AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(1): 763–774.
DOI:10.3934/math.2024039
Received: 13 October 2023
Revised: 22 November 2023
Accepted: 27 November 2023
Published: 04 December 2023

Research article

Fixed point results for almost $(\zeta - \theta_\rho)$-contractions on quasi metric spaces and an application

Gonca Durmaz Güngör[1] **and Ishak Altun**[2,*]

[1] Department of Mathematics, Faculty of Science, Çankırı Karatekin University, 18100 Çankırı, Turkey

[2] Department of Mathematics, Faculty of Engineering and Natural Science, Kirikkale University, Yahsihan 71450, Kirikkale, Turkey

* **Correspondence:** Email: ishakaltun@yahoo.com.

Abstract: This research paper investigated fixed point results for almost $(\zeta - \theta_\rho)$-contractions in the context of quasi-metric spaces. The study focused on a specific class of $(\zeta - \theta_\rho)$-contractions, which exhibit a more relaxed form of contractive property than classical contractions. The research not only established the existence of fixed points under the almost $(\zeta - \theta_\rho)$-contraction framework but also provided sufficient conditions for the convergence of fixed point sequences. The proposed theorems and proofs contributed to the advancement of the theory of fixed points in quasi-metric spaces, shedding light on the intricate interplay between contraction-type mappings and the underlying space's quasi-metric structure. Furthermore, an application of these results was presented, highlighting the practical significance of the established theory. The application demonstrated how the theory of almost $(\zeta - \theta_\rho)$-contractions in quasi-metric spaces can be utilized to solve real-world problems.

Keywords: quasi metric space; $(\zeta - \theta_\rho)$-contraction; left K-completeness
Mathematics Subject Classification: Primary 54H25; Secondary 47H10

1. Introduction

The notion of a quasi metric space represents an intriguing extension of the classical metric space, achieved by relaxing the requirement of symmetry. Among various alternatives to the metric space, the quasi metric space stands out as a particularly intuitive concept that finds direct applicability in real-world scenarios. A simple illustration of a quasi metric arises when considering the distance traveled by a commuter between their home and workplace in a city characterized by one-way streets and two-way roads. For additional and specific instances of quasi metrics, along with compelling fixed-point outcomes within this context, please references [4, 5, 7, 8, 12, 13, 15, 24, 26, 27, 31]. Over the past few decades, numerous papers on the fixed point theory have been published, many of which extend the established fixed point results in various ways: By altering the abstract space, by substituting the contraction condition with a milder one and so on and so forth. As a result, an inherent question arises: Can the existing outcomes be amalgamated in an uncomplicated manner? Several responses have been provided, and among these, a few of the most intriguing answers are related to the 'θ-contraction' and the 'simulation function'. The definition of the θ-contraction is given by Jleli et al. [20], while the concept of the simulation function is introduced by Khojasteh et al. [22]. Furthermore, using these functions, a multitude of single-valued fixed point results have been achieved in the standard metric space. In this work, we will explore the response to the inquiry: How can we amalgamate established fixed point theorems within the framework of a quasi-metric space by employing the θ-contraction and the simulation function? In order to provide the most effective solution, we will additionally make use of another auxiliary function known as an admissible mapping. It is notably intriguing that the admissible mapping possesses the capability to merge the fixed point theorems within a metric space coupled with a partially ordered set, as well as the associated fixed point propositions resulting from cyclic contractions or standard contractions. For a more comprehensive understanding, refer to sources such as [1, 2, 10, 11, 17, 18, 21, 23]. As a result, we have harmonized various fixed point outcomes within the framework of a quasi-metric space, utilizing both simulation functions, θ-contraction and admissible mappings.

2. Preliminaries

Now, review the definitions and notations related to quasi-metric space: $\Lambda \neq \emptyset$ and ρ are a function $\rho : \Lambda \times \Lambda \to \mathbb{R}$ such that for each $\omega, \gamma, \eta \in \Lambda$:

(i) $\rho(\omega, \omega) = 0$,
(ii) $\rho(\omega, \gamma) \leq \rho(\omega, \eta) + \rho(\eta, \gamma)$ (triangle inequality),
(iii) $\rho(\omega, \gamma) = \rho(\gamma, \omega) = 0$ implies $\omega = \gamma$,
(iv) $\rho(\omega, \gamma) = 0$ implies $\omega = \gamma$.

If (i) and (ii) conditions are satisfied, then ρ is called a quasi-pseudo metric (shortly qpm); if (i)–(iii) conditions are satisfied, then ρ is called quasi metric (shortly qm); in addition, if a qm ρ satisfies (iv), then ρ is called T_1-qm. It is evident that every metric is a T_1-qm, every T_1-qm is a qm and every qm is a qpm. Then, the pair (Λ, ρ) is also said to be a quasi-pseudo metric space (shortly qpms). Moreover, each qpm ρ on Λ generates a topology τ_ρ on Λ of the family of open balls as a base defined as follows:

$$\{B_\rho(\omega, \varepsilon) : \omega \in \Lambda \text{ and } \varepsilon > 0\},$$

where $B_\rho(\omega_0, \varepsilon) = \{\gamma \in \Lambda : \rho(\omega_0, \gamma) < \varepsilon\}$.

