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Research article

Pullback attractor for a nonautonomous parabolic Cahn-Hilliard phase-field system

Jean De Dieu Mangoubi*, **Mayeul Evrard Isseret Goyaud and Daniel Moukoko**

Faculté des Sciences et Technique, Université Marien Ngouabi, BP: 69, Brazzaville, Congo

* **Correspondence:** Email: mmangoubijean@gmail.com; Tel: +242-06-630-04-71.

Abstract: Our aim in this paper is to study generalizations of the Caginalp phase-field system based on a thermomechanical theory involving two temperatures and a nonlinear coupling. In particular, we prove well-posedness results. More precisely, the existence of a pullback attractor for a nonautonomous parabolic of type Cahn-Hilliard phase-field system. The pullback attractor is a compact set, invariant with respect to the cocycle and which attracts the solutions in the neighborhood of minus infinity, consequently the attractor pullback (or attractor retrograde) exhibits a infinite fractal dimension.

Keywords: attractor; pullback ω-limit compact; pullback condition; norm-to-weak continuous; nonautonomous parabolic Cahn-Hilliard phase-field system
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1. Introduction

We consider the following nonautonomous parabolic Cahn-Hilliard phase-field system

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \frac{\partial \alpha}{\partial t} + h(t,x) \tag{1.1}$$

$$\frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \tag{1.2}$$

with homogenous Dirichlet conditions

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \alpha|_{\partial\Omega} = \Delta\alpha|_{\partial\Omega} = 0 \tag{1.3}$$

and initial conditions

$$u|_{t=\tau} = u_0, \alpha|_{t=\tau} = \alpha_0, \tag{1.4}$$

where $\Omega \subset \mathcal{R}^n (1 \leqslant n \leqslant 3)$ is a bounded smooth domain, $h \in L^\infty(\mathbb{R}, L^2(\Omega))$, $\frac{\partial h}{\partial t} \in L^\infty(\mathbb{R}, L^2(\Omega))$ and

$$\int_{-\infty}^{t} e^{\lambda s} \|h(s)\|^2 ds < \infty \;\; for\ any\ t \in \mathbb{R}, \tag{1.5}$$

where λ is a positve constant which will be characterized later and

$$\|\frac{\partial h}{\partial t}(s)\|^2 \leqslant s^{c-1} \quad \forall\, c > 1, \;\; \forall s \in (0; \frac{1}{2}) \tag{1.6}$$

The nonlinear term f belonging to $C^2(\mathbb{R})$ which satisfies the following properties

$$f(0) = 0 \tag{1.7}$$

$$c_0|s|^q - c_1 \leqslant f(s)s \leqslant c_2|s|^q + c_3, \quad c_0, c_2, c_3 > 0, \; q \geqslant 3, \qquad \forall s \in \mathbb{R} \tag{1.8}$$

$$-c_4 \leqslant F(s) \leqslant f(s)s + c_5, \quad c_5, c_4 > 0, \qquad \forall s \in \mathbb{R} \tag{1.9}$$

$$|f'(s)| \leqslant \beta(|s|^{q-2} + 1), \quad \beta > 0, \quad \forall s \in \mathbb{R} \tag{1.10}$$

and

$$f'(s) \geqslant -c_6 \tag{1.11}$$

with

$$F(s) = \int_0^s f(\tau) d\tau.$$

At the last moment, autonomous dynamical systems and their attractors have been widely studied (see [8, 12, 16, 20]). However, non-autonomous infinite-dimensional dynamical systems are less well understood. The dance of the initial time τ is as important as the dependence of the final time t, and in this case we study the existence of exponential, uniform, pullback and exponential pullback attractors for this family (which replaces the semigroup for the case autonomous)., see ([17, 18]). However, in certain non-autonomous cases the trajectories can be unlimited when time tends towards infinity; the classical theory of uniform attractors is not applicable in such systems see ([2, 4, 22]). To deal with such problems, mathematicians have developed the theory of retrograde attractors (or cocycles) for non-autonomous dynamical systems (see [6, 7, 9, 13]), to study the behavior of system when $t - \tau$ tends to infinity. Pullback attractors are formulated in terms of a cocycle map on ϕ on a state X driven by an autonomous dynamical system on a parameter space p (see [1, 4, 15, 19]). Simply put, the pullback attractor is a family of non-empty compact subsets $\hat{\mathcal{A}} = \{A_p\}_{p \in P}$ of the space X parameterized by l 'base space Ω, which is $\hat{\mathcal{A}}$-invariant, i.e. $\phi(t, p, \mathcal{A}_p) = \mathcal{A}_{\theta_{t(p)}}$.

In this article, we use the concept of continuous norm-weak cocycle in Banach space and give a method to verify this kind of continuity. We obtain abstract results on the existence of pullback attractors of non-autonomous dynamical systems, however some authors have worked to prove the existence of pullback attractors, uniform attractors, pullback attractors with a single equation (see [1, 3, 5, 10, 21]). But, in this article we prove the existence of the pullback attractor with a system of equations using the same method as in [19], we get existence and uniqueness of solutions and we prove the existence of pullback attractors.

2. Notations

Throughout this paper we use the following notations: E is a Banach space with norm $||.||_E$ and the metric is d. B(E) is the set of all bounded subsets of E. Let $X, Y \subset E$; we denote by $d_H(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y)$ the Hausdorff semi-distance between X and Y and $N(X, \epsilon)$ the ϵ-neighborhood of X. Let E_1 and E_2 be Banach spaces, $E_1 \hookrightarrow E_2$ means that E_1 is embedded in E_2. $\mathbb{R}_\tau = [\tau, +\infty)$ and $\mathbb{R}_+ = \mathbb{R}_0$. $\rightarrow$ means the convergence in the strong topology and $\rightharpoonup$ means the convergence in the weak topology.

We denote by $||.||$ the usual L^2-norm (with associated product scalar $(.,.)$) and $||.||_{-1} = ||(-\Delta)^{\frac{-1}{2}}.||$, where $-\Delta$ denotes the minus Laplace operator with Dirichlet boundary conditions. More generally, $||.||_X$ denote the norm of Banach space X.

Throughout this paper, the same letters c, c', c''.... denote generally positive constants which may vary from line to line, or even in the same line. Similarly, the same letter Q denotes positive monotome increasing (with respect to each argument) and continuous functions which may vary from line to line.

2.1. Preliminaries and abstract results

Let (E, d) be a complete metric space, (P, ρ) be a metric space which will be called the parameter space, and let T, the time set, $\mathbb{R}_+$. $\theta : \mathbb{R} \times P \rightarrow P$ is a mapping, $\theta_t = \theta(t, .) : P \rightarrow P$ form a group, that is, θ satisfies

$$\theta_{t+\tau} = \theta_t.\theta_\tau \quad \forall t, \tau \in \mathbb{R}$$
$$\theta_0 = Id.$$

Definition 2.1.1. *A mapping $\phi : \mathbb{R}_+ \times P \times E \rightarrow E$ is said to be cocycle on E with respect to group θ, if*

$$1)\ \phi(0, p, x) = x \ \ \forall (p, x) \in P \times E$$
$$2)\ \phi(t + \tau, p, x) = \phi(t, \theta_\tau(p), \phi(\tau, p, x)), \quad \forall t, \tau \in \mathbb{R}_+, (p, x) \in P \times E$$

If $\phi : \mathbb{R}_+ \times P \times E \rightarrow E$ is continuous, ϕ is called a continuous cocycle on E with respect to group θ. The mapping $\pi : \mathbb{R}_+ \times P \times E \rightarrow P \times E$ defined by

$$\pi(t, p, x) = \phi(\theta_t(p), \phi(\tau, p, x)), \quad \forall t \in \mathbb{R}_+, (p, x) \in P \times E,$$

forms a semigroup on $P \times E$ and is called a skew-product flow.

