



Research article

On dominated multivalued operators involving nonlinear contractions and applications

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Abstract: The objective of this research is to establish new results for set-valued dominated mappings that meet the criteria of advanced locally contractions in a complete extended b -metric space. Additionally, we intend to establish new fixed point outcomes for a couple of dominated multi-functions on a closed ball that satisfy generalized local contractions. In this study, we present novel findings for dominated maps in an ordered complete extended b -metric space. Additionally, we introduce a new concept of multi-graph dominated mappings on a closed ball within these spaces and demonstrate some original results for graphic contractions equipped with a graphic structure. To demonstrate the uniqueness of our new discoveries, we verify their applicability in obtaining a joint solution of integral and functional equations. Our findings have also led to modifications of numerous classical and contemporary results in existing research literature.

Keywords: fixed point; advanced locally contraction; ordered extended b -metric space; dominated multi-functions; graph theory; integral equation; functional equation

1. Introduction and basic preliminaries

Fixed point theory is a branch of functional, analysis that focuses on studying mathematical mappings or operators that have at least one point that remains unchanged under their action. It is a popular area of research due to large applications in both applied and pure mathematics, such as contemporary optimization, control theory, numerical analysis, geometry in topology, dynamical systems and modeling. A fixed point theory is a fascinating area of mathematics that is fundamentally important. It serves as an important investigative and detecting tool in many fields. In addition to nonlinear and functional analysis, fixed point (abbreviated as FP) theory aims to advance economics, finance, computer science, and other disciplines in solving difficulties for matrix, integral, and fractional differential equations. This led to the development of the theory of FP as an analytical theory. The Banach FP theorem, the first well-known result in FP theory, was created by well-known mathematician Banach [11]. It is programmed to resolve differential, integral, and functional equations that are both linear and nonlinear in a number of different generalized spaces. The Banach's theorem can be applied in many different ways, each involving a distinct distance space and a separate set of contractive-type conditions that must be satisfied. Bakhtin [12], Czerwik [15,16], Demma et al. [18] and Elhamed et al. [19] investigated various extensions of the Banach's result in a metric and a b -MS. In addition to some new FP conclusions, Wardowski [42] offered a new generalization of Banach's contraction called F -contractions. Moreover, Agarwal et al. [2], Ahmed et al. [3], Alsulami et al. [6], Ameer et al. [9], Aydi et al. [10], Karapinar et al. [24] and Radcharoen et al. [34] showed different extensions of Wardowski's result [42] in different settings of metric spaces.

Nadler [28] developed the concept of set-valued contractive maps and shared his well-known finding that expanded the Banach FP result [11] for multi-valued mappings. Afterward, Acer et al. [1], Ali et al. [5], Altun et al. [8], Feng et al. [20], Jleli et al. [22], Miank et al. [27], Rasham et al. [35], Sgroi et al. [36] and Secelean et al. [40] discussed significant FP results concerning with multivalued mappings.

An extended b -metric space was initially proposed by Kamran et al. [23] who also showed certain FP theorems for self-mappings defined on these spaces. Additionally, Rasham et al. [36] proved FP theorems in a complete b -metric-like space by utilizing set-valued dominated locally F -contractions by employing the first condition of Wardowski's result. Results of FP on a closed ball for F -contractions and related applications regarding the systems of integral equations were established by various authors in [9,36,38].

We ensure the existence of various new generalized FP outcomes satisfying a local contraction on a closed ball defined in a complete extended b -metric space. Also, some new definitions and examples are introduced. Furthermore, we obtained new common FP results for α_* -dominated mappings in a complete extended b -metric space. Illustrative examples are given to validate our new acquired outcomes in which contractive conditions hold only on a closed-ball, but do not exist on the whole space. Moreover, applications for nonlinear systems of integral equations, functional equations and on graph theory are given to show the originality of our obtained outcomes.

Definition 1.1. [22] Let \mathcal{A} be a non-empty set and $s \geq 1$. The function $d: \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ is said a b -metric with coefficient s if the following conditions hold for all $g, x, e \in \mathcal{A}$;

- (i) $d(g, g) = 0$;
- (ii) $d(g, e) = 0 \Rightarrow g = e$;
- (iii) $d(g, e) = d(e, g)$;
- (iv) $d(g, e) \leq s[d(g, x) + d(x, e)]$.

Then, the pair (\mathcal{A}, d) is called a b -metric space, shortly as b -MS.

Example 1.2. [22] Suppose $\mathcal{A} = [0, 1]$. The function $d: \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$ defined by $d(h, t) = |h - t|^2$ for all $h, t \in \mathcal{A}$ is a b -metric with $s = 2$.

Definition 1.3. [22] Let (\mathcal{A}, d) be a b -MS.

- (i) The sequence $\{g_n\}$ in \mathcal{A} is convergent to g if for all $\varepsilon > 0$, there exists $S = S(\varepsilon) \in \mathbb{N}$ such that $d(g_n, g) < \varepsilon$, $\forall n \geq S$.
- (ii) A sequence $\{g_n\}$ in \mathcal{A} is called a Cauchy if for all $\varepsilon > 0$ there exists $S = S(\varepsilon) \in \mathbb{N}$ such that $d(g_n, g_m) < \varepsilon$, for all $n, m \geq S$.
- (iii) A b -metric space is complete if for every Cauchy sequence in \mathcal{A} is convergent to some point in \mathcal{A} .

Definition 1.4. [23] Let \mathcal{A} be a non-empty set and $\theta: \mathcal{A} \times \mathcal{A} \rightarrow [1, \infty)$ be a function. A mapping $d_\theta: \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ is said an extended b -metric if the following assumptions hold for all $x, y, z \in \mathcal{A}$;

- (i) $d_\theta(x, y) = 0$ iff $x = y$;
- (ii) $d_\theta(x, y) = d_\theta(y, x)$;
- (iii) $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

The pair (\mathcal{A}, d_θ) is called an extended b -metric space, shortly as, $EbMS$. Let $h \in \mathcal{A}$ and $r > 0$,

$\overline{B_{d_\theta}(g_0, r)} = \{q \in \mathcal{A} : d_\theta(q, h) \leq r\}$ is called a closed ball in the $EbMS$.

Example 1.5. [26] Let $\mathcal{A} = [0, \infty)$. Define $d_\theta: \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ by

$$d_\theta(h, e) = \begin{cases} 0, & \text{if } h = e; \\ 3, & \text{if } h \text{ or } e \in \{1, 2\}, h \neq e; \\ 5, & \text{if } h \neq e \in \{1, 2\}; \\ 1, & \text{otherwise} \end{cases}$$

Then, (\mathcal{A}, d_θ) is an $EbMS$ where $\theta: \mathcal{A} \times \mathcal{A} \rightarrow [1, \infty)$ is defined b

$$\theta(h, e) = h + e + 1, \text{ for all } h, e \in \mathcal{A}.$$

Definition 1.6. [23] Let (\mathcal{A}, d_θ) be an $EbMS$.

- (i) A sequence $\{g_n\}$ in \mathcal{A} converges to a limit point g if for each $\varepsilon > 0$ there exists $S = S(\varepsilon) \in \mathbb{N}$ such that $d_\theta(g_n, g) < \varepsilon$ for all $n \geq S$.
- (ii) A sequence $\{g_n\}$ in \mathcal{A} is said Cauchy if for all $\varepsilon > 0$ there exists $S = S(\varepsilon) \in \mathbb{N}$ such that $d_\theta(g_n, g_m) < \varepsilon$ for all $n, m \geq S$.
- (iii) If every Cauchy sequence in \mathcal{A} converges to some point $g \in \mathcal{A}$ then (\mathcal{A}, d_θ) is said to be complete.

Remark 1.7. [26] Every b -MS is an $EbMS$ with a constant function $\theta(x, z) = s$ for $s \geq 1$. However, it should be noted that the opposite statement is not always true in a general sense.