If ρ is a qm on Λ, then τ_ρ is a T_0 topology, and if ρ is a T_1-qm, then τ_ρ is a T_1 topology on Λ. If ρ is a qm and τ_ρ is a T_1 topology, then ρ is T_1-qm.

The mapping $\overline{\rho}$ defines

$$\overline{\rho}(\omega, \gamma) = \rho(\gamma, \omega)$$

as a qpm whenever ρ is a qpm on Λ. To find the fixed point, the most important part is to use the completeness of the metric space. However, since there is no symmetry conditions in a qm, there are many kinds of completeness in these spaces in the literature (see [9, 28, 30]).

Let (Λ, ρ) be a qms, then the convergence of a sequence $\{\omega_n\}$ to ω w. r. t. τ_ρ called ρ-convergence is defined as $\omega_n \overset{\rho}{\to} \omega$ if, and only if, $\rho(\omega, \omega_n) \to 0$. Similarly, the convergence of a sequence $\{\omega_n\}$ to ω w. r. t. $\tau_{\overline{\rho}}$ called $\overline{\rho}$-convergence is defined $\omega_n \overset{\overline{\rho}}{\to} \omega$ if, and only if, $\rho(\omega_n, \omega) \to 0$ for $\omega \in \Lambda$. A more detailed explanation of some essential metric properties can be found in [26]. Also, a sequence $\{\omega_n\}$ in Λ is called left (right) K-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, k \in \mathbb{N}$ with $n \geq k \geq n_0$ $(k \geq n \geq n_0)$, $\rho(\omega_k, \omega_n) < \varepsilon$. The left K-Cauchy property under ρ implies the right K-Cauchy property under $\overline{\rho}$. Assuming

$$\sum_{n=1}^{+\infty} \rho(\omega_n, \omega_{n+1}) < +\infty,$$

the sequence $\{\omega_n\}$ in the quasi-metric space (Λ, ρ) is left K-Cauchy.

In a metric space, every convergent sequence is indeed a Cauchy sequence, but since this may not hold true in qms, there have been several definitions of completeness. A qms (Λ, ρ) is said to be left (right) K (resp. M)-complete if every left (right) K-Cauchy sequence is ρ (resp. $\overline{\rho}$)-convergent.

Now, we explain the approach of α-admissibility as constructed by Samet et al. [29].

Let $\Lambda \neq \emptyset$, $\Upsilon : \Lambda \to \Lambda$ be a mapping and $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ be a function. In this context, Υ is said to be α-admissible if it satisfies the following condition:

$$\text{If } \alpha(\omega, \gamma) \geq 1, \text{ then } \alpha(\Upsilon\omega, \Upsilon\gamma) \geq 1.$$

By introducing the approach of α-admissibility, Samet et al. [29] was able to establish some general fixed point results that encompassed many well-known theorems of complete metric spaces.

In addition to these, in the study conducted by Jleli and Samet in [20], they led to the introduction of a new type of contractive mapping known as a θ-contraction. This θ-contraction serves as an attractive generalization within the field. To better understand this approach, let's review some notions and related results concerning θ-contraction.

The family of $\theta : (0, +\infty) \to (1, +\infty)$ functions that satisfy the following conditions can be denoted by the set Θ :

(θ_1) θ is nondecreasing;

(θ_2) Considering every sequence $\{\varkappa_n\} \subset (0, +\infty)$, $\lim_{n\to+\infty} \varkappa_n = 0^+$ if, and only if, $\lim_{n\to+\infty} \theta(\varkappa_n) = 1$;

(θ_3) There exists $0 < p < 1$ and $\beta \in (0, +\infty]$ such that $\lim_{\varkappa\to 0^+} \frac{\theta(\varkappa)-1}{\varkappa^p} = \beta$.

If we define $\theta(\varkappa) = e^{\sqrt{\varkappa}}$ for $\varkappa \leq 1$ and $\theta(\varkappa) = 9$ for $\varkappa > 1$, then $\theta \in \Theta$.

Let $\theta \in \Theta$ and (Λ, ρ) be a quasi metric space, then $\Upsilon : \Lambda \to \Lambda$ is said to be a θ-contraction if there exists $0 < \delta < 1$ such that

$$\theta(\rho(\Upsilon\omega, \Upsilon\gamma)) \leq [\theta(\rho(\omega, \gamma))]^\delta \tag{2.1}$$

for each $\omega, \gamma \in \Lambda$ with $\rho(\Upsilon\omega, \Upsilon\gamma) > 0$.

By choosing appropriate functions for θ, such as $\theta_1(\varkappa) = e^{\sqrt{\varkappa}}$ and $\theta_2(\varkappa) = e^{\sqrt{\varkappa e^{\varkappa}}}$, it is possible to obtain different types of nonequivalent contractions using (2.1).

Jleli and Samet proved that every θ-contraction on a complete metric space possesses a unique fixed point. This result provides a valuable insight into the uniqueness and existence of fixed points for a wide range of contractive mappings. If you are interested in exploring more papers and literature related to θ-contractions, there are several resources available (see [3, 19]).