Definition 2.1.2. *A family $\mathcal{A} = \{A_p\}_{p \in P}$ of nonempty compact sets of E is called a pullback attractor of the cocycle ϕ if it is ϕ-invariant, that is,*

$$\phi(t, p, A_p) = \mathcal{A}_{\theta_t(p)}, \quad \forall t \in \mathbb{R}_+, p \in P,$$

and pullback attracting, that is

$$\lim_{t\to+\infty} d_H(\phi(t,\theta_{-t}(p),B)A_p) = 0 \quad \forall B \in B(E), p \in P.$$

Theorem 2.1. *Let ϕ be a continuous cocycle on E with respect to a group θ of continuous mappings on P and let $\pi = (\theta, \phi)$ be the corresponding skew-product flow on $P \times E$. In addition, suppose that there is a nonempty compact subset B_0 of E and for $B \in B(E)$ there exists a $T(B) \in R_+$, which is independent of $p \in P$, such that*

$$\pi(t, P, B) \subset B_0, \quad \forall t > T(B).$$

then (1) there exists a unique pullback attractor $\mathcal{A} = \{A_p\}_{p\in P}$ of the cocycle ϕ on E, where

$$A_p = \bigcap_{\tau\in R_+} \overline{\bigcup_{t>\tau} \phi(t, \theta_{-t}(p), B_0)};$$

(2) there exists a global compact attractor $\hat{A}$ of the autonomous semi dynamical system ϕ on $P \times E$, where

$$\hat{A} = \bigcap_{\tau\in R_+} \overline{\bigcup_{t>\tau} \pi(t, P \times B_0)};$$

(3) assertions (1) and (2) above are equivalent, and

$$\hat{A} = \bigcup_{p\in P} \{p\} \times A_p.$$

See Crauel and Flandoli [7] and Schmalfub ([11, 14]) for the proof of assertion (1) and Cheban and Fakeeh [12] and Hale [19] for the proof assertion (2). Assertion (3) has been proved by Cheban [4]. Let $B \in B(E)$. Its Kuratowski measure of noncompactness $\alpha(B)$ is defined by

$$\alpha(B) = \inf\{\delta |\, \mathbf{B}\ admits\ a\ finite\ cover\ by\ set\ of\ diameter \leqslant \delta\}.$$

It has the following properties (see Hale [2], Sell and You [20]).

Lemma 2.1. *Let $B, B_1, B_2 \in B(E)$, then*

$$\begin{aligned}
&(1)\ \alpha(B) = 0 \Leftrightarrow \alpha(N(B,\epsilon)) \leqslant 2\epsilon \Leftrightarrow \overline{B}\ \ is\ compact;\\
&(2)\ \alpha(B_1 + B_2) \leqslant \alpha(B_1) + \alpha(B_2);\\
&(3)\ \alpha(B_1) \leqslant \alpha(B_2)\ \ whenever\ \ B_1 \subset B_2;\\
&(4)\ \alpha(B_1 \cup B_2) \leqslant \max\{\alpha(B_1), \alpha(B_2)\};\\
&(5)\ \alpha(\overline{B}) = \alpha(B);\\
&(6)\ if\ B\ is\ a\ ball\ of\ radius\ \epsilon\ \ then\ \ \alpha(B) \leqslant 2\epsilon. \qquad (2.1)
\end{aligned}$$

Lemma 2.2. *Let ... $\supset F_n \supset F_{n+1} \supset$ ... be a sequence of nonempty closed subsets of E such that $\alpha(F_n) \to 0$ as $n \to \infty$. Then $F = \bigcap_{n=1}^{\infty} F_n$ is nonempty and compact.*
We will characterize the existence of pullback attractor for the cocycle in term of concept of measure of noncompactness.

Definition 2.1.3. *Let ϕ be a cocycle on E with respect to group θ. We say that ϕ is a norm to-weak continuous cocycle on E if ϕ satisfies.*

$$\begin{aligned}
&(1)\ \ \phi(0,p,x) = x \quad \forall (p,x) \in P \times E;\\
&(2)\ \ \phi(\tau + t, p, x) = \phi(t, \theta_\tau(p), \phi(\tau, p, x));\\
&(3)\ \ \phi(t, p, x_n) \rightharpoonup \phi(t, p, x),\ if\ x_n \to x\ in\ E,\ \forall t \in R_+,\ p \in P.
\end{aligned}$$

Definition 2.1.4. *Let ϕ be a cocycle on E with respect to group θ. A set $B_0 \subset E$ is said to be uniformly absorbing set for ϕ, if for any $B \in B(E)$ there exists $T_0 = T_0(B) \in R_+$ such that*

$$\phi(t, p, B) \subset B_0 \ \ \forall t \geqslant T_0, \quad p \in P.$$

Definition 2.1.5. *Let ϕ be a cocycle on E with respect to group θ. We say that ϕ to be pullback w-limit compact if for any $B \in B(E),\ p \in P$,*

$$\lim_{t\to+\infty} \alpha\Big(\bigcup \phi(t, \theta_{-t}(p), B)\Big) = 0. \tag{2.2}$$

Definition 2.1.6. *Let ϕ be a cocycle on E with respect to group θ. Define the pullback w-limit set $\omega_p(B)$ of B by the following:*

$$\omega_p(B) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} \phi(t, \theta_{-t}(p), B).} \tag{2.3}$$

Remark 2.1. *It is easy to see that $y \in \omega_p(B)$ if and only if there are sequences $\{x_n\} \subset B,\ \{t_n\} \subset \mathbb{R}_+,\ t_n \to \infty$ such that $\phi(t_n, \theta_{t_n}(p), x_n) \to y (n \to \infty)$.*

Lemma 2.3. *If a cocycle on E with respect to group θ and ϕ is pullback w-limit compact, then for any $\{x_n\} \subset B \in B(E),\ p \in P,\ \{t_n\} \subset R_+,\ t_n \to +\infty$ as $n \to +\infty$ there exists a convergent subsequence of $\{\phi(t_n, \theta_{t_n}(p), x_n)\}$ whose limit lies in $\omega_p(B)$. To see the prove in [19].*

Theorem 2.2. *Let ϕ be a cocycle on E with respect to group θ. If ϕ is a norm-weak continuous and possesses a uniformly absorbing set B_0, then ϕ possesses a pullback attractor $\mathcal{A} = \{A_p\}_{p\in P}$, satisfying*

$$\mathcal{A} = \omega_p(B),\ \ \forall p \in P,$$

if and only if it is pullback w-limit compact.

To see the proof in [19].

Definition 2.1.7. *Let ϕ be a cocycle on E with respect to group θ. A cocycle ϕ is said to be satisfying pullback condition if for any $p \in P,\ B \in B(E)$ and $\epsilon > 0$, there exist $t_0 = t_0(p, B, \epsilon) \geqslant 0$ and a finite dimensional subspace E_1 of E such that*

$$(i)\ \ P\Big(\bigcup_{t\geqslant 0} \phi(t, \theta_{-t}(p), B)\Big)\ \textit{is bounded};$$

and

$$(ii)\ \ \|(I-P)(\bigcup_{t\geqslant 0}\phi(t,\theta_{-t}(p),x))\|\leqslant \epsilon,\ \ \forall x\in B,$$

where $P: E \longrightarrow E_1$ is a bounded projector.

Theorem 2.3. *Let E be a Banach space and let ϕ be a cocycle on E with respect to group θ. If cocycle ϕ satisfies pullback condition, then ϕ is pullback pullback w-limit compact. Moreover, let E is a uniformly convex Banach space, then ϕ is pullback pullback w-limit compact if and only if pullback condition holds true.*

See Y.J. Wang, C.K. Zhong, S.F. Zhou [22] for the proof of the theorem, and the theorem will be used in our consideration.
Now, we verify that a cocycle is norm-to-weak continuous for a system of two equations.

Theorem 2.4. *Let X, Y be two Banach space, X^*, Y^* be respectively their dual space. X is dense in Y, the injection $i : X \longrightarrow Y$ is continuous and adjoint $i^* : X^* \longrightarrow Y^*$ is dense, and ϕ is a norm-weak continuous cocycle on Y. Then ϕ is a norm-weak continuous cocycle on X if and only if for $p \in P,\ t \in \mathbb{R}_+,\ \phi(t, p, x)$ maps the compact set of X to be a bounded set of X.*

2.2. Main results

In order to prove the existence of pullback attractor we first show that the cocycle mapping is a norm-to-weak continuous , and we then demonstrat that the cocycle mapping possesses a uniformly absorbing set, and is a pullback w-limit compact.
Before proving those main results, we show the existence and uniqueness of solution relative to the system (1.1) – (1.4).

