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Research article

An algorithm for calculating spectral radius of s-index weakly positive tensors

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Abstract: In this paper, we introduced s-index weakly positive tensors and discussed the calculation of the spectral radius of this kind of nonnegative tensors. Using the diagonal similarity transformation of tensor and Perron-Frobenius theory of nonnegative tensor, the calculation method of the maximum H-eigenvalue of s-index weakly positive tensors was given. A variable parameter was introduced in each iteration of the algorithm, which is equivalent to a translation transformation of the tensor in each iteration to improve the calculation speed. At the same time, it was proved that the algorithm is linearly convergent for the calculation of the spectral radius of s-index weakly positive tensors. The final numerical example shows the effectiveness of the algorithm.

Keywords: irreducible nonnegative tensor; s-index weakly positive tensor; H-eigenvalue; numerical algorithm
Mathematics Subject Classification: 15A18, 15A69

1. Introduction

Consider an m-order n-dimensional square tensor $\mathcal{A}$ consisting of n^m entries in the real field $\mathbb{R}$:

$$\mathcal{A} = (a_{i_1 i_2 \cdots i_m}), a_{i_1 i_2 \cdots i_m} \in \mathbb{R}, 1 \le i_1, i_2, \cdots, i_m \le n.$$

If $a_{i_1 i_2 \cdots i_m} \ge 0, 1 \le i_1, i_2, \cdots, i_m \le n$, then $\mathcal{A}$ is called an m-order n-dimensional nonnegative tensor. Denote the set of all m-order n-dimensional nonnegative tensors as $\mathbb{R}_+^{[m,n]}$. $\mathbb{R}^n$, $\mathbb{R}_+^n$, $\mathbb{R}_{++}^n$ represents the set of all n-dimensional vectors, the set of all n-dimensional nonnegative vectors and the set of all n-dimensional positive vectors, respectively. Tensors play an important role in physics, engineering and mathematics. There are many application domains of tensors such as data analysis and mining, information science, image processing and computational biology [1–4].

In 2005, Qi [2] introduced the notion of eigenvalues of higher-order tensors and studied the existence of both complex and real eigenvalues and eigenvectors. Independently, in the same year, Lim [3] also defined eigenvalues and eigenvectors but restricted them to be real.

Qi [2] proposed the definition of H-eigenvalue. If a real number λ and a nonzero real vector $\mathbf{x} \in \mathbb{R}^n$ satisfy the following homogeneous polynomial equation:

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

where

$$\mathcal{A}\mathbf{x}^{m-1} = \left(\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \le i \le n}$$

is an n-dimensional vector and

$$(\mathbf{x}^{[m-1]})_i = x_i^{m-1},$$

then λ is an H-eigenalue of $\mathcal{A}$ and $\mathbf{x}$ is an H-eigenector of $\mathcal{A}$ associated with λ. Define the set of H-eigenvalues of $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ as $\sigma_H(\mathcal{A})$ and $\rho(\mathcal{A}) = \max_{\lambda \in \sigma_H(\mathcal{A})} |\lambda|$ as the H-spectral radius of tensor $\mathcal{A}$. Let $\sigma(\mathcal{A})$ be the set of eigenvalues of a tensor $\mathcal{A}$.

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$. Ng et al. [5] proposed the NQZ algorithm for the largest H-eigenalue of a nonnegative irreducible tensor.

Algorithm 1 ([5]) NQZ algorithm

Step 0. Choose $\mathbf{x}^{(0)} > 0, \mathbf{x}^{(0)} \in \mathbb{R}^n$. Let $\mathbf{y}^{(0)} = \mathcal{A}(\mathbf{x}^{(0)})^{m-1}$ and set $k := 0$.

Step 1. Compute

$$\mathbf{x}^{(k+1)} = \frac{(\mathbf{y}^{(k)})^{[\frac{1}{m-1}]}}{\|(\mathbf{y}^{(k)})^{[\frac{1}{m-1}]}\|},$$

$$\mathbf{y}^{(k+1)} = \mathcal{A}(\mathbf{x}^{(k+1)})^{m-1},$$

$$\underline{\lambda}_{k+1} = \min_{(\mathbf{x}^{(k+1)})_i > 0} \frac{(\mathbf{y}^{(k+1)})_i}{(\mathbf{x}^{(k+1)})_i^{m-1}},$$

$$\overline{\lambda}_{k+1} = \max_{(\mathbf{x}^{(k+1)})_i > 0} \frac{(\mathbf{y}^{(k+1)})_i}{(\mathbf{x}^{(k+1)})_i^{m-1}}.$$

Step 2. If $\overline{\lambda}_{k+1} = \underline{\lambda}_{k+1}$, stop. Otherwise, replace k by $k+1$ and go to Step 1.

Subsequently, the NQZ algorithm was proved to be convergent for primitive tensors in [6] and for weakly primitive tensors in [4], and the NQZ algorithm was shown to have an explicit linear convergence rate for essentially positive tensors in [7]. However, some examples [5] showed that it did not converge for some irreducible nonnegative tensors.

In 2010, Liu et al. [8] modified the NQZ algorithm and proposed the LZI algorithm. Let $\mathcal{E} = (\delta_{i_1 i_2 \cdots i_m})$ be the m-order n-dimensional unit tensor whose entries are

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & if \;\; i_1 = i_2 = \cdots = i_m, \\ 0, & otherwise. \end{cases}$$

Algorithm 2 ([8]) LZI algorithm

Step 0. Choose $\mathbf{x}^{(0)} > 0, \mathbf{x}^{(0)} \in \mathbb{R}^n$. Let $\mathcal{B} = \mathcal{A} + \rho\mathcal{E}$, where $\rho > 0$, and set $k := 0$.