On the other hand, Khojasteh et al. [22] introduced an innovative category of contractions through the utilization of the following concept of simulation functions. By employing the concept, they [22] established numerous fixed point theorems and demonstrated that numerous well-established findings in the literature stem directly from the outcomes they derived. Furthermore, using the simulation function, generalizations of many known theorems have been obtained (see [6, 25, 30]). To better understand this approach, let's review some notions and related results concerning simulation functions.

The function $\zeta : [0,+\infty)\times[0,+\infty) \to \mathbb{R}$ is said to be a simulation function that satisfies the following conditions and can be denoted by the set $\mathbb{Z}$:
(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$;
(ζ_3) if $\{t_n\}, \{s_n\}$ are sequence in $(0,+\infty)$ such that

$$\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n > 0,$$

then $\overline{\lim\limits_{n\to\infty}}\zeta(t_n, s_n) < 0$.

If we define $\zeta_1(t,s) = \psi(s) - \varphi(t)$ for all $t, s \geq 0$, where $\psi, \varphi : [0,+\infty) \to [0,+\infty)$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if, only if, $t = 0$ and $\psi(t) < t \leq \varphi(t)$ for all $t > 0$, then $\zeta \in \mathbb{Z}$.

3. The results

Our results are based on a novel approach that we have developed.

Let (Λ, ρ) be a qms, $\Upsilon : \Lambda \to \Lambda$ be a given mapping and $\alpha : \Lambda \times \Lambda \to [0,+\infty)$ be a function. We will consider the following set

$$\Upsilon_\alpha = \{(\omega,\gamma) \in \Lambda \times \Lambda : \alpha(\omega,\gamma) \geq 1 \text{ and } \rho(\Upsilon\omega, \Upsilon\gamma) > 0\}. \tag{3.1}$$

Definition 1. *Let (Λ, ρ) be a qms and $\Upsilon : \Lambda \to \Lambda$ be a mapping satisfying*

$$\rho(\omega,\gamma) = 0 \textit{ implies } \rho(\Upsilon\omega, \Upsilon\gamma) = 0. \tag{3.2}$$

Let $\alpha : \Lambda \times \Lambda \to [0,+\infty)$, $\zeta \in \mathbb{Z}$ and $\theta \in \Theta$ be three functions, then we say that Υ is an almost $(\zeta - \theta_\rho)$-contraction if there exists $0 < \delta < 1$ and $L \geq 0$ such that

$$\zeta\left(\alpha(\omega,\gamma)\,\theta(\rho(\Upsilon\omega,\Upsilon\gamma)), [\theta(M(\omega,\gamma) + LN(\omega,\gamma))]^\delta\right) \geq 0, \tag{3.3}$$

for each $(\omega,\gamma) \in \Upsilon_\alpha$, where

$$\begin{aligned} M(\omega,\gamma) &= \max\left\{\rho(\omega,\gamma), \rho(\Upsilon\omega,\omega), \rho(\Upsilon\gamma,\gamma), \frac{1}{2}\left[\rho(\Upsilon\omega,\gamma) + \rho(\omega,\Upsilon\gamma)\right]\right\}, \\ N(\omega,\gamma) &= \min\{\rho(\Upsilon\omega,\gamma), \rho(\omega,\Upsilon\gamma)\}. \end{aligned}$$

Before presenting our main results, let us recall some important remarks:

- If (Λ, ρ) is a T_1-qms, then every mapping $\Upsilon : \Lambda \to \Lambda$ satisfies the condition (3.2).

- It is clear from (3.1)–(3.3) that if Υ is an almost $(\zeta - \theta_\rho)$-contraction, then

$$\rho(\Upsilon\omega, \Upsilon\gamma) \leq M(\omega, \gamma) + LN(\omega, \gamma),$$

for each $\omega, \gamma \in \Lambda$ with $\alpha(\omega, \gamma) \geq 1$.

By utilizing the approach of the almost $(\zeta - \theta_\rho)$-contraction, we will now present the following theorem.

Theorem 1. *Let (Λ, ρ) be a Hausdorff right K-complete T_1-qms and let $\Upsilon : \Lambda \to \Lambda$ be τ_ρ-continuous, α-admissible and an almost $(\zeta - \theta_\rho)$-contraction. If there exists $\omega_0 \in \Lambda$ such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$, then Υ has a fixed point in Λ.*

Proof. Let $\omega_0 \in \Lambda$ be such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$. Define a sequence $\{\omega_n\}$ in Λ by $\omega_{n+1} = \Upsilon\omega_n$ for each n in $\mathbb{N}$. Since Υ is α-admissible, then $\alpha(\omega_{n+1}, \omega_n) \geq 1$ for each n in $\mathbb{N}$. If there exists $k \in \mathbb{N}$ with $\rho(\Upsilon\omega_k, \omega_k) = 0$, then $\omega_k = \Upsilon\omega_k$, since ρ is T_1-qm. Hence, ω_k is a fixed point of Υ. Presume $\rho(\Upsilon\omega_n, \omega_n) > 0$ for each n in $\mathbb{N}$. In this case, the pair (ω_{n+1}, ω_n) for each n in $\mathbb{N}$ belongs to Υ_α. Since Υ is an almost $(\zeta - \theta_\rho)$-contraction, we have

$$\zeta(\alpha(\omega_n, \omega_{n-1})\theta(\rho(\Upsilon\omega_n, \Upsilon\omega_{n-1})), [\theta(M(\omega_n, \omega_{n-1}) + LN(\omega_n, \omega_{n-1}))]^\delta) \geq 0,$$

and so from (ζ_2), we have

$$0 \leq [\theta(M(\omega_n, \omega_{n-1}) + LN(\omega_n, \omega_{n-1}))]^\delta - \alpha(\omega_n, \omega_{n-1})\,\theta(\rho(\omega_{n+1}, \omega_n)).$$

