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Research article

Applications of conjunctive complex fuzzy subgroups to Sylow theory

Aneeza Imtiaz [1], Hanan Alolaiyan[2], Umer Shuaib[1,*], Abdul Razaq[3,*] and Jia-Bao Liu[4]

[1] Department of Mathematics, Government College University, Faisalabad 38000, Pakistan
[2] Department of Mathematics, King Saud University, Riyadh 145111, Saudi Arabia
[3] Department of Mathematics, Division of Science and Technology, University of Education, Lahore 54770, Pakistan
[4] School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China

* **Correspondence:** Email: mumershuaib@gcuf.edu.pk; abdul.razaq@ue.edu.pk.

Abstract: Sylow's theorems are fundamental theorems in classical group theory that are of paramount importance. The extension of these theorems into diverse fuzzy contexts emerges as a compelling area of exploration. This study introduces the novel concept of the conjunctive complex fuzzy conjugate element within the conjunctive complex fuzzy subgroup of a group, elucidating numerous crucial properties of this concept. Additionally, it propounds the notion of the conjunctive complex fuzzy p-subgroup within the conjunctive complex fuzzy subgroup (CCFSG) and delineates various indispensable characteristics associated with this construct. Additionally, the paper formulates the conjunctive complex fuzzy version of the Cauchy theorem for finite groups. Lastly, it defines the concept of the conjunctive complex fuzzy Sylow p-subgroup for a finite group and conducts a generalization of Sylow's theorems within a conjunctive complex fuzzy environment.

Keywords: complex fuzzy subgroup; conjunctive complex fuzzy subgroup; conjunctive complex fuzzy Sylow p-subgroup
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1. Introduction

Sylow's theorems are a crucial component of crisp group theory, particularly in the context of categorizing finite simple groups. Finite groups have a significant role within the field of group theory. In the realm of finite group theory, these findings hold significant value as they serve as a noteworthy point of reference in relation to Lagrange's theorem, a widely recognized principle. By providing a means of examination, they facilitate the identification of subgroups with certain ordering. The fundamental concepts and methodologies of group theory, encompassing the pivotal Sylow theorems, have profound implications across several domains of scientific inquiry and technological advancements. The historical origins of these widely recognized views can be traced back to the sources cited in references [1,2].

In modern times, the fields of science and technology frequently encounter complex processes and phenomena that remain beyond comprehensive and accurate insight. Hence, it is crucial to integrate precise mathematical models into systems that demonstrate a significant degree of uncertainty. The motivation for the development of fuzzy set theory emerged from the necessity to expand traditional set theory in order to effectively address a certain objective. The framework presented herein offers a systematic approach for the development and evaluation of diverse models that effectively capture and tackle the inherent uncertainties within a given context. This theory is essential to the development of these models. Moreover, it facilitates our capacity to explore and adjust to the complex and unpredictable characteristics of systems within diverse scientific and technical fields.

Moreover, the applicability of fuzzy set theory has been proven across a diverse range of scientific disciplines and physical phenomena. Fuzzy sets (FS) have proven to be a flexible approach for effectively dealing with complex and uncertain situations across several areas. Nevertheless, it is possible that they may not consistently possess the capability to effectively communicate the intricate nature of the matter under consideration. FS predominantly focuses on membership functions that operate in a single dimension, hence posing challenges in representing intricate relationships and variables over many dimensions. Ordinary FS serves as a powerful mathematical tool in such situations. Fuzzy logic systems serve as a logical progression from the notion of FS, enabling the resolution of issues that are not amenable to traditional fuzzy sets. Complex fuzzy sets (CFS) have the ability to exhibit uncertainty in a more intricate way by integrating many dimensions or membership characteristics. This facilitates the thorough and effective examination of situations with intricate physical attributes and enhances an individual's ability to make knowledgeable judgments in difficult and confusing settings.

In contemporary society, the advancement of computer technology, the accessibility of high-speed processors, and the extensive utilization of programming languages have presented researchers with new prospects to explore and create algorithms that specifically address intricate physical phenomena in diverse scientific domains. The field of general operator theory provides a theoretical framework for comprehending the mathematical principles that form the foundation of several technical approaches utilized across different areas. The complex fuzzy environment demonstrates mathematical structures that may be readily elucidated within the context of general operator theory. By adopting this expansion, it becomes possible to create software programs that possess the potential to address a wide array of difficulties and make substantial contributions to advancements in several fields of study.

1.1. Literature review

Zadeh [3] laid the foundation for the notion of an FS, presenting it as a formidable tool to grapple with the intricacies inherent in navigating uncertainty within pragmatic contexts. Rosenfeld [4] initiated the seminal description of fuzzy subgroups (FSGs) and their algebraic properties. Subsequently, the introduction of the concept of level subgroups within the framework of FSG was accomplished [5]. The references [6–10] offer comprehensive information regarding the fundamental principles that underlie FSG. The topic of complex fuzzy numbers was introduced by Buckley in 1989 [11]. Based on this concept, the author [12] proposed the utilization of complex fuzzy numbers as a foundation for constructing a unique framework for differentiation. Furthermore, the previously mentioned author [13] elucidated several key characteristics of fuzzy contour integrals within the complex plane. Zhang [14] established some significant attributes pertaining to complex fuzzy numbers in 2012. In 2013, Ascia et al. [15] proposed a fuzzy processor designed to effectively handle intricate fuzzy inference systems. The concept of CFS was presented by Ramote et al. [16] in 2002 and demonstrated a comprehensive analysis of two novel operations: reflection and rotation. Furthermore, over a complex fuzzy space, the authors have recently devised the theories of complex fuzzy normal subgroups (CFNSG) [17], complex fuzzy hyperstructure [18] and CFSG [19]. Alsarahead and Ahmad [20] used the concept of CFS to develop the notion of CFSG in 2017. The concept of complex intuitionistic fuzzy sets was initially introduced in the publication [21]. Furthermore, [22, 23] have demonstrated the practical applications of this novel concept in decision-making scenarios. In 2018, a parallelity-preserving approximation was introduced in the realm of complex fuzzy operators [24], along with an exploration of multiple complex fuzzy geometric aggregation operators [25]. Two entropy metrics for CFS, investigating their rotational invariance were introduced in [26]. Dai et al. [27] presented distance measures between the interval-valued complex fuzzy sets. Abd Ulzeez et al. [28] defined a bipolar complex fuzzy distance measure in 2020, while [29] presented the CFS phenomenon based on the linear conjunctive operator. Current uses of CFS are evident in works cited in references [30–35].

1.2. Research gap in the existing literature, baseline information and innovative aspects of the study

Examining the previous publications, we determine that some advances have been made in the fields of classical and complex fuzzy group theory. Moreover, some results have been proven for CCFSG, but there are still many unanswered questions, such as:

1) Conjugacy relation in classical group theory is fundamental for understanding group structure, defining normal subgroups, characterizing elements into classes, and is pivotal in Sylow theory and various algebraic applications. This concept has been defined in the framework of FSG. However, in the CCFSG perspective, this notion has yet to be defined.
2) In classical group theory, it is well-known that the conjugacy relation is an equivalence relation. Furthermore, any two conjugate elements have the same order. In the existing literature, these results have been examined in classical fuzzy environments. The natural questions that come to mind are:
   a) Is the CCF conjugacy relation an equivalence relation?
   b) Do two CCF conjugate elements of a group G have the same CCF order?
3) Cauchy's theorem is essential in the process of classifying finite abelian groups. The existing body of knowledge lacks the CCF version of this important mathematical result.
4) Sylow's theorems are a set of three fundamental theorems in classical group theory that are of paramount importance in the study of finite groups, group actions and many other areas of algebra. In the literature, classical fuzzy variants of these theorems are available. However, within the framework of the CCFSG environment, the Sylow theorems have not yet been studied.

The primary aim of this study is to address the aforementioned unsolved problems and fill the existing knowledge gap within the field. Therefore, the findings presented in this study provide innovative insights into the examination of CCFSG.

