e-1	.207603e-1	.7950e-3	.225000e-1	.9447e-3
.900000	.114145e-1	.109568e-1	.4577e-3	.118750e-1	.4605e-3
1.	0.	0.	0.	0.	0.

Table 3. Comparison between the numerical solution $w_N(r)$ of BVP (1.1), the analytical lower and upper approximations $w_L(r), w_U(r)$ and the correspondent absolute errors (Err_L, Err_U) for the Hartmann number $H = 1$ and the parameter $\alpha = 0.1, 0.5$, where the condition $1 - \alpha w(0) > 0$ is satisfied.

$\alpha = \mathbf{0.1}$

r	$w_N(r)$	$w_L(r)$	ErrL	$w_U(r)$	ErrU
0.	.209556	.196500	.13056e-1	.250000	.40444e-1
.100000	.207590	.194535	.13055e-1	.247500	.39910e-1
.200000	.201675	.188640	.13035e-1	.240000	.38325e-1
.300000	.191768	.178815	.12953e-1	.227500	.35732e-1
.400000	.177788	.165060	.12728e-1	.210000	.32212e-1
.500000	.159628	.147375	.12253e-1	.187500	.27872e-1
.600000	.137147	.125760	.11387e-1	.160000	.22853e-1
.700000	.110169	.100215	.9954e-2	.127500	.17331e-1
.800000	.784875e-1	.707400e-1	.77475e-2	.900000e-1	.115125e-1
.900000	.418577e-1	.373350e-1	.45227e-2	.475000e-1	.56423e-2
1.	0.	0.	0.	0.	0.

$\alpha = \mathbf{0.5}$

r	$w_N(r)$	$w_1(r)$	Err1L	$w_2(r)$	ErrU
0.	.207008	.192272	.14736e-1	.250000	.42992e-1
.100000	.205084	.190350	.14734e-1	.247500	.42416e-1
.200000	.199293	.184582	.14711e-1	.240000	.40707e-1
.300000	.189583	.174968	.14615e-1	.227500	.37917e-1
.400000	.175862	.161509	.14353e-1	.210000	.34138e-1
.500000	.158008	.144204	.13804e-1	.187500	.29492e-1
.600000	.135862	.123054	.12808e-1	.160000	.24138e-1
.700000	.109229	.980590e-1	.111700e-1	.127500	.18271e-1
.800000	.778840e-1	.692180e-1	.86660e-2	.900000e-1	.121160e-1
.900000	.415702e-1	.365318e-1	.50384e-2	.475000e-1	.59298e-2
1.	0.	0.	0.	0.	0.

Table 4. Comparison between the numerical solution $w_N(r)$ of BVP (1.1), the analytical improved approximations $w_1(r,\alpha), w_2(r,\alpha)$ and the correspondent absolute errors (Err_1, Err_2) for the Hartmann number $H = 1$ and the parameter $\alpha = 0.1, 0.5, 2$, where the condition $1 - \alpha w(0) > 0$ is satisfied.

$\alpha = \mathbf{0.1}$

r	$w_N(r)$	$w_1(r,\alpha)$	Err1	$w_2(r,\alpha)$	Err2
.100000	.207590	.199537	.8053e-2	.205186	.2404e-2
.200000	.201675	.194159	.7516e-2	.200786	.889e-3
.300000	.191768	.184886	.6882e-2	.191662	.106e-3
.400000	.177788	.171679	.6109e-2	.178126	.338e-3
.500000	.159628	.154422	.5206e-2	.160176	.548e-3
.600000	.137147	.132939	.4208e-2	.137756	.609e-3
.700000	.110169	.107013	.3156e-2	.110732	.563e-3
.800000	.0784875	.0764193	.20682e-2	.0789155	.4280e-3
.900000	.0418577	.0408533	.10044e-2	.0420955	.2378 e-3
1.	0.	0.	0.	0.	0.

$\alpha = \mathbf{0.5}$

r	$w_N(r)$	$w_1(r,\alpha)$	Err1	$w_2(r,\alpha)$	Err2
.100000	.205084	.196035	.9049e-2	.208836	.3752e-2
.200000	.199293	.190634	.8659e-2	.202930	.3637e-2
.300000	.189583	.181566	.8017e-2	.192996	.3413e-2
.400000	.175862	.168708	.7154e-2	.178981	.3119e-2
.500000	.158008	.151894	.6114e-2	.160739	.2731e-2
.600000	.135862	.130917	.4945e-2	.138134	.2272e-2
.700000	.109229	.105533	.3696e-2	.110970	.1741e-2
.800000	.0778840	.0754618	.24222e-2	.0790521	.11681e-2
.900000	.0415702	.0403957	.11745e-2	.0421270	.5568e-3
1.	0.	0.	0.	0.	0.