Definition 1.8. [36] Let Q be a non-empty subset of \mathcal{A} and there exist an element l in \mathcal{A} . Then, $q \in Q$ is called a best approximation in Q if

$$d_\theta(l, Q) = d_\theta(l, q), \quad \text{where } d_\theta(l, Q) = \inf_{q \in Q} d_\theta(l, q).$$

Here, $\mathcal{P}(\mathcal{A})$ represents the collection of all subsets of \mathcal{A} that are compact.

Let ψ represent the collection of all non decreasing functions $\psi: [0, +\infty) \rightarrow [0, +\infty)$ for which $\sum_{k=1}^{+\infty} \psi^k(h) < +\infty$ and $\psi(h) < h$, where ψ^k denotes the k^{th} iterative term of ψ .

Definition 1.9. [37] Suppose $H_{d_\theta}: \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A}) \rightarrow \mathbb{R}^+$ is a function, defined by

$$H_{d_\theta}(M, N) = \max \left\{ \sup_{e \in M} (e, N), \sup_{f \in N} (M, f) \right\} \text{ for all } M, N \in \mathcal{P}(\mathcal{A}).$$

Then, H_{d_θ} is called a Pompeiu-Hausdorff *EbM* on $\mathcal{P}(\mathcal{A})$.

Definition 1.10. [37] Let \mathcal{A} be a non-empty set $K \subseteq \mathcal{A}$ and $\alpha: \mathcal{A} \times \mathcal{A} \rightarrow [0, +\infty)$. A mapping $S: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ satisfying

$$\alpha_*(Sg, Sh) = \inf\{\alpha(t, z): t \in Sg, z \in Sh\} \geq 1, \text{ whenever } \alpha(t, z) \geq 1, \text{ for all } t, z \in \mathcal{A}$$

is called α_* -admissible. The mapping $S: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ satisfying $\alpha_*(a, Sa) = \inf\{\alpha(a, h): h \in Sa\} \geq 1$ is said to be α_* -dominated on K .

Definition 1.11. [42] Let (\mathcal{A}, d) be a metric space. A function $L: \mathcal{A} \rightarrow \mathcal{A}$ is known as an \mathcal{F} -contraction if there exists $\tau > 0$ such that for each $\gamma, x \in \mathcal{A}$ with $d(L(\gamma), L(x)) > 0$, the following inequality holds:

$$\tau + \mathcal{F}(d(L(\gamma), L(x))) \leq \mathcal{F}(d(\gamma, x)).$$

Here, the function $\mathcal{F}: \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies the following assumptions:

(F1) \mathcal{F} is a strictly-increasing function;

(F2) $\lim_{i \rightarrow +\infty} \delta_i = 0$ if and only if $\lim_{i \rightarrow +\infty} \mathcal{F}(\delta_i) = -\infty$, for every positive sequence $\{\delta_i\}_{i=1}^{\infty}$;

(F3) For each $h \in (0, 1)$, $\lim_{j \rightarrow \infty} \delta_j^h \mathcal{F}(\delta_j) = 0$.

Example 1.12. [37] Let \mathcal{A} be a non-empty set and the function $\alpha: \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be given by

$$\alpha(c, q) = \begin{cases} 1 & \text{if } c > q, \\ \frac{1}{4} & \text{if } c \leq q. \end{cases}$$

Consider the mappings $G, R: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ defined as $Gr = [-4 + r, -3 + r]$ and $Rt = [-2 + t, -1 + t]$, respectively. Then G and R are α_* -dominated, but they are not α_* -admissible.

Lemma 1.13. Let (\mathcal{A}, d_θ) be an *EbMS* and $(\mathcal{P}(\mathcal{A}), H_{d_\theta})$ be an extended Hausdorff *b*-MS on $\mathcal{P}(\mathcal{A})$. Then, for all $U, W \in \mathcal{P}(\mathcal{A})$ and for each $u \in U$ such that $d_\theta(u, W) = d_\theta(u, h_u)$ where $h_u \in W$, the following holds:

$$H_{d_\theta}(U, W) \geq d_\theta(u, h_u).$$

Proof. If $H_{d_\theta}(U, W) = \sup_{u \in U} d_\theta(u, W)$, then $H_{d_\theta}(U, W) \geq d_\theta(u, h_u)$ for all $u \in U$. Since W is a

proximal set, for any $u \in \mathcal{A}$ there exists at least one element h_u in W that provides the best approximation to u and satisfies $d_\theta(u, W) = d_\theta(u, h_u)$. Now, we have $H_{d_\theta}(U, W) \geq d_\theta(u, h_u)$.

One writes

$$H_{d_\theta}(U, W) = \sup_{h \in W} d_\theta(U, h) \geq \sup_{u \in U} d_\theta(u, W) \geq d_\theta(u, h_u).$$

Hence, it is proved.

We will now present the key findings of research.

2. Main results

Let (\mathcal{A}, d_θ) be an *EbMS* with a function $\theta: \mathcal{A} \times \mathcal{A} \rightarrow [1, \infty)$. Let S and T be two multi-maps from \mathcal{A} to $\mathcal{P}(\mathcal{A})$. Let $g_1 \in S(g_0)$ so that $d_\theta(g_0, S(g_0)) = d_\theta(g_0, g_1)$. Let $g_2 \in T(g_1)$ be such that $d_\theta(g_1, T(g_1)) = d_\theta(g_1, g_2)$. By following this process, we obtain a sequence of sets $\{TS(g_n)\}$ in \mathcal{A} where $g_{2n+1} \in S(g_{2n})$ such that $g_{2n+2} \in T(g_{2n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Also, $d_\theta(g_{2n}, S(g_{2n})) = d_\theta(g_{2n}, g_{2n+1})$ and $d_\theta(g_{2n+1}, T(g_{2n+1})) = d_\theta(g_{2n+1}, g_{2n+2})$, then $\{TS(g_n)\}$ is a sequence in \mathcal{A} produced by g_0 . We mean by $x, y \in \{u\}$ that $x = u$ and $y = u$, define $D_\theta(x, y)$ by

$$D_\theta(x, y) = \max \left\{ d_\theta(x, y), d_\theta(x, S(x)), d_\theta(y, T(y)), \frac{d_\theta(x, S(x))d_\theta(x, T(y))}{1 + d_\theta(x, y)} \right\}.$$

Theorem 2.1. Let (\mathcal{A}, d_θ) be a complete *EbMS* with function $\theta: \mathcal{A} \times \mathcal{A} \rightarrow [1, \infty)$. Let $r > 0$, $g_0 \in \overline{B_{d_\theta}(g_0, r)} \subseteq \mathcal{A}$, $\alpha: \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ and $S, T: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ be semi α_* -dominated multivalued mappings on $\overline{B_{d_\theta}(g_0, r)}$. Suppose there are $\psi_\theta \in \Psi$, a constant $\tau > 0$, and \mathcal{F} a strictly increasing function, such that the following conditions hold:

$$i) \quad \tau + \mathcal{F}(H_{d_\theta}(S(x), T(y))) \leq \mathcal{F}(\psi_\theta(D_\theta(x, y))), \quad (2.1)$$

where $x, y \in \overline{B_{d_\theta}(g_0, r)} \cap \{TS(g_n)\}$, $\alpha(x, y) > 1$ and $H_{d_\theta}(S(x), T(y)) > 0$;

$$ii) \quad \sum_{i=0}^j \psi_\theta^i(d_\theta(g_0, S(g_0))) \prod_{i=0}^j \theta((g_0, g_{i+1})) \leq r. \quad (2.2)$$

where $\{TS(g_n)\}$ is a sequence in $\overline{B_{d_\theta}(g_0, r)}$, $\alpha(g_n, g_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$ and

$$\{TS(g_n)\} \rightarrow u \in \overline{B_{d_\theta}(g_0, r)}.$$

iii) (2.1) holds for $x, y \in \{u\}$, either $\alpha(g_n, u) \geq 1$ or $(u, g_n) \geq 1$, for all naturals.

