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Research article

Asymptotic behavior for a class of logarithmic wave equations with Balakrishnan-Taylor damping, nonlinear weak damping and strong linear damping

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Abstract: In spaces with any dimension, a class of logarithmic wave equations with Balakrishnan-Taylor damping, nonlinear weak damping and strong linear damping was considered

$$u_{tt} - M(t)\Delta u + (g * \Delta u)(t) + h(u_t)u_t - \Delta u_t + f(u) + u = u \ln |u|^k$$

with Dirichlet boundary condition. Under a set of specified assumptions, we established the existence of global weak solutions and elucidated the decay rate of the energy function for particular initial data. This contribution extended and surpassed prior investigations, as documented by A. Peyravi (2020), which omitted the consideration of Balakrishnan-Taylor damping and strong linear damping while being confined to three spatial dimensions. Our findings underscored the pivotal role of these overlooked damping factors. Furthermore, our demonstration underscored that Balakrishnan-Taylor damping, weak damping and strong damping collectively induced an exponential decay, although the precise nature of this decay was contingent upon the differentiable function associated with the memory damping term $\zeta(t)$. Consequently, the absence of the damping term in reference by A. Peyravi (2020) was unequivocally shown not to augment the decay rate. This insight enhanced our understanding of the nuanced dynamics involved and contributed to the refinement of existing models.

Keywords: global weak solutions; energy decay; logarithmic wave equation; Balakrishnan-Taylor damping
Mathematics Subject Classification: 35B40, 35Q35, 60H15

1. Introduction

In this paper, we focus on estimating the energy decay estimate of solutions for a logarithmic wave equation with Balakrishnan-Taylor damping, nonlinear damping and strong linear damping:

$$\begin{aligned}&u_{tt} - M(t)\Delta u + (g * \Delta u)(t) + h(u_t)u_t - \Delta u_t + f(u) + u = u \ln |u|^k, x \in \Omega, t > 0,\\&u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega,\\&u(x,t) = 0, x \in \partial\Omega, t \geq 0,\end{aligned} \tag{1.1}$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $M(t) = \xi_0 + \xi_1\|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)$, $h(s) = k_0 + k_1|s|^{m-1}$ where $k, k_0, k_1, \xi_0, \xi_1, \sigma$ and m are positive constants and $g : R^+ \to R^+, f : R \to R$ are functions that will be specified in (A1)–(A3). g represents the kernel of the memory term and

$$(g * v)(t) = \int_0^t g(t-s)v(s)ds.$$

In the absence of the terms $u, f(u), h(u_t)u_t, -\Delta u_t$ and $u \ln |u|^k$, problem (1.1) has been extensively studied in [1], with several results regarding the decay of energy in its solutions. When $f(u) = -|u|^p u$, it has been proven that the energy of the system is global in time and decays polynomially in [2]. Tatar proved an exponential decay result of the energy when there is $-\Delta u_t$, provided that the kernel g decays exponentially in [3]. Since then, various authors have explored results related to the existence and asymptotic behavior of a class of equations with Balakrishnan-Taylor damping (see [4–12] and their respective references). Many studies have also explored systems that include logarithmic terms. In [13], Han studied the global existence of weak solutions for the initial boundary value problem:

$$\begin{aligned}&u_{tt} - \Delta u + u - u \ln |u|^2 + u_t + u|u|^2 = 0, x \in \Omega, t \in (0,T),\\&u(x,t) = 0, x \in \partial\Omega,\\&u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega,\end{aligned} \tag{1.2}$$

where Ω is a smooth and bounded domain in R^3. If the nonlinear logarithmic term is $u \ln |u|^k (0 < k < 1)$, Hu et al. [14] investigated the exceptional growing-up solutions in H_0^1-norm. By constructing a Lyapunov function and implementing a suitable modification of the energy, they also estimated the exponential decay of solutions. In fact, model (1.2) is related to the following equation with logarithmic nonlinearity:

$$u_{tt} - \Delta u + u - \varepsilon u \ln |u|^2 = 0, (x,t) \in [a,b] \times (0,T). \tag{1.3}$$