Following an introductory discourse tracing the evolution of CCFSG, the subsequent trajectory of this paper unfolds as follows: Section 2 delves into a comprehensive exploration of the fundamental tenets underpinning CCFS and CCFSG, which constitute pivotal prerequisites for grasping the ensuing content of this paper. In Section 3, we expound upon the intricate constructs of the CCF conjugate element within the context of CCFSG, ascribed to a finite group G. The subsequent discourse, housed within Section 4, is dedicated to an intricate examination of the CCF p-subgroup of CCFSG. Within this purview, a compendium of algebraic properties intrinsic to this conceptual framework is meticulously expounded upon. This examination serves as a springboard for establishing the CCF variant of Cauchy's theorem, thereby demonstrating the utility of these concepts in extending classical results. In Section 5, we embark upon a scholarly exploration, introducing the domain of CCF conjugate subgroups (CCFCSG) and CCF Sylow p-subgroups. This endeavor is complemented by an intricate explication of the progressive concretization of the three renowned Sylow theorems within the overarching context of conjunctive complex fuzzification.

2. Basic definitions

This section provides essential prerequisite knowledge regarding the complex fuzzy environment in order to comprehend the novelty of the work described in this paper.

Definition 2.1. [29] Let U be a universe. A complex fuzzy set A is described as follows:

$$A = \{(m, \mu_A(m)) : m \in U\},$$

where μ_A is a complex valued function that maps each element $m \in U$ to a unit circle.

Definition 2.2. [29] Given a CFS A over the universe U and an element $\xi \in C^*$ expressed as: $\xi = \alpha e^{\delta}$, where $\alpha \in [0,1]$ and $\delta \in [0, 2\pi]$, the CCF set denoted as A^{ξ} with respect to CFS A is characterized by the following structure:

$$\mu_{A^{\xi}}(m) = min\left(r_A(m)e^{i\omega_A(m)}, \alpha e^{i\beta}\right) = min\{r_A(m), \alpha\}e^{i min(\omega_A(m), \beta)} = r_{A^{\xi}}(m)e^{i\omega_{A^{\xi}}(m)},\ \forall m \in U.$$

Here, the $r_{A^{\xi}}$ is a real valued function that maps each element $m \in U$ to a unit interval. Furthermore, the term $e^{i\omega_{A^{\xi}}}$ embodies a periodic function featuring a periodicity of 2π and $0 < arg_{A^{\xi}} \leq 2\pi$. Note that, $F^{\xi}(U)$ stands for the CCFS family of U.

Definition 2.3. [29] For any $A^{\xi} \in F^{\xi}(U)$, $\alpha \in [0,1]$ and $\delta \in [0.2\pi]$, the (α, δ)-cut set of A^{ξ} is described as $A^{\xi}_{(\alpha,\delta)} = \{m \in U : r_{A^{\xi}}(m) \geq \alpha, \omega_{A^{\xi}}(m) \geq \delta\}$.

Definition 2.4. [29] The level set $\Omega_{A^{\xi}}$ of A^{ξ} can be described as $\Omega_{A^{\xi}} = \{m \in U : r_{A^{\xi}}(m) = \alpha, \omega_{A^{\xi}}(m) = \delta\}$, where $0 \leq \alpha \leq 1, 0 \leq \delta \leq 2\pi$.

Definition 2.5. [29] Let $A^{\xi} \in F^{\xi}(U)$, $0 \leq \alpha \leq 1$ and $0 \leq \delta \leq 2\pi$. Then the subgroup $A^{\xi}_{(\alpha,\delta)}$ with $r_{A^{\xi}}(e) > \alpha$ and $\omega_{A^{\xi}}(e) > \delta$ is called the level subgroup of CCFSG of A^{ξ} and is denoted by $\ell_{(\alpha,\delta)}(A^{\xi})$.

Definition 2.6. [35] For any $A^{\xi} \in F^{\xi}(U)$, the support of A^{ξ} is defined as $A^{\xi}_{*} = \{m \in U : r_{A^{\xi}}(m) > 0, \omega_{A^{\xi}}(m) > 0\}$.

Definition 2.7. [29] Consider $A^{\xi}, B^{\xi} \in F^{\xi}(U)$. We establish the following definitions:

1) A^{ξ} is categorized as a homogeneous CCFS if $r_{A^{\xi}}(m) \leq r_{A^{\xi}}(n)$ implies $\omega_{A^{\xi}}(m) \leq \omega_{A^{\xi}}(n)$, $\forall m, n \in U$.
2) A^{ξ} is designated as a homogeneous CCFS with B^{ξ} if $r_{A^{\xi}}(m) \leq r_{B^{\xi}}(n)$ implies $\omega_{A^{\xi}}(m) \leq \omega_{B^{\xi}}(n)$, $\forall m, n \in U$.

Definition 2.8. [29] For CCFS A^{ξ} of a group G, we say that A^{ξ} is a CCFSG if A^{ξ} admits the subsequent conditions for all elements $m, n \in G$:

1) $\mu_{A^{\xi}}(mn) \geq min\,\{\mu_{A^{\xi}}(m), \mu_{A^{\xi}}(n)\}$.
2) $\mu_{A^{\xi}}(m^{-1}) \geq \mu_{A^{\xi}}(m)$.

Note that, the collection of CCFSG of G is denoted by $F^{\xi}(G)$.

Definition 2.9. [35] Consider $A^{\xi} \in F^{\xi}(G)$ and an element m of a finite group G. The least positive integer n is called CCF order of m (denoted by $\xi - CFO_{A^{\xi}}(m)$) if $\mu_{A^{\xi}}(m^n) = \mu_{A^{\xi}}(e)$.
Definition 2.10. [35] Let A^{ξ} be a CCFSG of G. The CCF order of A^{ξ} (written as $\xi - CFO(A^{\xi})$) is the smallest common multiple of CCF order of all elements of G.
Theorem 2.1. [35] If $\xi - CFO_{A^{\xi}}(m) = a$, then $\xi - CFO_{A^{\xi}}(m^b) = \frac{\xi - CFO_{A^{\xi}}(m)}{(a,b)}$, for some integer b.

3. Properties of conjugacy classes of conjunctive complex fuzzy subgroup of a group

In this segment, we introduce the notion of the CCF conjugate elements within the context of the CCFSG associated with a group G. We delve into a fundamental characteristic of this concept, and further explore the class equation pertaining to the CCFSG of a finite group.
Definition 3.1. Let $A^{\xi} \in F^{\xi}(G)$ and m, n be any elements of a finite group G. Then m is CCF conjugate to n (written as $m \sim_{A^{\xi}} n$) if there exists $e \neq x \in G$ such that $\mu_{A^{\xi}}(m) = \mu_{A^{\xi}}(x^{-1}nx)$.
Example 3.1. The CFSG A of $S_3 =< \alpha, \beta: \alpha^3 = \beta^2 = 1, \alpha\beta = \alpha^2\beta >$ is given by

$$\mu_A(m) = \begin{cases} 0.9e^{i1.9\pi}, & m \in \{e\}, \\ 0.7e^{i1.2\pi}, & m \in \{\alpha, \alpha^2\}, \\ 0.5e^{i\pi}, & m \in \{\beta, \alpha\beta, \alpha^2\beta\}. \end{cases}$$

The CCFSG A^{ξ} of S_3 corresponding to the value $\xi = 0.74e^{i1.5\pi}$ is given as

$$\mu_{A^{\xi}}(m) = \begin{cases} 0.9e^{i1.9\pi}, & m \in \{e\}, \\ 0.7e^{i1.2\pi}, & m \in \{\alpha, \alpha^2\}, \\ 0.5e^{i\pi}, & m \in \{\beta, \alpha\beta, \alpha^2\beta\}. \end{cases}$$

In view of Definition 3.1, we obtain $e \sim_{A^{\xi}} e$, $\alpha \sim_{A^{\xi}} \alpha^2$, $\beta \sim_{A^{\xi}} \alpha\beta$ and $\alpha^2\beta$.
The subsequent outcome delineates crucial algebraic characteristics of any two conjugate elements.
Theorem 3.1. Any two CCF conjugate elements of a group G have the same CCF order.
Proof. By applying the mentioned condition for any two conjugate elements $m, n \in G$, we have

$$\mu_{A^{\xi}}(m) = \mu_{A^{\xi}}(x^{-1}nx),\ x \in G.$$

Consider

$$\mu_{A^{\xi}}(m^2) = \mu_{A^{\xi}}(x^{-1}nx.x^{-1}nx) = \mu_{A^{\xi}}(x^{-1}n^2x).$$