Step 1. Compute

$$\mathbf{y}^{(k)} = \mathcal{B}(\mathbf{x}^{(k)})^{m-1},$$

$$\underline{\lambda}_k = \min_{(\mathbf{x}^{(k)})_i > 0} \frac{(\mathbf{y}^{(k)})_i}{(\mathbf{x}^{(k)})_i^{m-1}},$$

$$\overline{\lambda}_k = \max_{(\mathbf{x}^{(k)})_i > 0} \frac{(\mathbf{y}^{(k)})_i}{(\mathbf{x}^{(k)})_i^{m-1}}.$$

Step 2. If $\overline{\lambda}_k = \underline{\lambda}_k$, then let $\lambda = \overline{\lambda}_k$ and stop. Otherwise, compute

$$\mathbf{x}^{(k+1)} = \frac{(\mathbf{y}^{(k)})^{[\frac{1}{m-1}]}}{\|(\mathbf{y}^{(k)})^{[\frac{1}{m-1}]}\|},$$

replace k by $k + 1$ and go to Step 1.

Liu et al. [8] proved that the LZI algorithm is convergent for irreducible nonnegative tensors. In 2012, Zhang et al. [9] proved the linear convergence of the LZI algorithm for weakly positive tensors. Since then, there have been many studies on the calculation of the maximum eigenvalue of nonnegative tensors. For example, Yang and Ni [10] gave a nonlinear algorithm for calculating the maximum eigenvalue of symmetric tensors; another example, Zhang and Bu [11] gave a diagonal similar iterative algorithm for calculating the maximum H-eigenvalue of a class of generalized weakly positive tensors.

2. Preliminaries

In this section, we mainly introduce some related concepts and important properties of tensors and matrices. For a positive integer n, let $\langle n \rangle = \{1, 2, \ldots, n\}$.

Definition 2.1. *[12] An m-order n-dimensional tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $I \subset \langle n \rangle$ such that*

$$a_{i_1 i_2 \cdots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then $\mathcal{A}$ is irreducible.

Definition 2.2. *[13] A nonnegative matrix $\mathring{\mathcal{A}}$ is called the majorization associated to nonnegative tensor $\mathcal{A}$ if the (i, j)-th element of $\mathring{\mathcal{A}}$ is defined to be $a_{ij\cdots j}$ for any $i, j = 1, \cdots, n$.*

Definition 2.3. *[13] A nonnegative m-order n-dimensional tensor $\mathcal{A}$ is essentially positive if $\mathcal{A}\mathbf{x}^{m-1} \in \mathbb{R}^n_{++}$ for any nonzero $\mathbf{x} \in \mathbb{R}^n_+$.*

Definition 2.4. *[9] Let $\mathcal{A}$ be a nonnegative tensor of order m and dimension n. $\mathcal{A}$ is weakly positive if*

$$a_{ij\cdots j} > 0 \ for \ i \neq j \ and \ i, j \in \{1, 2, \cdots, n\}.$$

Definition 2.5. *[11] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$, then $\mathcal{A}$ is generalized weakly positive if there exists $i_0 \in \langle n \rangle$, such that $a_{i_0 j \cdots j} > 0, a_{j i_0 \cdots i_0} > 0$ for all $j \in \langle n \rangle \backslash \{i_0\}$.*

Definition 2.6. *[14] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$.*
(1) We call a nonnegative matrix $G(\mathcal{A})$ the representation associated to nonnegative tensor $\mathcal{A}$ if the (i, j)-th element of $G(\mathcal{A})$ is defined to be the summation of $a_{\{i i_2 \cdots i_m\}}$ with indices $\{i_2 \cdots i_m\} \ni j$.
(2) We call $\mathcal{A}$ weakly reducible if its representation $G(\mathcal{A})$ is a reducible matrix and weakly primitive if $G(\mathcal{A})$ is a primitive matrix. If $\mathcal{A}$ is not weakly reducible, then it is called weakly irreducible.

Definition 2.7. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$ and $\pi_{s-1}(i, j)$ be an arrangement of $s-1$ letters i and $m-s$ letters j. If there exists $s \in \langle m-1 \rangle$ and $i_0 \in \langle n \rangle$ for any $j \in \langle n \rangle$, $j \neq i_0$, such that $a_{i_0 \pi_{s-1}(i_0, j)} \neq 0$ and $a_{j \pi_{s-1}(j, i_0)} \neq 0$ hold, then $\mathcal{A}$ is called an s-index weakly positive tensor.*

For example, $\mathcal{A} = (a_{ijk}) \in \mathbb{R}_+^{[3,3]}$, where $a_{113} = 1, a_{223} = 4, a_{331} = 2, a_{332} = 5$ and $a_{i_1 i_2 i_3} \geq 0 (1 \leq i_1, i_2, i_3 \leq 3)$ elsewhere, then $\mathcal{A}$ is a two-index weakly positive tensor.

Remark 2.1. *The essentially positive tensors, the weakly positive tensors and the generalized weakly positive tensors are all one-index weakly positive tensors, which are special tensor classes of s-index weakly positive tensors..*

Theorem 2.1. *[15] For any nonnegative tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$, $\rho(\mathcal{A})$ is an eigenvalue with a nonnegative eigenvector $\mathbf{x} \in \mathbb{R}_+^n$ corresponding to it.*

Theorem 2.2. *[16] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$. $\rho(\mathcal{A})$ is the spectral radius of $\mathcal{A}$, then*

$$\min_{i \in \langle n \rangle} \sum_{i_2, \cdots, i_m = 1}^{n} a_{i i_2 \cdots i_m} \leq \rho(\mathcal{A}) \leq \max_{i \in \langle n \rangle} \sum_{i_2, \cdots, i_m = 1}^{n} a_{i i_2 \cdots i_m}.$$