Hence, from (θ_1) we obtain

$$\begin{aligned}
\theta(\rho(\omega_{n+1}, \omega_n)) &\leq [\theta(M(\omega_n, \omega_{n-1}) + LN(\omega_n, \omega_{n-1}))]^\delta \\
&= \left[\begin{array}{c} \theta(\max \left\{ \begin{array}{c} \rho(\omega_n, \omega_{n-1}), \rho(\omega_{n+1}, \omega_n), \rho(\omega_n, \omega_{n-1}), \\ \frac{1}{2}\left[\rho(\omega_{n+1}, \omega_{n-1}) + \rho(\omega_n, \omega_n)\right] \end{array} \right\} \\ +L \min\{\rho(\omega_{n+1}, \omega_{n-1}), \rho(\omega_n, \omega_n)\}) \end{array} \right]^\delta \\
&\leq \left[\theta(\max\{\rho(\omega_{n+1}, \omega_n), \rho(\omega_n, \omega_{n-1})\}\right]^\delta. \qquad (3.4)
\end{aligned}$$

If $\max\{\rho(\omega_{n+1}, \omega_n), \rho(\omega_n, \omega_{n-1})\} = \rho(\omega_{n+1}, \omega_n)$, using (3.4), we get

$$\theta(\rho(\omega_{n+1}, \omega_n)) \leq \left[\theta(\rho(\omega_{n+1}, \omega_n))\right]^\delta < \theta(\rho(\omega_{n+1}, \omega_n)),$$

which is a contradiction. Thus, $\max\{\rho(\omega_{n+1}, \omega_n), \rho(\omega_n, \omega_{n-1})\} = \rho(\omega_n, \omega_{n-1})$, and then we obtain

$$\theta(\rho(\omega_{n+1}, \omega_n)) \leq \left[\theta(\rho(\omega_n, \omega_{n-1}))\right]^\delta, \qquad (3.5)$$

for each n in $\mathbb{N}$. Denote $f_n = \rho(\omega_{n+1}, \omega_n)$ for n in $\mathbb{N}$, then $f_n > 0$ for each n in $\mathbb{N}$, and repeating this process by using (3.5) we have

$$1 < \theta(f_n) \leq \left[\theta(f_0)\right]^{\delta^n} \qquad (3.6)$$

for each n in $\mathbb{N}$. When taking the limit as $n \to +\infty$ in (3.6), we obtain

$$\lim_{n \to +\infty} \theta(f_n) = 1. \qquad (3.7)$$

Using (θ_2), we can deduce that $\lim_{n\to+\infty} f_n = 0^+$; thus, using (θ_3), there exists $p \in (0, 1)$ and $\beta \in (0, +\infty]$ such that

$$\lim_{n\to+\infty} \frac{\theta(f_n) - 1}{(f_n)^p} = \beta.$$

Presume that $\beta < +\infty$. In this case, let $F = \frac{\beta}{2} > 0$. Using the definition of the limit, there exists n_0 in $\mathbb{N}$ such that, for each $n_0 \le n$,

$$\left| \frac{\theta(f_n) - 1}{(f_n)^p} - \beta \right| \le F.$$

This implies that for each $n_0 \le n$,

$$\frac{\theta(f_n) - 1}{(f_n)^p} \ge \beta - F = F,$$

then, for each $n_0 \le n$,

$$n(f_n)^p \le Dn\left[\theta(f_n) - 1\right],$$

where $D = 1/F$.

Presume now that $\beta = +\infty$. Let $F > 0$ be an arbitrary positive number. Using the definition of the limit, there exists n_0 in $\mathbb{N}$ such that for each $n_0 \le n$,

$$\frac{\theta(f_n) - 1}{(f_n)^p} \ge F.$$

This implies that for each $n_0 \le n$,

$$n\left[f_n\right]^p \le Dn\left[\theta(f_n) - 1\right],$$

where $D = 1/F$.

Thus, in all cases, there exists $D > 0$ and n_0 in $\mathbb{N}$ such that

$$n\left[f_n\right]^p \le Dn\left[\theta(f_n) - 1\right],$$

for each $n_0 \le n$. Using (3.6), we obtain

$$n\left[f_n\right]^p \le Dn\left[\left[\theta(f_0)\right]^{\delta^n} - 1\right],$$

for each $n_0 \le n$. Letting $n \to +\infty$ from the last inequality, we have

$$\lim_{n\to+\infty} n\left[f_n\right]^p = 0.$$

Thus, there exists n_1 in $\mathbb{N}$ such that $n\left[f_n\right]^p \le 1$ for each $n \ge n_1$, so we have, for each $n \ge n_1$,

$$f_n \le \frac{1}{n^{1/p}}. \tag{3.8}$$

In order to show that $\{\omega_n\}$ is a right K-Cauchy sequence, consider m, n in $\mathbb{N}$ such that $m > n \ge n_1$. Using the triangular inequality for ρ and using (3.8), we have

$$\begin{aligned} \rho(\omega_m, \omega_n) &\le \rho(\omega_m, \omega_{m-1}) + \rho(\omega_{m-1}, \omega_{m-2}) + \cdots + \rho(\omega_{n+1}, \omega_n) \\ &= f_{m-1} + f_m + \cdots + f_n \end{aligned}$$

$$= \sum_{i=n}^{m-1} f_i \leq \sum_{i=n}^{+\infty} f_i \leq \sum_{i=n}^{+\infty} \frac{1}{i^{1/p}}.$$