Then, S, T have a mutual FP u in $\overline{B_{d_\theta}(g_0, r)}$.

Proof. Let $\{TS(g_n)\}$ be a sequence. From (2.2), we obtain

$$d_{\theta}(g_0, g_1) \leq \sum_{i=0}^j \psi_{\theta}^i(d_{\theta}(g_0, S(g_0))) \prod_{i=0}^j \theta((g_0, g_{i+1})) \leq r.$$

This implies that $g_1 \in \overline{B_{d_{\theta}}(g_0, r)}$. Let $g_2, \dots, g_j \in \overline{B_{d_{\theta}}(g_0, r)}$ for some $j \in \mathbb{N}$. If j is odd, then $j = 2i + 1$ for some $i \in \mathbb{N}$. As $S, T: A \rightarrow P(A)$ are semi α_* -dominated maps on $\overline{B_{d_{\theta}}(g_0, r)}$, thus $\alpha_*(g_{2i}, Sg_{2i}) \geq 1$. In addition, $\alpha_*(g_{2i+1}, Tg_{2i+1}) \geq 1$. As $\alpha_*(g_{2i}, Sg_{2i}) \geq 1$, this implies that $\inf\{\alpha(g_{2i}, l): l \in Sg_{2i}\} \geq 1$ and $g_{2i+1} \in Sg_{2i}$, with $\alpha(g_{2i}, g_{2i+1}) \geq 1$. Now, by using Lemma 1.13, we have

$$\begin{aligned} \tau + \mathcal{F}(d_{\theta}(g_{2i+1}, g_{2i+2})) &\leq \tau + \mathcal{F}(H_{d_{\theta}}(Sg_{2i}, Tg_{2i+1})) \leq \mathcal{F}(\psi_{\theta}(D_{\theta}(g_{2i}, g_{2i+1}))), \\ \tau + \mathcal{F}(d_{\theta}(g_{2i+1}, g_{2i+2})) &\leq \mathcal{F}\left[\psi_{\theta}\left(\max\left\{d_{\theta}(g_{2i}, g_{2i+1}), d_{\theta}(g_{2i}, Sg_{2i}), \right.\right.\right. \\ &\quad \left.\left.\left. d_{\theta}(g_{2i+1}, Tg_{2i+1}), \frac{d_{\theta}(g_{2i}, Sg_{2i})d_{\theta}(g_{2i+1}, Tg_{2i+1})}{1+d_{\theta}(g_{2i}, g_{2i+1})}\right\}\right)\right], \\ &\leq \mathcal{F}\left[\psi_{\theta}\left(\max\left\{d_{\theta}(g_{2i}, g_{2i+1}), d_{\theta}(g_{2i}, g_{2i+1}), \right.\right.\right. \\ &\quad \left.\left.\left. d_{\theta}(g_{2i+1}, g_{2i+2}), \frac{d_{\theta}(g_{2i}, g_{2i+1})d_{\theta}(g_{2i+1}, g_{2i+2})}{1+d_{\theta}(g_{2i}, g_{2i+1})}\right\}\right)\right], \\ &\leq \mathcal{F}[\psi_{\theta}(\max\{d_{\theta}(g_{2i}, g_{2i+1}), d_{\theta}(g_{2i+1}, g_{2i+2})\})]. \end{aligned}$$

If $\max\{d_{\theta}(g_{2i}, g_{2i+1}), d_{\theta}(g_{2i+1}, g_{2i+2})\} = d_{\theta}(g_{2i+1}, g_{2i+2})$, then

$$\tau + \mathcal{F}(d_{\theta}(g_{2i+1}, g_{2i+2})) \leq \mathcal{F}(\psi_{\theta}(d_{\theta}(g_{2i+1}, g_{2i+2}))).$$

Since \mathcal{F} is strictly increasing, we have

$$d_{\theta}(g_{2i+1}, g_{2i+2}) < \psi_{\theta}(d_{\theta}(g_{2i+1}, g_{2i+2})).$$

This not true due to the fact $\psi_{\theta}(u) < u$. So

$$\max\{d_{\theta}(g_{2i}, g_{2i+1}), d_{\theta}(g_{2i+1}, g_{2i+2})\} = d_{\theta}(g_{2i}, g_{2i+1}).$$

Hence, we have

$$d_{\theta}(g_{2i+1}, g_{2i+2}) < \psi_{\theta}(d_{\theta}(g_{2i}, g_{2i+1})). \quad (2.3)$$

As $\alpha_*(g_{2i-1}, Sg_{2i-1}) \geq 1$ and $g_{2i} \in Sg_{2i-1}$, $\alpha(g_{2i-1}, Sg_{2i}) \geq 1$. Now, by applying Lemma 1.13, we get

$$\tau + \mathcal{F}(d_{\theta}(g_{2i}, g_{2i+1})) \leq \tau + \mathcal{F}(H_{d_{\theta}}(Sg_{2i-1}, Tg_{2i})) \leq \mathcal{F}(\psi_{\theta}(D_{\theta}(g_{2i-1}, g_{2i}))),$$

$$\begin{aligned} \tau + \mathcal{F}(d_\theta(g_{2i}, g_{2i+1})) &\leq \mathcal{F} \left[\psi_\theta \left(\max \left\{ \begin{array}{l} d_\theta(g_{2i-1}, g_{2i})d_\theta(g_{2i-1}, Sg_{2i-1}), \\ d_\theta(g_{2i}, Tg_{2i}), \frac{d_\theta(g_{2i-1}, Sg_{2i-1}) \cdot d_\theta(g_{2i}, Tg_{2i})}{1+d_\theta(g_{2i-1}, g_{2i})} \end{array} \right\} \right) \right], \\ &\leq \mathcal{F} \left[\psi_\theta \left(\max \left\{ \begin{array}{l} d_\theta(g_{2i-1}, g_{2i}), d_\theta(g_{2i-1}, g_{2i}), \\ d_\theta(g_{2i}, g_{2i+1}), \frac{d_\theta(g_{2i-1}, g_{2i}) \cdot d_\theta(g_{2i}, g_{2i+1})}{1+d_\theta(g_{2i-1}, g_{2i})} \end{array} \right\} \right) \right], \\ &\leq \mathcal{F}[\psi_\theta(\max\{d_\theta(g_{2i-1}, g_{2i}), d_\theta(g_{2i}, g_{2i+1})\})]. \end{aligned}$$

If $\max\{d_\theta(g_{2i-1}, g_{2i}), d_\theta(g_{2i}, g_{2i+1})\} = d_\theta(g_{2i}, g_{2i+1})$, then

$$\tau + \mathcal{F}(d_\theta(g_{2i}, g_{2i+1})) \leq \mathcal{F}(\psi_\theta(d_\theta(g_{2i}, g_{2i+1}))).$$

Since \mathcal{F} is strictly increasing, we have

$$d_\theta(g_{2i}, g_{2i+1}) < \psi_\theta(d_\theta(g_{2i}, g_{2i+1})).$$

This is a contradiction due to the fact $\psi_\theta(u) < u$. Hence, we get

$$d_\theta(g_{2i}, g_{2i+1}) < \psi_\theta(d_\theta(g_{2i-1}, g_{2i})). \quad (2.4)$$

As ψ_θ is non-decreasing,

$$\psi_\theta(d_\theta(g_{2i}, g_{2i+1})) < \psi_\theta(\psi_\theta(d_\theta(g_{2i-1}, g_{2i}))).$$

By using above inequality in (2.3), we deduce that

$$d_\theta(g_{2i+1}, g_{2i+2}) < \psi_\theta^2(d_\theta(g_{2i-1}, g_{2i})).$$

Ongoing this process, we get,

$$d_\theta(g_{2i+1}, g_{2i+2}) < \psi_\theta^{2k+1}(d_\theta(g_0, g_1)). \quad (2.5)$$