Theorem 2.3. *[12] If $\mathcal{A}$ is an irreducible nonnegative tensor of order m and dimension n, then there exists $\lambda_0 > 0$ and $\mathbf{x}_0 > 0, \mathbf{x}_0 \in \mathbb{R}^n$ such that $\mathcal{A}\mathbf{x}_0^{m-1} = \lambda_0 \mathbf{x}_0^{[m-1]}$. Moreover, if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \lambda_0$. If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \leq \lambda_0$.*

Definition 2.8. *[17] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$, $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$. The tensors $\mathcal{A}$ and $\mathcal{B}$ are said to be diagonal similar if there exists some invertible diagonal matrix $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ of order n such that $\mathcal{B} = \mathcal{A} \times_1 D^{-(m-1)} \times_2 D \times_3 \cdots \times_m D$, where $b_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m} d_{i_1}^{-(m-1)} d_{i_2} \cdots d_{i_m}$.*

Theorem 2.4. *[17] If the two m-order n-dimensional tensors $\mathcal{A}$ and $\mathcal{B}$ are diagonal similar, then $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$.*

3. An algorithm and its convergence analysis

In 1981, Bunse [18] gave a diagonal similar iterative algorithm for calculating the maximum eigenvalue of irreducible nonnegative matrices. In 2008, Lv [19] further studied the diagonal similarity iterative algorithm for calculating the maximum eigenvalue of irreducible nonnegative matrices. In 2021, Zhang and Bu [11] gave a diagonal similar iterative algorithm for calculating the maximum H-eigenvalue of nonnegative tensors. In this paper, according to the construction idea of the algorithm in [19], a numerical algorithm for calculating the maximum H-eigenvalue and corresponding eigenvector of s-index weakly positive tensors is given.

Let $\mathcal{A} = \mathcal{A}^{(0)} = (a^{(0)}_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}_+$, $r_i(\mathcal{A}^{(0)}) = \sum\limits_{i_2,\cdots,i_m=1}^{n} a^{(0)}_{i i_2 \cdots i_m} (i \in \langle n \rangle)$ and $\overline{r}(\mathcal{A}^{(0)}) = \max\limits_{i\in\langle n\rangle} r_i(\mathcal{A}^{(0)})$. ε is a sufficiently small positive number. $\alpha_k \in \mathbb{R} (k = 0, 1, 2, \cdots)$ satisfies $\varepsilon - \min\limits_{i\in\langle n\rangle} a_{ii\cdots i} < \alpha_k \le \overline{r}(\mathcal{A}^{(0)})$.

Algorithm 3

Step 0. Given $\mathcal{A}^{(0)} = \mathcal{A} = (a_{i_1 i_2 \cdots i_m})$, $\varepsilon - \min\limits_{i\in\langle n\rangle} a_{ii\cdots i} < \alpha_k \le \overline{r}(\mathcal{A}^{(0)})$, $\varepsilon > 0$. Set $k := 0$.

Step 1. Compute

$$r_i(\mathcal{A}^{(k)}) = \sum_{i_2,\cdots,i_m=1}^{n} a^{(k)}_{i i_2 \cdots i_m}, \ i \in \langle n \rangle,$$

$$\overline{r}(\mathcal{A}^{(k)}) = \max_{i\in\langle n\rangle} r_i(\mathcal{A}^{(k)}), \ \underline{r}(\mathcal{A}^{(k)}) = \min_{i\in\langle n\rangle} r_i(\mathcal{A}^{(k)}).$$

Step 2. If $\overline{r}(\mathcal{A}^{(k)}) - \underline{r}(\mathcal{A}^{(k)}) < \varepsilon$, then $\rho(\mathcal{A}) = \frac{1}{2}(\overline{r}(\mathcal{A}^{(k)}) + \underline{r}(\mathcal{A}^{(k)}))$ and stop.

Step 3. Set

$$D^{(k)} = \left(\frac{\mathrm{diag}(r_1(\mathcal{A}^{(k)}), r_2(\mathcal{A}^{(k)}), \cdots, r_n(\mathcal{A}^{(k)})) + \alpha_k I}{\overline{r}(\mathcal{A}^{(k)}) + \alpha_k} \right)^{\frac{1}{m-1}},$$

$$\mathcal{A}^{(k+1)} = \mathcal{A}^{(k)} \times_1 (D^{(k)})^{-(m-1)} \times_2 D^{(k)} \times_3 \cdots \times_m D^{(k)},$$

and replace k by $k+1$, go to Step 1.

In the following, we will give the convergence condition of Algorithm 3.

Define $\mathcal{A}^{(k)} + \alpha_k \mathcal{E} = \mathcal{A}^{(k)}(\alpha_k) =: (a_{i_1 i_2 \cdots i_m}(\alpha_k))^n_{i_1, i_2, \cdots, i_m = 1}$.

Lemma 3.1. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}_+$. For tensor sequence $\mathcal{A}^{(l)} (l = 0, 1, 2, \cdots)$, we have $\overline{r}(\mathcal{A}^{(l)}) (l = 0, 1, 2, \cdots)$ as monotonically decreasing with lower bound, and $\underline{r}(\mathcal{A}^{(l)}) (l = 0, 1, 2, \cdots)$ as monotonically increasing with upper bound.*