By the convergence of the series $\sum_{i=1}^{+\infty} \frac{1}{i^{1/p}}$, we get $\rho(\omega_m, \omega_n) \to 0$ as $n \to +\infty$. This yields that $\{\omega_n\}$ is a right K-Cauchy sequence in the qms (Λ, ρ). Since (Λ, ρ) is a right K-complete, there exists $\eta \in \Lambda$ such that the sequence $\{\omega_n\}$ is ρ-converges to $\eta \in \Lambda$; that is, $\rho(\eta, \omega_n) \to 0$ as $n \to +\infty$. Since Υ is τ_ρ-continuous, then $\rho(\Upsilon\eta, \Upsilon\omega_n) = \rho(\Upsilon\eta, \omega_{n+1}) \to 0$ as $n \to +\infty$. Since Λ is Hausdorff, we get $\Upsilon\eta = \eta$. □

We may use the option to substitute the continuity assumption of Υ in Theorem 1 with the following hypothesis:

(R) If $\{\omega_n\}$ is a sequence in Λ such that $\alpha(\omega_{n+1}, \omega_n) \geq 1$ for all n in $\mathbb{N}$, when the distance $\rho(\omega, \omega_n) \to 0$, then $\alpha(\omega, \omega_n) \geq 1$ for all n in $\mathbb{N}$.

In the theorem below, it is assumed that the space (Λ, ρ) is Hausdorff; that is, τ_ρ is a Hausdorff topology, in which case it is clear that the limit of the convergent sequence is unique.

Theorem 2. *Let (Λ, ρ) be a Hausdorff right K-complete T_1-qms such that (R) holds, and let $\Upsilon : \Lambda \to \Lambda$ be an α-admissible and almost $(\zeta - \theta_\rho)$-contraction. If θ is continuous and there exists $\omega_0 \in \Lambda$ such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$, then Υ has a fixed point in Λ.*

Proof. Similar to the proof of Theorem 1, we obtain $\{\omega_n\}$ as a right K-Cauchy sequence in the qms (Λ, ρ). Since (Λ, ρ) is a right K-complete, there exists $\eta \in \Lambda$ such that the sequence $\{\omega_n\}$ is ρ-convergent to $\eta \in \Lambda$; that is, $\rho(\eta, \omega_n) \to 0$ as $n \to +\infty$. Hence, from (3.3), we have

$$\zeta\left(\alpha\left(\eta, \omega_n\right)\right) \theta(\rho(\Upsilon\eta, \Upsilon\omega_n)), \left[\theta(M(\eta, \omega_n) + LN(\eta, \omega_n))\right]^\delta \geq 0,$$

and so

$$\alpha\left(\eta, \omega_n\right) \theta(\rho(\Upsilon\eta, \Upsilon\omega_n)) \leq \left[\theta(M(\eta, \omega_n) + LN(\eta, \omega_n))\right]^\delta.$$

Hence, we have

$$\theta(\rho(\Upsilon\eta, \omega_{n+1})) \leq \left[\theta(M(\eta, \omega_n) + LN(\eta, \omega_n))\right]^\delta, \tag{3.9}$$

where

$$\begin{aligned} M(\eta, \omega_n) &= \max\left\{\rho(\eta, \omega_n), \rho(\Upsilon\omega_n, \omega_n), \rho(\Upsilon\eta, \eta), \frac{1}{2}\left[\rho(\Upsilon\eta, \omega_n) + \rho(\eta, \Upsilon\omega_n)\right]\right\}, \\ N(\eta, \omega_n) &= \min\left\{\rho(\Upsilon\eta, \omega_n), \rho(\eta, \Upsilon\omega_n,)\right\}. \end{aligned}$$

Letting $n \to +\infty$ from the given inequality, we have

$$\begin{aligned} \lim_{n\to+\infty} M(\eta, \omega_n) &= \rho(\Upsilon\eta, \eta), \\ \lim_{n\to+\infty} N(\eta, \omega_n) &= 0. \end{aligned}$$

Therefore, if $\rho(\Upsilon\eta, \eta) \neq 0$, from (3.9),

$$\theta(\rho(\Upsilon\eta, \eta)) \leq \theta(\rho(\Upsilon\eta, \eta))^\delta,$$

which is a contradiction. Hence $\rho(\Upsilon\eta, \eta) = 0$; that is, $\Upsilon\eta = \eta$. □

In Theorem 1, if we consider the approach of $\tau_{\overline{\rho}}$-continuity, we can derive the following theorem.