Instead if $j = 2k$, where $k = 1, 2, 3, \dots, \frac{j}{2}$, by following the same procedure and using (2.4), we get the given inequality as

$$d_\theta(g_{2i+1}, g_{2i+2}) < \psi_\theta^{2k}(d_\theta(g_0, g_1)). \quad (2.6)$$

Now, (2.5) and (2.6) collectively expressed as

$$d_\theta(g_j, g_{j+1}) < \psi_\theta^j(d_\theta(g_0, g_1)) \text{ for all } j \in \mathbb{N}. \quad (2.7)$$

Now, by using triangular inequality of *EbMS* also using (2.7), we have

$$\begin{aligned}
d_\theta(g_0, g_{j+1}) &\leq \theta(g_0, g_{j+1})d_\theta(g_0, g_1) + \theta(g_0, g_{j+1})\theta(g_1, g_{j+1})d_\theta(g_1, g_2) \\
&+ \cdots + \theta(g_0, g_{j+1})\theta(g_2, g_{j+1})\theta(g_3, g_{j+1}) \cdots \theta(g_{j-1}, g_{j+1})\theta(g_j, g_{j+1})d_\theta(g_j, g_{j+1}), \\
d_\theta(g_0, g_{j+1}) &\leq d_\theta(g_0, g_1)[\theta(g_0, g_{j+1}) \cdots \theta(g_{j-4}, g_{j+1})\theta(g_{j-4}, g_{j+1}) \\
&+ \theta(g_0, g_{j+1})\theta(g_1, g_{j+1}) \cdots \theta(g_{j-3}, g_{j+1})\theta(g_{j-2}, g_{j+1})\psi_\theta + \cdots + \\
&+ \theta(g_0, g_{j+1})\theta(g_1, g_{j+1}) \cdots \theta(g_{j-3}, g_{j+1})\theta(g_{j-2}, g_{j+1})\psi_\theta^j] \\
d_\theta(g_0, g_{j+1}) &\leq \sum_{i=0}^j \psi_\theta^i(d_\theta(g_0, S(g_0))) \prod_{i=0}^j \theta(g_0, g_{i+1}) \leq r.
\end{aligned}$$

Thus, $g_{j+1} \in \overline{B_{d_\theta}(g_0, r)}$. Hence, $g_n \in \overline{B_{d_\theta}(g_0, r)}$ for all $n \in \mathbb{N}$. Consequently, the sequence $\{TS(g_n)\} \rightarrow u \in \overline{B_{d_\theta}(g_0, r)}$. As $S, T: A \rightarrow P(A)$ are semi α_* -dominated maps on $\overline{B_{d_\theta}(g_0, r)}$, thus $\alpha_*(g_{2n}, Sg_{2n}) \geq 1$ and $\alpha_*(g_{2n+1}, Tg_{2n+1}) \geq 1$. This implies that $\alpha(g_n, g_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now, the Inequality (2.7) can be written as

$$d_\theta(g_n, g_{n+1}) < \psi_\theta^n(d_\theta(g_0, g_1)) \text{ for all } n \in \mathbb{N}. \quad (2.8)$$

By using the triangular inequality of the *EbMS* and (2.8), for $m > n$, we deduce that

$$\begin{aligned}
d_\theta(g_n, g_m) &\leq \theta(g_n, g_m)\psi_\theta^n(d_\theta(g_0, g_1)) + \theta(g_n, g_m)\theta(g_{n+1}, g_m)\psi_\theta^{n+1}(d_\theta(g_0, g_1)) \\
&+ \cdots + \theta(g_n, g_m)\theta(g_{n+1}, g_m)\theta(g_{n+2}, g_m) \cdots \theta(g_{m-2}, g_m)\theta(g_{m-1}, g_m)\psi_\theta^{m-1}(d_\theta(g_0, g_1)), \\
d_\theta(g_n, g_m) &\leq d_\theta(g_0, g_1)[\theta(g_1, g_m)\theta(g_2, g_m) \cdots \theta(g_{n-1}, g_m)\theta(g_n, g_m)\psi_\theta^n \\
&+ \theta(g_1, g_m)\theta(g_2, g_m) \cdots \theta(g_n, g_m)\theta(g_{n+1}, g_m)\psi_\theta^{n+1} + \cdots + \\
&+ \theta(g_1, g_m)\theta(g_2, g_m) \cdots \theta(g_n, g_m)\theta(g_{n+1}, g_m)\psi_\theta^{m-1}].
\end{aligned}$$

Since $\lim_{n, m \rightarrow \infty} \theta(g_n, g_m)\psi_\theta < 1$, the series $\sum_{n=0}^{\infty} \psi_\theta^n \prod_{i=1}^n \theta(g_i, g_m)$ converges for all $m \in \mathbb{N}$, by applying the ratio test, put

$$\Omega = \sum_{n=1}^{\infty} \psi_\theta^n \prod_{i=1}^n \theta(g_i, g_m), \quad \Omega_n = \sum_{j=1}^n \psi_\theta^j \prod_{i=1}^j \theta(g_i, g_m).$$

Therefore for $m > n$, the inequality mentioned above implies that

$$d_\theta(g_n, g_m) \leq d_\theta(g_0, g_1)[\Omega_{m-1} - \Omega_n].$$

Letting $n \rightarrow \infty$, we achieve that the sequence $\{TS(g_n)\}$ is Cauchy in $(\overline{B_{d_\theta}(g_0, r)}, d_\theta)$. As $\overline{B_{d_\theta}(g_0, r)}$ is a subspace of a complete *EbM*, hence, $\overline{B_{d_\theta}(g_0, r)}$ is also complete, and so there exists a limit point u in $\overline{B_{d_\theta}(g_0, r)}$ such that the sequence $\{TS(g_n)\}$ converges to u when $n \rightarrow \infty$, that is,

$$\lim_{n \rightarrow \infty} d_\theta(g_n, u) = 0. \quad (2.9)$$

Now, by using Lemma 1.13 and triangular inequality of *EbMS*, one writes

$$\begin{aligned} d_\theta(u, Tu) &\leq \theta(u, Tu)[d_\theta(u, g_{2n+1}) + d_\theta(g_{2n+1}, Tu)] \\ &\leq \theta(u, Tu)d_\theta(u, g_{2n+1}) + \theta(u, Tu)H_{d_\theta}(Sg_{2n}, Tu). \end{aligned}$$

By supposition, $\alpha(g_n, u) \geq 1$. Assume that $d_\theta(u, Tu) > 0$, then there exists a positive integer s so that $d_\theta(g_n, Tu) > 0$ for all $n \geq s$. For $n \geq s$, we have

$$d_\theta(u, Tu) < \theta(u, Tu) \left[d_\theta(u, g_{2n+1}) + \psi_\theta \left(\max \left\{ d_\theta(g_{2n}, u), d_\theta(g_{2n}, Sg_{2n}), d_\theta(u, Tu), \frac{d_\theta(g_{2n}, Sg_{2n})d_\theta(g_{2n}, Tu)}{1 + d_\theta(g_{2n}, u)} \right\} \right) \right].$$

Letting $n \rightarrow \infty$, and applying the Inequality (2.9), we obtain

$$d_\theta(u, Tu) < \theta(u, Tu)\psi_\theta(d_\theta(u, Tu)) < d_\theta(u, Tu).$$

This leads to a contradiction. Consequently, our assumption is false. Therefore, $d_\theta(u, Tu) = 0$ and so $u \in Tu$. Furthermore, using Lemma 1.13 and (2.9) we can prove that $u \in Su$. Hence, S and T have a common multi FP u in $\overline{B_{d_\theta}(g_0, r)}$.

Theorem 2.2. Let (\mathcal{A}, d_θ) be a complete *EbMS* with a function $\theta: \mathcal{A} \times \mathcal{A} \rightarrow [1, \infty)$ and $S: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ be a multi-map. Suppose that for some $\psi_\theta \in \Psi$ with constant $\tau > 0$ and \mathcal{F} a strictly increasing function satisfying the following:

$$\tau + \mathcal{F}(H_{d_\theta}(S(x), S(y))) \leq \mathcal{F}(\psi_\theta(D_\theta(x, y))), \quad (2.10)$$

where $x, y \in \{TS(g_n)\}$ and $H_{d_\theta}(S(x), T(y)) > 0$. Then, $\{TS(g_n)\} \rightarrow u \in \mathcal{A}$ and S has a common FP u in \mathcal{A} .