3. A priori estimates

The estimates derived in this section are formal, but they can easily be justified within a Galerkin scheme. In what follows, the Poincaré, Hölder and Young inequalities are extensively used, Without further referring to them.
We multiply (1.1) by $(-\Delta)^{-1}\dfrac{\partial u}{\partial t}$ and integrate over Ω. We have

$$\frac{d}{dt}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)dx\right) + \|\frac{\partial u}{\partial t}\|_{-1}^2 \leqslant 2\left(\frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t}\right) + c\|h\|^2. \tag{3.1}$$

Now we multiply (1.2) by $\dfrac{\partial \alpha}{\partial t}$ and integrate over Ω. We obtain

$$\frac{d}{dt}\|\nabla\alpha\|^2 + 2\|\frac{\partial \alpha}{\partial t}\|^2 = -2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}\right). \tag{3.2}$$

Summing (3.1) and (3.2), we find

$$\frac{dE_1}{dt} + \|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\|\frac{\partial \alpha}{\partial t}\|^2 \leqslant c\|h\|^2, \tag{3.3}$$

where

$$E_1 = \|\nabla u\|^2 + 2\int_\Omega F(u)dx + \|\nabla \alpha\|^2.$$

Multiplying (1.1) by $(-\Delta)^{-1}u$ and integrating over Ω, we find, owing to (1.9)

$$\frac{d}{dt}\|u\|_{-1}^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)dx \leqslant c + 2c_p^2\|\frac{\partial \alpha}{\partial t}\|^2 + c\|h\|^2. \tag{3.4}$$

Multiplying (1.2) by α and integrating over Ω, we have

$$\frac{d}{dt}\|\alpha\|^2 + \|\nabla \alpha\|^2 \leqslant \|\frac{\partial u}{\partial t}\|_{-1}^2. \tag{3.5}$$

Summing (3.3), δ_1(3.4) and δ_2(3.5), where δ_1 and $\delta_2 > 0$ are such that

$$1 - \delta_1 c_p^2 > 0,$$
$$1 - \delta_2 > 0,$$

we find, thanks to (1.9)

$$\frac{dE_2}{dt} + c\|\nabla u\|^2 + c\|\nabla \alpha\|^2 + c\int_\Omega F(u)dx + c\|\frac{\partial u}{\partial t}\|_{-1}^2 + c'\|\frac{\partial \alpha}{\partial t}\|^2 \leqslant c + c\|h\|^2 \tag{3.6}$$

where we have

$$E_2 = E_1 + \delta_1\|u\|_{-1}^2 + \delta_2\|\alpha\|^2 + c_4|\Omega| \geqslant 0.$$

Estimate (3.6) can be written as following

$$\frac{dE_2}{dt} + c\left(E_2 + \|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial \alpha}{\partial t}\|^2\right) \leqslant c + c\|h\|^2. \tag{3.7}$$

Applying Gronwall's lemma, we have

$$E_2(t) + c\int_0^t \left(\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial \alpha}{\partial t}\|^2\right) e^{c(s-t)}ds \leqslant E_2(0)e^{-ct} + c'. \tag{3.8}$$

Properties (1.8) allows to find the estimate

$$E_2(0) \leqslant c(\|\nabla u_0\|^2 + \|u_0\|_{L^q}^q + \|\nabla \alpha_0\|^2 + 1), \quad c > 0. \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\int_t^{t+1} \left(\|\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial u}{\partial t}\|_{-1}^2\right) ds \leqslant c'\left(\|\nabla u_0\|^2 + \|u_0\|_{L^q}^q + \|\nabla \alpha_0\|^2 + 1\right), \quad c' > 0, \quad t \geqslant 0. \tag{3.10}$$

Finally, more generally, for every $r > 0$

$$\int_t^{t+r} \left(\|\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial u}{\partial t}\|_{-1}^2\right) ds \leqslant c' \left(\|\nabla u_0\|^2 + \|u_0\|^q + \|\nabla \alpha_0\|^2 + 1\right) + c''(r), \quad c' > 0, \quad t \geqslant 0. \tag{3.11}$$

We multiply (1.1) by $\frac{\partial u}{\partial t}$ and integrate over Ω. We have

$$\frac{d}{dt}\|\Delta u\|^2 + \|\frac{\partial u}{\partial t}\|^2 \leqslant 2\|\Delta f(u)\|^2 + c\|h\|^2 - 2\left(\Delta \frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t}\right)$$

which yields, owing to the continuous embedding $H^2(\Omega) \subset C(\overline{\Omega})$,

$$\frac{d}{dt}\|\Delta u\|^2 + \|\frac{\partial u}{\partial t}\|^2 \leqslant Q(\|u\|_{H^2(\Omega)}) - 2\left(\Delta \frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t}\right) + c\|h\|^2. \tag{3.12}$$

Multiplying (1.2) by $-\Delta \frac{\partial \alpha}{\partial t}$ and integrating over Ω. we have

$$\frac{d}{dt}\|\Delta \alpha\|^2 + 2\|\nabla \frac{\partial \alpha}{\partial t}\|^2 = 2\left(\frac{\partial u}{\partial t}, \Delta \frac{\partial \alpha}{\partial t}\right). \tag{3.13}$$

Summing (3.12) and (3.13), thanks to $h \in L^\infty(\mathbb{R}; L^2(\Omega))$, we find

$$\frac{d}{dt}(\|\Delta u\|^2 + \|\Delta \alpha\|^2) + \|\frac{\partial u}{\partial t}\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2 \leqslant Q(\|u\|_{H^2(\Omega)}). \tag{3.14}$$

We set

$$y = \|\Delta u\|^2 + \|\Delta \alpha\|^2, \tag{3.15}$$

and we deduce from (3.14) an inequation of the form

$$y' \leqslant Q(y). \tag{3.16}$$

Let z be the solution to the following ordinary differential equation

$$z' = Q(z), \quad z(0) = y(0) = \|\Delta u_0\|^2 + \|\Delta \alpha_0\|^2. \tag{3.17}$$

It follows from the comparison principle that there exists $T_0 = T_0(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})$ belonging to, say $\left(0, \frac{1}{2}\right)$ such that

$$y(t) \leqslant z(t), \quad \forall t \in [0, T_0], \tag{3.18}$$

hence

$$\|\Delta u(t)\|^2 + \|\Delta \alpha(t)\|^2 \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}), \quad t \leqslant T_0. \tag{3.19}$$

We then differentiate (1.1) with respect to time and rewrite the resulting equation as

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t}-\Delta\frac{\partial u}{\partial t}+f'(u)\frac{\partial u}{\partial t}=\Delta\frac{\partial \alpha}{\partial t}-\frac{\partial}{\partial t}\frac{\partial u}{\partial t}+(-\Delta)^{-1}\frac{\partial h}{\partial t}. \tag{3.20}$$

Multiplying (3.20) by $t\dfrac{\partial u}{\partial t}$ and integrating over Ω, we have, owing to (1.10) and $u \in H^2(\Omega)$

$$\begin{aligned}&t\left((-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)+t\left(-\Delta\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)+t\left(f'(u)\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)\\&=t\left(\Delta\frac{\partial \alpha}{\partial t},\frac{\partial u}{\partial t}\right)-t\left(\frac{\partial}{\partial t}\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)+t\left((-\Delta)^{-1}\frac{\partial}{\partial t}h,\frac{\partial u}{\partial t}\right),\end{aligned} \tag{3.21}$$

which implies

$$\begin{aligned}&\frac{d}{dt}\left(t(\|\frac{\partial u}{\partial t}\|_{-1}^2+\|\frac{\partial u}{\partial t}\|^2)\right)+\frac{3}{2}t\|\nabla\frac{\partial u}{\partial t}\|^2\\&\leqslant ct(\|\frac{\partial u}{\partial t}\|^2+\|\nabla\frac{\partial \alpha}{\partial t}\|^2)+\|\frac{\partial u}{\partial t}\|_{-1}^2+\|\frac{\partial u}{\partial t}\|^2+ct\|\frac{\partial h}{\partial t}\|^2.\end{aligned} \tag{3.22}$$

Multiplying (1.2) by $-t\Delta\dfrac{\partial \alpha}{\partial t}$ and integrating over Ω, we have

$$\frac{d}{dt}(t\|\Delta\alpha\|^2)+t\|\nabla\frac{\partial \alpha}{\partial t}\|^2\leqslant t\|\nabla\frac{\partial u}{\partial t}\|^2+\|\Delta\alpha\|^2. \tag{3.23}$$

Add δ_3(3.22) and δ_4(3.23), where δ_3 and $\delta_4 > 0$ are such that

$$\begin{aligned}&\frac{1}{2}\delta_3-\delta_4>0,\\&\delta_4-c\delta_3>0,\end{aligned}$$

we find

$$\begin{aligned}&\frac{d}{dt}\left(t\left(\delta_3\|\frac{\partial u}{\partial t}\|_{-1}^2+\delta_3\|\frac{\partial u}{\partial t}\|^2+\delta_4\|\Delta\alpha\|^2\right)\right)+c_8t\|\nabla\frac{\partial u}{\partial t}\|^2+c_9t\|\nabla\frac{\partial \alpha}{\partial t}\|^2\\&\leqslant ct\|\frac{\partial u}{\partial t}\|_{-1}^2+\delta_3\|\frac{\partial u}{\partial t}\|_{-1}^2+\delta_3\|\frac{\partial u}{\partial t}\|^2+\delta_4\|\Delta\alpha\|^2+ct\|\frac{\partial h}{\partial t}\|^2,\\&\leqslant c'(t+1)\left(\delta_3\|\frac{\partial u}{\partial t}\|_{-1}^2+\delta_3\|\frac{\partial u}{\partial t}\|^2+\delta_4\|\Delta\alpha\|^2\right)+c''t\|\frac{\partial h}{\partial t}\|^2, c'>1.\end{aligned} \tag{3.24}$$