Applying the concept of mathematical induction to the equation above yields

$$\mu_{A^{\xi}}(m^k) = \mu_{A^{\xi}}(x^{-1}n^kx).$$

Moreover, suppose $\xi - CFO_{A^{\xi}}(m) = a$ and $\xi - CFO_{A^{\xi}}(n) = b$.
This implies that $\mu_{A^{\xi}}(n^a) = \mu_{A^{\xi}}(x^{-1}m^ax) = \mu_{A^{\xi}}(e)$ and $\mu_{A^{\xi}}(m^b) = \mu_{A^{\xi}}(x^{-1}n^bx) = \mu_{A^{\xi}}(e)$.
By applying the Theorem 2.1 in the above equation, we get $a|b$ and $b|a$.
Definition 3.2. The conjugacy class of an element $m \in G$ of a CCFSG A^{ξ} of a finite group G is denoted by $Cl_{A^{\xi}}(m)$ and is defined as $Cl_{A^{\xi}}(m) = \{n \in G: m \sim_{A^{\xi}} n\}$.
Example 3.2. In view of Example 3.1, we have

$$Cl_{A^{\xi}}(1) = \{1\}, Cl_{A^{\xi}}(\alpha) = \{\alpha, \alpha^2\}, Cl_{A^{\xi}}(\beta) = \{\beta, \alpha\beta, \alpha^2\beta\}.$$

Proposition 3.1. Show that the relation of CCF conjugacy between elements of CCFSG of G is an equivalence relation.

Proof. Reflexivity: Let $m \in G$, then $\mu_{A^\xi}(m) = \mu_{A^\xi}(e^{-1}me)$, where e is the identity element of G. Hence $m \sim m$ for all $m \in G$.

Symmetry: Consider $m \sim_{A^\xi} n$ so that there is an element $x \in G$, then we have

$$\mu_{A^\xi}(m) = \mu_{A^\xi}(x^{-1}nx) \text{ and } \mu_{A^\xi}(xmx^{-1}) = \mu_{A^\xi}(xx^{-1}nxx^{-1}).$$

This further implies that $\mu_{A^\xi}(x^{-1}mx) = \mu_{A^\xi}(n)$. This shows that $n \sim_{A^\xi} m$.

Transitivity: For any $m, n, p \in G$, consider $m \sim_{A^\xi} n$ and $n \sim_{A^\xi} p$, there exist two elements $x, y \in G$ such that $\mu_{A^\xi}(m) = \mu_{A^\xi}(x^{-1}nx)$ and $\mu_{A^\xi}(n) = \mu_{A^\xi}(y^{-1}py)$.

Now $\mu_{A^\xi}(x^{-1}nx) = \mu_{A^\xi}(x^{-1}y^{-1}pyx)$. This implies that $\mu_{A^\xi}(m) = \mu_{A^\xi}(x^{-1}y^{-1}pyx)$.

This shows that $\mu_{A^\xi}(m) = \mu_{A^\xi}\big((yx)^{-1}p(yx)\big)$. This means that $m \sim_{A^\xi} p$.

Remark 3.1. Let G be a commutative abelian group. Then CCF conjugacy class of $m \in G$ is a singleton set.

The subsequent finding delineates the conditions whereby the conjugacy class of two elements of CCFSG of a group G are equal.

Theorem 3.2. $Cl_{A^\xi}(m) = Cl_{A^\xi}(n)$ if and only if $m \sim_{A^\xi} n$.

Proof. Suppose that $m \sim_{A^\xi} n$. Consider $x \in Cl_{A^\xi}(m)$, then by using Definition 3.2, we have $x \sim_{A^\xi} m$. Since $x \sim_{A^\xi} m$ and $m \sim_{A^\xi} n$, then by the transitive property, we have $x \sim_{A^\xi} n$. Thus $x \in Cl_{A^\xi}(n)$. This shows that $Cl_{A^\xi}(m) \subseteq Cl_{A^\xi}(n)$. Similarly, we obtain $Cl_{A^\xi}(n) \subseteq Cl_{A^\xi}(m)$. Consequently, $Cl_{A^\xi}(m) = Cl_{A^\xi}(n)$.

Conversely, let $Cl_{A^\xi}(m) = Cl_{A^\xi}(n)$. This implies that $x \sim_{A^\xi} m$ and $x \sim_{A^\xi} n$, $x \in G$. Consequently, $m \sim_{A^\xi} n$.

Definition 3.3. Let $A^\xi \in F^\xi(G)$, then the centralizer of A^ξ (written as $\mathbb{C}(A^\xi)$) is described as $\mathbb{C}(A^\xi) = \{m \in G: \mu_{A^\xi}(mn) = \mu_{A^\xi}(nm), \forall n \in G\}$.

Lemma 3.1. For any elements m and n in G, $\mu_{A^\xi}(mn^{-1}) = \mu_{A^\xi}(e) \Leftrightarrow \mu_{A^\xi}(m) = \mu_{A^\xi}(n)$.

Proof. Assume that $\mu_{A^\xi}(mn^{-1}) = \mu_{A^\xi}(e)$. Consider $\mu_{A^\xi}(m) = \mu_{A^\xi}(mn^{-1}n) = \mu_{A^\xi}\big((mn^{-1})n\big) \geq \min\{\mu_{A^\xi}(mn^{-1}), \mu_{A^\xi}(n)\} = \min\{\mu_{A^\xi}(e), \mu_{A^\xi}(n)\}$. We get the following relation by applying the given facts in the above equation:

$$\mu_{A^\xi}(m) \geq \mu_{A^\xi}(n). \tag{3.1}$$

Similarly,

$$\mu_{A^\xi}(m) \leq \mu_{A^\xi}(n). \tag{3.2}$$

The application of (3.1) and (3.2) yield that $\mu_{A^\xi}(m) = \mu_{A^\xi}(n)$.

Conversely, suppose that $\mu_{A^\xi}(m) = \mu_{A^\xi}(n)$. This implies that $\mu_{A^\xi}(mn^{-1}) = \mu_{A^\xi}(nn^{-1}) = \mu_{A^\xi}(e)$.

Lemma 3.2. If $A^\xi \in F^\xi(G)$ and $T = \{m \in G: \mu_{A^\xi}(mnm^{-1}n^{-1}) = \mu_{A^\xi}(e),\ \forall n \in G\}$, then $T = \mathbb{C}(A^\xi)$.

Proof. Let $m \in T$, then for all $n \in G$, we get $\mu_{A^\xi}(mn(nm)^{-1}) = \mu_{A^\xi}(mnm^{-1}n^{-1}) = \mu_{A^\xi}(e)$.

We get the following relation by applying the Lemma 3.1 in the above equation: $\mu_{A^\xi}(mn) = \mu_{A^\xi}(nm),\ \forall n \in G$. This implies that $m \in \mathbb{C}(A^\xi)$. Thus,

$$T \subseteq \mathbb{C}(A^\xi). \tag{3.3}$$

Furthermore, if $m \in \mathbb{C}(A^\xi)$, then $\mu_{A^\xi}(mn) = \mu_{A^\xi}(nm)$.

We get the following relation by applying the Lemma 3.1 in the above equation:

$$\mu_{A^\xi}(mn(nm)^{-1}) = \mu_{A^\xi}(e), \forall m, n \in G.$$

It follows that $\mu_{A^\xi}(mnm^{-1}n^{-1}) = \mu_{A^\xi}(e)$. This shows that $m \in T$.

Thus,

$$\mathbb{C}(A^{\xi}) \subseteq T. \tag{3.4}$$

By comparing (3.3) and (3.4), we have $T = \mathbb{C}(A^{\xi})$.

Remark 3.2. It may be noted that the centralizer of an element of A^{ξ} is written as $\mathbb{C}_{A^{\xi}}(m)$ and is defined as: $\mathbb{C}_{A^{\xi}}(m) = \{n \in G: \mu_{A^{\xi}}(mnm^{-1}n^{-1}) = \mu_{A^{\xi}}(e)\}$.

Example 3.3. In view of Example 3.1, Definition 3.3 and Remark 3.2, we have

$$\mathbb{C}(A^{\xi}) = \{1\}, \mathbb{C}_{A^{\xi}}(1) = \{1\}, \mathbb{C}_{A^{\xi}}(\beta) = \{1, \beta\}, \mathbb{C}_{A^{\xi}}(\alpha) = \{1, \alpha, \alpha^2\}.$$

Lemma 3.3. Let $A^{\xi} \in F^{\xi}(G)$. If $m \in \mathbb{C}(A^{\xi})$, then $\mu_{A^{\xi}}(mn_1n_2 \dots n_k) = \mu_{A^{\xi}}(n_1mn_2 \dots n_k) = \cdots = \mu_{A^{\xi}}(n_1n_2 \dots n_km)$, $\forall n_1, n_2, \dots, n_k \in G$.