Here, the parameter ε measures the force of the nonlinear interaction. It has been experimentally demonstrated in quantum mechanics that the nonlinear effects are very small (see [15]). Bouhali Keltoum and Ellaggoune Fateh used weighted space to establish a general decay rate of solutions for the viscoelastic wave equation with logarithmic nonlinearities (see [16]):

$$u_{tt} - \phi(x)\left(\Delta_x u - \int_0^t g(t-s)\Delta_x u(s)ds\right) = u \ln |u|^k, \tag{1.4}$$

under convenient hypotheses on g and the initial data, the existence of a weak solution is associated to the equation. A recent work by Amir Peyravi [17] extended these results to:

$$u_{tt} - \Delta u + (g * \Delta u)(t) + h(u_t)u_t + u|u|^2 + u = u \ln |u|^k, x \in \Omega, t > 0, \tag{1.5}$$

where he proves a general stability of solutions. In the case where $k_1 = 0$, he also proves that the solutions will grow exponentially. Many literature references can be found on the internet. The remaining sections of this paper are organized as follows. In section two, we present some important Lemmas and assumptions. In sections three and four, we prove the existence of global weak solutions for the system and establish an exponential decay result for the energy.

2. Assumptions, notation, and preliminaries

In this section, we shall present some assumptions and lemmas that will be used throughout this work. We will use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $E_0^1(\Omega)$ with their usual products and norms. Additionally, we will use $(\cdot,\cdot)$ to denote the $L^2(\Omega)$-inner product.

Next, we will provide the assumptions for problem (1.1):

(A1) $g : R^+ \to R^+$ is a bounded C^1-function satisfying

$$\xi_0 - \int_t^{+\infty} g(s)ds = l_0 > 0,$$

.

(A2) Assume that there exists a positive nonincreasing differentiable function ζ for any $t \geq 0$ such that

$$g'(t) \leq -\zeta(t)g(t), \quad g(0) > 0 \text{ and } \int_t^{+\infty} \zeta(s)ds = +\infty.$$

(A3) $f : R \to R$ is continuously differentiable for any $s \in R$ such that

$$f(s) \geq 0, \quad \int_\Omega f(u)udx - \alpha \int_\Omega \int_0^u f(s)dsdx > 0,$$

where $0 < \alpha < 2$.

(A4) m satisfies $0 \leq m < +\infty$ for $n = 1, 2$, $1 \leq m < \frac{n+2}{n-2}$ for $n \geq 3$.

(A5) The constant k in (1.1) satisfies $0 < k < k_0$, such that

$$\frac{4\pi l_0}{k_0} = e^{-\frac{4}{k_0 n} - 2}.$$

Lemma 2.1. *(Logarithmic Sobolev inequality) [18]* Let u be any function in $H_0^1(\Omega)$ where $\Omega \subseteq R^n$ is a bounded smooth domain and let $a > 0$ be any number, then

$$2\int_\Omega |u|^2 \ln \frac{|u|}{\|u\|}dx + n(1 + \ln a)\|u\|^2 \leq \frac{a^2}{\pi} \int_\Omega |\nabla u|^2 dx.$$

Lemma 2.2. *(Sobolev-Poincaré inequality) [19]* Let $1 \leq p \leq \frac{2n}{n-2}$, if $n \geq 3$ or $1 \leq p < +\infty$, if $n = 1, 2$, the inequality

$$\|u\|_p \leq c_* \|\nabla u\| \quad for\ u \in H_0^1(\Omega),$$

holds with some positive constant c_*.

Definition 2.3. A function u is said to be a weak solution of problem (1.1) on $[0, T)$ if

$$u \in C\left([0,T), H_0^1(\Omega)\right), u_t \in C\left([0,T), L^2(\Omega)\right) \cap L^{m+1}\left([0,T) \times \Omega\right),$$

$$(u(x,0), u_t(x,0)) = (u_0(x), u_1(x)) \in H_0^1(\Omega) \times H_0^1(\Omega)$$

and u satisfies

$$(u_{tt}, \phi) + M(t)\left(\nabla u, \nabla\phi\right) - \int_\Omega \nabla\phi(x)\cdot \int_0^t g(t-s)\nabla u(s)dsdx + (h(u_t)u_t, \phi) + (\nabla u_t, \nabla\phi)$$
$$+ (f(u), \phi) + (u, \phi) = \left(u \ln|u|^k, \phi\right),$$

for all $\phi \in H_0^1(\Omega)$ and for almost all $t \in [0, T)$.

