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Research article

Operational algebraic properties and subsemigroups of semigroups in view of k-folded $\mathcal{N}$-structures

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Abstract: The concept of k-folded $\mathcal{N}$-structures (k-F$\mathcal{N}$Ss) is an essential concept to be considered for tackling intricate and tricky data. In this study, we want to broaden the notion of k-F$\mathcal{N}$S by providing a general algebraic structure for tackling k-folded $\mathcal{N}$-data by fusing the conception of semigroup and k-F$\mathcal{N}$S. First, we introduce and study some algebraic properties of k-F$\mathcal{N}$Ss, for instance, subset, characteristic function, union, intersection, complement and product of k-F$\mathcal{N}$Ss, and support them by illustrative examples. We also propose k-folded $\mathcal{N}$-subsemigroups (k-F$\mathcal{N}$SBs) and $\widetilde{\zeta}$-k-folded $\mathcal{N}$-subsemigroups ($\widetilde{\zeta}$-k-F$\mathcal{N}$SBs) in the structure of semigroups and explore some attributes of these concepts. Characterizations of subsemigroups are considered based on these concepts. Using the notion of k-folded $\mathcal{N}$-product, characterizations of k-F$\mathcal{N}$SBs are also discussed. Further, we obtain a necessary condition of a k-F$\mathcal{N}$SB to be a k-folded $\mathcal{N}$-idempotent. Finally, relations between k-folded $\mathcal{N}$-intersection and k-folded $\mathcal{N}$-product are displayed, and how the image and inverse image of a k-F$\mathcal{N}$SB become a k-F$\mathcal{N}$SB is studied.

Keywords: semigroups; $\mathcal{N}$-structure; k-folded $\mathcal{N}$-structure; k-folded $\mathcal{N}$-subsemigroup; k-folded $\mathcal{N}$-product; k-folded $\mathcal{N}$-idempotent
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1. Introduction

The field of algebra, which concentrates on the fundamental set-theoretic conceptions and procedures utilized in algebra, is known as general algebra. It is the foundation of other fields of algebra, including differential algebra, algebraic graph theory and algebraic geometry. As a type of algebra, semigroups are quite helpful in numerous domains containing control problems, sociology, biology, dynamical systems, stochastic differential equations, etc. The term "semigroup" was used to give a title for some structures that weren't groups but emerged through the growth of outcomes.

Obscurity, uncertainty and imprecision are typical aspects of real-world situations. The conventional mathematical techniques for handling ambiguity and doubtfulness fall short in dealing with these features. Some of the innovative methods utilized to address these restrictions include multipolar (fuzzy) sets, $\mathcal{N}$-structures, etc. In 1965, Zadeh [1] adopted the idea of fuzzy structures and created a grade of membership that is a positive fuzzy value in the interval $[0, 1]$ for each ordinary item. In 2009, Jun et al. [2] proposed $\mathcal{N}$-structures and created a grade of membership for each ordinary item that is a negative fuzzy value in the range [-1, 0]. To deal with polarity, Chen and colleagues [3] suggested the grade of membership, which is a k-tuple positive fuzzy value for each item, to offer the notion of polarity fuzziness structures. In the context of multipolar (fuzzy) sets, Bashir et al. [4] presented and studied subsemigroups and several types of ideals of semigroups under polarity of fuzziness structures. In $\mathcal{N}$-structures, Abdullah and Fawad Ali [5] formulated the idea of $\mathcal{N}$-fuzziness filters in BE-algebras and investigated some connected assets. Rattana and Chinram [6], Khan et al. [7] and Rangsuk et al. [8] explored neutrosophic $\mathcal{N}$-structures and their uses in semigroups, UP-algebras and n-ary groupoids, respectively. In [9], Jana et al. discussed several aspects related to fuzziness algebraic structures. In addition, some extensions of fuzziness structures like bipolar and Intuitionistic fuzziness structures were linked to BCK(BCI/G)-algebras (see [10–16]). Following that, polarity of fuzziness models and $\mathcal{N}$-structures were linked to algebraic structures and real-life domains (see [17–23]).

Although the previous mathematical methods can deal with informational ambiguities and uncertainties, none of them is capable of handling the negative form of multi-polarity that frequently appears in real-world situations. In addition, since the multi-polar fuzziness structure, presented by Chen et al. [3], deals primarily with multi-positive data, we believe that we need a scientific approach to handle multi-negative data. If multi-positive data reflects the data of the current world, it may be thought that multi-negative data represents the afterlife. As a generalization of $\mathcal{N}$-structure and as a tool for dealing with data from the hereafter, Gon Lee et al. devised the so-called k-F$\mathcal{N}$S, which is suitable for processing multi-negative data, and applied it to the BCH-algebras (see [24]). A k-F$\mathcal{N}$S $\widetilde{\prod}$ over K is an object having the form $\widetilde{\Pi} = \{\langle \alpha, (q_j \circ \widetilde{\Pi})(\alpha) \mid \alpha \in K\}$, where the function $\widetilde{\Pi} : K \to [-1, 0]^k$ represents the degree of multi-positive membership for all $\alpha \in K$. The concept of k-F$\mathcal{N}$Ss was first presented by Gon Lee et al. [24] in 2021, which is a combination between $\mathcal{N}$-structures and multi-polar fuzziness structures. Since no negative version of multipolar fuzziness semigroups has been proposed thus far, we now feel compelled to address multipolar fuzziness negative versions in the context of semigroups.

In this paper, we study k-F$\mathcal{N}$Ss of semigroups. Some fundamental definitions and conceptions, such as semigroups, $\mathcal{N}$-structures and k-polar fuzziness structures, are provided in Section 2. These definitions will aid us to discuss our study. In Section 3, we study and discuss some algebraic properties of k-F$\mathcal{N}$Ss, for instance, subset, characteristic function, union, intersection, complement and product of k-F$\mathcal{N}$Ss, and support them using illustrative examples. In Section 4, we propose the concept of k-F$\mathcal{N}$SBs and explore some attributes and characterizations of this concept. Also, we study how the image and inverse image of a k-F$\mathcal{N}$SB become a k-F$\mathcal{N}$SB. We discuss the characterizations of k-F$\mathcal{N}$SBs through the idea of k-folded $\mathcal{N}$-product. Further, we obtain a necessary condition of a k-F$\mathcal{N}$SB to be a k-folded $\mathcal{N}$-idempotent. We display the relation between k-folded $\mathcal{N}$-intersection and

k-folded $\mathcal{N}$-product. In Section 5, we propose the idea of $\widetilde{\zeta}$-k-F$\mathcal{N}$SBs in the structure of semigroups and explore some related properties and characterizations of it. At last, the finding and some future research directions of this study are offered in Section 6.

2. Preliminaries and basic definitions

This section collects some fundamental notations and definitions of semigroups, $\mathcal{N}$-structures and k-F$\mathcal{N}$Ss needed later. Throughout the current manuscript.

- We use the semigroup K as the domain of discourse (universe set).
- We use the symbols I_n, I_n^k and $j \in k$ instead of $[-1, 0]$, $[-1, 0]^k$ and $j = 1, 2, ..., k$, respectively.

2.1. Fundamentals on semigroups

Here, we present a subsimigroup of semigroups and homomorphisms semigroups.

A semigroup K is a non-empty set together with an associative binary operation. If $T, S \subseteq K$, then the multiplication of T and S is defined as:

$$TS = \{ts \in K \mid t \in T \text{ and } s \in S\}.$$

A non-empty subset T of K is a subsemigroup of K if $TT \subseteq T$. That is, $t_1t_2 \in T, \forall t_1, t_2 \in T$.

Definition 2.1. A mapping $\Psi : K \to H$ of semigroups K and H is a homomorphism if $\Psi(\alpha\beta) = \Psi(\alpha)\Psi(\beta)\ \forall \alpha, \beta \in K$.

2.2. Fundamentals on $\mathcal{N}$-structures and k-polar fuzziness structures

Let $\mathbb{F}(K, I_n)$ be the collection of functions from K to I_n. An element of $\mathbb{F}(K, I_n)$ is said to be a negative valued function ($\mathcal{N}$-function on K) from K to I_n. An ordered pair (K, Λ) of K is an $\mathcal{N}$-structure, and Λ is an $\mathcal{N}$-function over K.