Definition 2.3. Let \mathcal{A} be a non-empty set, \preceq be a partial order on \mathcal{A} and $G \subseteq \mathcal{A}$. We assume that $y \preceq L$ for each $x \in L$, we have $y \preceq x$. Functions $S, T: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ are said dominated on G if $y \preceq Sy$ and Ty for all $y \in G \subseteq \mathcal{A}$. If $G = \mathcal{A}$, then $S, T: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ are totally dominated.

Now, we present the result for a hybrid coupled multivalued dominated maps on $\overline{B_{d_\theta}(g_0, r)}$ in a complete *EbMS*.

Theorem 2.4. Let $(\mathcal{A}, \preceq, d_\theta)$ be an ordered complete *EbMS* with a function $\theta: \mathcal{A} \times \mathcal{A} \rightarrow [1, \infty)$. Let $r > 0$, $g_0 \in \overline{B_{d_\theta}(g_0, r)} \subseteq \mathcal{A}$ and $S, T: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ be multi dominated mappings on $\overline{B_{d_\theta}(g_0, r)}$. Suppose that there are $\psi_\theta \in \Psi$, a constant $\tau > 0$, and a strictly increasing function \mathcal{F} satisfying the following:

$$i) \quad \tau + \mathcal{F}\left(H_{d_\theta}(S(x), T(y))\right) \leq \mathcal{F}\left(\psi_\theta(D_\theta(x, y))\right), \quad (2.11)$$

where $x, y \in \overline{B_{d_\theta}(g_0, r)} \cap \{TS(g_n)\}$, $y \preceq x$ and $H_{d_\theta}(S(x), T(y)) > 0$;

$$ii) \quad \sum_{i=0}^j \psi_\theta^i(d_\theta(g_0, S(g_0))) \prod_{i=0}^j \theta((g_0, g_{i+1})) \leq r. \quad (2.12)$$

where $\{TS(g_n)\}$ is a sequence in $\overline{B_{d_\theta}(g_0, r)}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\theta > 1$. Then, $\{TS(g_n)\} \rightarrow u \in \overline{B_{d_\theta}(g_0, r)}$. Also, if (2.11) holds for $x, y \in \{u\}$, either $g_n \preceq u$ or $u \preceq g_n$ for all naturals, then, S, T have a mutual FP u in $\overline{B_{d_\theta}(g_0, r)}$.

Proof. Let $\alpha: \mathcal{A} \times \mathcal{A} \rightarrow [0, +\infty)$ be a map defined by $\alpha(x, y) = 1$ for each $x \in \overline{B_{d_\theta}(g_0, r)}$, $x \preceq y$ and $\alpha(x, y) = 0$ otherwise. S and T are multi dominated maps on $\overline{B_{d_\theta}(g_0, r)}$, so $x \preceq S(x)$ and $y \preceq T(y)$ for every $x \in \overline{B_{d_\theta}(g_0, r)}$. This shows that $x \preceq z$ for each $z \in S(x)$ and $x \preceq p$ for each $p \in T(y)$. So, $\alpha(x, z) = 1$ for every $z \in S(x)$ and $\alpha(x, p) = 1$ for each $p \in T(y)$. This signifies that $\inf\{\alpha(x, y): y \in S(x)\} = 1$ and $\inf\{\alpha(x, y): y \in T(y)\} = 1$. So, $\alpha_*(x, S(x)) = 1$, $\alpha_*(x, T(y)) = 1$ for every $x \in \overline{B_{d_\theta}(g_0, r)}$. So, $S, T: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ are α_* -dominated multi maps on $\overline{B_{d_\theta}(g_0, r)}$. Moreover, (2.12) can be written as

$$\tau + \mathcal{F}(H_{d_\theta}(S(x), T(y))) \leq \mathcal{F}(\psi_\theta(D_\theta(x, y))),$$

for all $x, y \in \overline{B_{d_\theta}(g_0, r)} \cap \{TS(g_n)\}$, $\alpha(x, y) \geq 1$. Also inequality (2.11) holds. Then from Theorem 2.1, we have $\{TS(g_n)\}$ is a sequence in $\overline{B_{d_\theta}(g_0, r)}$ and $\{TS(g_n)\} \rightarrow u \in \overline{B_{d_\theta}(g_0, r)}$. Now, $g_n, u \in \overline{B_{d_\theta}(g_0, r)}$ and either $g_n \preceq u$ or $u \preceq g_n$ signifies that either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$. Consequently, all conditions of Theorem 2.1 hold. Hence, both the maps S and T have a common multi FP u in $\overline{B_{d_\theta}(g_0, r)}$.

We left with the result without using the condition of closed balls in an ordered complete *EbMS*. In the upcoming result, by using single multi-maps which are only defined on the whole

space instead on a closed ball, we present the given result.

Theorem 2.5. Let $(\mathcal{A}, \preceq, d_\theta)$ be an ordered complete *EbMS* with a function $\theta: \mathcal{A} \times \mathcal{A} \rightarrow [1, \infty)$. Let $S, T: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ be the multi dominated maps on \mathcal{A} . Suppose that there are $\psi_\theta \in \Psi$, a constant $\tau > 0$ and \mathcal{F} a strictly increasing function satisfying the following:

$$\tau + \mathcal{F}(H_{d_\theta}(S(x), T(y))) \leq \mathcal{F}(\psi_\theta(D_\theta(x, y))), \quad (2.13)$$

where $x, y \in \{TS(g_n)\}$, $y \preceq x$. Then, for all $n \in \mathbb{N} \cup \{0\}$, $\{TS(g_n)\} \rightarrow u \in \mathcal{A}$. If (2.13) sustains for u and either $g_n \preceq u$ or $u \preceq g_n$ for all naturals, then, u is the mutual FP of S, T .

Example 2.6. Let $A = \mathbb{R}^+ \cup \{0\}$ and $d_\theta: A \times A \rightarrow [0, \infty)$ be a complete *EbMS* defined as

$$d_\theta(x, y) = (x - y)^2 \text{ for all } x, y \in A,$$

where $\theta: A \times A \rightarrow [1, \infty)$ is given as $\theta(x, y) = 2 > 1$. Define $S, T: A \times A \rightarrow P(A)$ by

$$S(k) = \begin{cases} \left[\frac{k}{3}, \frac{2k}{3} \right] & \text{if } k \in \left[0, \frac{13}{3} \right] \cap A \\ [k, k + 1] & \text{if } k \in \left(\frac{13}{3}, \infty \right) \cap A \end{cases},$$

and

$$T(w) = \begin{cases} \left[\frac{w}{4}, \frac{3w}{4} \right] & \text{if } w \in \left[0, \frac{13}{3} \right] \cap A \\ [w + 1, w + 3] & \text{if } w \in \left(\frac{13}{3}, \infty \right) \cap A \end{cases}.$$

Suppose that $k_0 = \frac{1}{3}$ and $r = 16$, then $\overline{B_{d_\theta}(k_0, r)} = \left[0, \frac{13}{3} \right] \cap A$. Now, we define $d_\theta(k_0, S(k_0)) =$