2.1. Global weak solutions

According to the logarithmic Sobolev inequality and by using the Galerkin method, similar to the proof in [17], we can establish the existence of global weak solutions for (1.1).

First, we define the following functionals to support our main results.

$$I(t) = I(u(t)) = \left(\xi_0 - \int_0^t g(s)ds\right)\|\nabla u\|^2 + \frac{\xi_1}{2}\|\nabla u\|^4 + \left(g \circ \nabla u\right)(t) + \|u\|^2 - \int_\Omega u^2 \ln|u|^k dx, \tag{2.1}$$

$$J(t) = J(u(t)) = \frac{k}{4}\|u\|^2 + \int_\Omega \int_0^u f(s)dsdx + \frac{1}{2}I(t), \tag{2.2}$$

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(t), \tag{2.3}$$

for $u \in H_0^1(\Omega), t \geq 0$, where

$$\left(g \circ \nabla u\right)(t) = \int_0^t g(t-s) \int_\Omega |\nabla u(s) - \nabla u(t)|^2 dxds.$$

Lemma 3.1. Suppose that assumption (A2) holds, then the energy functional $E(t)$ is a nonincreasing function and

$$\begin{aligned} \frac{d}{dt}E(t) =& \frac{1}{2}\left(g' \circ \nabla u\right)(t) - \frac{1}{2}g(t)\|u\|^2 - \alpha(\nabla u, \nabla u_t)^2 \\ &- k_0\|u_t\|^2 - k_1\|u_t\|_{m+1}^{m+1} - \|\nabla u_t\|^2 < 0. \end{aligned} \tag{2.4}$$

Proof. By multiplying Eq (1.1) by u_t and integrating by parts over Ω, we obtain (3.4), where we utilize assumption (A2).

Theorem 3.2. Let $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in H_0^1(\Omega)$ be given. Assume that (A1)–(A5) hold, then there exists a weak solution u for (1.1) such that

$$u \in L^\infty\left(0, +\infty; H_0^1(\Omega)\right), u_t \in L^\infty\left(0, +\infty; L^2(\Omega)\right).$$

Proof. From the definition of $E(t)$, Lemmas 2.1 and 3.1, we have

$$\begin{aligned}E(0) \geq E(t) \geq & \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(\xi_0 - \int_0^t g(s)ds\right)\|\nabla u\|^2 + \frac{\xi_1}{4}\|\nabla u\|^4 \\ & + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega u^2 \ln|u|^k dx \\ \geq & \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(l_0 - \frac{ka^2}{4\pi}\right)\|\nabla u\|^2 + \frac{1}{2}\left(1 + \frac{kn + kn\ln a}{2} - k\ln\|u\|\right)\|u\|^2,\end{aligned} \tag{2.5}$$

where $t \in [0, T)$ and for some $T > 0$. Choosing $\sqrt{\frac{4\pi l_0}{k}} > a > e^{-\frac{2}{kn}-1}$, will make

$$l_0 - \frac{ka^2}{4\pi} > 0,$$

and

$$1 + \frac{kn + kn\ln a}{2} > 0.$$

This selection is possible thanks to (A5), so we get

$$\|u_t\|^2 + \|\nabla u\|^2 + \|u\|^2 \leq C(1 + \|u\|^2 \ln\|u\|^2). \tag{2.6}$$

Using the Cauchy-Schwarz inequality, we get

$$\|u(t)\|^2 \leq 2\|u(0)\|^2 + 2\left\|\int_0^t \frac{\partial u}{\partial s}(s)\mathrm{d}s\right\|^2 \leq 2\|u(0)\|^2 + 2T\int_0^t \|u_t(s)\|^2 ds,$$

hence, inequality (3.6) gives

$$\|u(t)\|^2 \leq 2C_T\left(1 + \int_0^t \|u\|^2 \ln\|u\|^2 \mathrm{d}s\right).$$

Applying the Logarithmic Gronwall inequality to the last inequality, we obtain the following estimate

$$\|u(t)\|^2 \leq 2Ce^{2CT}.$$

Hence, from inequality (3.6) it follows that:

$$\|u_t\|^2 + \|\nabla u\|^2 + \|u\|^2 \leq C_T. \tag{2.7}$$

Finally, by (3.7), the theorem can be proven using the standard Galerkin method, as in [13].

3. Energy decay

In this section, we will prove the exponential decay of solutions for problem (1.1).