Chen and co-workers [24] propounded the conceptualization of a k-polar fuzziness structure as follows:

Definition 2.2. Let k be a finite number, where $k \geq 1$. By a k-polar fuzziness structure over $K \neq \phi$, we mean a mapping $\widetilde{\Pi} : K \to [0, 1]^k$.

That is, $\widetilde{\Pi}(\alpha) = \Big((q_1 \circ \widetilde{\Pi})(\alpha), (q_2 \circ \widetilde{\Pi})(\alpha), ..., (q_k \circ \widetilde{\Pi})(\alpha)\Big)$, where $(q_j \circ \widetilde{\Pi})(\alpha) \in [0, 1]$ for $j \in k$ and $\alpha \in K$.

Example 2.1. Let $K = \{u, y, z, w\}$ be a set. Define $\Lambda : K \to I_n$ as

$$\Lambda(\alpha) = \begin{cases} -0.5, & \text{if } \alpha = u; \\ -0.8, & \text{if } \alpha = y; \\ -0.9, & \text{if } \alpha = z; \\ -0.9, & \text{if } \alpha = w. \end{cases}$$

Then,

$$\Lambda = \{\langle u, -0.5\rangle, \langle y, -0.8\rangle, \langle z, -0.9\rangle, \langle w, -0.9\rangle\}$$

is an $\mathcal{N}$-structure over K. Also, if we define $\widetilde{\Pi} : K \to [0,1]^3$ as

$$\widetilde{\Pi}(\alpha) = \begin{cases} (0.3, 0.4, 0.5), & \text{if } \alpha = u; \\ (0.6, 0.7, 0.8), & \text{if } \alpha = y; \\ (0.6, 0.8, 0.9), & \text{if } \alpha = z; \\ (0.8, 0.9, 0.9), & \text{if } \alpha = w. \end{cases}$$

Then,

$$\widetilde{\Pi} = \{\langle u, (0.3, 0.4, 0.5)\rangle, \langle y, (0.6, 0.7, 0.8)\rangle, \langle z, (0.6, 0.8, 0.9)\rangle, \langle w, (0.8, 0.9, 0.9)\rangle\}$$

is a 3-polar fuzziness structure over K.

3. Operational properties of k-F$\mathcal{N}$Ss

Here, we define and study some operational properties of k-F$\mathcal{N}$Ss, for instance, subset, characteristic function, union, intersection, complement and product of k-F$\mathcal{N}$Ss, and provide them by illustrative examples.

Gon Lee and co-workers [24] propounded the conceptualization of k-F$\mathcal{N}$Ss as follows:

Definition 3.1. Let K be a non-empty set. By a k-F$\mathcal{N}$S over K, we mean a function $\widetilde{\Pi} : K \to I_n^k$, where $k \in \mathbb{N}$ a finite number. If $\alpha \in K$, then

$$\widetilde{\Pi}(\alpha) = \Big((q_1 \circ \widetilde{\Pi})(\alpha), (q_2 \circ \widetilde{\Pi})(\alpha), ..., (q_k \circ \widetilde{\Pi})(\alpha)\Big),$$

where $q_j : I_n^k \to I_n$ is the j-th projection $\forall j \in k$.

A k-F$\mathcal{N}$S $\widetilde{\Pi}$ may be expressed as the following notation:

$$\widetilde{\Pi} = \{\langle \alpha, (q_j \circ \widetilde{\Pi})(\alpha)\rangle \mid \alpha \in K\} = \{\langle \alpha, \Big((q_1 \circ \widetilde{\Pi})(\alpha), (q_2 \circ \widetilde{\Pi})(\alpha), ..., (q_k \circ \widetilde{\Pi})(\alpha)\Big)\rangle \mid \alpha \in K\}.$$

Let $\widetilde{\omega} = (\omega_1, \omega_2, ..., \omega_k), \widetilde{\varpi} = (\varpi_1, \varpi_2, ..., \varpi_k) \in I_n^k$, we describe the orders $\lesssim$ and $\lnsim$ on I_n^k as follows: $\forall j \in k$,

- $\widetilde{\omega} \lesssim \widetilde{\varpi} \Leftrightarrow \omega_j \leq \varpi_j$,
- $\widetilde{\omega} \lnsim \widetilde{\varpi} \Leftrightarrow \omega_j \gneqq \varpi_j$.

Example 3.1. Let $K = \{u, y, z, w\}$ be a set. Define $\widetilde{\Pi} : K \to I_n^k$ as

$$\widetilde{\Pi}(\alpha) = \begin{cases} (-0.7, -0.6, -0.5), & \text{if } \alpha = u; \\ (-0.4, -0.3, -0.2), & \text{if } \alpha = y; \\ (-0.4, -0.2, -0.1), & \text{if } \alpha = z; \\ (-0.2, -0.1, -0.1), & \text{if } \alpha = w. \end{cases}$$

Then,

$$\widetilde{\Pi} = \{\langle u, (-0.7, -0.6, -0.5)\rangle, \langle y, (-0.4, -0.3, -0.2)\rangle, \langle z, (-0.4, -0.2, -0.1)\rangle, \langle w, (-0.2, -0.1, -0.1)\rangle\}$$

is a 3-F$\mathcal{N}$S over K.

Definition 3.2. Let $\widetilde{\Pi}$ be k-F$\mathcal{N}$S in K. Then,

$$V(\widetilde{\Pi}, \widetilde{\eta}) = \{\alpha \in K \mid \widetilde{\Pi}(\alpha) \leq \widetilde{\eta}\},$$

where $\widetilde{\eta} = (\eta_1, \eta_2, ..., \eta_k) \in I_n^k$, that is,

$$V(\widetilde{\Pi}, \widetilde{\eta}) = \{\alpha \in K \mid (q_j \circ \widetilde{\Pi})(\alpha) \leq \eta_j \forall j \in k\}$$

is called a k-folded $\mathcal{N}$-level structure of $\widetilde{\Pi}$. It is clear that $V(\widetilde{\Pi}, \widetilde{\eta}) = \bigcap_{j=1}^{k} V(\widetilde{\Pi}, \widetilde{\eta})^j$, where $V(\widetilde{\Pi}, \widetilde{\eta})^j = \{\alpha \in K \mid (q_j \circ \widetilde{\Pi})(\alpha) \leq \eta_j\}$.

Definition 3.3. Let $\widetilde{\Pi}$ and $\widetilde{\Upsilon}$ be two k-F$\mathcal{N}$Ss over K. If for all $\alpha \in K$, $\widetilde{\Pi}(\alpha) \geq \widetilde{\Upsilon}(\alpha)$, that is, $(q_j \circ \widetilde{\Pi})(\alpha) \geq (q_j \circ \widetilde{\Upsilon})(\alpha)$, then $\widetilde{\Pi}$ is a k-folded $\mathcal{N}$-substructure of $\widetilde{\Upsilon}$ and written as $\widetilde{\Pi} \subseteq \widetilde{\Upsilon}$. We say $\widetilde{\Pi} = \widetilde{\Upsilon} \Leftrightarrow \widetilde{\Pi} \subseteq \widetilde{\Upsilon}$ and $\widetilde{\Upsilon} \subseteq \widetilde{\Pi}$.

Definition 3.4. Let $\phi \neq T \subseteq K$. Then, the k-folded $\mathcal{N}$-characteristic function of T is a function $\widetilde{C}_T = \{\langle \alpha, (q_j \circ \widetilde{C}_T)(\alpha) \mid \alpha \in T \rangle\}$ defined as:

$$(q_j \circ \widetilde{C}_T)(\alpha) = \begin{cases} \widetilde{-1}, & \text{if } \alpha \in T, \\ \widetilde{0}, & \text{if } \alpha \notin T, \end{cases}$$

for any $\alpha \in T$ and $j \in k$.