Proof. Notice that

$$\begin{aligned} r_i(\mathcal{A}^{(l+1)}) &= \sum_{i_2,\cdots,i_m=1}^{n} a^{(l+1)}_{i i_2 \cdots i_m} = \sum_{i_2,\cdots,i_m=1}^{n} a^{(l)}_{i i_2 \cdots i_m} \frac{\prod_{j=2}^{m} \left(r_{i_j}(\mathcal{A}^{(l)}) + \alpha_l \right)^{\frac{1}{m-1}}}{r_i(\mathcal{A}^{(l)}) + \alpha_l} \\ &= \sum_{i_2,\cdots,i_m=1}^{n} a_{i i_2 \cdots i_m}(\alpha_l) \frac{\prod_{j=2}^{m} \left(r_{i_j}(\mathcal{A}^{(l)}) + \alpha_l \right)^{\frac{1}{m-1}}}{r_i(\mathcal{A}^{(l)}) + \alpha_l} - \alpha_l \\ &\le \sum_{i_2,\cdots,i_m=1}^{n} a_{i i_2 \cdots i_m}(\alpha_l) \frac{\overline{r}(\mathcal{A}^{(l)}) + \alpha_l}{r_i(\mathcal{A}^{(l)}) + \alpha_l} - \alpha_l \\ &= \overline{r}(\mathcal{A}^{(l)}), \end{aligned}$$

then $\overline{r}(\mathcal{A}^{(l+1)}) \le \overline{r}(\mathcal{A}^{(l)}), l = 0, 1, 2, \cdots$. Therefore, $\overline{r}(\mathcal{A}^{(l)})$ is monotonically decreasing. Similarly, it can be proved that $\underline{r}(\mathcal{A}^{(l+1)}) \ge \underline{r}(\mathcal{A}^{(l)}), l = 0, 1, 2, \cdots$, so $\underline{r}(\mathcal{A}^{(l)})$ is monotonically increasing. By $\underline{r}(\mathcal{A}^{(l)}) \le \overline{r}(\mathcal{A}^{(l)})$ we can obtain that $\overline{r}(\mathcal{A}^{(l)}) (l = 0, 1, 2, \cdots)$ is monotonically decreasing with lower bound $\underline{r}(\mathcal{A}^{(0)})$, and $\underline{r}(\mathcal{A}^{(l)}) (l = 0, 1, 2, \cdots)$ is monotonically increasing with upper bound $\overline{r}(\mathcal{A}^{(0)})$.

Lemma 3.2. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$ be an s-index weakly positive tensor, then it is an irreducible tensor.*

Proof. Because $\mathcal{A}$ is an s-index weakly positive tensor, by Definition 2.7, there are $s \in \langle m-1 \rangle$ and $i_0 \in \langle n \rangle$ such that $a_{i_0 \pi_{s-1}(i_0, j)} \neq 0$, $a_{j\pi_{s-1}(j,i_0)} \neq 0$, $j \in \langle n \rangle, j \neq i_0$. That is, for any $j \in I \subset \langle n \rangle$, there is $i_0 \in \langle n \rangle \backslash I$ such that $a_{j\pi_{s-1}(j,i_0)} \neq 0$. By Definition 2.1, $\mathcal{A}$ is an irreducible tensor.

Lemma 3.3. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$ be an s-index weakly positive tensor. That is, there are $s \in \langle m-1 \rangle$ and $i_0 \in \langle n \rangle$ for any $j \in \langle n \rangle$, $j \neq i_0$, such that $a_{i_0\pi_{s-1}(i_0,j)} \neq 0$ and $a_{j\pi_{s-1}(j,i_0)} \neq 0$ hold, then for any $k \in \mathbb{N}$,*

$$a^{(k)}_{j\pi_{s-1}(j,i_0)}(\alpha_k) \geq \min\{\frac{\tilde{a}^2}{\overline{r}(\mathcal{A}^{(0)})}, \varepsilon\}, \text{ where } \tilde{a} = \min_{j \in \langle n \rangle \backslash \{i_0\}} \{a_{i_0\pi_{s-1}(i_0,j)}, a_{j\pi_{s-1}(j,i_0)}\}.$$

Proof. If $a_{j\pi_{s-1}(j,i_0)} \neq 0$, then $a^{(k)}_{j\pi_{s-1}(j,i_0)} \neq 0, a^{(k)}_{i_0 i_0 \cdots i_0} = a_{i_0 i_0 \cdots i_0}, k = 0, 1, 2, \cdots$. In the case of $j \neq i_0$, it can be obtained from Lemma 3.1 that

$$\begin{aligned} \overline{r}(\mathcal{A}^{(0)}) &\geq \overline{r}(\mathcal{A}^{(k)}) \geq a^{(k)}_{j\pi_{s-1}(j,i_0)} = a^{(k-1)}_{j\pi_{s-1}(j,i_0)} \cdot \frac{(r_{i_0}(\mathcal{A}^{(k-1)})+\alpha_{k-1})^{m-s}}{(r_j(\mathcal{A}^{(k-1)})+\alpha_{k-1})^{m-s}} = \cdots \\ &= a^{(0)}_{j\pi_{s-1}(j,i_0)} \cdot \frac{\prod_{t=0}^{k-1}(r_{i_0}(\mathcal{A}^{(t)})+\alpha_t)^{m-s}}{\prod_{t=0}^{k-1}(r_j(\mathcal{A}^{(t)})+\alpha_t)^{m-s}} \geq \tilde{a} \cdot \frac{\prod_{t=0}^{k-1}(r_{i_0}(\mathcal{A}^{(t)})+\alpha_t)^{m-s}}{\prod_{t=0}^{k-1}(r_j(\mathcal{A}^{(t)})+\alpha_t)^{m-s}}, \end{aligned} \tag{3.1}$$

where $\tilde{a} = \min_{j \in \langle n \rangle \backslash \{i_0\}} \{a_{i_0\pi_{s-1}(i_0,j)}, a_{j\pi_{s-1}(j,i_0)}\}$, then

$$\frac{\prod_{t=0}^{k-1}(r_{i_0}(\mathcal{A}^{(t)}) + \alpha_t)^{m-s}}{\prod_{t=0}^{k-1}(r_j(\mathcal{A}^{(t)}) + \alpha_t)^{m-s}} \leq \frac{\overline{r}(\mathcal{A}^{(0)})}{\tilde{a}}.$$