Lemma 4.1. *[17]* Let $E : R^+ \to R^+$ be a nonincreasing function and $\psi : R^+ \to R^+$ be a C^2-increasing function such that $\psi(0) = 0$ and $\lim_{t\to+\infty} \psi(t) = +\infty$. Assume that there exist $c > 0$ for which

$$\int_t^{+\infty} \psi'(s)E(s)ds \leq cE(t), \quad \forall t \geq 0,$$

then

$$E(t) \leq \lambda E(0)e^{-\omega\psi(t)},$$

for some positive constants ω and λ.

Lemma 4.2. Suppose that (A1)–(A3) hold, and let $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in H_0^1(\Omega)$ be given and satisfy

$$\|u_0(x)\| < e^{\frac{n\ln a+n-1}{2}+\frac{1}{k}},$$

$$0 < E(0) < G\left(e^{\frac{n\ln a+n-1}{2}+\frac{1}{k}}\right).$$

Then, $\|u(x,t)\| < e^{\frac{n\ln a+n-1}{2}+\frac{1}{k}}$ for all $t \in [0,T)$, where $G(\|u\|) := \left(\frac{1}{2} + \frac{kn+kn\ln a}{4}\right)\|u\|^2 - \frac{k}{2}\|u\|^2 \ln \|u\|$ and $0 < a < \frac{2\sqrt{kl_0\pi}}{k}$.

Proof. By (3.3), (A1), (A2) and Lemma 2.1, we have

$$\begin{aligned} E(t) &\geq J(t) \geq \frac{1}{2}I(t) \\ &\geq \frac{1}{2}\left(\xi_0 - \int_0^t g(s)ds\right)\|\nabla u\|^2 + \frac{\xi_1}{4}\|\nabla u\|^4 + \frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega u^2 \ln|u|^k dx \\ &\geq \frac{1}{2}\left(l_0 - \frac{ka^2}{4\pi}\right)\|\nabla u\|^2 + \frac{1}{2}\left(1 + \frac{kn+kn\ln a}{2} - k\ln\|u\|\right)\|u\|^2. \end{aligned} \tag{3.1}$$

Let $0 < a < \frac{2\sqrt{kl_0\pi}}{k}$, and by (3.4) we get

$$E(t) \geq \left(\frac{1}{2} + \frac{kn+kn\ln a}{4}\right)\|u\|^2 - \frac{k}{2}\|u\|^2 \ln\|u\|. \tag{3.2}$$

By definition of $G(\|u\|)$, we can show that $G(\|u\|)$ is increasing on $(0, e^{\frac{n\ln a+n-1}{2}+\frac{1}{k}})$, decreasing on $(e^{\frac{n\ln a+n-1}{2}+\frac{1}{k}}, +\infty)$ and $G(\|u\|) \to -\infty$ as $\|u\| \to +\infty$. Moreover,

$$\max_{0<\|u\|<+\infty} G(\|u\|) = G\left(e^{\frac{n\ln a+n-1}{2}+\frac{1}{k}}\right).$$

By the the continuity u and using reduction to absurdity (see Lemma 3.3 in [17]), we obtain $\|u(x,t)\| < e^{\frac{n\ln a+n-1}{2}+\frac{1}{k}}$ for all $t \in [0,T)$ under the assumptions.

Lemma 4.3. Under the assumptions of Lemma 4.2, for $t \in [0,T)$ we have

$$\begin{aligned} &\|u_t\|^2 \leq 2E(t) \leq 2E(0), \\ &\|u\|^2 \leq \frac{4}{k}E(t) \leq \frac{4}{k}E(0), \\ &\int_\Omega \int_0^u f(s)dsdx \leq E(t) \leq E(0), \\ &\|\nabla u\|^4 \leq \frac{4}{\xi_1}E(t) \leq \frac{4}{\xi_1}E(0), \\ &\|\nabla u\|^2 \leq \frac{8\pi}{4\pi l_0 - ka^2}E(t) \leq \frac{8\pi}{4\pi l_0 - ka^2}E(0). \end{aligned} \tag{3.3}$$

Proof. Under the assumptions of Lemma 4.2, we know $I(t) \geq 0$ for all $t \in [0, T)$. By (4.1) for all $t \in [0, T)$, we have

$$I(t) \geq \left(l_0 - \frac{ka^2}{4\pi}\right)\|\nabla u\|^2 + \left(1 + \frac{kn + kn\ln a}{2} - k\ln\|u\|\right)\|u\|^2 \geq \left(l_0 - \frac{ka^2}{4\pi}\right)\|\nabla u\|^2 + \frac{k}{2}\|u\|^2 \geq 0. \quad (3.4)$$

Therefore, we have $E(t) \geq 0$ and $J(t) \geq 0$. Similar to Remark 3.5 in [17], by utilizing (3.4), (4.1) and (4.4), we can obtain (4.3) for $t \in [0, T)$.