We apply gronwall's lemma and we obtain, owing to (1.6)

$$\|\frac{\partial u}{\partial t}(t)\|^2\leqslant Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)}),\quad t\in[0,\ T_0]. \tag{3.25}$$

Equation (1.2) implies

$$\frac{\partial \alpha}{\partial t}=-\frac{\partial u}{\partial t}+\Delta\alpha \tag{3.26}$$

and owing to (3.19) and 3.25), we have

$$\|\frac{\partial \alpha}{\partial t}\|^2 \leqslant \left(\|\frac{\partial u}{\partial t}\|^2 + \|\Delta\alpha\|^2\right)$$
$$\leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}), \quad t \in [0,\ T_0], \tag{3.27}$$

Multiplying (3.20) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we have

$$\frac{d}{dt}\left(\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial u}{\partial t}\|^2\right) + \frac{1}{2}\|\nabla\frac{\partial u}{\partial t}\|^2 \leqslant c(\|\frac{\partial u}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2) + c\|\frac{\partial h}{\partial t}\|^2. \tag{3.28}$$

Multiplying (1.2) by $-\Delta\frac{\partial \alpha}{\partial t}$ and integrating over Ω, we have

$$\frac{d}{dt}\|\Delta\alpha\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 \leqslant \|\nabla\frac{\partial u}{\partial t}\|^2. \tag{3.29}$$

Summing δ_5(3.28) and δ_6(3.29) where δ_5 and $\delta_6 > 0$ are such that

$$\frac{1}{2}\delta_5 - \delta_6 > 0,$$
$$\delta_6 - c\delta_5 > 0,$$

we find

$$\frac{d}{dt}\left(\delta_5\|\frac{\partial u}{\partial t}\|_{-1}^2 + \delta_6\|\frac{\partial u}{\partial t}\|^2 + \delta_6\|\Delta\alpha\|^2\right) + c\|\nabla\frac{\partial u}{\partial t}\|^2 + c\|\nabla\frac{\partial \alpha}{\partial t}\|^2$$
$$\leqslant c\|\frac{\partial u}{\partial t}\|^2 + c\|\frac{\partial h}{\partial t}\|^2. \tag{3.30}$$

Applying Gronwall's lemma, we deduce from (3.19), (3.25), (3.30) the following estimates

$$\|\frac{\partial u}{\partial t}(t)\|^2 \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}),\ t \geqslant T_0 \tag{3.31}$$

and

$$\|\Delta\alpha(t)\|^2 \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}),\ t \geqslant T_0. \tag{3.32}$$

Thanks to (3.31) and (3.32), from (3.26) we have

$$\|\frac{\partial \alpha}{\partial t}(t)\|^2 \leqslant \left(\|\frac{\partial \alpha}{\partial t}(t)\|^2 + \|\Delta\alpha\|^2\right)$$
$$\leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}), \quad t \geqslant T_0. \tag{3.33}$$

We now rewrite (1.1) in the form

$$-\Delta u + f(u) = L_u(t) \ \ u = 0 \ on\ \Gamma, \quad for\ \ t \geqslant T_0, \tag{3.34}$$

where

$$L_u(t) = -(-\Delta)^{-1}\frac{\partial u}{\partial t} + \frac{\partial \alpha}{\partial t}, \tag{3.35}$$

satisfies, owing to (3.31) and (3.33), the following estimate

$$\|L_u(t)\| \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}), \quad t \geqslant T_0. \tag{3.36}$$

Multiplying (3.34) by u, and integrating over Ω, we have, using (1.10)

$$\|\nabla u\|^2 \leqslant c\|L_u(t)\|^2 + c', \quad t \geqslant T_0. \tag{3.37}$$

Then multiplying (3.34) by $-\Delta u$ and integrating over Ω, we have, owing to (1.9)

$$\|\Delta u\|^2 \leqslant c(\|L_u(t)\|^2 + \|\nabla u\|^2), \quad t \geqslant T_0. \tag{3.38}$$

We thus deduce from (3.36), (3.37) and (3.38) that

$$\|\Delta u(t)\|^2 \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}), \quad t \geqslant T_0, \tag{3.39}$$

Combining (3.32) and (3.39), we have

$$\|\Delta u(t)\|^2 + \|\Delta\alpha(t)\|^2 \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}), \quad t \geqslant T_0. \tag{3.40}$$

We finally deduce from (3.25) and (3.31)

$$\|\frac{\partial u}{\partial t}(t)\|^2 \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}), \quad t \geqslant 0 \tag{3.41}$$

$$\tag{3.42}$$

and from (3.19) and (3.40)

$$\|\Delta u(t)\|^2 + \|\Delta\alpha(t)\|^2 \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}), \quad t \geqslant 0. \tag{3.43}$$

Multiplying (3.20) by $t\frac{\partial^2 u}{\partial t^2}$ and integrating over Ω, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(t\|\nabla\frac{\partial u}{\partial t}\|^2) + t\|\frac{\partial^2 u}{\partial t^2}\|_{-1}^2\\ &\leqslant t\left(f'(u)\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}\right) + t\left(\Delta\frac{\partial \alpha}{\partial t}, \frac{\partial^2 u}{\partial t^2}\right) - t\|\frac{\partial^2 u}{\partial t^2}\|^2 + t\left((-\Delta)^{-1}\frac{\partial h}{\partial t}, \frac{\partial^2 u}{\partial t^2}\right) + \frac{1}{2}\|\nabla\frac{\partial u}{\partial t}\|^2\end{aligned}$$

which implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(t\|\nabla\frac{\partial u}{\partial t}\|^2) + t\|\frac{\partial^2 u}{\partial t^2}\|_{-1}^2 + t\|\frac{\partial^2 u}{\partial t^2}\|^2\\ &\leqslant 4t\|f'(u)\frac{\partial u}{\partial t}\|^2 + \frac{t}{8}\|\frac{\partial^2 u}{\partial t^2}\|^2 + \frac{t}{8}\|\frac{\partial^2 u}{\partial t^2}\|^2 + 4t\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + 4t\|\frac{\partial h}{\partial t}\|_{-1}^2 + \frac{t}{2}\|\frac{\partial^2 u}{\partial t^2}\|_{-1}^2 + \frac{1}{2}\|\nabla\frac{\partial u}{\partial t}\|^2\end{aligned}$$

and gives

$$\frac{d}{dt}(t\|\nabla\frac{\partial u}{\partial t}\|^2) + t\|\frac{\partial^2 u}{\partial t^2}\|_{-1}^2 + \frac{3t}{4}\|\frac{\partial^2 u}{\partial t^2}\|^2$$
$$\leqslant 4t\|f'(u)\frac{\partial u}{\partial t}\|^2 + 8t\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + ct\|\frac{\partial h}{\partial t}\|^2 + \|\nabla\frac{\partial u}{\partial t}\|^2$$
$$\leqslant ct\|\frac{\partial u}{\partial t}\|^2 + 8t\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + ct\|\frac{\partial h}{\partial t}\|^2 + \|\nabla\frac{\partial u}{\partial t}\|^2. \tag{3.44}$$

Now we then differentiate (1.2) with respect to time and rewrite the resulting equation as

$$\frac{\partial}{\partial t}\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t} = -\frac{\partial}{\partial t}\frac{\partial u}{\partial t}. \tag{3.45}$$

Multiplying (3.45) by $-t\Delta\frac{\partial \alpha}{\partial t}$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}(t\|\nabla\frac{\partial \alpha}{\partial t}\|^2) + t\|\Delta\frac{\partial \alpha}{\partial t}\|^2 = t\left(\frac{\partial^2 u}{\partial t^2}, \Delta\frac{\partial \alpha}{\partial t}\right) + \frac{1}{2}\|\nabla\frac{\partial \alpha}{\partial t}\|^2$$

which implies

$$\frac{d}{dt}(t\|\nabla\frac{\partial \alpha}{\partial t}\|^2) + t\|\Delta\frac{\partial \alpha}{\partial t}\|^2 \leqslant t\|\frac{\partial^2 u}{\partial t^2}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2. \tag{3.46}$$