Then, by

$$\frac{\prod_{t=0}^{k-1}(r_{i_0}(\mathcal{A}^{(t)}) + \alpha_t)^{m-s}}{\prod_{t=0}^{k-1}(r_j(\mathcal{A}^{(t)}) + \alpha_t)^{m-s}} \cdot \frac{\prod_{t=0}^{k-1}(r_j(\mathcal{A}^{(t)}) + \alpha_t)^{m-s}}{\prod_{t=0}^{k-1}(r_{i_0}(\mathcal{A}^{(t)}) + \alpha_t)^{m-s}} = 1,$$

we have

$$\frac{\prod_{t=0}^{k-1}(r_{i_0}(\mathcal{A}^{(t)}) + \alpha_t)^{m-s}}{\prod_{t=0}^{k-1}(r_j(\mathcal{A}^{(t)}) + \alpha_t)^{m-s}} \geq \frac{\tilde{a}}{\overline{r}(\mathcal{A}^{(0)})},$$

then $a^{(k)}_{i_0\pi_{s-1}(i_0,j)} \geq \frac{\tilde{a}^2}{\overline{r}(\mathcal{A}^{(0)})} (j \neq i_0)$ can be obtained by (3.1).

In the case of $j = i_0$, it holds that $a^{(k)}_{i_0 i_0 \cdots i_0}(\alpha_k) = a^{(k)}_{i_0 i_0 \cdots i_0} + \alpha_k \geq \min_{i \in \langle n \rangle} a_{ii\cdots i} - \min_{i \in \langle n \rangle} a_{ii\cdots i} + \varepsilon = \varepsilon$, then $a^{(k)}_{j\pi_{s-1}(j,i_0)}(\alpha_k) \geq \min\{\frac{\tilde{a}^2}{\overline{r}(\mathcal{A}^{(0)})}, \varepsilon\}$.

Theorem 3.1. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$ be an s-index weakly positive tensor. That is, there are $s \in \langle m-1 \rangle$ and $i_0 \in \langle n \rangle$ for any $j \in \langle n \rangle$, $j \neq i_0$, such that $a_{i_0 \pi_{s-1}(i_0,j)} \neq 0$ and $a_{j\pi_{s-1}(j,i_0)} \neq 0$ hold, then for Algorithm 3,*

$$\overline{r}(\mathcal{A}^{(k+1)}) - \underline{r}(\mathcal{A}^{(k+1)}) \leq \alpha(\overline{r}(\mathcal{A}^{(k)}) - \underline{r}(\mathcal{A}^{(k)})),$$

where $\alpha = 1 - \frac{\hat{a}^2}{2\overline{r}(\mathcal{A}^{(0)})}$, $\overline{r}(\mathcal{A}^{(0)}) = \max\limits_{i \in \langle n \rangle} r_i(A)$, $\hat{a} = \min\limits_{j \in \langle n \rangle \setminus \{i_0\}} \{\frac{\tilde{a}^2}{\overline{r}(\mathcal{A}^{(0)})}, \varepsilon\}$, then we can get

$$\lim_{k\to\infty} \overline{r}(\mathcal{A}^{(k)}) = \lim_{k\to\infty} \underline{r}(\mathcal{A}^{(k)}) = \rho(\mathcal{A}).$$

Proof. Let $A^{[0]} = (a_{ij}^{[0]})_{n\times n}$, $a_{i_0 j} = a_{i_0\pi_{s-1}(i_0,j)}$, $a_{ji_0} = a_{j\pi_{s-1}(j,i_0)}$, $j \in \langle n \rangle$ and zero elsewhere, then $A^{[0]}$ is an irreducible matrix. Let $\overline{r}(\mathcal{A}^{(k+1)}) = r_p^{(k+1)}$, $\underline{r}(\mathcal{A}^{(k+1)}) = r_q^{(k+1)}$, then

$$\begin{aligned}\overline{r}(\mathcal{A}^{(k+1)}) - \underline{r}(\mathcal{A}^{(k+1)}) &= r_p^{(k+1)} - r_q^{(k+1)} \\ &= \sum_{i_2,\cdots,i_m=1}^{n} \left(\frac{a_{pi_2\cdots i_m}^{(k)}}{r_p^{(k)} + \alpha_k} - \frac{a_{qi_2\cdots i_m}^{(k)}}{r_q^{(k)} + \alpha_k} \right) \prod_{j=2}^{m} (r_{i_j}(\mathcal{A}^{(k)}) + \alpha_k)^{\frac{1}{m-1}}.\end{aligned}$$

Denote $I = \{i_2 \cdots i_m | i_2, \cdots, i_m \in \langle n \rangle\}$ and

$$I(k) = \left\{ i_2 \cdots i_m | \frac{a_{pi_2\cdots i_m}^{(k)}}{r_p^{(k)} + \alpha_k} \geq \frac{a_{qi_2\cdots i_m}^{(k)}}{r_q^{(k)} + \alpha_k} \right\},$$