Lemma 4.4. Under the assumptions of Lemma 4.2 and assuming (A4) holds, for any $\varepsilon > 0$ and $\delta > 0$ small enough, as well as any $0 < \alpha < \frac{2}{2+k}$, we have the following

$$\begin{aligned}&\left[\alpha - \varepsilon\left(\frac{2\sigma}{\xi_1} + \frac{8\pi}{4\pi l_0 - ka^2} + \frac{2k_0}{k}\right) - \frac{k_1\left(8\pi\delta^2 c_*{}^2\right)^{\frac{m+1}{2}}(E(0))^{\frac{m-1}{2}}}{(m+1)(4\pi l_0 - ka^2)^{\frac{m+1}{2}}}\right]E(t)\\ &\leq -\int_\Omega uu_{tt}dx - \left(\frac{m\delta^{-\frac{m+1}{m}}}{m+1} + \frac{1+2\sigma}{2\varepsilon\sigma} + \frac{\alpha}{2k_0}\right)E'(t) + \left(\frac{\xi_0 - l_0}{2\varepsilon} + \frac{\alpha}{2}\right)(g \circ \nabla u)(t).\end{aligned} \quad (3.5)$$

Proof. Multiplying the equation in (1.1) by u and integrating over Ω, we get

$$\begin{aligned}&\int_\Omega uu_{tt}dx + \left(\xi_0 - \int_0^t g(s)ds\right)\|\nabla u\|^2 + \xi_1\|\nabla u\|^4 + \left(\sigma\|\nabla u\|^2 + 1\right)(\nabla u, \nabla u_t)\\ &+ \int_\Omega \nabla u(t)\int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right)dsdx + \int_\Omega uu_t h(u_t)dx\\ &+ \int_\Omega f(u)udx + \|u\|^2 - \int_\Omega u^2\ln|u|^k dx = 0.\end{aligned} \quad (3.6)$$

For any $0 < \alpha < \frac{2}{2+k}$, from (3.3) and (4.6) we have

$$\begin{aligned}\alpha E(t) =& -\int_\Omega uu_{tt}dx + \left(\frac{\alpha}{2} - 1\right)\left(\xi_0 - \int_0^t g(s)ds\right)\|\nabla u\|^2 + \xi_1\left(\frac{\alpha}{4} - 1\right)\|\nabla u\|^4\\ &- \left(\sigma\|\nabla u\|^2 + 1\right)(\nabla u, \nabla u_t) - \int_\Omega \nabla u(t)\int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right)dsdx\\ &+ \frac{\alpha}{2}(g \circ \nabla u)(t) - \int_\Omega uu_t h(u_t)dx + \frac{\alpha}{2}\|u_t\|^2 + \alpha\int_\Omega\int_0^u f(s)dsdx - \int_\Omega f(u)udx\\ &+ \left[\frac{\alpha}{2}\left(1 + \frac{k}{2}\right) - 1\right]\|u\|^2 + k\left(1 - \frac{\alpha}{2}\right)\int_\Omega u^2\ln|u|dx.\end{aligned} \quad (3.7)$$

Next, we will estimate some terms on the righthand side of (4.7). For the fourth term on the righthand side of (4.7), we can utilize the ε-Young inequality to obtain

$$\begin{aligned}&-\left(\sigma\|\nabla u\|^2 + 1\right)(\nabla u, \nabla u_t)\\ \leq&\frac{\sigma\varepsilon}{2}\|\nabla u\|^4 + \frac{1}{2\varepsilon}(\nabla u, \nabla u_t)^2 + \frac{\varepsilon}{2}\|\nabla u\|^2 + \frac{1}{2\varepsilon}\|\nabla u_t\|^2\\ \leq&\frac{\sigma\varepsilon}{2}\|\nabla u\|^4 + \frac{4\pi\varepsilon}{4\pi l_0 - ka^2}E(t) - \frac{1+\sigma}{2\varepsilon\sigma}E'(t)\\ \leq&\left(\frac{2\sigma\varepsilon}{\xi_1} + \frac{4\pi\varepsilon}{4\pi l_0 - ka^2}\right)E(t) - \frac{1+\sigma}{2\varepsilon\sigma}E'(t).\end{aligned} \quad (3.8)$$