Add δ_7(3.44) and δ_8(3.46) where δ_7 and $\delta_8 > 0$ are such that

$$\delta_8 - 8\delta_7 > 0,$$
$$\frac{3}{4}\delta_7 - \delta_8 > 0,$$

we obtain

$$\frac{d}{dt}\left(t(\delta_7\|\nabla\frac{\partial u}{\partial t}\|^2 + \delta_8\|\nabla\frac{\partial \alpha}{\partial t}\|^2)\right) + c't\|\frac{\partial^2 u}{\partial t^2}\|_{-1}^2 + c''t\|\frac{\partial^2 u}{\partial t^2}\|^2 + c'''t\|\Delta\frac{\partial \alpha}{\partial t}\|^2$$
$$\leqslant ct\|\nabla\frac{\partial u}{\partial t}\|^2 + \|\nabla\frac{\partial u}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + ct\|\frac{\partial h}{\partial t}\|^2$$

which yields

$$\frac{d}{dt}\left(t(\delta_7\|\nabla\frac{\partial u}{\partial t}\|^2 + \delta_8\|\nabla\frac{\partial \alpha}{\partial t}\|^2)\right) + c't\|\frac{\partial^2 u}{\partial t^2}\|_{-1}^2 + c''t\|\frac{\partial^2 u}{\partial t^2}\|^2 + c'''t\|\Delta\frac{\partial \alpha}{\partial t}\|^2$$
$$\leqslant c(t+1)\left(\delta_7\|\nabla\frac{\partial u}{\partial t}\|^2 + \delta_8\|\nabla\frac{\partial \alpha}{\partial t}\|^2\right) + ct\|\frac{\partial h}{\partial t}\|^2,\ c',\ c \geqslant 0. \tag{3.47}$$

We apply gronwall's lemma and we obtain

$$t\left(\delta_7\|\nabla\frac{\partial u(t)}{\partial t}\|^2 + \delta_8\|\nabla\frac{\partial \alpha(t)}{\partial t}\|^2\right) + \int_0^t s(\|\frac{\partial^2 u}{\partial t^2}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2)ds$$

$$\leqslant te^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}), \tag{3.48}$$

which yields

$$\|\nabla\frac{\partial u(t)}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha(t)}{\partial t}\|^2 \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}). \tag{3.49}$$

Multiplying (3.20) by $\frac{\partial^2 u}{\partial t^2}$ and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\frac{\partial u}{\partial t}\|^2) + \|\frac{\partial^2 u}{\partial t^2}\|_{-1}^2 \\
&= -\left(f'(u)\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}\right) + \left(\Delta\frac{\partial \alpha}{\partial t}, \frac{\partial^2 u}{\partial t^2}\right) - \|\frac{\partial^2 u}{\partial t^2}\|^2 + \left((-\Delta)^{-1}\frac{\partial h}{\partial t}, \frac{\partial^2 u}{\partial t^2}\right)
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\frac{\partial u}{\partial t}\|^2) + \|\frac{\partial^2 u}{\partial t^2}\|_{-1}^2 \\
&\leqslant 4\|f'(u)\frac{\partial u}{\partial t}\|^2 + \frac{1}{8}\|\frac{\partial^2 u}{\partial t^2}\|^2 + 4\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + \frac{1}{8}\|\frac{\partial^2 u}{\partial t^2}\|^2 + \frac{1}{2}\|\frac{\partial h}{\partial t}\|_{-1}^2 + \frac{1}{2}\|\frac{\partial^2 u}{\partial t^2}\|_{-1}^2 - \|\frac{\partial^2 u}{\partial t^2}\|^2
\end{aligned}$$

and gives

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla\frac{\partial u}{\partial t}\|^2) + \|\frac{\partial^2 u}{\partial t^2}\|_{-1}^2 + \frac{3}{4}\|\frac{\partial^2 u}{\partial t^2}\|^2 \\
&\leqslant 4\|f'(u)\|_{L^\infty}\|\frac{\partial u}{\partial t}\|^2 + 8\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + c\|\frac{\partial h}{\partial t}\|^2 \\
&\leqslant c\|\frac{\partial u}{\partial t}\|^2 + 8\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + c\|\frac{\partial h}{\partial t}\|^2.
\end{aligned} \tag{3.50}$$

Multiplying (3.45) by $-\Delta\frac{\partial \alpha}{\partial t}$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\frac{\partial \alpha}{\partial t}\|^2) + \|\Delta\frac{\partial \alpha}{\partial t}\|^2 = \left(\frac{\partial^2 u}{\partial t^2}, \Delta\frac{\partial \alpha}{\partial t}\right)$$

which implies

$$\frac{d}{dt}(\|\nabla\frac{\partial \alpha}{\partial t}\|^2) + \|\Delta\frac{\partial \alpha}{\partial t}\|^2 \leqslant \|\frac{\partial^2 u}{\partial t^2}\|^2. \tag{3.51}$$

Add δ_9(3.50) and δ_{10}(3.51) where δ_9 and $\delta_{10} > 0$ are such that

$$\begin{aligned}
&\delta_{10} - 8\delta_9 > 0, \\
&\frac{3}{4}\delta_9 - \delta_{10} > 0,
\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{d}{dt}(\delta_9\|\nabla\frac{\partial u}{\partial t}\|^2+\delta_{10}\|\nabla\frac{\partial \alpha}{\partial t}\|^2)+c'\|\frac{\partial^2 u}{\partial t^2}\|_{-1}^2+c''\|\frac{\partial^2 u}{\partial t^2}\|^2+c'''\|\Delta\frac{\partial \alpha}{\partial t}\|^2\\&\leqslant c\|\frac{\partial u}{\partial t}\|^2+c\|\frac{\partial h}{\partial t}\|^2.\end{aligned}\tag{3.52}$$

Integrating from 0 to t, we find

$$\int_0^t\|\Delta\frac{\partial \alpha}{\partial t}\|^2ds\leqslant\delta_9\|\nabla\frac{\partial u}{\partial t}(0)\|^2+\delta_{10}\|\nabla\frac{\partial \alpha}{\partial t}(0)\|^2+c\int_0^t\|\frac{\partial u}{\partial t}\|^2dt+c\int_0^t\|\frac{\partial h}{\partial t}\|^2dt,\quad for\ t\geqslant 0.\tag{3.53}$$

Combining (3.49) and (3.53), we have, owing to (3.42)

$$\int_0^t\|\Delta\frac{\partial \alpha}{\partial t}\|^2ds\leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)}).\ \ for\ \ t\geqslant 0\tag{3.54}$$

4. Existence and uniqueness of solutions

Theorem 4.1. *(Existence) We assume* $(u_0,\alpha_0)\in(H^2(\Omega)\cap H_0^1(\Omega))^2$*, then the system* (1.1) – (1.4) *possesses at least one solution* (u,α) *such as* $(u,\alpha)\in(L^\infty\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right))^2$, $\frac{\partial u}{\partial t}\in L^2\left(0,T;H^{-1}(\Omega)\right)$ *and* $\frac{\partial \alpha}{\partial t}\in L^2\left(0,T;H_0^1(\Omega)\right)$, $\forall T>0$.

The proof is based on a priory estimate (3.43) obtained previously and on a standard Galerkin scheme.

Theorem 4.2. *(Uniqueness) Let the assumptions of Theorem 4.1 hold. Then, the problem (1.1)-(1.4) possesses a unique solution* (u,α) *such as* $(u,\alpha)\in(L^\infty\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right))^2$, $\dfrac{\partial u}{\partial t}\in L^2\left(0,T;H^{-1}(\Omega)\right)$ *and* $\frac{\partial \alpha}{\partial t}\in L^2\left(0,T;H_0^1(\Omega)\right)$, $\forall T>0$.

Proof. Let $\left(u^{(1)},\alpha^{(1)}\right)$ and $\left(u^{(2)},\alpha^{(2)}\right)$ be two solutions of the problem (1.1)-(1.4) with initial data $\left(u_0^{(1)},\alpha_0^{(1)}\right)$ and $\left(u_0^{(2)},\alpha_0^{(2)}\right)\in(H^2(\Omega)\cap H_0^1(\Omega))^2$. We set $u=u^{(1)}-u^{(2)}$ and $\alpha=\alpha^{(1)}-\alpha^{(2)}$, then (u,α) is one solution of the following

$$\frac{\partial u}{\partial t}+\Delta^2u-\Delta(f(u^{(1)})-f(u^{(2)}))=-\Delta\frac{\partial \alpha}{\partial t}\tag{4.1}$$

$$\frac{\partial \alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}\tag{4.2}$$

with homogenous Dirichlet conditions

$$u|_{\partial\Omega}=\alpha|_{\partial\Omega}=\Delta u|_{\partial\Omega}=\Delta\alpha|_{\partial\Omega}=0$$

and initial conditions

$$\begin{aligned}u|_{t=0}&=u_0=u_0^{(1)}-u_0^{(2)}\\\alpha|_{t=0}&=\alpha_0=\alpha_0^{(1)}-\alpha_0^{(2)}\end{aligned}$$

Multiply (4.2) by $\frac{\partial \alpha}{\partial t}$ and integrate over Ω. We get

$$\frac{d}{dt}\|\nabla \alpha\|^2 + 2\|\frac{\partial \alpha}{\partial t}\|^2 = -2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}\right). \tag{4.3}$$

We multiply (4.1) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and integrate over Ω. We obtain

$$\frac{d}{dt}\|\nabla u\|^2 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\left(f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t}\right) = 2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}\right) \tag{4.4}$$