Theorem 3. *Let (Λ, ρ) be a right M-complete T_1-qms such that $(\Lambda, \tau_{\overline{\rho}})$ is Hausdorff and $\Upsilon : \Lambda \to \Lambda$ is an α-admissible and almost $(\zeta - \theta_\rho)$-contraction. Presume that Υ is $\tau_{\overline{\rho}}$-continuous. If there exists $\omega_0 \in \Lambda$ such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$, then Υ has a fixed point in Λ.*

Proof. Similar to the proof of Theorem 1, we can take iterative sequence $\{\omega_n\}$ as right K-Cauchy. Since (Λ, ρ) is right M-complete, there exists $\eta \in \Lambda$ such that $\{\omega_n\}$ is $\overline{\rho}$-convergent to η; that is, $\rho(\omega_n, \eta) \to 0$ as $n \to +\infty$. Using $\tau_{\overline{\rho}}$-continuity of Υ, we get $\rho(\Upsilon\omega_n, \Upsilon\eta) = \rho(\omega_{n+1}, \Upsilon\eta) \to 0$ as $n \to +\infty$. Since $(\Lambda, \tau_{\overline{\rho}})$ is Hausdorff, we get $\eta = \Upsilon\eta$. □

Based on the outcomes we have derived, we present diverse fixed point conclusions within the existing literature, and we can derive the following corollaries:

Corollary 1. *Let (Λ, ρ) be a Hausdorff right K-complete T_1-qms and $\Upsilon : \Lambda \to \Lambda$ be given a mapping that satisfies*

$$\alpha(\omega, \gamma)\, \theta(\rho(T\omega, T\gamma)) \leq \left[\theta(M(\omega, \gamma) + LN(\omega, \gamma)\right]^{\delta}, \tag{3.10}$$

for each $\omega, \gamma \in \Lambda$, where $0 < \delta < 1$ and $L \geq 0$. Presume that Υ is α-admissible and τ_ρ-continuous or (R) holds. If θ is continuous and there exists $\omega_0 \in \Lambda$ such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$, then Υ has a fixed point in Λ.

Proof. It suffices to take a simulation function $\zeta(t, s) = ks - t$ for all $s, t \geq 0$ in Theorem 1, (resp. Theorem 2). □

Corollary 2 (see Durmaz and Altun [14])**.** *Let (Λ, ρ) be a Hausdorff right K-complete T_1-qms and $\Upsilon : \Lambda \to \Lambda$ be given a mapping that satisfies*

$$\theta(\rho(T\omega, T\gamma)) \leq \left[\theta(M(\omega, \gamma)\right]^{\delta}, \tag{3.11}$$

for each $\omega, \gamma \in \Lambda$, where $0 < \delta < 1$. Presume that Υ is τ_ρ-continuous or (R) holds with θ as continuous, then Υ has a fixed point in Λ.

Proof. It suffices to choose the mapping $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ such that $\alpha(\omega, \gamma) \geq 1$ for all $\omega, \gamma \in \Lambda$ and $L = 0$ with $\zeta(t, s) = ks - t$ for all $s, t \geq 0$ in Theorem 1. □

Remark 1. *By considering the notion of left completeness in the sense of K, M and Smyth, we can extend similar fixed point results to the setting of qms.*

4. Application

In this part, we propose a novel application in which we demonstrate the existence and uniqueness of the solution to a fractional boundary value problem (FBVP) using Theorem 1: Here, for continuous functions $a : [0, 1] \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, we consider the FBVP given as

$$\begin{cases} D^{\alpha}_{0^+}\xi(t) + a(t)f(\xi(t)) = 0, \ t \in (0, 1), \\ \xi(0) = D^{\beta}_{0^+}\xi(1) = 0, \end{cases} \tag{4.1}$$

where $\alpha \in (1,2]$, $\beta \in [0,1]$ and $D^{\gamma}_{0^+}$ is Riemann-Liouville derivative of order γ. It is well known that the operator $D^{\gamma}_{0^+}$ is defined as, for positive integer n and $\gamma \in (n-1, n]$,

$$D^{\gamma}_{0^+}\xi(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\gamma-1}\xi(s)ds$$

for a function $\xi : [0,1] \to \mathbb{R}$, provided the righthand side exists. It is demonstrated in [16] that (4.1) is equivalent to the following integral equation:

$$\xi(t) = \int_0^1 G(t,s)a(s)f(\xi(s))ds,\ 0 \le t \le 1, \tag{4.2}$$

where $G(t,s)$ is the associated Green's function defined by

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases}$$

Define an operator $\Upsilon : C[0,1] \to C[0,1]$ by

$$\Upsilon\xi(t) = \int_0^1 G(t,s)a(s)f(\xi(s))ds.$$

Hence, η is a solution of (4.1) whenever it is a fixed point of Υ.

Let (Λ, ρ) be the T_1-qms, where $\Lambda = C[0,1]$ and ρ is defined by

$$\rho(\xi,\eta) = \max\left\{\sup_{t\in[0,1]}\{\xi(t)-\eta(t)\}, 2\sup_{t\in[0,1]}\{\eta(t)-\xi(t)\}\right\}.$$

In this case (Λ, ρ) is both Hausdorff and right K-complete.