For the fifth term in the righthand side of (4.7), we use the Cauchy-Schwarz inequality and the ε-Young inequality and we obtain

$$\begin{aligned}
&\left|-\int_\Omega \nabla u(t)\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)dsdx\right|\\
&\leq \|\nabla u(t)\|\left[\int_\Omega\left(\int_0^t g(t-s)\left|\nabla u(t)-\nabla u(s)\right|ds\right)^2 dx\right]^{\frac{1}{2}}\\
&\leq \frac{\varepsilon}{2}\|\nabla u(t)\|^2+\frac{1}{2\varepsilon}\left(\int_0^t g(s)ds\right)\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2 dsdx\\
&\leq \frac{\varepsilon}{2}\|\nabla u(t)\|^2+\frac{\xi_0-l_0}{2\varepsilon}\left(g\circ\nabla u\right)(t)\\
&\leq \frac{4\pi\varepsilon}{4\pi l_0-ka^2}E(t)+\frac{\xi_0-l_0}{2\varepsilon}\left(g\circ\nabla u\right)(t).
\end{aligned}\tag{3.9}$$

For the seventh term, similar to (3.14) in [17], by the ε-Young inequality, Lemma 2.2 and (A4), we have

$$\begin{aligned}
&\left|-\int_\Omega u u_t h(u_t)dx\right|\\
&\leq \frac{k_0}{2}\left(\varepsilon\|u\|^2+\frac{1}{\varepsilon}\|u_t\|^2\right)+\frac{k_1}{m+1}\left(\delta^{m+1}\|u\|_{m+1}^{m+1}+m\delta^{-\frac{m+1}{m}}\|u_t\|_{m+1}^{m+1}\right)\\
&\leq \frac{k_0}{2}\left(\varepsilon\|u\|^2+\frac{1}{\varepsilon}\|u_t\|^2\right)+\frac{k_1\delta^{m+1}{c_*}^{m+1}(8\pi E(0))^{\frac{m-1}{2}}}{(m+1)(4\pi l_0-ka^2)^{\frac{m-1}{2}}}\|\nabla u\|^2+\frac{k_1 m\delta^{-\frac{m+1}{m}}}{m+1}\|u_t\|_{m+1}^{m+1}\\
&\leq \left[\frac{2k_0\varepsilon}{k}+\frac{k_1\left(8\pi\delta^2{c_*}^2\right)^{\frac{m+1}{2}}(E(0))^{\frac{m-1}{2}}}{(m+1)(4\pi l_0-ka^2)^{\frac{m+1}{2}}}\right]E(t)-\left(\frac{1}{2\varepsilon}+\frac{m\delta^{-\frac{m+1}{m}}}{m+1}\right)E'(t).
\end{aligned}\tag{3.10}$$

By using the assumption of (A3), we know

$$\alpha\int_\Omega\int_0^u f(s)dsdx-\int_\Omega f(u)udx-\alpha\int_\Omega\int_0^u f(s)dsdx<0.\tag{3.11}$$

Therefore, (4.11) can add to the right of (4.7) and be discarded.

For the last term in equality (4.7), by using the Logarithmic Sobolev inequality, we find

$$\begin{aligned}
k\left(1-\frac{\alpha}{2}\right)\int_\Omega u^2\ln|u|dx &= k\left(1-\frac{\alpha}{2}\right)\left(\int_\Omega u^2\ln\frac{|u|}{\|u\|}dx+\|u\|^2\ln\|u\|\right)\\
&\leq k\left(1-\frac{\alpha}{2}\right)\left(\|u\|^2\ln\|u\|+\frac{a^2}{4\pi}\|\nabla u\|^2-\frac{n+n\ln a}{2}\|u\|^2\right).
\end{aligned}\tag{3.12}$$