Now summing (4.3) and (4.4), we find

$$\begin{aligned} \frac{dE_5}{dt} + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\|\frac{\partial \alpha}{\partial t}\|^2 &= -2\left(f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t}\right) \\ &\leqslant 2\|\nabla(f(u^{(1)}) - f(u^{(2)}))\|\|\frac{\partial u}{\partial t}\|_{-1} \end{aligned}$$

$$\frac{dE_5}{dt} + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\|\frac{\partial \alpha}{\partial t}\|^2 \leqslant \|\nabla(f(u^{(1)}) - f(u^{(2)}))\|^2. \tag{4.5}$$

where

$$E_5 = \|\nabla u\|^2 + \|\nabla \alpha\|^2,$$

Furthermore, owing to $u^{(1)}, u^{(2)} \in L^\infty\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right)$

$$\begin{aligned} &\|\nabla(f(u^{(1)}) - f(u^{(2)}))\| \\ &= \left\|\nabla\left(\int_0^1 f'(u^{(1)} + s(u^{(2)} - u^{(1)}))dsu\right)\right\| \\ &\leqslant \left\|\int_0^1 f'(u^{(1)} + s(u^{(2)} - u^{(1)}))ds\nabla u\right\| \\ &+ \left\|u\int_0^1 f''(u^{(1)} + s(u^{(2)} - u^{(1)}))(\nabla u^{(1)} + s(\nabla u^{(2)} - \nabla u^{(1)}))ds\right\| \\ &\leqslant Q(\|u_{0_2}\|_{H^2(\Omega)}, \|\alpha_{0_2}\|_{H^2(\Omega)}, \|u_{0_1}\|_{H^2(\Omega)}, \|\alpha_{0_1}\|_{H^2(\Omega)})(\|\nabla u\| + \|\|u\|_{L^4}(\|\nabla u^{(2)}\|_{L^4} + \|\nabla u^{(1)}\|_{L^4})) \\ &\leqslant Q(\|u_{0_2}\|_{H^2(\Omega)}, \|\alpha_{0_2}\|_{H^2(\Omega)}, \|u_{0_1}\|_{H^2(\Omega)}, \|\alpha_{0_1}\|_{H^2(\Omega)})\|\nabla u\|. \end{aligned} \tag{4.6}$$

Integrating (4.6) in (4.5), we have

$$\frac{dE_5}{dt} + \|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\|\frac{\partial \alpha}{\partial t}\|^2 \leq Q(\|u_{0_2}\|_{H^2(\Omega)}, \|\alpha_{0_2}\|_{H^2(\Omega)}, \|u_{0_1}\|_{H^2(\Omega)}, \|\alpha_{0_1}\|_{H^2(\Omega)})\|\nabla u\|^2. \tag{4.7}$$

Hence the uniqueness, as well as the continuous dependence with respect to the initial data. □

Theorem 4.3. *We assume that the function $h(t)$ is translation bounded in $L^2_{Loc}(\mathbb{R}; L^2(\Omega))$, i.e.*

$$|h|_b^2 = \sup_{l\in\mathbb{R}} \int_l^{l+1} \|h(s)\|^2 ds < +\infty. \tag{4.8}$$

Let (u, α) be the weak solution of (1.2)-(1.4) such that u and $\alpha \in L^\infty\left(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega)\right)$. Then for all $t \geqslant \tau$, the following estimates are hold

$$\|\nabla u(t)\|^2 + \|\nabla \alpha(t)\|^2 \leqslant \left(\|\nabla u(\tau)\|^2 + \|\nabla \alpha(\tau)\|^2\right) e^{-\lambda(t-\tau)} + R_1^2 \tag{4.9}$$

$$\tag{4.10}$$

and

$$\begin{aligned} &\int_\tau^t (\|\Delta u(s)\|^2 + \|\Delta \alpha(s)\|^2) e^{-\lambda(s-\tau)} ds \\ &\leqslant \left(1 + \lambda(t-\tau))(\|\nabla u(\tau)\|^2 + \|\nabla \alpha(\tau)\|^2\right) e^{-\lambda(t-\tau)} + 2R_1^2 e^{-\lambda(t-\tau)} \end{aligned} \tag{4.11}$$

with

$$R_1^2 = 3\lambda^{-1}(1 - e^{-\lambda})^{-1}|h|_b^2 + Q(\|u_\tau\|_{H_0^2(\Omega)}, \|\alpha_\tau\|_{H_0^2(\Omega)})e^{ct} \tag{4.12}$$

where λ is the first eigenvalue of $-\Delta$ with zero boundary condition.

Proof. Multiplying (1.2) by $-\Delta\alpha$ and integrating over Ω, we obtain

$$\frac{d}{dt}\|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 \leqslant \|\frac{\partial u}{\partial t}\|^2. \tag{4.13}$$

Multiplying (1.1) by $-\Delta u$ and integrating over Ω, we have

$$-\left(\frac{\partial u}{\partial t}, \Delta u\right) - \left(\Delta^2 u, \Delta u\right) + (\Delta f(u), \Delta u) = \left(\Delta\frac{\partial \alpha}{\partial t}, \Delta u\right) - (h, \Delta u)$$

which implies

$$\frac{d}{dt}\|\nabla u\|^2 + 2\|\nabla\Delta u\|^2 \leqslant 3(\|f'(u)\nabla u\|^2 + \|\nabla\frac{\partial\alpha}{\partial t}\|^2) + \frac{1}{3}\|\nabla\Delta u\|^2 + \frac{3}{\lambda^2}\|h\|^2 + \frac{1}{3}\|\nabla\Delta u\|^2 + \frac{1}{3}\|\nabla\Delta u\|^2,$$

which gives

$$\frac{d}{dt}\|\nabla u\|^2 + \lambda\|\nabla u\|^2 \leqslant 3(\|f'(u)\nabla u\|^2 + \|\nabla\frac{\partial\alpha}{\partial t}\|^2) + \frac{3}{\lambda}\|h\|^2. \tag{4.14}$$

Summing (4.13) and (4.14), we find

$$\frac{d}{dt}\left(\|\nabla u\|^2 + \|\nabla\alpha\|^2\right) + \lambda\left(\|\nabla u\|^2 + \|\nabla\alpha\|^2\right) \leqslant 3(\|\frac{\partial u}{\partial t}\|^2 + \|\nabla\frac{\partial\alpha}{\partial t}\|^2 + \|f'(u)\nabla u\|^2) + 3\lambda^{-1}\|h\|^2. \tag{4.15}$$

Applying Gronwall's lemma from τ to t, we have, thanks to (3.39) and (3.49)

$$\|\nabla u(t)\|^2 + \|\nabla\alpha(t)\|^2$$

$$\leqslant \left(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2\right) e^{-\lambda(t-\tau)} + 3\lambda^{-1}\int_\tau^t \|h(s)\| e^{-\lambda(t-s)} ds$$
$$+ 3\int_\tau^t (\|\frac{\partial u}{\partial t}(s)\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}(s)\|^2) e^{-\lambda(t-s)} ds + c\int_\tau^t \|\nabla u\|^2 e^{-\lambda(t-s)} ds$$
$$\leqslant \left(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2\right) e^{-\lambda(t-\tau)} + 3\lambda^{-1}\int_\tau^t \|h(s)\| e^{-\lambda(t-s)} ds$$
$$+ \int_\tau^t Q(\|u_\tau\|_{H_0^2(\Omega)}, \|\alpha_\tau\|_{H_0^2(\Omega)}) e^{cs} ds + c\int_\tau^t \|\Delta u\|^2 e^{-\lambda(t-\tau)} ds$$
$$\leqslant \left(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2\right) e^{-\lambda(t-\tau)} + 3\lambda^{-1}\int_\tau^t \|h(s)\|^2 e^{-\lambda(t-s)} ds$$
$$+ e^{ct} Q(\|u_\tau\|_{H_0^2(\Omega)}, \|\alpha_\tau\|_{H_0^2(\Omega)}). \tag{4.16}$$

Hence, noting that

$$\int_\tau^t \lambda^{-1}\|h(s)\|^2 e^{-\lambda(t-s)} dt$$
$$\leqslant \lambda^{-1} \lim_{n\to+\infty}\left(1 + e^{-\lambda} + e^{-2\lambda} + e^{-3\lambda} + ... + e^{-n\lambda}\right) \sup_{t\in R}\int_t^{t+1} \|h(s)\|^2 ds$$
$$\leqslant \lambda^{-1} \lim_{n\to+\infty}\left[\frac{1 - e^{-(n+1)\lambda}}{1 - e^{-\lambda}}\right] \sup_{t\in\mathbb{R}}\int_t^{t+1} \|h(s)\|^2 ds$$
$$\leqslant \left[\lambda^{-1}.\frac{1}{1 - e^{-\lambda}}\right] \sup_{t\in\mathbb{R}}\int_t^{t+1} \|h\|^2 dt$$
$$\leqslant \lambda^{-1}(1 - e^{-\lambda})^{-1}|h|_b^2. \tag{4.17}$$