Proof. We prove the result by induction on k. Suppose $m \in \mathbb{C}(A^{\xi})$. Then, $\mu_{A^{\xi}}(mn_1n_2) = \mu_{A^{\xi}}(n_1n_2m)$, $\forall n_1, n_2 \in G$.

Assume that $\mu_{A^{\xi}}(mn_1n_2 \dots n_k) = \mu_{A^{\xi}}(n_1mn_2 \dots n_k)$=... $= \mu_{A^{\xi}}(n_1n_2 \dots n_km)$, $\forall n_1, n_2, \dots, n_k \in G$.

Consider

$$\begin{aligned}(mn_1n_2 \dots .(n_kn_{k+1})) &= \mu_{A^{\xi}}(n_1mn_2 \dots .(n_kn_{k+1})) \\ &\vdots \\ &= \mu_{A^{\xi}}(n_1n_2 \dots .m(n_kn_{k+1})) \\ &= \mu_{A^{\xi}}(n_1n_2 \dots .(n_kn_{k+1})m),\end{aligned}$$

and

$$\begin{aligned}\mu_{A^{\xi}}(m(n_1n_2) \dots .n_kn_{k+1}) &= \mu_{A^{\xi}}((n_1n_2)m \dots .n_kn_{k+1}) \\ &\vdots \\ &= \mu_{A^{\xi}}((n_1n_2) \dots .n_kmn_{k+1}) \\ &= \mu_{A^{\xi}}((n_1n_2) \dots .n_kn_{k+1}m), \ \forall n_1, n_2, \dots n_k \in G.\end{aligned}$$

This completes the proof.

Theorem 3.3. If $A^{\xi} \in F^{\xi}(G)$, then $\mathbb{C}(A^{\xi})$ is a subgroup of G.

Proof. For any element m in $\mathbb{C}(A^{\xi})$, we get $\mu_{A^{\xi}}(mxm^{-1}x^{-1}) = \mu_{A^{\xi}}(e)$, $\forall x \in G$. Consider

$$\begin{aligned}\mu_{A^{\xi}}((mn)x(mn)^{-1}x^{-1}) &= \mu_{A^{\xi}}(mnxn^{-1}m^{-1}x^{-1}) \\ &= \mu_{A^{\xi}}(mnn^{-1}xm^{-1}x^{-1}) \\ &= \mu_{A^{\xi}}(mxm^{-1}x^{-1}) \\ &= \mu_{A^{\xi}}(e).\end{aligned}$$

This shows that $mn \in \mathbb{C}(A^{\xi})$.

Furthermore, consider

$$\begin{aligned}\mu_{A^{\xi}}(m^{-1}x(m^{-1})^{-1}x^{-1}) &= \mu_{A^{\xi}}(m^{-1}xmx^{-1}) \\ &= \mu_{A^{\xi}}(m^{-1}mxx^{-1}) \\ &= \mu_{A^{\xi}}(e).\end{aligned}$$

Thus, $m^{-1} \in \mathbb{C}(A^{\xi})$. Consequently, $\mathbb{C}(A^{\xi})$ is a subgroup of G.

Remark 3.3. Assume that $A^{\xi} \in F^{\xi}(G)$, then:

1) If A^{ξ} is a conjunctive complex fuzzy normal subgroup CCFNSG of a group G, then $\mathbb{C}(A^{\xi}) \trianglelefteq G$.
2) If A^{ξ} is a CCFSG of an abelian group G, then $\mathbb{C}(A^{\xi}) = G$.

Theorem 3.4. Let G be a finite group and $A^{\xi} \in F^{\xi}(G)$, then $|Cl_{A^{\xi}}(m)| = \frac{|G|}{|\mathbb{C}_{A^{\xi}}(m)|}, m \in G$.

Proof. Let $H = \{x_1\mathbb{C}_{A^\xi}(m), x_2\mathbb{C}_{A^\xi}(m), x_3\mathbb{C}_{A^\xi}(m), \dots, x_n\mathbb{C}_{A^\xi}(m): x_i \in G, i = 1,2, \dots n\}$ be the collection of all disjoint cosets of $\mathbb{C}_{A^\xi}(m)$ in G. The left decomposition of G as a disjoint union of cosets of $\mathbb{C}_{A^\xi}(m)$ in G is given by $G = \bigcup_{i=1}^{n} x_i\mathbb{C}_{A^\xi}(m), x_i \in G$. This implies that $O(G) = n.O\left(\mathbb{C}_{A^\xi}(m)\right)$. Define a mapping $\phi: H \to Cl_{A^\xi}(m)$ by

$$\phi\left(x\mathbb{C}_{A^\xi}(m)\right) = \mu_{A^\xi}(x^{-1}mx).$$

Note that ϕ is well-defined, since for $x, y \in G$, we have

$$x\mathbb{C}_{A^\xi}(m) = y\mathbb{C}_{A^\xi}(m).$$

This implies that $x^{-1}y \in \mathbb{C}_{A^\xi}(m)$. By using Definition 3.3, we have

$$\mu_{A^\xi}(m(x^{-1}y)m^{-1}(x^{-1}y)^{-1}) = \mu_{A^\xi}(e).$$

This shows that $\mu_{A^\xi}(x^{-1}mx) = \mu_{A^\xi}(y^{-1}my)$. Consequently, $\phi\left(x\mathbb{C}_{A^\xi}(m)\right) = \phi\left(y\mathbb{C}_{A^\xi}(m)\right)$. Let $x, y \in G$, then $\phi\left(x\mathbb{C}_{A^\xi}(m)\right) = \phi\left(y\mathbb{C}_{A^\xi}(m)\right)$. This implies that $\mu_{A^\xi}(x^{-1}mx) = \mu_{A^\xi}(y^{-1}my)$. This further implies that $\mu_{A^\xi}(m(x^{-1}y)m^{-1}(x^{-1}y)^{-1}) = \mu_{A^\xi}(e)$. This means that $x^{-1}y \in \mathbb{C}_{A^\xi}(m)$, implying that $x\mathbb{C}_{A^\xi}(m) = y\mathbb{C}_{A^\xi}(m)$. Thus, φ is injective.

Furthermore, it is easy to show that ϕ is onto. Therefore, there exists a bijective mapping between H and $Cl_{A^\xi}(m)$. Hence $O(H) = O\left(Cl_{A^\xi}(m)\right)$. Consequently, $\left|Cl_{A^\xi}(m)\right| = \frac{|G|}{\left|\mathbb{C}_{A^\xi}(m)\right|}$.

Corollary 3.1. The cardinality of the conjugacy class of an element of CCFSG A^ξ divides the order of G.
Proof. The required outcome is a consequence of the well-known result that the centralizer is a subgroup of a finite group G and in view of Langrange's theorem its index divides the order of the finite group G. It may be noted that the index of $Cl_{A^\xi}(m)$ is infact the number of disjoint left cosets of $\mathbb{C}_{A^\xi}(m)$ in G.

Definition 3.4. Let $A^\xi \in F^\xi(G)$, then the normalizer A^ξ (written as $\mathbb{N}(A^\xi)$) is described as follows:

$$\mathbb{N}(A^\xi) = \{m \in G: \mu_{A^\xi}(m) = \mu_{A^\xi}(n^{-1}mn),\ \forall n \in G\}.$$

Example 3.4. In the light of Example 3.1 and Definition 3.4, we have $\mathbb{N}(A^\xi) = \{1, \alpha, \alpha^2\}$.
Remark 3.4. Suppose that $A^\xi \in F^\xi(G)$, then:
1) If A^ξ is a CCFNSG of a group G, then $\mathbb{N}(A^\xi) = G$.
2) $\mathbb{C}(A^\xi) \trianglelefteq \mathbb{N}(A^\xi)$.