Hence, by (4.8)–(4.12), $0 < \alpha < \frac{2}{2+k}$ and $0 < a < \frac{2\sqrt{kl_0\pi}}{k}$, equality (4.7) converts to

$$\begin{aligned}
&\left[\alpha - \varepsilon\left(\frac{2\sigma}{\xi_1} + \frac{8\pi}{4\pi l_0 - ka^2} + \frac{2k_0}{k}\right) - \frac{k_1\left(8\pi\delta^2 {c_*}^2\right)^{\frac{m+1}{2}} (E(0))^{\frac{m-1}{2}}}{(m+1)(4\pi l_0 - ka^2)^{\frac{m+1}{2}}}\right] E(t) \\
&\le -\int_\Omega u u_{tt} dx - \left(\frac{m\delta^{-\frac{m+1}{m}}}{m+1} + \frac{1+2\sigma}{2\varepsilon\sigma} + \frac{\alpha}{2k_0}\right) E'(t) + \left(\frac{\xi_0 - l_0}{2\varepsilon} + \frac{\alpha}{2}\right) (g \circ \nabla u)(t) \\
&\quad + \left[k\left(1 - \frac{\alpha}{2}\right)\left(\ln \|u\| - \frac{n(1+\ln a)}{2}\right) - \frac{1}{2}\right] \|u\|^2.
\end{aligned} \tag{3.13}$$

Inequality (4.4) implies that

$$k\left(1 - \frac{\alpha}{2}\right)\left(\ln \|u\| - \frac{n(1+\ln a)}{2}\right) - \frac{1}{2} < \left(1 - \frac{\alpha}{2}\right)\left(1 - \frac{k}{2}\right) - \frac{1}{2} \le 0. \tag{3.14}$$

Therefore, inequality (4.5) holds.

Theorem 4.5. Suppose that the assumptions (A1)–(A4) hold. Let $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in H_0^1(\Omega)$ be given and satisfy

$$\|u_0(x)\| < e^{\frac{n \ln a + n - 1}{2} + \frac{1}{k}},$$
$$0 < E(0) < G\left(e^{\frac{n \ln a + n - 1}{2} + \frac{1}{k}}\right).$$

Then, there exists two positive constants C_1 and C_2 such that

$$E(t) \le C_1 E(0) e^{-C_2 \int_0^t \zeta(s) ds} \tag{3.15}$$

holds for all $t > 0$, where functions $G(\cdot)$ and a are given in *Lemma 4.2*.

Proof. Under the assumptions, we shall have (4.5) hold, so by multiplying both sides of inequality (4.5) by $\zeta(t)$ and integrating on $[S, T]$, $0 \le S < T < +\infty$, we get

$$\begin{aligned}
&\left[\alpha - \varepsilon\left(\frac{2\sigma}{\xi_1} + \frac{8\pi}{4\pi l_0 - ka^2} + \frac{2k_0}{k}\right) - \frac{k_1\left(8\pi\delta^2 {c_*}^2\right)^{\frac{m+1}{2}} (E(0))^{\frac{m-1}{2}}}{(m+1)(4\pi l_0 - ka^2)^{\frac{m+1}{2}}}\right] \int_S^T \zeta(t) E(t) dt \\
&\le -\int_S^T \zeta(t) \int_\Omega u u_{tt} dx dt - \left(\frac{m\delta^{-\frac{m+1}{m}}}{m+1} + \frac{1+2\sigma}{2\varepsilon\sigma} + \frac{\alpha}{2k_0}\right) \int_S^T \zeta(t) E'(t) dt \\
&\quad + \left(\frac{\xi_0 - l_0}{2\varepsilon} + \frac{\alpha}{2}\right) \int_S^T \zeta(t) (g \circ \nabla u)(t) dt.
\end{aligned} \tag{3.16}$$