$d_\theta\left(\frac{1}{3}, \left[\frac{1}{9}, \frac{2}{9}\right]\right) = d_\theta\left(\frac{1}{3}, \frac{1}{9}\right) = \frac{4}{81}$. By doing so, we derive a sequence $\{TS(k_n)\} = \left\{\frac{1}{3}, \frac{1}{9}, \frac{1}{36}, \dots\right\}$ in A

generated by k_0 . Let $\psi_\theta(t) = \frac{4}{5}t$ and $a = 1$. Define $\alpha: A \times A \rightarrow [0, \infty)$ by

$$\alpha(k, w) = \begin{cases} 1 & \text{if } k > w, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Now if $k, w \in \overline{B_{d_\theta}(k_0, r)} \cap \{TS(k_n)\}$ with $\alpha(k, w) \geq 1$, we have

$$H_{d_\theta}(S(k), T(w)) = \max \left\{ \sup_{\alpha \in S(k)} d_\theta(\alpha, T(w)), \sup_{b \in T(w)} d_\theta(S(k), b) \right\},$$

$$= \max \left\{ \sup_{\alpha \in S(k)} d_\theta\left(\alpha, \left[\frac{w}{4}, \frac{3w}{4}\right]\right), \sup_{b \in T(w)} d_\theta\left(\left[\frac{k}{3}, \frac{2k}{3}\right], b\right) \right\},$$

$$= \max \left\{ \sup_{\alpha \in S(k)} d_\theta\left(\frac{2k}{3}, \left[\frac{w}{4}, \frac{3w}{4}\right]\right), \sup_{b \in T(w)} d_\theta\left(\left[\frac{k}{3}, \frac{2k}{3}\right], \frac{3w}{4}\right) \right\},$$

$$\begin{aligned}
&= \max \left\{ d_{\theta} \left(\frac{2k}{3}, \frac{w}{4} \right), d_{\theta} \left(\frac{k}{3}, \frac{3w}{4} \right) \right\}, \\
&= \max \left\{ \left(\frac{2k}{3} - \frac{w}{4} \right)^2, \left(\frac{k}{3} - \frac{3w}{4} \right)^2 \right\}, \\
&\leq \psi_{\theta} \left(\max \left\{ \frac{(k-w)^2, \left(\frac{2k}{3} \right)^2, \left(\frac{3w}{4} \right)^2}{k^2 w^2} \right\}, \frac{4\{1 + (k-w)^2\}}{4\{1 + (k-w)^2\}} \right), \\
&\leq \psi_{\theta}(D_{\theta}(k, w)).
\end{aligned}$$

This means that for $\tau \in (0, \frac{12}{95}]$ and for a strictly increasing function $\mathcal{F}(s) = \ln s$, we have

$$\tau + \mathcal{F}(H_{d_{\theta}}(S(k), T(w))) \leq \mathcal{F}(\psi_{\theta}(D_{\theta}(k, w))).$$

Note $\alpha(6,5) \geq 1$. But, we get

$$\tau + \mathcal{F}(H_{d_{\theta}}(S(6), T(5))) > \mathcal{F}(\psi_{\theta}(D_{\theta}(k, w))).$$

So condition (2.1) is not fulfilled on A . Furthermore, for all $i \in \mathbb{N} \cup \{0\}$,

$$\sum_{i=0}^j \psi_{\theta}^i(d_{\theta}(g_0, S(g_0))) \prod_{i=0}^j \theta((g_0, g_{i+1})) = \frac{4}{81} \times 2 \sum_{i=0}^n \left(\frac{4}{5}\right)^i < 16 \leq r.$$

Hence, S and T satisfy all the restrictions of Theorem 2.1 for $k, w \in \overline{B_{d_{\theta}}(k_0, r)} \cap \{TS(k_n)\}$ with $\alpha(k, w) \geq 1$. Therefore S, T have a mutual FP.

3. Results for graph theory

Existence of FP results for multi graph dominated mappings of Theorem 2.1 will be demonstrated in this section. In a metric space with a graph, Jachymski [21] obtained a significant conclusion regarding the contraction mappings. FP results for graph contractions were reported by Hussain et al. [14], Rasham et al. [37,38].

Definition 3.1 Let $\mathcal{A} \neq \{\varphi\}$ and $G = (\mathcal{E}(G), \mathcal{C}(G))$ be a graph so that $V(G) = \mathcal{A}, B \subseteq \mathcal{A}$. A multi-map $S: \mathcal{A} \rightarrow P(\mathcal{A})$ is said as a multi graph dominated on B if $(x, y) \in \mathcal{C}(G)$, for each $y \in S(x)$ and $y \in \mathcal{A}$.

Theorem 3.2 Let (A, d_{θ}) be a complete *EbMS* with a function $\theta: A \times A \rightarrow [1, \infty]$. Let $r > 0$, $f_0 \in \overline{B_{d_{\theta}}(f_0, r)} \subseteq A$. Suppose the following restrictions are fulfilled:

(i) $S, T: A \rightarrow P(A)$ are multi- graph dominated maps on $\overline{B_{d_{\theta}}(f_0, r)} \cap \{TS(f_n)\}$.

(ii) There exist $\tau > 0$ and a strictly increasing function \mathcal{F} satisfying the following:

$$\tau + \mathcal{F}(H_{d_\theta}(S(x), T(y))) \leq \mathcal{F}(\psi_\theta(D_\theta(x, y))), \quad (3.1)$$

where $x, y \in \overline{B_{d_\theta}(f_0, r)} \cap \{TS(f_n)\}$, $(x, y) \in \mathcal{C}(G)$ and $H_{d_\theta}(S(x), T(y)) > 0$.

$$(iii) \sum_{i=0}^j \psi_\theta^i(d_\theta(g_0, S(g_0))) \prod_{i=0}^j \theta((g_0, g_{i+1})) \leq r, \quad (3.2)$$

where $\{TS(f_n)\}$ is a sequence in $\overline{B_{d_\theta}(f_0, r)}$, for each $n \in \mathbb{N} \cup \{0\}$.

Then $\{TS(f_n)\} \rightarrow u \in \overline{B_{d_\theta}(f_0, r)}$, where $(f_n, f_{n+1}) \in \mathcal{C}(G)$ and $(f_n, f_{n+1}) \in \{TS(f_n)\}$. Moreover, if (3.1) is fulfilled for $x, y \in \{u\}$ either $(f_n, u) \in \mathcal{C}(G)$ or $(u, f_n) \in \mathcal{C}(G)$ for every $n \in \mathbb{N} \cup \{0\}$, then, S and T have a mutual FP u in $\overline{B_{d_\theta}(f_0, r)}$.

Proof. Define $\alpha : A \times A \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \in \overline{B_{d_\theta}(f_0, r)}(x, y) \in \mathcal{C}(G), \\ 0, & \text{otherwise.} \end{cases}$$

The mappings S and T are semi graph dominated on $\overline{B_{d_\theta}(f_0, r)}$, then for $x \in \overline{B_{d_\theta}(f_0, r)}$, $(x, y) \in \mathcal{C}(G)$ for every $y \in S(x)$ and $(x, y) \in \mathcal{C}(G)$ for each $y \in T(y)$. So, $\alpha(x, y) = 1$ for every $y \in S(x)$ and $\alpha(x, y) = 1$ for every $y \in T(y)$. This means that $\inf\{\alpha(x, y) : y \in S(x)\} = 1$ and $\inf\{\alpha(x, y) : y \in T(x)\} = 1$. Therefore, $\alpha_*(x, S(x)) = 1$ and $\alpha_*(x, T(x)) = 1$ for each $x \in \overline{B_{d_\theta}(f_0, r)}$. So, $S, T : A \rightarrow P(A)$ are α_* -dominated maps on $\overline{B_{d_\theta}(f_0, r)}$. Now, (3.1) can be expressed as

$$\tau + \mathcal{F}\left(H_{d_\theta}(S(x), T(y))\right) \leq \mathcal{F}\left[\psi_\theta\left(\max\left\{d_\theta(x, y), d_\theta(x, S(x)), d_\theta(y, T(y)), \frac{d_\theta(x, S(x))d_\theta(x, T(y))}{1+d_\theta(x, y)}\right\}\right)\right],$$

whenever $x, y \in \overline{B_{d_\theta}(f_0, r)} \cap \{TS(f_n)\}$, $\alpha(x, y) \geq 1$ and $H_{d_\theta}(S(x), T(y)) > 0$. Also, (iii) holds.