Definition 3.5. Let $\widetilde{\Pi}$ and $\widetilde{\Upsilon}$ be two k-F$\mathcal{N}$Ss in K. Then, their union and intersection, respectively, are also a k-F$\mathcal{N}$S in K, defined as, for all $\alpha \in K$,

$$\widetilde{\Pi} \cup \widetilde{\Upsilon} = \{\langle \alpha, (q_j \circ (\widetilde{\Pi} \cup \widetilde{\Upsilon}))(\alpha) \rangle \mid \alpha \in K\},$$

and

$$\widetilde{\Pi} \cap \widetilde{\Upsilon} = \{\langle \alpha, (q_j \circ (\widetilde{\Pi} \cap \widetilde{\Upsilon}))(\alpha) \rangle \mid \alpha \in K\},$$

where $(q_j \circ (\widetilde{\Pi} \cup \widetilde{\Upsilon}))(\alpha) = \inf\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Upsilon})(\alpha)\} = \Big(((q_1 \circ \widetilde{\Pi}) \wedge (q_1 \circ \widetilde{\Upsilon}))(\alpha), ((q_2 \circ \widetilde{\Pi}) \wedge (q_2 \circ \widetilde{\Upsilon}))(\alpha), ..., ((q_k \circ \widetilde{\Pi}) \wedge (q_k \circ \widetilde{\Upsilon}))\Big)(\alpha)$, and $(q_j \circ (\widetilde{\Pi} \cap \widetilde{\Upsilon}))(\alpha) = \sup\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Upsilon})(\alpha) = \Big(((q_1 \circ \widetilde{\Pi}) \vee (q_1 \circ \widetilde{\Upsilon}))(\alpha), ((q_2 \circ \widetilde{\Pi}) \vee (q_2 \circ \widetilde{\Upsilon}))(\alpha), ..., ((q_k \circ \widetilde{\Pi}) \vee (q_k \circ \widetilde{\Upsilon}))\Big)(\alpha)$.

Example 3.2. Let $K = \{u, y, z, w\}$. Then,

$$\begin{aligned} \widetilde{\Pi} = \Big\{ & \langle u, (-0.8, -0.6, -0.4, -0.2) \rangle, \langle y, (-0.3, -0.5, -0.4, -0.1) \rangle \\ & \langle z, (-0.2, -0.4, -0.6, -0.8) \rangle, \langle w, (-0.3, -0.5, -0.7, -0.9) \rangle \Big\}, \end{aligned}$$

and

$$\begin{aligned} \widetilde{\Upsilon} = \Big\{ & \langle u, (-0.7, -0.5, -0.3, -0.1) \rangle, \langle y, (-0.4, -0.3, -0.2, -0.1) \rangle \\ & \langle z, (-0.8, -0.4, -0.1, 0.0) \rangle, \langle w, (-0.1, -0.2, -0.3, -0.4) \rangle \Big\}, \end{aligned}$$

are 4-F$\mathcal{N}$Ss in K. The union of $\widetilde{\Pi}$ and $\widetilde{\Upsilon}$ is

$$\widetilde{\Pi} \cup \widetilde{\Upsilon} = \Big\{ \langle u, (-0.8, -0.6, -0.4, -0.2) \rangle, \langle y, (-0.4, -0.5, -0.4, -0.1) \rangle$$

$$\langle z,(-0.8,-0.4,-0.6,-0.8)\rangle,\langle w,(-0.3,-0.5,-0.7,-0.9)\rangle\Big\},$$

and the intersection of $\widetilde{\Pi}$ and $\widetilde{\Upsilon}$ is

$$\widetilde{\Pi}\cap\widetilde{\Upsilon}=\Big\{\langle u,(-0.7,-0.5,-0.3,-0.1)\rangle,\langle y,(-0.3,-0.3,-0.2,-0.1)\rangle$$
$$\langle z,(-0.2,-0.4,-0.1,-0.8)\rangle,\langle w,(-0.1,-0.2,-0.3,-0.4)\rangle\Big\}.$$

Obviously, $\widetilde{\Pi}\cup\widetilde{\Upsilon}$ and $\widetilde{\Pi}\cap\widetilde{\Upsilon}$ are 4-F$\mathcal{N}$Ss in K.

Definition 3.6. The complement $\widetilde{\Pi}^c=\{\langle\alpha,\big((q_1\circ\widetilde{\Pi})^c(\alpha),(q_2\circ\widetilde{\Pi})^c(\alpha),...,(q_k\circ\widetilde{\Pi})^c(\alpha)\big)\rangle\mid\alpha\in K\}$ of a k-F$\mathcal{N}$S $\widetilde{\Pi}=\{\langle\alpha,\big((q_1\circ\widetilde{\Pi})(\alpha),(q_2\circ\widetilde{\Pi})(\alpha),...,(q_k\circ\widetilde{\Pi})(\alpha)\big)\rangle\mid\alpha\in K\}$ is defined by:

$$\widetilde{\Pi}^c=\{\langle\alpha,\big(-1-(q_1\circ\widetilde{\Pi}(\alpha),-1-(q_2\circ\widetilde{\Pi})(\alpha),...,-1-(q_k\circ\widetilde{\Pi})(\alpha)\big)\rangle\mid\alpha\in K\}.$$

Example 3.3. In Example 3.2. The complement of a 4-F$\mathcal{N}$S $\widetilde{\Pi}$ is

$$\widetilde{\Pi}^c=\{\langle u,(-0.2,-0.4,-0.6,-0.8)\rangle,\langle y,(-0.7,-0.5,-0.6,-0.9)\rangle$$
$$\langle z,(-0.8,-0.6,-0.4,-0.2)\rangle,\langle w,(-0.7,-0.5,-0.3,-0.1)\rangle\}.$$

Definition 3.7. Let $\widetilde{\Pi}$ and $\widetilde{\Upsilon}$ be two k-F$\mathcal{N}$Ss over K. Then, the k-folded $\mathcal{N}$- product of $\widetilde{\Pi}$ and $\widetilde{\Upsilon}$ is defined to be a k-F$\mathcal{N}$S over K,

$$\widetilde{\Pi}\circledast\widetilde{\eta}=\{\langle\alpha,(q_j\circ(\widetilde{\Pi}\circledast\widetilde{\eta}))(\alpha)\rangle\mid\alpha\in K,j\in k\rangle\},$$

where

$$(q_j\circ(\widetilde{\Pi}\circledast\widetilde{\Upsilon}))(\alpha)=\begin{cases}\bigwedge\limits_{\alpha=\beta\delta}\Big\{\sup\{(q_j\circ\widetilde{\Pi})(\beta),(q_j\circ\widetilde{\Upsilon})(\delta)\}\Big\}, & \text{if } \exists\beta,\delta\in K \text{ such that } \alpha=\beta\delta,\\ \widetilde{0}, & \text{otherwise .}\end{cases}$$

Theorem 3.8. *For any k-F$\mathcal{N}$Ss $\widetilde{\Pi}$, $\widetilde{\Upsilon}$ and $\widetilde{\Theta}$ over K, we have*

1) $\widetilde{\Pi}\cup(\widetilde{\Upsilon}\cap\widetilde{\Theta})=(\widetilde{\Pi}\cup\widetilde{\Upsilon})\cap(\widetilde{\Pi}\cup\widetilde{\Theta})$.
2) $\widetilde{\Pi}\cap(\widetilde{\Upsilon}\cup\widetilde{\Theta})=(\widetilde{\Pi}\cap\widetilde{\Upsilon})\cup(\widetilde{\Pi}\cap\widetilde{\Theta})$.
3) $\widetilde{\Pi}\circledast(\widetilde{\Upsilon}\cup\widetilde{\Theta})=(\widetilde{\Pi}\circledast\widetilde{\Upsilon})\cup(\widetilde{\Pi}\circledast\widetilde{\Theta})$.
4) $\widetilde{\Pi}\circledast(\widetilde{\Upsilon}\cap\widetilde{\Theta})=(\widetilde{\Pi}\circledast\widetilde{\Upsilon})\cap(\widetilde{\Pi}\circledast\widetilde{\Theta})$.