$\alpha = \mathbf{2}$

r	$w_N(r)$	$w_1(r,\alpha)$	Err1	$w_2(r,\alpha)$	Err2
.100000	.192891	.168128	.24763e-1	.203655	.10764e-1
.200000	.187737	.164286	.23451e-1	.198055	.10318e-1
.300000	.179037	.157647	.21390e-1	.188629	.9592e-2
.400000	.166629	.147888	.18741e-1	.175240	.8611e-2
.500000	.150299	.134609	.15690e-1	.157709	.7410e-2
.600000	.129790	.117366	.12424e-1	.135827	.6037e-2
.700000	.104815	.95707e-1	.9108e-2	.109360	.4545e-2
.800000	.0750666	.069192	.58746e-2	.0780648	.29982e-2
.900000	.0402316	.037413	.28186e-2	.0416928	.14612e-2
1.	0.	0.	0.	0.	0.

Table 5. Comparison of the numerical solution $w_N(r)$ of BVP (1.1) with the analytical lower and upper approximations $w_L(r), w_U(r)$, as well as the large-α perturbation solution $w_{Mackee}(r;\alpha_l)$ obtained by Mckee et al. [4]. The absolute errors for the given Hartmann number ($H = 0.5$) and parameter values ($\alpha = 4, 6$) are represented as (Err_L, Err_U, Err_{Mc}), respectively, where the condition $1 - \alpha w(0) > 0$ is satisfied.

$\alpha = \mathbf{4}$

r	$w_N(r)$	$w_1(r,\alpha)$	ErrL	$w_2(r,\alpha)$	ErrU	$w_{Mc}(r;\alpha_l)$	Err_{Mc}
.100000	.584777e-1	.582372e-1	.2405e-3	.585857e-1	.1080e-3	1.79527e-1	1.21049e-1
.200000	.567454e-1	.565160e-1	.2294e-3	.568488e-1	.1034e-3	1.75283e-1	1.18538e-1
.300000	.538517e-1	.536402e-1	.2115e-3	.539482e-1	.965e-4	1.68147e-1	1.14295e-1
.400000	.497880e-1	.496003e-1	.1877e-3	.498750e-1	.870e-4	1.58003e-1	1.08215e-1
.500000	.445421e-1	.443828e-1	.1593e-3	.446177e-1	.756e-4	1.44674e-1	1.00132e-1
.600000	.380996e-1	.379717e-1	.1279e-3	.381617e-1	.621e-4	1.27874e-1	.897744e-1
.700000	.304437e-1	.303484e-1	.953e-4	.304906e-1	.469e-4	1.07126e-1	.766823e-1
.800000	.215553e-1	.214929e-1	.624e-4	.215863e-1	.310e-4	.81558e-1	.600027e-1
.900000	.114145e-1	.113841e-1	.304e-4	.114294e-1	.149e-4	.492106e-1	.377961e-1
1.	0.	0.	0.	0.	0.	0.	0.

$\alpha = \mathbf{8}$

r	$w_N(r)$	$w_1(r,\alpha)$	ErrL	$w_2(r,\alpha)$	ErrU	$w_{Mc}(r;\alpha_l)$	Err_{Mc}
.100000	.574628e-1	.569142e-1	.5486e-3	.577780e-1	.3152e-3	.951555e-1	.376927e-1
.200000	.557864e-1	.552678e-1	.5186e-3	.560878e-1	.3014e-3	.925702e-1	.367838e-1
.300000	.529807e-1	.525092e-1	.4715e-3	.532596e-1	.2789e-3	.882483e-1	.352676e-1
.400000	.490296e-1	.486180e-1	.4116e-3	.492784e-1	.2488e-3	.821569e-1	.331273e-1
.500000	.439126e-1	.435693e-1	.3433e-3	.441250e-1	.2124e-3	.742545e-1	.303419e-1
.600000	.376062e-1	.373353e-1	.2709e-3	.377778e-1	.1716e-3	.644684e-1	.268622e-1
.700000	.300859e-1	.298879e-1	.1980e-3	.302140e-1	.1281e-3	.526801e-1	.225942e-1
.800000	.213269e-1	.211994e-1	.1275e-3	.214109e-1	.840e-4	.386706e-1	.173437e-1
.900000	.113056e-1	.112445e-1	.611e-4	.113463e-1	.407e-4	.219515e-1	.106459e-1
1.	0.	0.	0.	0.	0.	0.	0.

distilled water, and microfluidic technologies exhibit small Hartmann numbers, so the approximate solution obtained in this study can serve as a suitable model to investigate the solutions of (1.1) in these cases. Further, estimates for the central velocity $w(0)$ were established, and it was found that sufficiently large α yields a very small Hartmann number. On the other hand, sufficiently large Hartmann numbers imply $\beta \longrightarrow \dfrac{1}{1+\alpha}$, which agrees with [3].

Furthermore, this study also confirmed the fact that finding lower and upper bounds for solutions to differential equations is an important mathematical and analytical tool that can provide valuable insights in understanding and modeling various physical and engineering systems. The methods used in this paper are expected to contribute to the development of more effective and efficient strategies for solving similar problems in the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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