Definition 3.5. The class equation of CCFSG A^ξ of a finite group G is defined as

$$|G| = \sum_{m \in G}\left|Cl_{A^\xi}(m)\right|.$$

Example 3.5. In light of Example 3.2, the class equation of CCFSG S_3 is given as

$$|G| = \left|Cl_{A^\xi}(1)\right| + \left|Cl_{A^\xi}(\alpha)\right| + \left|Cl_{A^\xi}(\beta)\right| = 1 + 2 + 3 = 6.$$

4. Algebraic characteristics of conjunctive complex fuzzy p-subgroups

This section introduces the idea of CCF p-subgroup of CCFSG and explores the several algebraic aspects associated with this phenomenon. In addition, we demonstrate the CCF variant of the Cauchy theorem.
Definition 4.1. A CCFSG A^ξ of a group G is a CCF p-subgroup if $\xi - CFO_{A^\xi}(m)$ is a power of prime $p,\ \forall m \in G$.

Theorem 4.1. Let A^{ξ} be a CCFNSG of a finite group G, then the set $G^{\xi} = \{m \in G: \mu_{A^{\xi}}(m) = \mu_{A^{\xi}}(e)\}$ is normal in G.

We establish a condition under which a CCFSG is a CCF p-subgroup in the following result:

Theorem 4.2. Consider a CCFSG A^{ξ} of a finite group G such that $G^{\xi} = \{m \in G: \mu_{A^{\xi}}(m) = \mu_{A^{\xi}}(e)\}$ is normal in G, then A^{ξ} is a CCF p-subgroup if and only if $G/G^{\xi}$is a p-group.

Proof. In light of Definition 4.1 and for any element $m \in G$, we have $\xi - CFO_{A^{\xi}}(m) = p^q$ for some non-negative integer q and so $m^{p^k} \in G^{\xi}$. Thus G/G^{ξ} is a p-group. Conversely, let G/G^{ξ} be a p-group. If $m \in G$, then $m^{p^k} \in G^{\xi}$ for some nonnegative integer q and so $\mu_{A^{\xi}}\left(m^{p^k}\right) = \mu_{A^{\xi}}(e)$. Consequently, A^{ξ} is a CCF p-subgroup.

Theorem 4.3. If $\xi - CFO_{A^{\xi}}(m) = ab$ for some coprime positive integers a and b, then there exist $m_1, m_2 \in G$ such that $m = m_1 m_2 = m_2 m_1$, $\xi - CFO_{A^{\xi}}(m_1) = a$ and $\xi - CFO_{A^{\xi}}(m_2) = b$.

Proof. Assume that $\xi - CFO_{A^{\xi}}(m) = ab$. Since $(a, b) = 1$, then there exist integers x and y such that $ax + by = 1$. Here, $(a, y) = (b, x) = 1$. Let $m_1 = m^{ay}$ and $m_2 = m^{bx}$, then $m = m_1 m_2 = m_2 m_1$. By using Theorem 2.1, we have $\xi - CFO_{A^{\xi}}(m_1) = \xi - CFO_{A^{\xi}}(m^{ay}) = b$ and $\xi - CFO_{A^{\xi}}(m_2) = \xi - CFO_{A^{\xi}}(m^{bx}) = a$.

Theorem 4.4. (Conjunctive complex fuzzification of Cauchy theorem) Let G be a finite group and $A^{\xi} \in F^{\xi}(G)$ and $\xi - CFO\left(A^{\xi}\right) = p^r q$, where p is prime and $(p, q) = 1$, then there is an element $m \in G$ such that $\xi - CFO_{A^{\xi}}(m) = p^s$, for each nonnegative integer $s \leq r$.

Proof. Since $\xi - CFO(A^{\xi})$ is the greatest common divisor of $\xi - CFO_{A^{\xi}}(m)$, where $m \in G$ there is an element m in G such that $\xi - CFO_{A^{\xi}}(m) = p^s$. Applying the induction approach to s and utilizing the Cauchy theorem in classical group theory, we can establish the existence of m in G such that $\xi - CFO_{A^{\xi}}(m) = p^s$.

Corollary 4.1. If A^{ξ} is a CCFSG of an abelian group G and $\xi - CFO\left(A^{\xi}\right) = ab$ for some $a, b \in Z$, then there is an element m in G such that $\xi - CFO_{A^{\xi}}(m) = a$.

Remark 4.1. Let A^{ξ} be a CCFSG of a group G and p be a prime. Then $H_P = \{m \in G: \xi - CFO_{A^{\xi}}(m, p) = 1\}$ and $L_P = \{m \in G: \xi - CFO_{A^{\xi}}(m \text{ is a power of the prime } p)\}$ are subgroups of G.

Theorem 4.5. Let $A^{\xi} \in F^{\xi}(G)$ such that the CCF index of A^{ξ} is p, where p is the smallest prime divisor of the order of G, then A^{ξ} is a CCFNSG of G.

Proof. Consider the subgroup H of G having index p as follows: $H = \left\{m \in G: \mu_{A^{\xi}}(m) = \mu_{A^{\xi}}(e)\right\}$.

Then, the group G acts on the set of all left cosets of G by H denoted by $G/H = \left\{m_1 H, m_2 H, m_3 H, \ldots m_p H\right\}$ by the left multiplication. The corresponding permutation representation of the action of G on the set G/H is interpreted as follows: $\varphi: G \to sym(G/H)$ with $ker\varphi = core(H)$. In view of the first fundamental isomorphism theorem of the classical groups, we have the quotient group $G/Core(H)$ is isomorphic to a subgroup of $sym(G/H)$. Thus, by means of Lagrange's theorem, we have $O(G/Core(H))$ divides $p!$. Since $O(G/H) = p$, it follows that $O(G/Core(H))$ divides $(p-1)!$. But as the $O(H)$ divides the $O(G)$, we obtain the following relation $H = Core(H)$. Otherwise, we have a contradiction against the minimality of the prime p. Note that, $H \lhd G$ as $Core(H)$ is normal in G. Moreover, G/H is abelian. Then, $mH = nH = nHmH$. This implies that $mnH = nmH$. Thus, $mn = nm$. Hence, $\mu_{A^{\xi}}(mn) = \mu_{A^{\xi}}(nm)$, $\forall m, n \in G$. Consequently, A^{ξ} is a CCFNSG of G.

Corollary 4.2. If the CCF index of A^{ξ} is 2, then A^{ξ} is CCFNSG of G.

Definition 4.2. Let A^{ξ} be any CCFSG of a finite group G and $H = \left\{m \in G: \mu_{A^{\xi}}(m) = \mu_{A^{\xi}}(e)\right\}$. Then, A^{ξ} is CCF abelian if H is an abelian subgroup of G.

Theorem 4.6. A CCFSG A^{ξ} is CCF abelian if $\xi - CFO\left(A^{\xi}\right) = p^2$, where p is a prime.

Proof. The proof is derived from the straightforward implementation of Definition 4.1.

5. Conjunctive complex fuzzification of Sylow's theorems

In this section, we present the notion of CCF conjugate subgroup (CCFCSG) and the CCF Sylow p-subgroup in the context of finite group G. In addition, we demonstrate the conjunctive complex fuzzification of all three Sylow's theorems.

Definition 5.1. Suppose that $A^{\xi}, B^{\xi} \in F^{\xi}(G)$, then A^{ξ} is CCF conjugate (CCFCSG) to B^{ξ} if there exists $m \in G$ such that $\mu_{A^{\xi}}(n) = \mu_{B^{\xi}}(m^{-1}nm),\ \forall n \in G$.

Theorem 5.1. Let $A^{\xi}, B^{\xi} \in F^{\xi}(G)$, then A^{ξ} and B^{ξ} are CCFCSG of G if and only if $A^{\xi} = B^{\xi}$.

Proof. Assume that A^{ξ} and B^{ξ} are CCFCSG of G. In light of Definition 5.1, we get $\mu_{A^{\xi}}(n) = \mu_{B^{\xi}}(m^{-1}nm),\ \forall n \in G$ and $\mu_{A^{\xi}}(mn) = \mu_{B^{\xi}}(mm^{-1}nm)$. This implies that $\mu_{A^{\xi}}(mn) = \mu_{B^{\xi}}(nm)$. For some $m = e \in G$, we have $\mu_{A^{\xi}}(en) = \mu_{B^{\xi}}(ne)$. This further implies that $\mu_{A^{\xi}}(n) = \mu_{B^{\xi}}(n)$. Consequently, $A^{\xi} = B^{\xi}$.