Proof. 1) and 2) are straightforward.
3) Let $\widetilde{\Pi}$, $\widetilde{\Upsilon}$ and $\widetilde{\Theta}$ be any k-F$\mathcal{N}$Ss over K and let $\alpha\in K$. If $\alpha\neq\beta\delta$, then, $\big((q_j\circ(\widetilde{\Pi}\circledast(\widetilde{\Upsilon}\cup\widetilde{\Theta}))\big)(\alpha)=\widetilde{0}=\big((q_j\circ(\widetilde{\Pi}\circledast\widetilde{\Upsilon}))\cup(q_j\circ(\widetilde{\Pi}\circledast\widetilde{\Theta}))\big)(\alpha)$ for $j\in k$. Therefore, $\widetilde{\Pi}\circledast(\widetilde{\Upsilon}\cup\widetilde{\Theta})=(\widetilde{\Pi}\circledast\widetilde{\Upsilon})\cup(\widetilde{\Pi}\circledast\widetilde{\Theta})$. Assume that $\alpha=\beta\delta$ for some $\beta,\delta\in K$. Then,

$$\big((q_j\circ(\widetilde{\Pi}\ \circledast\ (\widetilde{\Upsilon}\cup\widetilde{\Theta}))\big)(\alpha)$$

$$
\begin{aligned}
&= \bigwedge_{\alpha=\beta\delta} \left\{ \sup\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ (\widetilde{\Upsilon} \cup \widetilde{\Theta}))(\delta)\} \right\} \\
&= \bigwedge_{\alpha=\beta\delta} \left\{ \sup\left\{(q_j \circ \widetilde{\Pi})(\beta), \inf\left\{(q_j \circ \widetilde{\Upsilon})(\delta), (q_j \circ \widetilde{\Theta})(\delta)\right\}\right\} \right\} \\
&= \bigwedge_{\alpha=\beta\delta} \left\{ \sup\inf\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Upsilon})(\delta)\right\}, \sup\inf\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Theta})(\delta)\right\} \right\} \\
&= \bigwedge_{\alpha=\beta\delta} \left\{ \inf\left\{ \sup\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Upsilon})(\delta)\right\}, \sup\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Theta})(\delta)\right\} \right\} \right\} \\
&= \inf\left\{ \bigwedge_{\alpha=\beta\delta} \left\{ \sup\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Upsilon})(\delta)\right\}, \sup\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Theta})(\delta)\right\} \right\} \right\} \\
&= \inf\left\{(q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Upsilon}))(\alpha), (q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Theta}))(\alpha)\right\} \\
&= (\widetilde{\Pi} \circledast \widetilde{\Upsilon}) \cup (\widetilde{\Pi} \circledast \widetilde{\Theta}).
\end{aligned}
$$

4) Let $\widetilde{\Pi}$, $\widetilde{\Upsilon}$ and $\widetilde{\Theta}$ be any k-F$\mathcal{N}$Ss over K and let $\alpha \in K$. If $\alpha \neq \beta\delta$, then

$$
\left((q_j \circ (\widetilde{\Pi} \circledast (\widetilde{\Upsilon} \cap \widetilde{\Theta}))\right)(\alpha) = \widetilde{0} = \left((q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Upsilon})) \cap (q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Theta}))\right)(\alpha)
$$

for $j \in k$. Therefore, $\widetilde{\Pi} \circledast (\widetilde{\Upsilon} \cap \widetilde{\Theta}) = (\widetilde{\Pi} \circledast \widetilde{\Upsilon}) \cap (\widetilde{\Pi} \circledast \widetilde{\Theta})$. Assume that $\alpha \neq \beta\delta$ for some $\beta, \delta \in K$. Then,

$$
\begin{aligned}
&\left((q_j \circ (\widetilde{\Pi} \circledast (\widetilde{\Upsilon} \cap \widetilde{\Theta}))\right)(\alpha) \\
&= \bigwedge_{\alpha=\beta\delta} \left\{ \sup\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ (\widetilde{\Upsilon} \cap \widetilde{\Theta}))(\delta)\} \right\} \\
&= \bigwedge_{\alpha=\beta\delta} \left\{ \sup\left\{(q_j \circ \widetilde{\Pi})(\beta), \sup\left\{(q_j \circ \widetilde{\Upsilon})(\delta), (q_j \circ \widetilde{\Theta})(\delta)\right\}\right\} \right\} \\
&= \bigwedge_{\alpha=\beta\delta} \left\{ \sup\sup\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Upsilon})(\delta)\right\}, \sup\sup\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Theta})(\delta)\right\} \right\} \\
&= \bigwedge_{\alpha=\beta\delta} \left\{ \sup\left\{ \sup\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Upsilon})(\delta)\right\}, \sup\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Theta})(\delta)\right\} \right\} \right\} \\
&= \sup\left\{ \bigwedge_{\alpha=\beta\delta} \left\{ \sup\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Upsilon})(\delta)\right\}, \sup\left\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Theta})(\delta)\right\} \right\} \right\} \\
&= \sup\left\{(q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Upsilon}))(\alpha), (q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Theta}))(\alpha)\right\} \\
&= (\widetilde{\Pi} \circledast \widetilde{\Upsilon}) \cap (\widetilde{\Pi} \circledast \widetilde{\Theta}).
\end{aligned}
$$

□

Theorem 3.9. *If $\widetilde{\Pi}$, $\widetilde{\Upsilon}$, $\widetilde{\Theta}$ and $\widetilde{\Lambda}$ are k-F$\mathcal{N}$Ss over K, if $\widetilde{\Pi} \subseteq \widetilde{\Theta}$ and $\widetilde{\Upsilon} \subseteq \widetilde{\Lambda}$, then $\widetilde{\Pi} \circledast \widetilde{\Upsilon} \subseteq \widetilde{\Theta} \circledast \widetilde{\Lambda}$.*

Proof. Let $\alpha \in K$. If $\alpha \neq \beta\delta$ for $\beta, \delta \in K$, then clearly $\widetilde{\Pi} \circledast \widetilde{\Upsilon} \subseteq \widetilde{\Theta} \circledast \widetilde{\Lambda}$. Assume that $\alpha = \beta\delta$ for some $\beta, \delta \in K$. Then,

$$
\left((q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Upsilon}))\right)(\alpha) = \bigwedge_{\alpha=\beta\delta} \left\{ \sup\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Upsilon})(\delta)\} \right\}
$$

$$\geq \bigwedge_{\alpha=\beta\delta} \left\{ \sup\{(q_j \circ \widetilde{\Theta})(\beta), (q_j \circ \widetilde{\Lambda})(\delta)\} \right\}$$
$$= (q_j \circ (\widetilde{\Theta} \circledast \widetilde{\Lambda}))(\alpha).$$

Therefore, $\widetilde{\Pi} \circledast \widetilde{\Upsilon} \subseteq \widetilde{\Theta} \circledast \widetilde{\Lambda}$. □

Theorem 3.10. *For any k-FNSs $\widetilde{\Pi}$, $\widetilde{\Upsilon}$ and $\widetilde{\Theta}$ over K, if $\widetilde{\Pi} \subseteq \widetilde{\Upsilon}$, then $\widetilde{\Pi} \circledast \widetilde{\Theta} \subseteq \widetilde{\Upsilon} \circledast \widetilde{\Theta}$ and $\widetilde{\Theta} \circledast \widetilde{\Pi} \subseteq \widetilde{\Theta} \circledast \widetilde{\Upsilon}$.*

Proof. Let $\alpha \in K$. If $\alpha \neq \beta\delta$ for $\beta, \delta \in K$, then clearly $\widetilde{\Pi} \circledast \widetilde{\Theta} \subseteq \widetilde{\Upsilon} \circledast \widetilde{\Theta}$. Assume that $\alpha = \beta\delta$ for some $\beta, \delta \in K$. Then,

$$\begin{aligned}
\left((q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Theta}))\right)(\alpha) &= \bigwedge_{\alpha=\beta\delta} \left\{ \sup\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Theta})(\delta)\} \right\} \\
&\geq \bigwedge_{\alpha=\beta\delta} \left\{ \sup\{(q_j \circ \widetilde{\Upsilon})(\beta), (q_j \circ \widetilde{\Theta})(\delta)\} \right\} \\
&= (q_j \circ (\widetilde{\Upsilon} \circledast \widetilde{\Theta}))(\alpha).
\end{aligned}$$

Therefore, $\widetilde{\Pi} \circledast \widetilde{\Theta} \subseteq \widetilde{\Upsilon} \circledast \widetilde{\Theta}$. Similarly, we can show that $\widetilde{\Theta} \circledast \widetilde{\Pi} \subseteq \widetilde{\Theta} \circledast \widetilde{\Upsilon}$. □

Theorem 3.11. *For any non-empty subsets T and S of K, we have*

1) $\widetilde{C}_T \circledast \widetilde{C}_S = \widetilde{C}_{TS}$.
2) $\widetilde{C}_T \cup \widetilde{C}_S = \widetilde{C}_{T \cup S}$.
3) $\widetilde{C}_T \cap \widetilde{C}_S = \widetilde{C}_{T \cap S}$.