Inserting (4.17) into (4.16), we find

$$\begin{aligned}\|\nabla u(t)\|^2 + \|\nabla\alpha(t)\|^2 &\leqslant \left(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2\right) e^{-\lambda(t-\tau)} + 3\lambda^{-1}(1 - e^{-\lambda})^{-1}|h|_b^2 \\ &+ Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}) e^{ct} \\ &\leqslant R_1^2 + \left(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2\right) e^{-\lambda(t-\tau)}\end{aligned} \tag{4.18}$$

We have proved the estimate (4.10)

We are now proving inequality (4.12). Multiply (4.18) by $\lambda e^{\lambda(t-\tau)}$ and integrating between τ to t, we find

$$\lambda\int_\tau^t (\|\nabla u(s)\|^2 + \|\nabla\alpha(s)\|^2) e^{\lambda(s-\tau)} ds$$
$$\leqslant \lambda\left(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2\right)\int_\tau^t ds + \lambda R_1^2\int_\tau^t e^{\lambda(s-\tau)} ds + \lambda Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})\int_\tau^t e^{ct} e^{\lambda(s-\tau)} ds$$
$$\leqslant \lambda(t-\tau)\left(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2\right) + Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}) e^{ct}(e^{c'(t-\tau)} - 1) + R_1^2 e^{\lambda(t-\tau)}$$
$$\leqslant \lambda(t-\tau)\left(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2\right) + Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}) e^{ct}(e^{c'(t-\tau)} - 1) + R_1^2 e^{\lambda(t-\tau)} \tag{4.19}$$

Multiplying (1.1) by $-\Delta u$ and integrating over Ω, we have

$$\frac{d}{dt}\|\nabla u\|^2 + 2\|\nabla\Delta u\|^2 \leqslant 2\left(f'(u)\nabla u, \nabla\Delta u\right) + 2\|\nabla\frac{\partial\alpha}{\partial t}\|\|\nabla\Delta u\| + 2\|h\|\|\Delta u\|$$

which gives

$$\frac{d}{dt}\|\nabla u\|^2 + \lambda\|\Delta u\|^2 \leqslant 3\|f'(u)\nabla u\|^2 + 3\|\nabla\frac{\partial\alpha}{\partial t}\|^2 + \frac{3}{\lambda}\|h\|^2.$$

which implies, thanks to (3.39) and (3.49)

$$\begin{aligned}\frac{d}{dt}\|\nabla u\|^2 + \|\Delta u\|^2 &\leqslant 3\|f'(u)\|_{L^\infty}\|\nabla u\|^2 + 3\|\nabla\frac{\partial\alpha}{\partial t}\|^2 + \frac{3}{\lambda}\|h\|^2,\\ &\leqslant 3\|f'(u)\|_{L^\infty}\|\Delta u\|^2 + 3\|\nabla\frac{\partial\alpha}{\partial t}\|^2 + 3\lambda^{-1}\|h\|^2,\\ &\leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{ct} + 3\lambda^{-1}\|h(t)\|^2. \end{aligned} \tag{4.20}$$

Multiplying (1.2) by $-\Delta\alpha$ and integrating over Ω, we obtain

$$\left(\frac{\partial\alpha}{\partial t}, \Delta\alpha\right) + (\Delta\alpha, \Delta\alpha) = \left(\frac{\partial u}{\partial t}, \Delta\alpha\right)$$

which implies

$$\frac{d}{dt}\|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 \leqslant \lambda\|\frac{\partial u}{\partial t}\|^2$$

then

$$\frac{d}{dt}\|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 \leqslant \|\nabla\frac{\partial u}{\partial t}\|^2. \tag{4.21}$$

Summing (4.20) and (4.21) and thanks to (3.49), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla u\|^2 + \|\nabla\alpha\|^2\right) + \left(\|\Delta u\|^2 + \|\Delta\alpha\|^2\right)\\ &\leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{ct} + 3\lambda^{-1}\|h\|^2. \end{aligned} \tag{4.22}$$

Multiplying (4.22) by $e^{\lambda(t-\tau)}$, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|^2 + \|\nabla\alpha\|^2)e^{\lambda(t-\tau)} + (\|\Delta u\|^2 + \|\Delta\alpha\|^2)e^{\lambda(t-\tau)}\\ &\leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{ct}e^{\lambda(t-\tau)} + 3\lambda^{-1}\|h\|^2e^{\lambda(t-\tau)} + \lambda(\|\nabla u\|^2 + \|\nabla\alpha\|^2)e^{\lambda(t-\tau)}.\end{aligned}$$

Integrating from τ to t, we obtain

$$(\|\nabla u(t)\|^2 + \|\nabla\alpha(t)\|^2)e^{\lambda(t-\tau)} + \int_\tau^t(\|\Delta u(s)\|^2 + \|\Delta\alpha(s)\|^2)e^{\lambda(s-\tau)}ds$$

$$\leqslant (\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2) + \int_\tau^t 3\lambda^{-1}\|h(s)\|^2 e^{\lambda(s-\tau)}ds$$
$$+ \lambda\int_\tau^t (\|\nabla u(s)\|^2 + \|\nabla\alpha(s)\|^2)e^{\lambda(s-\tau)}ds + Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})\int_\tau^t e^{cs}e^{\lambda(s-\tau)}ds \tag{4.23}$$

which implies

$$(\|\nabla u(t)\|^2 + \|\nabla\alpha(t)\|^2)e^{\lambda(t-\tau)} + \int_\tau^t (\|\Delta u(s)\|^2 + \|\Delta\alpha(s)\|^2)e^{\lambda(s-\tau)}ds$$
$$\leqslant (\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2) + \int_\tau^t 3\lambda^{-1}\|h(s)\|^2 e^{\lambda(s-\tau)}ds + \lambda\int_\tau^t (\|\nabla u(s)\|^2 + \|\nabla\alpha(s)\|^2)e^{\lambda(s-\tau)}ds$$
$$+ Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{ct}(e^{\lambda(t-\tau)} - 1), \tag{4.24}$$

therefore using (4.19), we find

$$(\|\nabla u(t)\|^2 + \|\nabla\alpha(t)\|^2)e^{\lambda(t-\tau)} + \int_\tau^t (\|\Delta u(s)\|^2 + \|\Delta\alpha(s)\|^2)e^{\lambda(s-\tau)}ds$$
$$\leqslant (\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2) + \int_\tau^t 3\lambda^{-1}\|h\|e^{\lambda(s-\tau)}ds + \lambda(t-\tau)\left(\|u(\tau)\|^2 + \|\alpha(\tau)\|^2\right) + R_1^2e^{\lambda(t-\tau)}$$
$$+ Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{ct}(e^{\lambda(t-\tau)} - 1)$$
$$\leqslant (1 + \lambda(t-\tau))(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2) + 3\lambda^{-1}|h|_b e^{\lambda(t-\tau)}$$
$$+ R_1^2e^{\lambda(t-\tau)} + Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{ct}e^{\lambda(t-\tau)}$$
$$\leqslant (1 + \lambda(t-\tau))(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2) + R_1^2e^{\lambda(t-\tau)} + R_1^2e^{\lambda(t-\tau)}$$
$$\leqslant (1 + \lambda(t-\tau))(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2) + 2R_1^2e^{\lambda(t-\tau)}. \tag{4.25}$$

We have proved (4.12). □

Theorem 4.4. *Verify that the function $h(t)$ is translation bounded in $L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and $f(u)$ satisfies conditions (1.8), (1.9). Then for every weak solution (u, α) of (1.1)-(1.4) such as u and $\alpha \in L^\infty\left(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega)\right)$, the following inequality holds for $t > \tau$*

$$(t-\tau)(\|\Delta u(t)\|^2 + \|\Delta\alpha(t)\|^2)e^{\lambda(t-\tau)}$$
$$\leqslant (1 + 2(t-\tau) + (t-\tau)^2)c'\left(\|\nabla u(\tau)\|^2 + \|\nabla\alpha(\tau)\|^2\right) + R_2^2, \tag{4.26}$$

where $R_2^2 = (1 + t - \tau)R_1^2$ is monotone function of $|h|_b^2$, $\|u_0\|_{H^2(\Omega)}$ and $\|\alpha_0\|_{H^2(\Omega)}$.