Now we can state the following theorem:

Theorem 4. *The FBVP (4.1) has a solution under the following assumption: For all $\xi, \eta \in \Lambda$,*

$$\max\left\{\sup_{s\in[0,1]}\{f(\xi(s))-f(\eta(s))\}, 2\sup_{s\in[0,1]}\{f(\eta(s))-f(\xi(s))\}\right\} \le \rho(\xi,\eta)$$

and

$$M(\alpha-1)^{\alpha-1} < \alpha(\alpha-\beta)^{\alpha}\Gamma(\alpha),$$

where $M = \|a\|_\infty$.

Proof. First of all, we know by Lemma 3.1 of [16] that $G(t,s) \ge 0$ for all $t, s \in [0,1]$ and

$$\sup_{t\in[0,1]}\int_0^1 G(t,s)ds = \frac{(\alpha-1)^{\alpha-1}}{\alpha(\alpha-\beta)^{\alpha}\Gamma(\alpha)}.$$

Consider the operator $\Upsilon : C[0,1] \rightarrow C[0,1]$ defined by

$$\Upsilon\xi(t) = \int_0^1 G(t,s)a(s)f(\xi(s))ds,$$

then for any $\xi, \zeta \in C[0,1]$, we have

$$\begin{aligned}
\rho(\Upsilon\xi, \Upsilon\eta) &= \max\left\{ \sup_{t\in[0,1]} \{\Upsilon\xi(t) - \Upsilon\eta(t)\}, 2\sup_{t\in[0,1]} \{\Upsilon\eta(t) - \Upsilon\xi(t)\} \right\} \\
&= \max\left\{ \begin{array}{l} \sup_{t\in[0,1]}\left\{\int_0^1 G(t,s)a(s)f(\xi(s))ds - \int_0^1 G(t,s)a(s)f(\eta(s))ds\right\}, \\ 2\sup_{t\in[0,1]}\left\{\int_0^1 G(t,s)a(s)f(\eta(s))ds - \int_0^1 G(t,s)a(s)f(\xi(s))ds\right\} \end{array} \right\} \\
&= \max\left\{ \begin{array}{l} \sup_{t\in[0,1]}\left\{\int_0^1 G(t,s)a(s)\{f(\xi(s)) - f(\eta(s))\}ds\right\}, \\ 2\sup_{t\in[0,1]}\left\{\int_0^1 G(t,s)a(s)\{f(\eta(s)) - f(\xi(s))\}ds\right\} \end{array} \right\} \\
&\leq M\max\left\{ \begin{array}{l} \sup_{t\in[0,1]}\left\{\int_0^1 G(t,s)\rho(\xi,\eta)ds\right\}, \\ \sup_{t\in[0,1]}\left\{\int_0^1 G(t,s)\rho(\xi,\eta)ds\right\} \end{array} \right\} \\
&= M\rho(\xi,\eta)\sup_{t\in[0,1]}\left\{\int_0^1 G(t,s)ds\right\} \\
&= M\rho(\xi,\eta)\frac{(\alpha-1)^{\alpha-1}}{\alpha(\alpha-\beta)^{\alpha}\Gamma(\alpha)} \\
&= \frac{M(\alpha-1)^{\alpha-1}}{\alpha(\alpha-\beta)^{\alpha}\Gamma(\alpha)}\rho(\xi,\eta).
\end{aligned}$$

Therefore, Υ is an $(\zeta - \theta_\rho)$-contraction with the functions $\alpha(\xi,\eta) = 1$, $\zeta(t,s) = ks - t$ and $\theta(x) = e^{\sqrt{x}}$. The other conditions of Theorem 1 are clearly satisfied. Consequently, there exists $\zeta \in C[0,1]$, which is fixed point of the operator Υ. Hence, the (4.1) has a solution in $C[0,1]$. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

References

1. M. Ali, T. Kamran, N. Shahzad, Best proximity point for α-ψ-proximal contractive multimaps, *Abstr. Appl. Anal.*, **2014** (2014), 181598. http://dx.doi.org/10.1155/2014/181598
2. I. Altun, N. Al-Arifi, M. Jleli, A. Lashin, B. Samet, A new concept of (α, F_d)-contraction on quasi metric space, *J. Nonlinear Sci. Appl.*, **9** (2016), 3354–3361.