Proof. Let $\alpha \in K$. If $\alpha \in TS$, then $(q_j \circ \widetilde{C}_T)(\alpha) = \widetilde{-1}$ for $j \in k$ and $\alpha = \beta\delta$ for some $\beta \in T$ and $\delta \in S$. Thus,

$$\begin{aligned}
\left((q_j \circ (\widetilde{C}_T \circledast \widetilde{C}_S))\right)(\alpha) &= \bigwedge_{\alpha=\beta\delta} \left(\sup\{(q_j \circ \widetilde{C}_T)(\alpha), (q_j \circ \widetilde{C}_S)(\beta)\} \right) \\
&\leq \sup\{(q_j \circ \widetilde{C}_T)(\beta), (q_j \circ \widetilde{C}_S)(\delta)\} \\
&= (q_j \circ \widetilde{C}_{TS})(\alpha) = \widetilde{-1}.
\end{aligned}$$

Therefore, $\widetilde{C}_T \circledast \widetilde{C}_S = \widetilde{C}_{TS}$.

Assume that $\alpha \notin TS$, then $(q_j \circ \widetilde{C}_{TS})(\alpha) = \widetilde{0}$ for $j \in k$. Let $\alpha, \beta \in K$ such that $\alpha = \beta\delta$, since if $\beta \notin T$ or $\delta \notin S$. If $\beta \notin T$, then,

$$\begin{aligned}
\left((q_j \circ (\widetilde{C}_T \circledast \widetilde{C}_S))\right)(\alpha) &= \bigwedge_{\alpha=\beta\delta} \left(\sup\,(q_j \circ \widetilde{C}_T)(\alpha), (q_j \circ \widetilde{C}_S)(\beta) \right) \\
&\leq \sup\{(q_j \circ \widetilde{C}_T)(\beta), (q_j \circ \widetilde{C}_S)(\delta)\} \\
&= \sup\{\widetilde{0}, (q_j \circ \widetilde{C}_S)(\delta)\} \\
&= (q_j \circ \widetilde{C}_{TS})(\alpha) = \widetilde{0}.
\end{aligned}$$

Similarly, if $\delta \notin S$, then,

$$\begin{aligned}\Big((q_j \circ (\widetilde{C}_T \circledast \widetilde{C}_S))\Big)(\alpha) &= \bigwedge_{\alpha=\beta\delta} \Big(\sup\{(q_j \circ \widetilde{C}_T)(\alpha), (q_j \circ \widetilde{C}_S)(\beta)\}\Big) \\ &\leq \sup\{(q_j \circ \widetilde{C}_T)(\beta), (q_j \circ \widetilde{C}_S)(\delta)\} \\ &= \sup\{(q_j \circ \widetilde{C}_T)(\delta), \widetilde{0}\} \\ &= (q_j \circ \widetilde{C}_{TS})(\alpha) = \widetilde{0}.\end{aligned}$$

In each case, we have $\Big((q_j \circ (\widetilde{C}_T \circledast \widetilde{C}_S))\Big)(\alpha) = (q_j \circ (\widetilde{C}_T \circledast \widetilde{C}_S))(\alpha) = \widetilde{0}$. Therefore, $\widetilde{C}_T \circledast \widetilde{C}_S = \widetilde{C}_{TS}$. □

2) and 3) are straightforward, so the proof is omitted.

4. k-folded $\mathcal{N}$-subsemigroups

Here, we apply the notion of a k-F$\mathcal{N}$S to the subsemigroups of a semigroup and we will characterize these subsemigroups in terms of k-F$\mathcal{N}$Ss.

Definition 4.1. A k-F$\mathcal{N}$S $\widetilde{\Pi}$ over K is a k-F$\mathcal{N}$SB of K if the assertion (S1) is valid: $\forall \alpha, \beta \in K$, where

$$(S1) \quad \widetilde{\Pi}(\alpha\beta) \leq \sup\{\widetilde{\Pi}(\alpha), \widetilde{\Pi}(\beta)\},$$

that is,

$$(q_j \circ \widetilde{\Pi})(\alpha\beta) \leq \sup\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Pi})(\beta)\},$$

for each $j \in k$.

Theorem 4.2. *A k-F$\mathcal{N}$S $\widetilde{\Pi}$ over K is a k-F$\mathcal{N}$SB of K if and only if its non-empty k-folded $\mathcal{N}$-level structure $V(\widetilde{\Pi}, \widetilde{\eta})$ is a subsemigroup of K for all $\widetilde{\eta} \in I_n^k$.*

Proof. Assume that $\widetilde{\Pi}$ is a k-F$\mathcal{N}$SB of K and $V(\widetilde{\Pi}, \widetilde{\eta}) \neq \phi$ for all $\widetilde{\eta} \in I_n^k$. Let $\alpha, \beta \in V(\widetilde{\Pi}, \widetilde{\eta})$. Then, $(q_j \circ \widetilde{\Pi})(\alpha) \leq \widetilde{\eta}_j$ and $(q_j \circ \widetilde{\Pi})(\beta) \leq \widetilde{\eta}_j$, for all $j \in k$. It follows that

$$(q_j \circ \widetilde{\Pi})(\alpha\beta) \leq \sup\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Pi})(\beta)\} \leq \widetilde{\eta}_j$$

for $j \in k$. Hence, $\alpha\beta \in \bigcap_{j=1}^{k} V(\widetilde{\Pi}, \widetilde{\eta})^j = V(\widetilde{\Pi}, \widetilde{\eta})$. Therefore, $V(\widetilde{\Pi}, \widetilde{\eta})$ is a subsemigroup of K.

Conversely, let $\widetilde{\Pi}$ be a k-F$\mathcal{N}$S over K such that its non-empty k-folded $\mathcal{N}$-level structure $V(\widetilde{\Pi}, \widetilde{\eta})$ is a subsemigroup of K for all $\widetilde{\eta} \in I_n^k$. Assume that assertion (S1) is not valid, i.e., $\exists \alpha, \beta \in K$ such that $\widetilde{\Pi}(\alpha\beta) > \sup\{\widetilde{\Pi}(\alpha), \widetilde{\Pi}(\beta)\}$. Then,

$$(q_j \circ \widetilde{\Pi})(\alpha\beta) > \sup\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Pi})(\beta)\},$$

for $j \in k$. If we take

$$\zeta_j = \sup\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Pi})(\beta)\},$$

for $j \in k$, then $\alpha \in V(\widetilde{\Pi}, \widetilde{\zeta})^j$ and $\beta \in V(\widetilde{\Pi}, \widetilde{\zeta})^j$. Since $V(\widetilde{\Pi}, \widetilde{\zeta})^j$ is a subsemigroup of K for $j \in k$, it follows that $\alpha\beta \in V(\widetilde{\Pi}, \widetilde{\zeta})^j$ and $(q_j \circ \widetilde{\Pi})(\alpha\beta) \leq \zeta_j$. This is a contradiction and thus $\widetilde{\Pi}$ is a k-F$\mathcal{N}$SB of K. □

Theorem 4.3. *The intersection of two k-F$\mathcal{N}$SBs is also a k-F$\mathcal{N}$SB.*

Proof. Let $\widetilde{\Pi}$ and $\widetilde{\Upsilon}$ be k-F$\mathcal{N}$SBs of K. $\forall\alpha,\beta \in K$ and $j \in k$, we have

$$\begin{aligned}
&\Big((q_j \circ \widetilde{\Pi}) \cap (q_j \circ \widetilde{\Upsilon})\Big)(\alpha\beta) \\
&\qquad = \sup\Big\{(q_j \circ \widetilde{\Pi})(\alpha\beta), (q_j \circ \widetilde{\Pi})(\alpha\beta)\Big\} \\
&\qquad \leq \sup\Big\{ \sup\Big\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Pi})(\beta)\Big\}, \sup\Big\{(q_j \circ \widetilde{\Upsilon})(\alpha), (q_j \circ \widetilde{\Upsilon})(\beta)\Big\}\Big\} \\
&\qquad = \sup\Big\{ \sup\Big\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Upsilon})(\alpha)\Big\}, \sup\Big\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Upsilon})(\beta)\Big\}\Big\} \\
&\qquad = \sup\Big\{\Big((q_j \circ \widetilde{\Pi}) \cap (q_j \circ \widetilde{\Upsilon})\Big)(\alpha), \Big((q_j \circ \widetilde{\Pi}) \cap (q_j \circ \widetilde{\Upsilon})\Big)(\beta)\Big\}.
\end{aligned}$$

Hence, $\widetilde{\Pi} \cap \widetilde{\Upsilon}$ is a k-F$\mathcal{N}$SB of K. □

Corollary 4.4. *If* $\{\widetilde{\Pi}_i \mid i \in \mathbb{N}\} = \{(q_j \circ \widetilde{\Pi})_i \mid i \in \mathbb{N}\}$ *is a family of k-F$\mathcal{N}$SBs of K, then so* $\bigcap_{i\in\mathbb{N}}(q_j \circ \widetilde{\Pi})_i$ *for* $j \in k$.