Now, $(f_n, u) \in \overline{B_{d_\theta}(f_0, r)}$ and $(f_n, u) \in \mathcal{C}(G)$ or $(u, f_n) \in \mathcal{C}(G)$ such that $\alpha(f_n, u) \geq 1$ or $\alpha(u, f_n) \geq 1$. So, all the restrictions of Theorem 2.1 are fulfilled. Hence, by means of Theorem 2.1, S, T have a mutual FP u in $\overline{B_{d_\theta}(f_0, r)}$ and $d_\theta(u, u) = 0$.

4. Application to integral equations

Numerous authors have utilized distinct generalized contractions in various distance spaces to establish conditions that are both required and enough for a range of linear and nonlinear integrals, including Volterra types within the framework of FP theory. To access more up-to-date

FP results that incorporate applications of integral inclusions, please refer to the following references [4,7,17,30,31,36].

Theorem 4.1. Let (\mathcal{A}, d_θ) be a complete *EbMS* with a function $\theta: \mathcal{A} \times \mathcal{A} \rightarrow [1, \infty)$. Let $S, T: \mathcal{A} \rightarrow \mathcal{A}$ be self-mappings. Suppose there are $\psi_\theta \in \Psi$, constant $\tau > 0$, and a strictly increasing function, such that

$$\tau + \mathcal{F}(H_{d_\theta}(S(x), T(y))) \leq \mathcal{F}(\psi_\theta(D_\theta(x, y))), \quad (4.1)$$

where $x, y \in \{TS(g_n)\}$ and $H_{d_\theta}(S(x), T(y)) > 0$. Then, $\{TS(g_n)\} \rightarrow u \in \mathcal{A}$ for all $n \in \mathbb{N} \cup \{0\}$.

Likewise if (4.1) sustains for u , then u becomes the FP of S and T in \mathcal{A} .

Proof. The proof of Theorem 4.1 is equivalent to the proof of Theorem 2.1, stated in different terms. In this section, we demonstrate the utilization of Theorem 2.1 by showcasing its application in the context of Volterra-type integral equations

$$g(k) = \int_0^k H_1(k, h, g(h)) dh, \quad (4.2)$$

$$p(k) = \int_0^k H_2(k, h, p(h)) dh, \quad (4.3)$$

for each $k \in [0, 1]$. To solve (4.2) and (4.3), let A denote the collection of continuous functions defined on the closed interval $[0, 1]$ to non-negative real numbers, denoted by $C([0, 1], \mathbb{R}_+)$. We describe the norm for $g \in C([0, 1], \mathbb{R}_+)$ as $\|g\|_\tau = \sup_{k \in [0, 1]} [|g(k)| e^{-\tau k}]$, wherever $\tau > 0$ is chosen arbitrary. Define

$$d_\theta(g, p) = [\sup_{k \in [0, 1]} \{|g(k) - p(k)| e^{-\tau k}\}]^2 = \|g - p\|_\tau^2$$

for each $p \in C([0, 1], \mathbb{R}_+)$. The space $(C([0, 1], \mathbb{R}_+), d_\theta)$ attains the completeness and satisfies the conditions of a complete *EbMS*.

Now, we will prove this theorem to derive a solution of integral Eqs (4.2) and (4.3).

Theorem 4.2. Suppose that the given assumptions hold:

- (i) $H_1, H_2: [0, 1] \times [0, 1] \times C[0, 1], \mathbb{R}_+ \rightarrow \mathbb{R}$;
- (ii) Define $S, T: C[0, 1], \mathbb{R}_+ \rightarrow C[0, 1], \mathbb{R}_+$ by

$$S(k) = \int_0^k H_1(k, h, g(h)) dh,$$

$$T(k) = \int_0^k H_2(k, h, p(h)) dh.$$

Assume there exists $\tau > 0$ such that

$$|H_1(k, \hbar, g(\hbar)) - H_2(k, \hbar, p(\hbar))| \leq \frac{\tau M(g,p)}{[\tau\sqrt{\|M(g,p)\|_\tau+1}]^2}$$

for each $k, \hbar \in [0,1]$ and $g, p \in C([0,1], \mathbb{R}_+)$, where

$$M(g(\hbar), p(\hbar)) = \sup \left\{ \psi_\theta \left(\begin{array}{c} [|g(\hbar) - p(\hbar)|]^2, [|g(\hbar) - S(\hbar)|]^2, [|g(\hbar) - T(\hbar)|]^2, \\ \frac{[|g(\hbar) - S(\hbar)|]^2 [|g(\hbar) - T(\hbar)|]^2}{1 + [|g(\hbar) - p(\hbar)|]^2} \end{array} \right) \right\}.$$

Then (4.2) and (4.3) have a unique solution.

Proof. By supposition (ii), one writes

$$\begin{aligned} |S(k) - T(k)| &\leq \int_0^k |H_1(k, \hbar, g(\hbar)) - H_2(k, \hbar, p(\hbar))| d\hbar, \\ &\leq \int_0^k \frac{\tau M(g,p) e^{-\tau\hbar} e^{\tau\hbar}}{[\tau\sqrt{\|M(g,p)\|_\tau+1}]^2} d\hbar, \\ &\leq \int_0^k \frac{\tau M(g,p) e^{\tau\hbar}}{[\tau\sqrt{\|M(g,p)\|_\tau+1}]^2} d\hbar, \\ &\leq \frac{\tau M(g,p)}{[\tau\sqrt{\|M(g,p)\|_\tau+1}]^2} \int_0^k e^{\tau\hbar} d\hbar, \\ &\leq \frac{\|M(g,p)\|_\tau e^{\tau k}}{[\tau\sqrt{\|M(g,p)\|_\tau+1}]^2}. \end{aligned}$$

That is,

$$\begin{aligned} |S(k) - T(k)| e^{-\tau k} &\leq \frac{\|M(g,p)\|_\tau}{[\tau\sqrt{\|M(g,p)\|_\tau+1}]^2}, \\ \|S(k) - T(k)\|_\tau &\leq \frac{\|M(g,p)\|_\tau}{[\tau\sqrt{\|M(g,p)\|_\tau+1}]^2}. \end{aligned}$$

Taking square root on both sides,

$$\begin{aligned} \sqrt{\|S(k) - T(k)\|_\tau} &\leq \frac{\sqrt{\|M(g,p)\|_\tau}}{[\tau\sqrt{\|M(g,p)\|_\tau+1}]^2}, \\ \sqrt{\|S(k) - T(k)\|_\tau} &\leq \frac{\sqrt{\|M(g,p)\|_\tau}}{\tau\sqrt{\|M(g,p)\|_\tau+1}}, \\ \frac{\tau\sqrt{\|M(g,p)\|_\tau+1}}{\sqrt{\|M(g,p)\|_\tau}} &\leq \frac{1}{\sqrt{\|S(k)-T(k)\|_\tau}}, \\ \tau + \frac{1}{\sqrt{\|M(g,p)\|_\tau}} &\leq \frac{1}{\sqrt{\|S(k)-T(k)\|_\tau}}, \end{aligned}$$

which shows that

$$\tau - \frac{1}{\sqrt{\|S(k)-T(k)\|_\tau}} \leq -\frac{1}{\sqrt{\|M(g,p)\|_\tau}}.$$

Thus, all assumptions of Theorem 2.1 are met for $\mathcal{F}(g) = \frac{-1}{\sqrt{g}}, g > 0$ and $d_\theta(g, p) = \|g - p\|_\tau^2$.

Consequently, (4.2) and (4.3) possess only one solution.