Conversely, suppose that $A^{\xi} = B^{\xi}$, which implies that $\mu_{A^{\xi}}(m) = \mu_{B^{\xi}}(m),\ \forall m \in G$. This implies that $\mu_{A^{\xi}}(n) = \mu_{B^{\xi}}(e^{-1}ne),\ \forall n, \in G$. This concludes that, A^{ξ} and B^{ξ} are CCFCSG of G.

Corollary 5.1. If A^{ξ} and B^{ξ} are any two CCFCSG of G, then $\xi - CFO(A^{\xi}) = \xi - CFO(B^{\xi})$.

Example 5.1. The CFSGs A and B of D_4 are defined as follows:

$$\mu_A(m) = \begin{cases} 0.8e^{i1.8\pi}, & if\ m = 1, \\ 0.72e^{i1.3\pi}, & if\ m = \{\alpha^2, \beta, \alpha^2\beta\}, \\ 0.4e^{i0.9\pi}, & otherwise, \end{cases}$$

and

$$\mu_B(m) = \begin{cases} 0.8e^{i1.8\pi}, & if\ m = 1, \\ 0.72e^{i1.3\pi}, & if\ m = \{\alpha^2, \alpha\beta, \alpha^3\beta\}, \\ 0.4e^{i0.9\pi}, & otherwise. \end{cases}$$

Then, the CCFSGs A^{ξ} and B^{ξ} of G corresponding to the value $\xi = 0.75e^{i1.5\pi}$ are given as

$$\mu_{A^{\xi}}(m) = \begin{cases} 0.75e^{i1.5\pi}, & if\ m = 1, \\ 0.72e^{i1.3\pi}, & if\ m = \{\alpha^2, \beta, \alpha^2\beta\}, \\ 0.4e^{i0.9\pi}, & otherwise, \end{cases}$$

and

$$\mu_{B^{\xi}}(m) = \begin{cases} 0.75e^{i1.5\pi}, & if\ m = 1, \\ 0.72e^{i1.3\pi}, & if\ m = \{\alpha^2, \alpha\beta, \alpha^3\beta\}, \\ 0.4e^{i0.9\pi}, & otherwise. \end{cases}$$

Clearly, $\mu_{A^{\xi}}(\alpha^2) = \mu_{B^{\xi}}((\alpha^3\beta)\alpha^2(\alpha^3\beta))$.

Definition 5.2. Let $m \in G$ and $A^{\xi} \in F^{\xi}(G)$. Then, the set $C\ell(A^{\xi}) = \left\{m^{-1}A^{\xi}_{(\alpha,\delta)}m : m \in G\right\}$ is called the class of CCFCSG to A^{ξ}.

Example 5.2. Consider the CFSG A of D_5 as follows:

$$\mu_A(m) = \begin{cases} 0.8e^{i1.8\pi}, & if\ m = 1, \\ 0.72e^{i1.3\pi}, & if\ m = \{\alpha\beta\}, \\ 0.4e^{i0.9\pi}, & if\ m \in \{\alpha, \alpha^2, \alpha^3, \alpha^4, \beta, \alpha^2\beta, \alpha^3\beta, \alpha^4\beta\}. \end{cases}$$

Then, the CCFSG A^{ξ} of G corresponding to the value $\xi = 0.75e^{i1.5\pi}$ is given as

$$\mu_{A^{\xi}}(m) = \begin{cases} 0.75e^{i1.5\pi}, & if\ m = 1, \\ 0.72e^{i1.3\pi}, & if\ m = \{\alpha\beta\}, \\ 0.5e^{i\pi}, & if\ m \in \{\alpha, \alpha^2, \alpha^3, \alpha^4, \beta, \alpha^2\beta, \alpha^3\beta, \alpha^4\beta\}. \end{cases}$$

In view of Definition 2.3, we have $A^{\xi}_{(\alpha,\delta)} = \{1, \alpha\beta\}$.

The required class of CCFCSG of A^{ξ} is obtained as $C\ell(A^{\xi}) = \{C_1, C_2, C_3, C_4, C_5\}$, where $C_1 = \{1, \alpha\beta\}$, $C_2 = \{1, \beta\}$, $C_3 = \{1, \alpha^2\beta\}$, $C_4 = \{1, \alpha^3\beta\}$, $C_5 = \{1, \alpha^4\beta\}$.

Remark 5.1. For any $A^{\xi} \in F^{\xi}(G)$, we have $O[C\ell(A^{\xi})] = \frac{O(G)}{O[\mathbb{N}(A^{\xi})]}$, where $\mathbb{N}(A^{\xi})$ is the normalizer of A^{ξ} in G.

The subsequent example demonstrates the algebraic concepts mentioned in the aforementioned finding.

Example 5.3. The application of Definition 3.4 and Corollary 5.1 gives that $O[C\ell(A^{\xi})] = \frac{10}{2} = 5$.

Definition 5.3. A CCFSG A^{ξ} of a finite group G is called a CCF Sylow p-subgroup (written as CCF$\mathcal{S}_p$SG) if the support set A^{ξ^*} is a Sylow p-subgroup of G.

The following remark gives an alternative definition of CCF Sylow p-subgroup of a finite group G.

Remark 5.2. Let A^{ξ} be a CCFSG of a finite group G, then A^{ξ} is called CCF$\mathcal{S}_p$SG, if one of the level subgroups of A^{ξ} is Sylow p-subgroup of G.

Example 5.4. Consider the CFSG A of the group $G \times H = \{(\alpha, \beta): \alpha \in G, \beta \in H\}$, where $G =< \alpha: \alpha^5 = 1 >$ and $H =< \beta: \beta^2 = 1 >$ is defined as

$$\mu_A(m) = \begin{cases} 0.93e^{i1.8\pi}, & if\ m \in \{(1,1)\}, \\ 0.7e^{i1.1\pi}, & if\ m \in < (\alpha, 1) > -\{(1,1)\}, \\ 0.4e^{i0.7\pi}, & if\ m \in G \times H - < (\alpha, 1). \end{cases}$$

Then, the CCFSG A^{ξ} of $G \times H$ for the value $\xi = 0.8e^{i1.4\pi}$ is defined as follows:

$$\mu_{A^{\xi}}(m) = \begin{cases} 0.8e^{i1.4\pi}, & if\ m \in \{(1,1)\}, \\ 0.7e^{i1.1\pi}, & if\ m \in < (\alpha, 1) > -\{(1,1)\}, \\ 0.4e^{i0.7\pi}, & if\ m \in G \times H - < (\alpha, 1). \end{cases}$$

In view of Definition 2.5, we have $\ell_{(\alpha,\delta)}(A^{\xi}) =< (\alpha, 1) >$, which is a Sylow 5-subgroup of $G \times H$. Thus, A^{ξ} is a CCF$\mathcal{S}_5$SG of $G \times H$.

The following theorem establishes the CCF version of Sylow's first theorem.

Theorem 5.2. (Conjunctive complex fuzzification of Sylow's first theorem) Let A^{ξ} be a CCFSG of a finite group G, where $O(G) = p^a b$, p is a prime and a, b are positive integers with (a, b) =1. Let $A^{\xi^*} = H$ be a support of A^{ξ}such that $p|o(H)$. Then, there exists a CCF$\mathcal{S}_p$SG B^{ξ} of G such that $B^{\xi} \subseteq A^{\xi}$ in H.

Proof. If $H = A^*_{\xi}$ is a Sylow p-subgroup, then there is no further proof required. We proceed under the assumption that H is not a Sylow p-subgroup of G. Suppose that $\alpha = \{r_{A^{\xi}}(m): r_{A^{\xi}}(m) > \alpha, m \in G\}$ and $\delta = \{\omega_{A^{\xi}}(m): \omega_{A^{\xi}}(m) > \delta, m \in G\}$. Clearly, $0 \le \alpha \le 1$ and $0 \le \delta \le 2\pi$. Since G is finite, therefore, $A^{\xi}_{(\alpha,\delta)} = A^{\xi^*} = H$. Given that p divides the order of H, we can apply Sylow's first theorem to assert the existence of a Sylow p-subgroup H_1 of H. By our assumptions, $O(H_1) = p^k$, where $1 \le k \le a$. Additionally, H_1 is contained in a subgroup H_2 of G. We will now define a CCFS B^{ξ} of G as follows:

$$\mu_{B^\xi}(m) = \begin{cases} 1e^{i2\pi}, & if\ m = e, \\ \alpha e^{i\delta}, & if\ m \in H_1 - \{e\}, \\ \alpha' e^{i\delta'}, & if\ m \in H_2 - H_1, \\ 0, & if\ m \in G - H_2, \end{cases}$$

where $0 \leq \alpha' \leq 1$ and $0 \leq \delta' \leq 2\pi$. Note that B^ξ is a CCFSG of G such that $B^\xi \subseteq A^\xi$ in H. Therefore, $B^{\xi^*} = H_2$ is a Sylow p-subgroup of G. Moreover, in view of Definition 5.3, we have B^ξ is a CCF$\mathcal{S}_p$SG of G.