then

$$\begin{aligned}\overline{r}(\mathcal{A}^{(k+1)}) - \underline{r}(\mathcal{A}^{(k+1)}) &= r_p^{(k+1)} - r_q^{(k+1)} \\ &= \sum_{i_2,\cdots,i_m \in I(k)} \left(\frac{a_{pi_2\cdots i_m}^{(k)}}{r_p^{(k)} + \alpha_k} - \frac{a_{qi_2\cdots i_m}^{(k)}}{r_q^{(k)} + \alpha_k} \right) \prod_{j=2}^{m} (r_{i_j}(\mathcal{A}^{(k)}) + \alpha_k)^{\frac{1}{m-1}} \\ &\quad + \sum_{i_2,\cdots,i_m \in I\setminus I(k)} \left(\frac{a_{pi_2\cdots i_m}^{(k)}}{r_p^{(k)} + \alpha_k} - \frac{a_{qi_2\cdots i_m}^{(k)}}{r_q^{(k)} + \alpha_k} \right) \prod_{j=2}^{m} (r_{i_j}(\mathcal{A}^{(k)}) + \alpha_k)^{\frac{1}{m-1}} \\ &\leq (\overline{r}(\mathcal{A}^{(k)}) + \alpha_k) \sum_{i_2,\cdots,i_m \in I(k)} \left(\frac{a_{pi_2\cdots i_m}^{(k)}}{r_p^{(k)} + \alpha_k} - \frac{a_{qi_2\cdots i_m}^{(k)}}{r_q^{(k)} + \alpha_k} \right) \\ &\quad + (\underline{r}(\mathcal{A}^{(k)}) + \alpha_k) \sum_{i_2,\cdots,i_m \in I\setminus I(k)} \left(\frac{a_{pi_2\cdots i_m}^{(k)}}{r_p^{(k)} + \alpha_k} - \frac{a_{qi_2\cdots i_m}^{(k)}}{r_q^{(k)} + \alpha_k} \right)\end{aligned}$$

$$
\begin{aligned}
&= (\overline{r}(\mathcal{A}^{(k)}) + \alpha_k) \sum_{i_2,\cdots,i_m \in I(k)} \left(\frac{a^{(k)}_{pi_2\cdots i_m}}{r^{(k)}_p + \alpha_k} - \frac{a^{(k)}_{qi_2\cdots i_m}}{r^{(k)}_q + \alpha_k} \right) \\
&\quad - (\underline{r}(\mathcal{A}^{(k)}) + \alpha_k) \sum_{i_2,\cdots,i_m \in I(k)} \left(\frac{a^{(k)}_{pi_2\cdots i_m}}{r^{(k)}_p + \alpha_k} - \frac{a^{(k)}_{qi_2\cdots i_m}}{r^{(k)}_q + \alpha_k} \right) \\
&= (\overline{r}(\mathcal{A}^{(k)}) - \underline{r}(\mathcal{A}^{(k)})) \\
&\quad \cdot \left(1 - \left(\sum_{i_2,\cdots,i_m \in I\setminus I(k)} \frac{a^{(k)}_{pi_2\cdots i_m}}{r^{(k)}_p + \alpha_k} + \sum_{i_2,\cdots,i_m \in I(k)} \frac{a^{(k)}_{qi_2\cdots i_m}}{r^{(k)}_q + \alpha_k} \right) \right) \\
&\le (\overline{r}(\mathcal{A}^{(k)}) - \underline{r}(\mathcal{A}^{(k)})) \left(1 - \frac{\sum\limits_{i_2,\cdots,i_m \in I\setminus I(k)} a^{(k)}_{pi_2\cdots i_m} + \sum\limits_{i_2,\cdots,i_m \in I(k)} a^{(k)}_{qi_2\cdots i_m}}{2\overline{r}(\mathcal{A}^{(0)})} \right).
\end{aligned}
$$

Therefore, either $\sum\limits_{i_2,\cdots,i_m \in I\setminus I(k)} a^{(k)}_{pi_2\cdots i_m}$ includes $a^{(k)}_{p\pi_{s-1}(p,i_0)}$ or $\sum\limits_{i_2,\cdots,i_m \in I(k)} a^{(k)}_{qi_2\cdots i_m}$ includes $a^{(k)}_{q\pi_{s-1}(q,i_0)}$, then

$$
\begin{aligned}
\sum_{i_2,\cdots,i_m \in I\setminus I(k)} a^{(k)}_{pi_2\cdots i_m} + \sum_{i_2,\cdots,i_m \in I(k)} a^{(k)}_{qi_2\cdots i_m} &\ge \min\{a^{(k)}_{i_0\cdots i_0}(\alpha_k), a^{(k)}_{p\pi_{s-1}(p,i_0)}(\alpha_k), a^{(k)}_{q\pi_{s-1}(q,i_0)}(\alpha_k)\} \\
&\ge \min\{\frac{\tilde{a}^2}{\overline{r}(\mathcal{A}^{(0)})}, \varepsilon\} =: \hat{a}.
\end{aligned}
$$

Thus,

$$
\overline{r}(\mathcal{A}^{(k+1)}) - \underline{r}(\mathcal{A}^{(k+1)}) \le \alpha(\overline{r}(\mathcal{A}^{(k)}) - \underline{r}(\mathcal{A}^{(k)})),
$$

where $\alpha = 1 - \frac{\hat{a}}{2\overline{r}(\mathcal{A}^{(0)})}$. Therefore,

$$
\overline{r}(\mathcal{A}^{(k)}) - \underline{r}(\mathcal{A}^{(k)}) \le \alpha(\overline{r}(\mathcal{A}^{(k-1)}) - \underline{r}(\mathcal{A}^{(k-1)})) \le \cdots \le \alpha^k(\overline{r}(\mathcal{A}^{(0)}) - \underline{r}(\mathcal{A}^{(0)})).
$$

Note that $0 < \alpha = 1 - \frac{\hat{a}}{2\overline{r}(\mathcal{A}^{(0)})} < 1$, and we can obtain $\lim\limits_{k\to\infty}(\overline{r}(\mathcal{A}^{(k)}) - \underline{r}(\mathcal{A}^{(k)})) = 0$. From Lemma 3.1,

$$
\lim_{k\to\infty} \overline{r}(\mathcal{A}^{(k)}) = \lim_{k\to\infty} \underline{r}(\mathcal{A}^{(k)}) = \rho(\mathcal{A}).
$$

Corollary 3.1. *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}_+, \varepsilon > 0$. If $\mathcal{A}$ is an s-index weakly positive tensor, then by Algorithm 3 there must be*

$$
K = \left[\frac{\log(\frac{\varepsilon}{\overline{r}(\mathcal{A}^{(0)}) - \underline{r}(\mathcal{A}^{(0)})})}{\log(\alpha)} \right] + 1
$$

that satisfies $\overline{r}(\mathcal{A}^{(K)}) - \underline{r}(\mathcal{A}^{(K)}) < \varepsilon$, where α is defined in Theorem 3.1.