5. Application to functional equations

In dynamic programming, for the solution of functional equations we present an application in this section. Let P as well as Q be two Banach spaces, $l \subseteq P, m \subseteq Q$ and

$$d_\theta: l \times m \rightarrow l,$$

$$h, v: l \times m \rightarrow \mathbb{R},$$

$$S, T: l \times m \times \mathbb{R} \rightarrow \mathbb{R}.$$

For more results on dynamic programming see [37,39]. Suppose that l and m appear for decisions spaces. The problem related to dynamic programming is to find out a result of the following equations:

$$f(\delta) = \sup_{\delta \in m} \left\{ h(\delta, \varphi) + S(\delta, \varphi, f(d_\theta(\delta, \varphi))) \right\}, \quad (5.1)$$

$$g(\delta) = \sup_{\varphi \in \Omega} \left\{ v(\theta, \varphi) + T(\delta, \varphi, g(d_\theta(\delta, \varphi))) \right\}, \quad (5.2)$$

for $\delta \in l$. We want to show that (5.1) and (5.2) have a unique solution. Assume $R(l)$ symbolizes the set of all positive valued functions on l . Consider,

$$d_\theta(x, u) = [\sup_{\delta \in N} \{|x(\delta) - u(\delta)|\}]^2 = \|x - u\|_\tau^2, \quad (5.3)$$

for all $x, u \in R(l)$. With this setting, $(R(l), d_\theta)$ becomes a complete EbMS with $\theta(x, u) = 2$. The following restrictions are assumed to verify the following:

(i) S, T, h and v are bounded.

(ii) For $\delta \in l, u \in R(l)$, let $G, H: R(l) \rightarrow R(l)$ be multi-maps, so that

$$Gx(\delta) = \sup_{\delta \in m} \left\{ h(\delta, \varphi) + S(\delta, \varphi, x(d_\theta(\delta, \varphi))) \right\}, \quad (5.4)$$

$$Hx(\delta) = \sup_{\delta \in m} \left\{ v(\delta, \varphi) + T(\delta, \varphi, x(d_\theta(\delta, \varphi))) \right\}. \quad (5.5)$$

Suppose there exists $\tau > 0$, and for all $(\delta, \varphi) \in l \times m, u \in R(l), t \in l$ such that

$$S(\delta, \varphi, x(t)) + T(\delta, \varphi, v(t)) \leq M(u, v) \left(e^{M(u,v) - |P_{U_1}(\delta) - A_{U_2}(\delta)|^2 - \tau} \right), \quad (5.6)$$

where

$$M(x, u) = \sup \left\{ \psi_{\theta} \left(\left([|x(t) - u(t)]^2, [|x(t) - Gx(t)]^2, [|u(t) - Hu(t)]^2 \right), \frac{[|x(t) - Gx(t)]^2 [|x(t) - Hu(t)]^2}{1 + [|x(t) - u(t)]^2} \right) \right\}.$$

Theorem 5.1. Assume that (i), (ii), and (5.6) hold. Then, (5.1) and (5.2) have a distinctive, mutual and bounded solution in $R(l)$.

Proof. Take any $c > 0$. From (5.4) and (5.5), there are $u_1, u_2 \in R(l)$, and $\varphi_1, \varphi_2 \in m$ such that

$$Gu_1 < h(\delta, \varphi_1) + S(\delta, \varphi_1, u_1(d_{\theta}(\delta, \varphi_1))) + c, \quad (5.7)$$

$$Hu_2 < h(\delta, \varphi_2) + T(\delta, \varphi_2, u_2(d_{\theta}(\delta, \varphi_2))) + c. \quad (5.8)$$

By the definition of supremum, we obtain

$$Gu_1 < h(\delta, \varphi_1) + L(\delta, \varphi_1, u_1(d_{\theta}(\delta, \varphi_1))) \quad (5.9)$$

$$Hu_2 < h(\delta, \varphi_2) + M(\delta, \varphi_2, u_2(d_{\theta}(\delta, \varphi_2))). \quad (5.10)$$

Then, from (5.6), (5.7) and (5.10), we have

$$|Gu_1(\delta) - Hu_2(\delta)|^2 \leq e^{S(\delta, \varphi_1, u_1(d_{\theta}(\delta, \varphi_1))) - T(\delta, \varphi_2, u_2(d_{\theta}(\delta, \varphi_2)))},$$

$$|Gu_1(\delta) - Hu_2(\delta)|^2 \leq M(x, u) (e^{M(x, u) - |Gu_1(\delta) - Hu_2(\delta)|^2 - \tau} + c).$$

Since $c > 0$ is arbitrary, we obtain

$$|Gu_1(\delta) - Hu_2(\delta)|^2 \leq M(x, u) (e^{M(x, u) - |Gu_1(\delta) - Hu_2(\delta)|^2 - \tau}),$$

$$\frac{|Gu_1(\delta) - Hu_2(\delta)|^2}{M(x, u)} \leq e^{-\tau} (e^{M(x, u) - |Gu_1(\delta) - Hu_2(\delta)|^2}).$$

That is,

$$e^{\tau} \cdot \frac{|Gu_1(\delta) - Hu_2(\delta)|^2}{M(x, u)} \leq e^{M(x, u)} \cdot e^{-|Gu_1(\delta) - Hu_2(\delta)|^2}.$$

Taking antilog on both sides,

$$\ln \left(e^{\tau} \cdot \frac{|Gu_1(\delta) - Hu_2(\delta)|^2}{M(x, u)} \right) \leq \ln (e^{M(x, u)} \cdot e^{-|Gu_1(\delta) - Hu_2(\delta)|^2}),$$

$$\ln e^{\tau} + \ln \frac{|Gu_1(\delta) - Hu_2(\delta)|^2}{M(x, u)} \leq \ln e^{M(x, u)} - \ln e^{-|Gu_1(\delta) - Hu_2(\delta)|^2}.$$

That is,

$$\tau + \ln \frac{|Gu_1(\delta) - Hu_2(\delta)|^2}{M(x, u)} \leq M(x, u) - |Gu_1(\delta) - Hu_2(\delta)|^2.$$

This implies that

$$\tau + \ln(|Gu_1(\delta) - Hu_2(\delta)|^2) + |Gu_1(\delta) - Hu_2(\delta)|^2 \leq \ln(M(x, u)) + M(x, u).$$

So, all the restrictions of Theorem 2.1 are fulfilled for $F(v) = \ln(v^2 + v); v > 0$ and $d_\theta(x, u) = \|x - u\|_\tau^2$. Therefore, G and H have a distinct, mutual and bounded solution of (5.1) and (5.2).

6. Conclusions

The aim of this research is to introduce new FP theorems for set-valued dominated mappings that satisfy the advanced nonlinear contractions in a complete EbMS. Additionally, we establish novel FP results for a pair of dominated multi-functions on a closed ball that meets the conditions of generalized local nonlinear contractions. We provide new and unique findings for dominated maps in an ordered complete EbMS. We also propose a new concept of multi-graph dominated mappings on a closed ball in these spaces and present some new results for graphic contractions endowed with a graphic structure. We provide examples to validate our newly acquired outcomes, which demonstrate that contractive conditions hold only on a closed-ball and not on the whole space. Furthermore, we provide applications for nonlinear systems of integral equations and functional equations to illustrate the originality of our obtained outcomes. We have expanded and broadened the scope of various findings that have previously been reported in the literature. Our work builds upon and encompasses the contributions of several prior studies, including those conducted by Rasham et al. [35–39], Wordowski [42], Acar et al. [1], Altun et al. [8], Nashine et al. [30,31] and several established classical results [2,5,13,23,25,26,29,32,33,40,41,43]. In summary, our results provide a more comprehensive understanding of the topic at hand by incorporating and extending upon previous findings.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Availability of data and materials

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflict of interest

The authors affirm that they have no conflicts of interest to disclose.

Authors' contributions

T. R and N. N out lined the problem to write this research manuscript. M. S observed novel results for set-valued dominated operators satisfying an advanced local contraction and made a few changes in the application section of this article. R. P. A wrote the complete original draft for this article, H. A. reviewed-wrote and edited manuscript, T. R.; adminstrated the project and M. D. L. S; acquired the funds. The final version of the manuscript has been thoroughly reviewed and approved by all authors.

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