Theorem 5.3. A CCFCSG of a CCF$\mathcal{S}_p$SG is a CCF$\mathcal{S}_p$SG subgroup of a group G.

Proof. Let A^ξ be a CCF$\mathcal{S}_p$SG of a group G and B^ξ be a CCFCSG to A^ξ. In the light of Definition 5.1, we have $\mu_{B^\xi}(n) = \mu_{A^\xi}(m^{-1}nm),\ m \in G$. This implies that $B^{\xi^*} = m^{-1}A^{\xi^*}m$. As A^ξ is CCF$\mathcal{S}_p$SG, therefore there is a Sylow p-subgroup H of G contained in A^{ξ^*}. Moreover, in view of Sylow's second theorem, $m^{-1}Hm$ being a conjugate of H is itself a Sylow p-subgroup of G. Further, $m^{-1}Hm$ is contained in $m^{-1}A^{\xi^*}m = B^{\xi^*}$. Consequently, B^ξ is a CCF$\mathcal{S}_p$SG.

Remark 5.3. Two distinct CCF$\mathcal{S}_p$SG need not be CCF conjugate to each other.

In the following example, we detail above algebraic feature.

Example 5.5. Consider the Sylow 2-subgroups H_1 and H_2 of D_6 as follows:

$$H_1 =< \alpha^3, \beta >= \{1, \alpha^3, \beta, \alpha^3\beta\} \text{ and } H_2 =< \alpha^3, \alpha\beta >= \{1, \alpha^3, \alpha\beta, \alpha^4\beta\}.$$

The CFSG A and B of D_6 are defined as follows:

$$\mu_A(m) = \begin{cases} 0.93e^{i1.9\pi}, & if\ m = 1, \\ 0.77e^{i1.1\pi}, & if\ m \in < \alpha^3, \beta > -\{1\}, \\ 0.5e^{i0.7\pi}, & otherwise, \end{cases}$$

and

$$\mu_B(m) = \begin{cases} 0.93e^{i1.9\pi}, & if\ m = 1, \\ 0.77e^{i1.1\pi}, & if\ m \in < \alpha^3, \alpha\beta > -\{1\}, \\ 0.6e^{i0.8\pi}, & otherwise. \end{cases}$$

Then, the CCFSG A^ξ and B^ξ of D_6 corresponding to the value $\xi = 0.81e^{i1.5\pi}$ are given as

$$\mu_{A^\xi}(m) = \begin{cases} 0.81e^{i1.5\pi}, & if\ m = 1, \\ 0.77e^{i1.1\pi}, & if\ m \in < \alpha^3, \beta > -\{1\}, \\ 0.5e^{i0.7\pi}, & otherwise. \end{cases}$$

and

$$\mu_{B^\xi}(m) = \begin{cases} 0.81e^{i1.5\pi}, & if\ m = 1, \\ 0.77e^{i1.1\pi}, & if\ m \in < \alpha^3, \alpha\beta > -\{1\}, \\ 0.6e^{i0.8\pi}, & otherwise. \end{cases}$$

Clearly, A^ξ and B^ξ are CCF$\mathcal{S}_2$SG of D_6.

But A^ξ is not CCFCSG to B^ξ because $\mu_{A^\xi}(\beta) \neq \mu_{B^\xi}((\alpha^4\beta)^{-1}\beta(\alpha^4\beta))$.

The subsequent result describes the CCF variant of Sylow's second theorem.

Theorem 5.4. (Conjunctive complex fuzzification of Sylow's second theorem) For any two CCF$\mathcal{S}_p$SG A^ξ and B^ξ having the same images such that $B^\xi_{(\alpha,\delta)} = m^{-1}A^\xi_{(\alpha,\delta)}m$, for all $m \in G$. Then A^ξ and B^ξ are CCFCSG to each other.

Proof. Let $B^{\xi}_{(\alpha,\delta)} = m^{-1}A^{\xi}_{(\alpha,\delta)}m$, for all $m \in G$, then,

$$\begin{aligned}&\{n \in G: r_{B^{\xi}}(n) \geq \alpha, \omega_{B^{\xi}}(n) \geq \delta, 0 \leq \alpha \leq 1, 0 \leq \delta \leq 2\pi\}\\ &= m^{-1}\{n \in G: r_{A^{\xi}}(n) \geq \alpha, \omega_{A^{\xi}}(n) \geq \delta, 0 \leq \alpha \leq 1, 0 \leq \delta \leq 2\pi\}m\\ &= \{m^{-1}nm \in G: r_{B^{\xi}}(n) \geq \alpha, \omega_{B^{\xi}}(n) \geq \delta, 0 \leq \alpha \leq 1, 0 \leq \delta \leq 2\pi\}.\end{aligned}$$

This implies that $x \in B^{\xi}_{(\alpha,\delta)}$ and $m^{-1}nm \in A^{\xi}_{(\alpha,\delta)}$.

This further implies that $\mu_{B^{\xi}}(n) = \mu_{A^{\xi}}(m^{-1}nm)$.

Hence, A^{ξ} and B^{ξ} are CCFCSG to each other.

Example 5.6. The CFSG A and B of D_6 are defined as follows:

$$\mu_A = \begin{cases} 0.99e^{i1.99\pi}, & if\ m = 1, \\ 0.75e^{i1.2\pi}, & if\ m = \{\alpha^3, \beta, \alpha^3\beta\}, \\ 0.5e^{i0.7\pi}, & otherwise, \end{cases}$$

and

$$\mu_B = \begin{cases} 0.99e^{i1.99\pi}, & if\ m = 1, \\ 0.75e^{i1.2\pi}, & if\ m = \{\alpha^3, \alpha\beta, \alpha^4\beta\}, \\ 0.5e^{i0.7\pi}, & otherwise. \end{cases}$$

Then, the CCFSG A^{ξ} and B^{ξ} of D_6 corresponding to the value $\xi = 0.79e^{i1.5\pi}$ are given as

$$\mu_{A^{\xi}} = \begin{cases} 0.79e^{i1.5\pi}, & if\ m = 1, \\ 0.75e^{i1.2\pi}, & if\ m = \{\alpha^3, \beta, \alpha^3\beta\}, \\ 0.5e^{i0.7\pi}, & otherwise, \end{cases}$$

and

$$\mu_{B^{\xi}} = \begin{cases} 0.79e^{i1.5\pi}, & if\ m = 1, \\ 0.75e^{i1.2\pi}, & if\ m = \{\alpha^3, \alpha\beta, \alpha^4\beta\}, \\ 0.5e^{i0.7\pi}, & otherwise. \end{cases}$$

Clearly, $A^{\xi}_{(0.75,1.2\pi)} = \{1, \alpha^3, \beta, \alpha^3\beta\}$ and $B^{\xi}_{(0.75,1.2\pi)} = \{1, \alpha^3, \alpha\beta, \alpha^4\beta\}$.

Now, let $m = \alpha^2\beta$, then,

$$\begin{aligned}B^{\xi}_{(0.75,1.2\pi)} &= (\alpha^2\beta)^{-1}A^{\xi}_{(0.75,1.2\pi)}\alpha^2\beta\\ &= (\alpha^2\beta)^{-1}\{1, \alpha^3, \beta, \alpha^3\beta\}\alpha^2\beta\\ &= \{1, \alpha^3, \alpha\beta, \alpha^4\beta\}.\end{aligned}$$

Hence, $B^{\xi}_{(0.75,1.2\pi)} = m^{-1}A^{\xi}_{(0.75,1.2\pi)}m$. Hence, in view of Theorem 5.4, we have A^{ξ} and B^{ξ} are CCFCSG to each other.