From Definitions 2.3–2.5, we can see that the essentially positive tensors, the weakly positive tensors and the generalized weakly positive tensors are all s-index weakly positive tensors, so we have the following corollary.

Corollary 3.2. *If* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$ *is an essentially positive tensor or a weakly positive tensor, then*

$$\overline{r}(\mathcal{A}^{(k+1)}) - \underline{r}(\mathcal{A}^{(k+1)}) \le \alpha(\overline{r}(\mathcal{A}^{(k)}) - \underline{r}(\mathcal{A}^{(k)})),$$

where $\alpha = 1 - \frac{\hat{a}}{2\overline{r}(\mathcal{A}^{(0)})}$, $\overline{r}(\mathcal{A}^{(0)}) = \max_{i\in\langle n\rangle} r_i(\mathcal{A}), \hat{a} = \min\{\frac{\tilde{a}^2}{\overline{r}(\mathcal{A}^{(0)})}, \varepsilon\}$, $\tilde{a} = \min_{j\neq i}\{a_{ij\cdots j}\}$ *and, thus,*

$$\lim_{k\to\infty} \overline{r}(\mathcal{A}^{(k)}) = \lim_{k\to\infty} \underline{r}(\mathcal{A}^{(k)}) = \rho(\mathcal{A}).$$

Corollary 3.2 further confirms the linear convergence of the LZI algorithm for weakly essentially positive tensors in Theorem 4.1 in [9].

From above, the inclusion relationship is shown among irreducible tensor, primitive tensor, s-index weakly positive tensor, generalized weakly positive tensor, weakly positive tensor and essentially positive tensor sets in Figure 1.

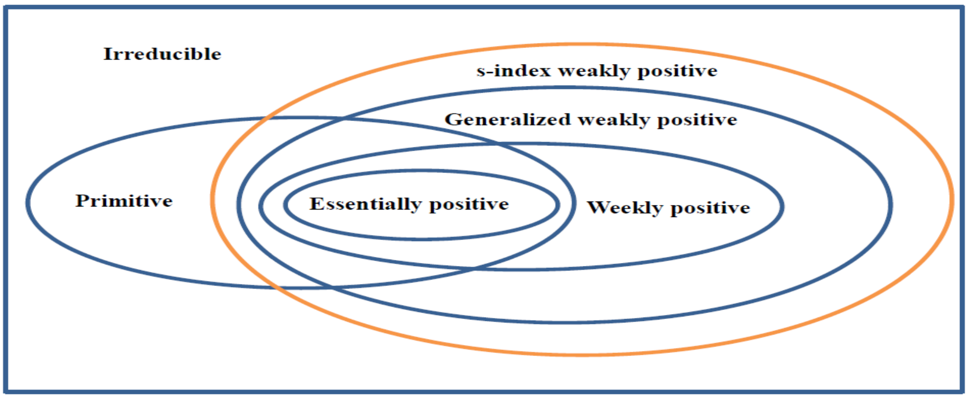

Figure 1. Relations among six classes of nonnegative tensors.

Theorem 3.2. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$ *be an s-index weakly positive tensor. For the positive diagonal matrix* $D_i (i = 0, 1, 2, \cdots)$ *in the construction process of Algorithm 3, define* $D^{(t)} = \prod_{i=0}^{t} D_i$*, then* $\lim_{k\to\infty} D^{(t)}$ *exists and denote it as* $\hat{D}$*. Then,* $\mathbf{x} = \hat{D}\mathbf{e} \in \mathbb{R}_{++}^n$ *satisfies* $\mathcal{A}\mathbf{x}^{m-1} = \rho(\mathcal{A})\mathbf{x}^{[m-1]}$*, where* $\mathbf{e} = (1, 1, \cdots, 1)^T \in \mathbb{R}_{++}^n$.

Proof. From the construction of Algorithm 3, it is known that the sequence of diagonal elements $(D^{(t)})_{jj} (j = 1, 2, \cdots, n)$ of the positive diagonal matrix $D^{(t)} (j = 1, 2, \cdots, n)$ monotonically decreases with a lower bound, so $\lim_{k\to\infty} D^{(t)}$ exists. According to Lemma 3.1, the sequence of each element corresponding to tensor sequence $\{\mathcal{A}^{(k)}\}_{k=0}^{\infty}$ is nonnegative and has an upper bound, so there is a convergent tensor subsequence of $\{\mathcal{A}^{(k)}\}_{k=0}^{\infty}$ marked as $\{\mathcal{A}^{(k_l)}\}_{l=0}^{\infty}$, and denote $\lim_{k\to\infty} \mathcal{A}^{(k_l)} = \hat{\mathcal{A}}$. Take the limit on both sides of $\mathcal{A}^{(k_{l+1})} = \mathcal{A} \times_1 (D^{(k_l)})^{m-1} \times_2 D^{(k_l)} \times_3 \cdots \times_m D^{(k_l)}$ to get

$$\begin{aligned}\lim_{k\to\infty} \mathcal{A}^{(k_{l+1})} &= \lim_{k\to\infty} (\mathcal{A} \times_1 (D^{(k_l)})^{m-1} \times_2 D^{(k_l)} \times_3 \cdots \times_m D^{(k_l)}) \\ &= \mathcal{A} \times_1 (\lim_{k\to\infty} D^{(k_l)})^{m-1} \times_2 \lim_{k\to\infty} D^{(k_l)} \times_3 \cdots \times_m \lim_{k\to\infty} D^{(k_l)};\end{aligned}$$

that is,

$$\hat{\mathcal{A}} = \mathcal{A} \times_1 (\hat{D})^{m-1} \times_2 \hat{D} \times_3 \cdots \times_m \hat{D}.$$