Theorem 4.5. *A k-F$\mathcal{N}$Ss over K is a k-F$\mathcal{N}$SB* $\Leftrightarrow \widetilde{\Pi} \circledast \widetilde{\Pi} \subseteq \widetilde{\Pi}$.

Proof. Suppose $\widetilde{\Pi}$ is a k-F$\mathcal{N}$SB of K and let $\alpha \in K$. Consider that α is not an element in K, then $(q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Pi}))(\alpha) = (0, 0, ..., 0) = (q_j \circ \widetilde{\Pi})(\alpha)$ for $j \in k$. Hence, $\widetilde{\Pi} \circledast \widetilde{\Pi} \subseteq \widetilde{\Pi}$. Otherwise, there exist $\beta, \delta \in K$ such that $\alpha = \beta\delta$. Then,

$$\begin{aligned}
((q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Pi})(\alpha) &= \bigwedge_{\alpha=\beta\delta} \Big\{ \sup\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Pi})(\delta)\}\Big\} \\
&\geq \bigwedge_{\alpha=\beta\delta} (q_j \circ \widetilde{\Pi})(\beta\delta) \\
&= (q_j \circ \widetilde{\Pi})(\alpha),
\end{aligned}$$

for $j \in k$. Thus, $\widetilde{\Pi} \circledast \widetilde{\Pi} \subseteq \widetilde{\Pi}$.

Conversely, let $\widetilde{\Pi}$ be a k-F$\mathcal{N}$S over K such that $\widetilde{\Pi} \circledast \widetilde{\Pi} \subseteq \widetilde{\Pi}$. Let $\alpha,\beta \in K$ and $\delta = \alpha\beta$. Then,

$$\begin{aligned}
(q_j \circ \widetilde{\Pi})(\alpha\beta) &= (q_j \circ \widetilde{\Pi})(\delta) \\
&\leq (q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Pi}))(\delta) \\
&= \bigwedge_{\alpha=\beta\delta} \Big\{ \sup\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Pi})(\beta)\}\Big\} \\
&\leq \sup\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Pi})(\beta)\},
\end{aligned}$$

for $j \in k$. Thus, $\widetilde{\Pi}$ is a k-F$\mathcal{N}$SB of K. □

Theorem 4.6. *Let K be a semigroups with identity e and let* $\widetilde{\Pi}$ *be a k-F$\mathcal{N}$S over K such that* $\widetilde{\Pi}(e) \leq \widetilde{\Pi}(\alpha)\forall\alpha \in K$, *that is,* $(q_j \circ \widetilde{\Pi})(e) \leq (q_j \circ \widetilde{\Pi})(\alpha)$ $\forall\alpha \in K, j \in k$. *If* $\widetilde{\Pi}$ *is a k-F$\mathcal{N}$SB of K, then* $\widetilde{\Pi}$ *is a k-folded $\mathcal{N}$-idempotent, i.e.,* $\widetilde{\Pi} \circledast \widetilde{\Pi} = \widetilde{\Pi}$.

Proof. For every $\alpha \in K$, we get

$$(q_j \circ (\widetilde{\Pi} \circledast \widetilde{\Pi}))(\alpha) = \bigwedge_{\alpha=\beta\delta} \Big\{ \sup\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Pi})(\delta)\}\Big\}$$

$$\begin{aligned} &\leq \sup\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Pi})(e)\} \\ &= (q_j \circ \widetilde{\Pi})(\alpha), \end{aligned}$$

for $j \in k$. Thus, $\widetilde{\Pi} \subseteq \widetilde{\Pi} \circledast \widetilde{\Pi}$. Since $\widetilde{\Pi} \circledast \widetilde{\Pi} \subseteq \widetilde{\Pi}$ by Theorem 4.5, we have $\widetilde{\Pi} \circledast \widetilde{\Pi} = \widetilde{\Pi}$, i.e., $\widetilde{\Pi}$ is a k-folded $\mathcal{N}$-idempotent. □

Let $\Psi : K \to H$ be a function of sets. If $\widetilde{\Pi}_H = \{\langle \beta, (q_j \circ \widetilde{\Pi}_H)(\beta) \rangle \mid \beta \in H\}$ is a k-F$\mathcal{N}$S of H, then the preimage of $\widetilde{\Pi}_H$ under Ψ is defined to be a k-F$\mathcal{N}$S $\Psi^{-1}(\widetilde{\Pi}_H) = \{\langle \alpha, \Psi^{-1}(q_j \circ \widetilde{\Pi}_H)(\alpha) \rangle \mid \alpha \in K\}$ of K, where $\Psi^{-1}(q_j \circ \widetilde{\Pi}_H)(\alpha) = (q_j \circ \widetilde{\Pi}_H)(\Psi(\alpha))$ for all $\alpha \in K$.

Theorem 4.7. *Let $\Psi : K \to H$ be a homomorphism of semigroups. If $\widetilde{\Pi}_H$ is a k-F$\mathcal{N}$SB of H, then $\Psi^{-1}(\widetilde{\Pi}_H)$ is a k-F$\mathcal{N}$SB of K.*

Proof. Let $\alpha, \beta \in K$. For any $j \in k$, we get

$$\begin{aligned} \Psi^{-1}(q_j \circ \widetilde{\Pi}_H)(\alpha\beta) &= (q_j \circ \widetilde{\Pi}_H)(\Psi(\alpha\beta)) \\ &= (q_j \circ \widetilde{\Pi}_H)(\Psi(\alpha)\Psi(\beta)) \\ &\leq \sup\{(q_j \circ \widetilde{\Pi}_H)(\Psi(\alpha)), (q_j \circ \widetilde{\Pi}_H)(\Psi(\beta))\} \\ &= \sup\{\Psi^{-1}(q_j \circ \widetilde{\Pi}_H)(\alpha), \Psi^{-1}(q_j \circ \widetilde{\Pi}_H)(\beta)\}. \end{aligned}$$

Hence, $\Psi^{-1}(\widetilde{\Pi}_H)$ is a k-F$\mathcal{N}$SB of K. □

Let $\Psi : K \to H$ be a function of sets. If $\widetilde{\Pi}_K = \{\langle \alpha, (q_j \circ \widetilde{\Pi}_K)(\alpha) \rangle \mid \alpha \in K\}$ is a k-F$\mathcal{N}$S of K, then the image of $\widetilde{\Pi}_K$ under Ψ is defined to be a k-F$\mathcal{N}$S $\Psi(\widetilde{\Pi}_K) = \{\langle \beta, \Psi(q_j \circ \widetilde{\Pi}_H)(\beta) \rangle \mid \beta \in H\}$ of H, where $\Psi(q_j \circ \widetilde{\Pi}_K)(\beta) = \bigwedge_{\alpha \in \Psi^{-1}(\beta)} \{(q_j \circ \widetilde{\Pi}_K)(\alpha)$ for all $j \in k$.

Theorem 4.8. *Let $\Psi : K \to H$ be an onto homomorphism of semigroups and let $\widetilde{\Pi}_K$ be a k-F$\mathcal{N}$SB of K such that*

$$(\forall Z \subseteq K)(\exists \alpha_\circ)\Big((q_j \circ \widetilde{\Pi}_K)(\alpha_\circ) = \bigwedge_{z \in Z} (q_j \circ \widetilde{\Pi}_K)(z)\Big). \tag{4.1}$$

If $\widetilde{\Pi}_K$ is a k-F$\mathcal{N}$SB of K, then $\Psi(\widetilde{\Pi}_K)$ is a k-F$\mathcal{N}$SB of H.