3. I. Altun, H. Hançer, G. Mınak, On a general class of weakly Picard operators, *Miskolc Math. Notes*, **16** (2015), 25–32. http://dx.doi.org/10.18514/MMN.2015.1168
4. A. Arutyunov, A. Greshnov, (q_1, q_2)-quasimetric spaces. Covering mappings and coincidence points, *Izv. Math.*, **82** (2018), 245. http://dx.doi.org/10.1070/IM8546
5. A. Arutyunov, A. Greshnov, (q_1, q_2)-quasimetric spaces. Covering mappings and coincidence points. A review of the results, *Fixed Point Theor.*, **23** (2022), 473–486. http://dx.doi.org/10.24193/fpt-ro.2022.2.03
6. H. Aydi, A. Felhi, E. Karapinar, F. Alojail, Fixed points on quasi-metric spaces via simulation functions and consequences, *J. Math. Anal.*, **9** (2018), 10–24.
7. J. Brzdek, E. Karapinar, A. Petruşel, A fixed point theorem and the Ulam stability in generalized *dq*-metric spaces, *J. Math. Anal. Appl.*, **467** (2018), 501–520. http://dx.doi.org/10.1016/j.jmaa.2018.07.022
8. S. Cobzaş, Completeness in quasi-metric spaces and Ekeland variational principle, *Topol. Appl.*, **158** (2011), 1073–1084. http://dx.doi.org/10.1016/j.topol.2011.03.003
9. S. Cobzaş, *Functional analysis in asymmetric normed spaces*, Basel: Springer, 2013. http://dx.doi.org/10.1007/978-3-0348-0478-3
10. G. Durmaz, G. Mınak, I. Altun, Fixed point results for α-ψ-contractive mappings including almost contractions and applications, *Abstr. Appl. Anal.*, **2014** (2014), 869123. http://dx.doi.org/10.1155/2014/869123
11. A. Farajzadeh, M. Delfani, Y. Wang, Existence and uniqueness of fixed points of generalized F-contraction mappings, *J. Math.*, **2021** (2021), 6687238. http://dx.doi.org/10.1155/2021/6687238
12. Y. Gaba, Startpoints and $(\alpha$-$\gamma)$-contractions in quasi-pseudometric spaces, *J. Math.*, **2014** (2014), 709253. http://dx.doi.org/10.1155/2014/709253
13. A. Greshnov, V. Potapov, About coincidence points theorems on 2-step Carnot groups with 1-dimensional centre equipped with Box-quasimetrics, *AIMS Mathematics*, **8** (2023), 6191–6205. http://dx.doi.org/10.3934/math.2023313
14. G. Güngör, I. Altun, Some fixed point results for α-admissible mappings on quasi metric space via θ-contractions, *Mathematical Sciences and Applications E-Notes*, **12** (2024), 12–19. http://dx.doi.org/10.36753/mathenot.1300609
15. T. Hicks, Fixed point theorems for quasi-metric spaces, *Math. Japonica*, **33** (1988), 231–236.
16. C. Hollon, J. Neugebauer, Positive solutions of a fractional boundary value problem with a fractional derivative boundary condition, *Conference Publications*, **2015** (2015), 615–620. http://dx.doi.org/10.3934/proc.2015.0615
17. N. Hussain, E. Karapınar, P. Salimi, F. Akbar, α-admissible mappings and related fixed point theorems, *J. Inequal. Appl.*, **2013** (2013), 114. http://dx.doi.org/10.1186/1029-242X-2013-114
18. N. Hussain, C. Vetro, F. Vetro, Fixed point results for α-implicit contractions with application to integral equations, *Nonlinear Anal.-Model.*, **21** (2016), 362–378. http://dx.doi.org/10.15388/NA.2016.3.5
19. M. Jleli, E. Karapinar, B. Samet, Further generalizations of the Banach contraction principle, *J. Inequal. Appl.*, **2014** (2014), 439. http://dx.doi.org/10.1186/1029-242X-2014-439

20. M. Jleli, B. Samet, A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, **2014** (2014), 38. http://dx.doi.org/10.1186/1029-242X-2014-38

21. E. Karapınar, B. Samet, Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, *Abstr. Appl. Anal.*, **2012** (2012), 793486. http://dx.doi.org/10.1155/2012/793486

22. F. Khojasteh, S. Shukla, S. Radenovic, A new approach to the study of fixed point theory for simulation functions, *Filomat*, **29** (2015), 1189–1194. http://dx.doi.org/10.2298/FIL1506189K

23. P. Kumam, C. Vetro, F. Vetro, Fixed points for weak α-ψ-contractions in partial metric spaces, *Abstr. Appl. Anal.*, **2013** (2013), 986028. http://dx.doi.org/10.1155/2013/986028

24. A. Latif, S. Al-Mezel, Fixed point results in quasimetric spaces, *Fixed Point Theor. Appl.*, **2011** (2011), 178306. http://dx.doi.org/10.1155/2011/178306

25. M. Olgun, T. Alyildiz, Ö. Biçer, A new aspect to Picard operators with simulation functions, *Turk. J. Math.*, **40** (2016), 832–837. http://dx.doi.org/10.3906/mat-1505-26

26. I. Reilly, P. Subrahmanyam, M. Vamanamurthy, Cauchy sequences in quasi-pseudo-metric spaces, *Monatsh. Math.*, **93** (1982), 127–140. http://dx.doi.org/10.1007/BF01301400

27. B. Rhoades, A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.*, **226** (1977), 257–290. http://dx.doi.org/10.2307/1997954

28. S. Romaguera, Left K-completeness in quasi-metric spaces, *Math. Nachr.*, **157** (1992), 15–23. http://dx.doi.org/10.1002/mana.19921570103

29. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal.-Theor.*, **75** (2012), 2154–2165. http://dx.doi.org/10.1016/j.na.2011.10.014

30. H. Şimsek, M. Yalçın, Generalized Z-contraction on quasi metric spaces and a fixed point result, *J. Nonlinear Sci. Appl.*, **10** (2017), 3397–3403. http://dx.doi.org/10.22436/jnsa.010.07.03

31. W. Wilson, On quasi-metric spaces, *Am. J. Math.*, **53** (1931), 675–684. http://dx.doi.org/10.2307/2371174