It can be seen from Theorem 2.4 that $r_i(\hat{\mathcal{A}}) = \rho(\mathcal{A}), i \in \langle n \rangle$; therefore

$$\rho(\mathcal{A})(\hat{D}\mathbf{e})^{m-1} = \mathcal{A}(\hat{D}\mathbf{e})^{[m-1]},$$

where $\mathbf{e} = (1, 1, \cdots, 1)^T \in \mathbb{R}^n_{++}$. Denote $\mathbf{x} = \hat{D}\mathbf{e}$, then $\mathbf{x} = \hat{D}\mathbf{e} \in \mathbb{R}^n_{++}$, and $\mathcal{A}\mathbf{x}^{m-1} = \rho(\mathcal{A})\mathbf{x}^{[m-1]}$.

4. Numerical examples

In this section, to show the effectiveness of Algorithm 3, we compare it with the LZI algorithm. For the parameter α_k in the algorithm, we selected different values and compared the corresponding results.

Example 4.1. *Let* $\mathcal{A} = (a_{i_1i_2i_3i_4i_5}) \in \mathbb{R}^{[m,n]}_{+} (m = 5)$*, where* $a_{111jj} = 1, a_{jjj11} = 1$ *for all* $j \in \langle n \rangle \backslash \{1\}$ *and zero elsewhere.*

In the experiment, $\alpha_k = (\overline{r}(\mathcal{A}^{(k)}) - \underline{r}(\mathcal{A}^{(k)}))/mn$ in Algorithm 3, and $\mathbf{x}^{(0)} = \mathbf{e} = (1, 1, \cdots, 1)^T \in \mathbb{R}^n_{++}$, $\rho = 1$ in the LZI algorithm. We terminated the iteration when one of the conditions below was met:

(1) $\overline{r}(\mathcal{A}^{(k)}) - \underline{r}(\mathcal{A}^{(k)}) < 10^{-8}$.

(2) The number of iteration exceeds 10^4.

Some numerical results are given in Table 1, where $\rho(\mathcal{A})$ denotes the H-spectral radius of $\mathcal{A}$, Iter denotes the iteration of the algorithms and Time(s) denotes the CPU time (in seconds) used when the conditions (1) are met.

Table 1 shows a comparison between Algorithm 3 and the LZI algorithm given in [8], with the same error and the number of iterations and calculation time significantly reduced, which further verifies that our proposed algorithm is more efficient.

Table 1. The comparison of the Algorithm 3 and LZI algorithm.

n	Algorithm	Iter	Time(s)	$\rho(\mathcal{A})$
5	Algorithm 3	3	0.0996	2
5	LZI	18	0.2412	2
10	Algorithm 3	3	2.3991	3
10	LZI	15	5.6626	3
15	Algorithm 3	3	17.5646	3.7414
15	LZI	14	40.6961	3.7414
20	Algorithm 3	3	73.7770	4.3589
20	LZI	13	157.9810	4.3589
25	Algorithm 3	3	224.9131	4.8990
25	LZI	13	489.2777	4.8990
30	Algorithm 3	3	598.0174	5.3852
30	LZI	12	1148.8	5.3852

Example 4.2. *Consider a random tensor* $\mathcal{A} \in \mathbb{R}_+^{[m,n]}(m = 3)$*, whose all entries of random values drawn from the standard uniform distribution on (0, 1).*

Obviously, this is an s-index weakly positive tensor ($s = 1$ or 2). Choose $\varepsilon = 10^{-8}$ and the termination conditions are the same as in Example 4.1. Take different values for α_k in Algorithm 3, and the corresponding results are shown in Table 2, where $\overline{r} - \underline{r} = \overline{r}(\mathcal{A}^{(k)}) - \underline{r}(\mathcal{A}^{(k)})$.

Table 2. The comparison of different values of α_k in Algorithm 3.

	$\alpha_k = (\overline{r} - \underline{r})/mn$		$\alpha_k = \overline{r} - \underline{r}$		$\alpha_k = n(\overline{r} - \underline{r})$		$\alpha_k = mn(\overline{r} - \underline{r})$	
n	iter	Time(s)	iter	Time(s)	iter	Time(s)	iter	Time(s)
5	7	0.0142	7	0.0006	9	0.0255	15	0.0333
10	6	0.0010	6	0.0009	10	0.0014	14	0.0056
20	6	0.0044	6	0.0124	9	0.0091	13	0.0179
40	5	0.0225	6	0.0516	10	0.0513	14	0.0725
60	5	0.0670	5	0.0621	10	0.1305	15	0.1917
80	5	0.1702	5	0.1622	10	0.3074	16	0.4595
100	5	0.3940	5	0.3183	10	0.5547	15	0.8285

From the data in Table 2, it can be seen that when the value of α_k is different, there are differences in the number of iteration steps and operation times. The calculation time is almost the same, but the difference in iteration steps is quite significant, so selecting the appropriate α_k will improve the efficiency of the algorithm.

5. Conclusions

In this paper, a class of s-index weakly positive tensors was defined and a diagonal similar iterative algorithm for the maximum H-eigenvalue of such tensors was given. In the algorithm, a variable parameter was introduced in each iteration, which is equivalent to a translation transformation for each iteration of the tensor. Compared with the LZI algorithm, the number of iterations and time of calculation have great advantages. It was also proved that the algorithm has linearly convergence for s-index weakly positive tensors.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflict of interest.

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