Proof. Let $r, s \in H$. Then, $\Psi^{-1}(r) \neq \phi$ and $\Psi^{-1}(s) \neq \phi$ in K, so from (4.1) $\exists \alpha_r \in \Psi^{-1}(r)$ and $\alpha_s \in \Psi^{-1}(s)$ such that

$$(q_j \circ \widetilde{\Pi}_K)(\alpha_r) = \bigwedge_{z \in \Psi^{-1}(r)} (q_j \circ \widetilde{\Pi}_K)(z),$$

and

$$(q_j \circ \widetilde{\Pi}_K)(\alpha_s) = \bigwedge_{w \in \Psi^{-1}(s)} (q_j \circ \widetilde{\Pi}_K)(w).$$

Thus,

$$\begin{aligned} \Psi(q_j \circ \widetilde{\Pi}_K)(rs) &= \bigwedge_{\alpha \in \Psi^{-1}(rs)} (q_j \circ \widetilde{\Pi}_K)(\alpha) \\ &\leq \sup\{(q_j \circ \widetilde{\Pi}_K)(\alpha_r), (q_j \circ \widetilde{\Pi}_K)(\beta_s)\} \end{aligned}$$

$$= \sup\Big\{\bigwedge_{z\in\Psi^{-1}(r)}(q_j\circ\widetilde{\Pi}_K)(z), \bigwedge_{w\in\Psi^{-1}(s)}(q_j\circ\widetilde{\Pi}_K)(w)\Big\}$$
$$= \sup\{\Psi(q_j\circ\widetilde{\Pi}_K)(r), \Psi(q_j\circ\widetilde{\Pi}_K)(s)\}.$$

Hence, $\Psi(\widetilde{\Pi}_K)$ is a k-F$\mathcal{N}$SB of H. □

5. $\widetilde{\zeta}$-k-folded $\mathcal{N}$-subsemigroups

Here, we present the concept of $\widetilde{\zeta}$-k-F$\mathcal{N}$SBs and consider several results related to this concept.

Definition 5.1. A k-F$\mathcal{N}$S over a universe K is a $\widetilde{\zeta}$-k-F$\mathcal{N}$SB of K if the following assertion is valid: $\forall$ $\alpha,\beta\in K$,

$$(S2)\quad \widetilde{\Pi}(\alpha\beta)\leq \sup\{\widetilde{\Pi}(\alpha),\widetilde{\Pi}(\beta),\widetilde{\zeta}\},$$

that is,

$$(q_j\circ\widetilde{\Pi})(\alpha\beta)\leq \sup\{(q_j\circ\widetilde{\Pi})(\alpha),(q_j\circ\widetilde{\Pi})(\beta),\zeta_j\},$$

for $j\in k$ and $\widetilde{\zeta}\in I_n^k$.

Example 5.1. Let K be a semigroup of four elements $\{e,y,z,w\}$ with the following multiplication table:

.	e	y	z	w
e	e	e	e	e
y	e	y	e	y
z	e	e	z	z
w	e	y	z	w

Let $\widetilde{\Pi}$ be a 3-F$\mathcal{N}$S over K which is given as:

$$\widetilde{\Upsilon} = \Big\{\langle e,(-0.40,-0.25,-0.25)\rangle,\langle y,(-0.30,-0.25,-0.25)\rangle,$$
$$\langle z,(-0.20,-0.20,-0.20)\rangle,\langle d,(-0.10,-0.10,-0.10)\rangle\Big\}.$$

Then, $\widetilde{\Pi}$ is a $\widetilde{\zeta}$-3-F$\mathcal{N}$SBs over K with $\widetilde{\zeta}=(-0.40,-0.30,-0.30)$.

Theorem 5.2. *Let $\widetilde{\Pi}$ be a $\widetilde{\zeta}$-k-F$\mathcal{N}$SB of K. If $\widetilde{\Pi}(\alpha)\geq\widetilde{\zeta}$, that is, $(q_j\circ(q_j\circ\widetilde{\Pi})(\alpha)\geq\zeta_j$ for all $\alpha\in K$ and $j\in k$, then $\widetilde{\Pi}$ is a k-F$\mathcal{N}$SB over K.*

Proof. Straightforward. □

Theorem 5.3. *A k-F$\mathcal{N}$S over K is a $\widetilde{\zeta}$-k-F$\mathcal{N}$SB of K if and only if its non-empty k-folded $\mathcal{N}$-level structure $V(\widetilde{\Pi},\widetilde{\eta})$ is a subsemigroup of K, for all $\widetilde{\eta},\widetilde{\zeta}\in I_n^k$, whenever $\widetilde{\eta}\geq\widetilde{\zeta}$, i.e., $\eta_j\geq\zeta_j$ for $j\in k$.*

Proof. Assume that $\widetilde{\Pi}$ is a $\widetilde{\zeta}$-k-F$\mathcal{N}$SB over K and $V(\widetilde{\Pi},\widetilde{\eta})\neq\phi$ for all $\widetilde{\eta}\in I_n^k$. Let $\alpha,\beta\in V(\widetilde{\Pi},\widetilde{\eta})$. Then, $(q_j\circ\widetilde{\Pi})(\alpha)\leq\widetilde{\eta}_j$ and $(q_j\circ\widetilde{\Pi})(\beta)\leq\widetilde{\eta}_j$ for all $j\in k$. Then,

$$q_j\circ\widetilde{\Pi}(\alpha\beta)\leq\sup\{q_j\circ\widetilde{\Pi}(\alpha),q_j\circ\widetilde{\Pi}(\beta),\zeta_j\}\leq sup\{\eta_j,\zeta_j\}=\eta_j,$$

for all $j\in k$. Hence, $\alpha\beta\in\bigcap_{j=1}^k V(\widetilde{\Pi},\widetilde{\eta})^j=V(\widetilde{\Pi},\widetilde{\eta})$. Therefore, $V(\widetilde{\Pi},\widetilde{\eta})$ is a subsemigroup of K.

Conversely, let $\widetilde{\Pi}$ be k-F$\mathcal{N}$S over K such that its non-empty k-folded $\mathcal{N}$-level structure $V(\widetilde{\Pi},\widetilde{\eta})$ is a subsemigroup of K for all $\widetilde{\eta} \in I_n^k$. Assume that assertion (S2) is not valid, i.e., $\exists\alpha,\beta \in K$ such that $\widetilde{\Pi}(\alpha\beta) > \sup\{\widetilde{\Pi}(\alpha),\widetilde{\Pi}(\beta),\widetilde{\zeta}\}$. Then,

$$q_j \circ \widetilde{\Pi}(\alpha\beta) > \sup\{q_j \circ \widetilde{\Pi}(\alpha), q_j \circ \widetilde{\Pi}(\beta), \zeta_j\},$$

for $j \in k$. If we take

$$\vartheta_j = \sup\{q_j \circ \widetilde{\Pi}(\alpha), q_j \circ \widetilde{\Pi}(\beta), \zeta_j\},$$

for $j \in k$, then $\alpha \in V(\widetilde{\Pi},\widetilde{\vartheta})^j, \beta \in V(\widetilde{\Pi},\widetilde{\vartheta})^j$ and $\vartheta_j \geq \zeta_j$. Since $V(\widetilde{\Pi},\widetilde{\vartheta})^j$ is a subsemigroup of K for $j \in k$, it follows that $\alpha\beta \in V(\widetilde{\Pi},\widetilde{\vartheta})^j$. Hence, $q_j \circ \widetilde{\Pi}(\alpha\beta) \leq \vartheta_j$. This is a contradiction and thus $\widetilde{\Pi}$ is a $\widetilde{\zeta}$-k-F$\mathcal{N}$SB of K. □

Theorem 5.4. *If $\widetilde{\Pi}$ and $\widetilde{\Upsilon}$ are an $\widetilde{\zeta}$-k-F$\mathcal{N}$SB and an $\widetilde{\varrho}$-k-F$\mathcal{N}$SB, respectively, of K for any $\widetilde{\zeta},\widetilde{\varrho} \in I_n^k$, then their intersection is an $\widetilde{\varpi}$-k-F$\mathcal{N}$SB of K for $\widetilde{\varpi} = \sup\{\widetilde{\zeta},\widetilde{\varrho}\}$.*