Next, we will estimate the righthand side of (4.16). By applying the same approach as in [1], utilizing Lemma 4.2 and assuming (A4), we can estimate the first and second terms on the righthand side of (4.16) as follows:

$$\begin{aligned}
-\int_S^T \zeta(t) \int_\Omega u u_{tt} dx dt &\le \left(\frac{6}{k} + 3\right) \zeta(S) E(S) - \frac{1}{k_0} \int_S^T \zeta(t) E'(t) dt \\
&\le \left(\frac{6}{k} + \frac{1}{k_0} + 3\right) \zeta(0) E(S).
\end{aligned} \tag{3.17}$$

$$
\begin{aligned}
&-\left(\frac{m\delta^{-\frac{m+1}{m}}}{m+1}+\frac{1+2\sigma}{2\varepsilon\sigma}+\frac{\alpha}{2k_0}\right)\int_S^T \zeta(t)E'(t)dt \\
&\leq -\left(\frac{m\delta^{-\frac{m+1}{m}}}{m+1}+\frac{1+2\sigma}{2\varepsilon\sigma}+\frac{\alpha}{2k_0}\right)\zeta(t)E(t)\Big|_S^T+\left(\frac{m\delta^{-\frac{m+1}{m}}}{m+1}+\frac{1+2\sigma}{2\varepsilon\sigma}+\frac{\alpha}{2k_0}\right)\int_S^T \zeta'(t)E(t)dt \qquad (3.18)\\
&\leq \left(\frac{m\delta^{-\frac{m+1}{m}}}{m+1}+\frac{1+2\sigma}{2\varepsilon\sigma}+\frac{\alpha}{2k_0}\right)\zeta(0)E(S).
\end{aligned}
$$

Exploiting (A2) to estimate the third term, we obtain

$$
\begin{aligned}
\left(\frac{\xi_0-l_0}{2\varepsilon}+\frac{\alpha}{2}\right)\int_S^T \zeta(t)\left(g\circ\nabla u\right)(t)dt &\leq -\left(\frac{\xi_0-l_0}{2\varepsilon}+\frac{\alpha}{2}\right)\int_S^T \left(g'\circ\nabla u\right)(t)dt \\
&\leq -\left(\frac{\xi_0-l_0}{\varepsilon}+\alpha\right)\int_S^T E'(t)dt \qquad (3.19)\\
&\leq \left(\frac{\xi_0-l_0}{\varepsilon}+\alpha\right)E(S).
\end{aligned}
$$

Hence, we conclude from (4.17)–(4.19) that

$$
\begin{aligned}
&\left[\alpha-\varepsilon\left(\frac{2\sigma}{\xi_1}+\frac{8\pi}{4\pi l_0-ka^2}+\frac{2k_0}{k}\right)-\frac{k_1\left(8\pi\delta^2 c_*{}^2\right)^{\frac{m+1}{2}}(E(0))^{\frac{m-1}{2}}}{(m+1)(4\pi l_0-ka^2)^{\frac{m+1}{2}}}\right]\int_S^T \zeta(t)E(t)dt \qquad (3.20)\\
&\leq\left[\left(\frac{6}{k}+3+\frac{m\delta^{-\frac{m+1}{m}}}{m+1}+\frac{1+2\sigma}{2\varepsilon\sigma}+\frac{1}{k_0}\left(1+\frac{\alpha}{2}\right)\right)\zeta(0)+\frac{\xi_0-l_0}{\varepsilon}+\alpha\right]E(S).
\end{aligned}
$$

Therefore, according to Lemma 4.1 as $T\to+\infty$, there exists two constants C_1 and C_2 such that (4.15) holds.

4. Conclusions

In the realm of n-dimensional space, our exploration delved into the logarithmic Wave equation featuring Balakrishnan-Taylor damping, both nonlinear weak damping, and potent linear damping. By leveraging the logarithmic inequality and the energy integral inequality as elucidated in [17], we not only established the existence of a global solution for the system but also illuminated its dynamics. From a physics standpoint, the Balakrishnan-Taylor damping becomes a pivotal tool for scrutinizing oscillations and waves in fluid mediums and unraveling the intricacies of their decay processes within the fluid. The physical essence of this damping mechanism lies in its adept depiction of energy dissipation and the decay of oscillations resulting from viscosity in fluid motion. Conversely, the nonlinear weak damping and strong linear damping mechanisms find their utility in portraying energy dissipation within systems subjected to frictional forces. Additionally, the introduction of memory damping introduces another layer of complexity, influencing the stability of the system. The inclusion of memory damping allows for an in-depth analysis of the system's response to external excitation and how stability evolves in the presence of internal energy dissipation. Mathematically, Balakrishnan-Taylor damping, weak damping, and strong damping can all lead to exponential decay, but the specific form of decay is intricately tied to the differentiable function associated with the memory damping term.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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