The subsequent result describes the CCF version of Sylow's third theorem.

Theorem 5.5. (Conjunctive complex fuzzification of Sylow's third theorem) Let G be a finite group of order is $p^a b$, where p represents a prime and a, b are positive integers such that p and b are relatively prime. If F is the number of CCF$\mathcal{S}_p$SG with respect to $A^{\xi}_{(\alpha,\delta)}$. Then, the total number T of CCF$\mathcal{S}_p$SG is congruent to $F(mod p)$ and T $|p^a b\psi$.

Proof. According to Sylow's third theorem, the number k of Sylow p-subgroups H of G is a divisor of the order of G and equal to 1 $modulo\ p$. Each such Sylow p-subgroup with A^{ξ^*} will give rise to F number of CCF$\mathcal{S}_p$SG. Therefore, the total number of CCF$\mathcal{S}_p$SG of G is F.k. From this fact and the note preceding the theorem, it is clear that the total number T of CCF$\mathcal{S}_p$SG is congruent to $F(mod p)$ and divides $p^a bF$.

Example 5.7. Consider the alternating group of degree 5 having order 60=$2^2 \times 3.5$. By means of Sylow's third theorem, there are 5 Sylow 2-subgroups of A_5, namely N_1, N_2, N_3, N_4 and N_5. One can easily see from Figure 1 that N_1 gives $F = 16$ CCF$\mathcal{S}_2$SG. Then, T $= 16 \times 5 = 80$ is the total number of CCF$\mathcal{S}_2$SG of A_5. Moreover, it is quite evident that $80 \equiv 16(mod 2)$ and $80|2^2 \times 3.5 \times 16$.

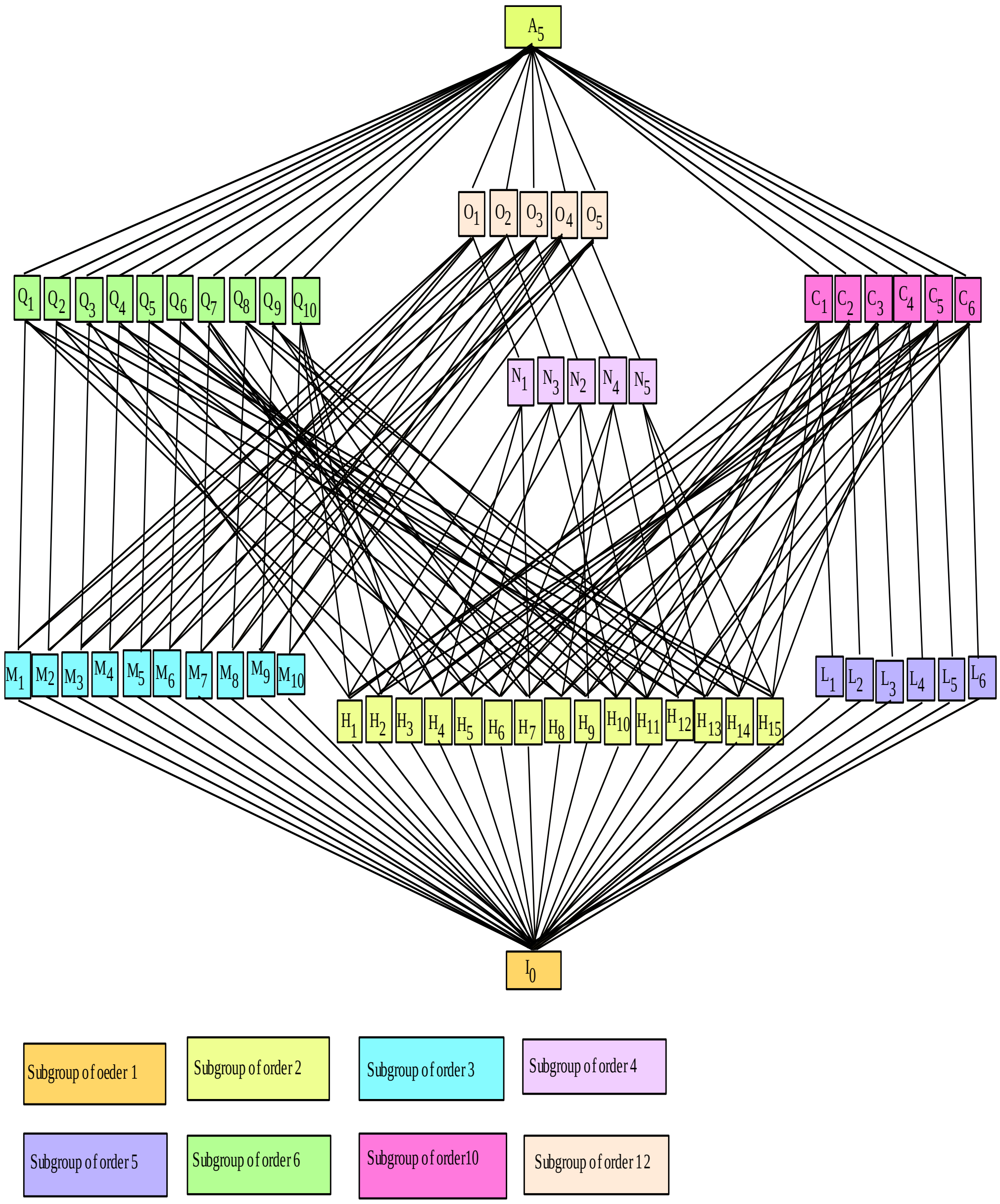

Figure 1. Diagrammatic view of subgroups of A_5.

For $p = 3$ there are 10 Sylow 3-subgroups of A_5, namely, M_1–M_{10}. One can easily see from Figure 1 that M_1 gives $F = 8$ CCF$\mathcal{S}_3$SG. Then, T $= 8 \times 10 = 80$ is the total number of CCF$\mathcal{S}_3$SG of A_5. It is quite evident that $80 \equiv 8(mod 3)$ and $80|2^2 \times 3 \times 5 \times 8$.

Similarly, for $p = 5$ there are 6 Sylow 5-subgroups of A_5, namely, L_1–L_6. One can easily see from Figure 1 that L_1 gives $F = 4$ CCF$\mathcal{S}_5$SG. Thus, T $= 4 \times 6 = 24$. It is quite evident that $24 \equiv 4(mod 5)$ and $24|2^2 \times 3 \times 5 \times 4$.

6. Comparative analysis and limitations of the current work

The paper emphasizes the lack of investigation into CCFSG from a group-theoretic perspective, highlighting the need for this specific analysis. The introduction of the CCF conjugate element and the exploration of various concepts related to CCFSG, including the Cauchy theorem and Sylow theorems, are presented as innovative contributions. FSG is a special case of CCFSG, therefore the results presented in this study are valid for FSG. However, we cannot apply these results directly to complex q-rung orthopair fuzzy subgroups, complex picture fuzzy subgroups, complex spherical fuzzy subgroups and complex fuzzy soft subgroups. Hence, it is essential to conduct specific investigations into these generalized structures. This is the main limitation of our research.

7. Conclusions

The concept of the CCF conjugate element of a CCFSG of a group has been introduced, and many important properties of this idea have been studied in this paper. The idea of the class equation of the CCFSG of a finite group has been initiated. The CCF p-subgroup of the CCFSG has been defined and the study of this ideology has been established by proving many elementary structural attributes of this concept. Moreover, the conjunctive complex fuzzification of the Cauchy theorem of a finite group has been developed in this article. In addition, the phenomena of the CCFCSG and the CCF Sylow p-subgroup have been presented. Furthermore, the study of these ideologies has been extended to propose the conjunctive complex fuzzification of the three Sylow's theorems for a finite group. One of our prime aims in future work will be to address the limitations of this work in an efficient manner by extending this study to more generalized environments of CFS like complex q-rung orthopair fuzzy subgroups, complex picture fuzzy subgroups, complex spherical fuzzy subgroups and complex fuzzy soft subgroups. Moreover, the aim of forthcoming undertakings will be centered on the advancement of a comprehensive decision analysis tool that integrates the linear conjunctive operator. Furthermore, the concept of the CCFSG will be utilized to present a suitable method for tackling the difficulties in the areas of transportation networks, web graphs, model reduction, randomized algorithms, symmetry exploitation, convex optimization, design concept evaluation and the assessment of key engineering characteristics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

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