Proof. For every $\alpha,\beta \in K$ and $j \in k$, we have

$$\begin{aligned}
&\Big((q_j \circ \widetilde{\Pi}) \cap (q_j \circ \widetilde{\Upsilon})\Big)(\alpha\beta)\\
&\quad= \sup\Big\{(q_j \circ \widetilde{\Pi})(\alpha\beta), (q_j \circ \widetilde{\Pi})(\alpha\beta)\Big\}\\
&\quad\leq \sup\Big\{\sup\Big\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Pi})(\beta), \zeta_j\Big\}, \sup\Big\{(q_j \circ \widetilde{\Upsilon})(\alpha), (q_j \circ \widetilde{\Upsilon})(\beta), \varrho_j\Big\}\Big\}\\
&\quad\leq \sup\Big\{\sup\Big\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Pi})(\beta), \varpi_j\Big\}, \sup\Big\{(q_j \circ \widetilde{\Upsilon})(\alpha), (q_j \circ \widetilde{\Upsilon})(\beta), \varpi_j\Big\}\Big\}\\
&\quad= \sup\Big\{\sup\Big\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Upsilon})(\alpha), \varpi_j\Big\}, \sup\Big\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Upsilon})(\beta), \varpi_j\Big\}\Big\}\\
&\quad= \sup\Big\{\sup\Big\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Upsilon})(\alpha)\Big\}, \sup\Big\{(q_j \circ \widetilde{\Pi})(\beta), (q_j \circ \widetilde{\Upsilon})(\beta)\Big\}, \varpi_j\Big\}.
\end{aligned}$$

Hence, $\widetilde{\Pi} \cap \widetilde{\Upsilon}$ is a $\widetilde{\zeta}$-k-F$\mathcal{N}$SB of K. □

Theorem 5.5. *Let $\widetilde{\Pi}$ be an $\widetilde{\zeta}$- k-F$\mathcal{N}$SB of K. If $\widetilde{\varpi} = \bigvee_{\alpha\in K}\Big\{\widetilde{\Pi}(\alpha)\Big\}$, i.e., $\varpi_j = \bigvee_{\alpha\in K}\Big\{(q_j \circ \widetilde{\Pi})(\alpha)$, for $j \in k\Big\}$, then the set $\Delta = \{\alpha \in K \mid \widetilde{\Pi}(\alpha) \leq \sup\{\widetilde{\varpi},\widetilde{\zeta}\}$, that is, $\Delta = \{\alpha \in K \mid (q_j \circ \widetilde{\Pi})(\alpha) \leq \sup\{\varpi_j,\zeta_j\}$ for $j \in k\}$ is a subsemigroup of K.*

Proof. Let $\alpha,\beta \in \Delta\forall\alpha,\beta \in K$. Then, for $j \in k$

$$\begin{aligned}
(q_j \circ \widetilde{\Pi})(\alpha) &\leq \sup\{\varpi_j, \zeta_j\}\\
&= \sup\Big\{\bigvee_{\alpha\in K}\{(q_j \circ \widetilde{\Pi})(\alpha)\}, \zeta_j\Big\},
\end{aligned}$$

and

$$\begin{aligned}
(q_j \circ \widetilde{\Pi})(\beta) &\leq \sup\{\varpi_j, \zeta_j\}\\
&= \sup\Big\{\bigvee_{\alpha\in K}\{(q_j \circ \widetilde{\Pi})(\beta)\}, \zeta_j\Big\}.
\end{aligned}$$

Thus,

$$(q_j \circ \widetilde{\Pi})(\alpha\beta) \leq \sup\{(q_j \circ \widetilde{\Pi})(\alpha), (q_j \circ \widetilde{\Pi})(\beta), \zeta_j\}$$

$$\begin{aligned}
&\leq \sup\left\{\sup\{\varpi_j,\zeta_j\},\sup\{\varpi_j,\zeta_j\},\zeta_j\right\}\\
&= \sup\{\varpi_j,\zeta_j\}.
\end{aligned}$$

So $\alpha\beta \in \Delta$. Hence, Δ is a subsemigroup of K. □

For a map $\Psi : K \to H$ of semigroups and k-F$\mathcal{N}$S $\widetilde{\Pi} = \{\langle\alpha, (q_j \circ \widetilde{\Pi})(\alpha)\rangle \mid \alpha \in K\}$ of H. Define a new k-F$\mathcal{N}$S $\widetilde{\Pi}^{\widetilde{\zeta}} = \{\langle\alpha, (q_j \circ \widetilde{\Pi})^{\widetilde{\zeta}}(\alpha)\rangle \mid \alpha \in K\}$ of K such that $(q_j \circ \widetilde{\Pi})^{\widetilde{\zeta}}(\alpha) = \sup\{(q_j \circ \widetilde{\Pi})(\Psi(\alpha)), \zeta_j\}$, where $(q_j \circ \widetilde{\Pi})^{\widetilde{\zeta}} : K \to I_n^k$ and $j \in k$.

Theorem 5.6. *Let $\Psi : K \to H$ be a homomorphism of semigroups. If a k-F$\mathcal{N}$S $\widetilde{\Pi}$ of H is a $\widetilde{\zeta}$-k-F$\mathcal{N}$SB of H, then $\widetilde{\Pi}^{\widetilde{\zeta}}$ is a $\widetilde{\zeta}$-k-F$\mathcal{N}$SB of K.*

Proof. Let $\alpha, \beta \in K$ and $j = 1, 2, ..., k$. Then,

$$\begin{aligned}
(q_j \circ \widetilde{\Pi})^{\widetilde{\zeta}}(\alpha\beta) &= \sup\{(q_j \circ \widetilde{\Pi})(\Psi(\alpha\beta)), \zeta_j\}\\
&= \sup\{(q_j \circ \widetilde{\Pi})(\Psi(\alpha)\Psi(\beta)), \zeta_j\}\\
&\leq \sup\left\{\sup\{(q_j \circ \widetilde{\Pi})(\Psi(\alpha)), (q_j \circ \widetilde{\Pi})(\Psi(\beta))\}, \zeta_j\right\}\\
&= \sup\left\{\sup\{(q_j \circ \widetilde{\Pi})(\Psi(\alpha)), \zeta_j\}, \sup\{(q_j \circ \widetilde{\Pi})(\Psi(\beta)), \zeta_j\}, \zeta_j\right\}\\
&= \sup\{(q_j \circ \widetilde{\Pi})^{\widetilde{\zeta}}(\Psi(\alpha), (q_j \circ \widetilde{\Pi})^{\widetilde{\zeta}}(\Psi(\beta), \zeta_j\}.
\end{aligned}$$

Thus, $\widetilde{\Pi}^{\widetilde{\zeta}}$ is a $\widetilde{\zeta}$-k-F$\mathcal{N}$SB of K. □

6. Conclusions

The idea of k-F$\mathcal{N}$S being a new framework containing the negative data may be utilized to explain and solve real-life challenges more easily like the multi polarity fuzziness structures and $\mathcal{N}$-structures. In this research, we studied some algebraic properties of k-F$\mathcal{N}$Ss, such as subset, characteristic function, union, intersection, complement and product of k-F$\mathcal{N}$Ss, and supported them by illustrative examples. We also originated k-F$\mathcal{N}$SBs and $\widetilde{\zeta}$-k-F$\mathcal{N}$SBs in the structure of semigroups and probed some attributes and characteristics of these concepts. Based on k-folded $\mathcal{N}$-product, we discussed some characterizations of k-F$\mathcal{N}$SBs. Further, we obtained a necessary condition of a k-F$\mathcal{N}$SB to be a k-folded $\mathcal{N}$-idempotent. Finally, we displayed the relations between k-folded $\mathcal{N}$-intersection and k-folded $\mathcal{N}$-product, and studied how the image and inverse image of a k-F$\mathcal{N}$SB become a k-F$\mathcal{N}$SB. In our future study, we are planning to build some additional theories on this structure. We will apply this principle to characterize some algebraic structures by their left and right ideals. Moreover, for the applications of k-F$\mathcal{N}$S, we will apply this platform to real-world issues and attempt to demonstrate these issues in greater detail.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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