Proof. Without loss of generality, we can assume that $f(0) = 0$. Otherwise, we can replace $f(u)$ and $h(t, x)$ by $\tilde{f}(u) = f(u) - f(0)$ and $\tilde{h}(t, x) = h(t, x) - f(0)$ respectively. The functions $\tilde{f}$ and $\tilde{h}$ satisfy the same condition.
Multiplying (1.2) by $\Delta^2\alpha$ and integrating over Ω, we obtain

$$\left(\frac{\partial\alpha}{\partial t}, \Delta^2\alpha\right) - \left(\Delta\alpha, \Delta^2\alpha\right) = -\left(\frac{\partial u}{\partial t}, \Delta^2\alpha\right)$$

which implies

$$\frac{d}{dt}\|\Delta\alpha\|^2 + 2\|\nabla\Delta\alpha\|^2 \leqslant 2\|\nabla\frac{\partial u}{\partial t}\|\|\nabla\Delta\alpha\|,$$

which implies

$$\frac{d}{dt}\|\Delta\alpha\|^2 + \lambda\|\Delta\alpha\|^2 \leqslant \|\nabla\frac{\partial u}{\partial t}\|^2. \tag{4.27}$$

Multiplying (1.1) by $\Delta^2 u$ and integrating over Ω, we have

$$\frac{d}{dt}\|\Delta u\|^2 + 2\|\Delta^2 u\|^2 \leqslant 2\|\Delta f(u)\|\|\Delta^2 u\| + 2\|\Delta\frac{\partial\alpha}{\partial t}\|\|\Delta^2 u\| + 2\|h\|\|\Delta^2 u\|,$$

which implies

$$\begin{aligned}\frac{d}{dt}\|\Delta u\|^2 + \lambda^2\|\Delta u\|^2 &\leqslant 3\|f''(u)(\nabla u)^2 + f'(u)\Delta u\|^2 + 3\|\Delta\frac{\partial\alpha}{\partial t}\|^2 + \|h(t)\|^2,\\ &\leqslant 6\|f''(u)(\nabla u)^2\|^2 + 6\|f'(u)\Delta u\|^2 + 3\|\Delta\frac{\partial\alpha}{\partial t}\|^2 + 3\|h\|^2,\\ &\leqslant 6\|f''(u)\|_{L^\infty}\|\nabla u\|_{L^4}^4 + 6\|f'(u)\|_{L^\infty}\|\Delta u\|^2 + 3\|\Delta\frac{\partial\alpha}{\partial t}\|^2 + 3\|h\|^2,\\ &\leqslant c'\|\Delta u\|^2 + 3\|\Delta\frac{\partial\alpha}{\partial t}\|^2 + Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{ct} + 3\|h\|^2,\end{aligned}$$

which yields

$$\frac{d}{dt}\|\Delta u\|^2 + \lambda\|\Delta u\|^2 \leqslant c'\|\Delta u\|^2 + 3\|\Delta\frac{\partial\alpha}{\partial t}\|^2 + Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{ct} + 3\|h\|^2. \tag{4.28}$$

Summing (4.27) and (4.28), we find

$$\begin{aligned}&\frac{d}{dt}\left(\|\Delta u(t)\|^2 + \|\Delta\alpha(t)\|^2\right) + \lambda(\|\Delta u(t)\|^2 + \|\Delta\alpha(t)\|^2)\\ &\leqslant c'(\|\Delta u(t)\|^2 + \|\Delta\alpha(t)\|^2) + 3\|\Delta\frac{\partial\alpha}{\partial t}\|^2 + 3\|h(t)\|^2 + Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{ct}.\end{aligned} \tag{4.29}$$

Multiplying (4.29) by $(t-\tau)e^{\lambda(t-\tau)}$, thanks to (3.49), we find

$$\begin{aligned}&\frac{d}{dt}\left((t-\tau)e^{\lambda(t-\tau)}(\|\Delta u(t)\|^2 + \|\Delta\alpha(t)\|^2)\right)\\ &\leqslant (\|\Delta u(t)\|^2 + \|\Delta\alpha(t)\|^2)e^{\lambda(t-\tau)} + c'(\|\Delta u(t)\|^2 + \|\Delta\alpha(t)\|^2)(t-\tau)e^{\lambda(t-\tau)}\\ &+ 3\|\Delta\frac{\partial\alpha}{\partial t}\|^2(t-\tau)e^{\lambda(t-\tau)} + Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})(t-\tau)e^{ct}e^{\lambda(t-\tau)} + 3\|h(t)\|^2(t-\tau)e^{\lambda(t-\tau)}\\ &\leqslant (1 + 3c(t-\tau))(\|\Delta u\|^2 + \|\Delta\alpha\|^2)e^{\lambda(t-\tau)} + 3\|\Delta\frac{\partial\alpha}{\partial t}\|^2(t-\tau)e^{\lambda(t-\tau)}\\ &+ Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})(t-\tau)e^{ct}e^{\lambda(t-\tau)} + 3\|h(t)\|^2(t-\tau)e^{\lambda(t-\tau)}.\end{aligned} \tag{4.30}$$

Integrating (4.30) from τ to t and thanks to (3.48), we obtain

$$\begin{aligned}
&(t-\tau)(\|\Delta u\|^2+\|\Delta\alpha\|^2)e^{\lambda(t-\tau)}\\
&\leqslant (1+c'(t-\tau))\int_\tau^t(\|\Delta u\|^2+\|\Delta\alpha\|^2)e^{\lambda(s-\tau)}ds\\
&+\lambda^{-2}Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)})e^{ct}\left[(\lambda(t-\tau)-1)\,e^{\lambda(t-\tau)}+1\right]\\
&+3(t-\tau)|h(t)|_b^2(1-e^{-\lambda})^{-1}e^{\lambda(t-\tau)}.
\end{aligned}\tag{4.31}$$

Now using estimate (4.12) we have

$$\begin{aligned}
&(t-\tau)(\|\Delta u\|^2+\|\Delta\alpha\|^2)e^{\lambda(t-\tau)}\\
&\leqslant (1+c'(t-\tau))\Big((1+\lambda(t-\tau)(\|\nabla u(\tau)\|^2+\|\nabla\alpha(\tau)\|^2\Big)e^{\lambda(t-\tau)}+(1+\lambda)R_1^2e^{\lambda(t-\tau)})\\
&+\lambda^{-2}Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)})e^{\lambda c}\left[(\lambda(t-\tau)-1)\,e^{\lambda(t-\tau)}+1\right]+3(t-\tau)|h|_b^2(1+\lambda^{-1})e^{\lambda(t-\tau)}\\
&\leqslant (1+c'(t-\tau))\Big((1+\lambda(t-\tau)(\|\nabla u(\tau)\|^2+\|\nabla\alpha(\tau)\|^2\Big)+(1+c'(t-\tau))(1+\lambda)R_1^2e^{\lambda(t-\tau)}\\
&+\lambda^{-2}Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)})e^{ct}\left[(\lambda(t-\tau)-1)\,e^{\lambda(t-\tau)}+1\right]+3\lambda(t-\tau)|h|_b^2[\lambda^{-1}(1+\lambda^{-1})]e^{\lambda(t-\tau)}\\
&\leqslant (1+c'(t-\tau))\Big((1+\lambda(t-\tau)(\|\nabla u(\tau)\|^2+\|\nabla\alpha(\tau)\|^2\Big)+(1+c'(t-\tau))(1+\lambda)R_1^2e^{\lambda(t-\tau)}\\
&+(1+\lambda)(t-\tau)R_1^2e^{\lambda(t-\tau)}+\lambda^{-2}Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)})e^{ct}\left[(\lambda(t-\tau)-1)\,e^{\lambda(t-\tau)}+1\right]\\
&\leqslant (1+c'(t-\tau))\Big((1+\lambda(t-\tau)(\|\nabla u(\tau)\|^2+\|\nabla\alpha(\tau)\|^2\Big)+(1+c'(t-\tau))(1+\lambda)R_1^2e^{\lambda(t-\tau)}\\
&+(1+\lambda)((t-\tau)+1)R_1^2e^{\lambda(t-\tau)}+\lambda^{-2}Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)})e^{ct}\left[(\lambda(t-\tau)-1)\,e^{\lambda(t-\tau)}+1\right]\\
&\leqslant c'(1+(t-\tau)+(t-\tau)^2)\Big(\|\nabla u(\tau)\|^2+\|\nabla\alpha(\tau)\|^2\Big)+\lambda^{-2}Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)})e^{ct}\left[(\lambda(t-\tau)-1)\,e^{\lambda(t-\tau)}+1\right]\\
&+(1+t-\tau)R_1^2e^{\lambda(t-\tau)}.
\end{aligned}\tag{4.32}$$

We then have prove (4.26). □

5. Existence of pullback attractor

For the system (1.1)-(1.4), we now give a fixed symbol $h_0(t)$ and take the symbol space $P=\{h_0(t+l)/l\in\mathbb{R}\}$, $\theta: P\longrightarrow P$, $\theta_t(p)=p(t+.,.)$. By Theorem 4.2 we define a cocycle on $(H^2(\Omega)\cap H_0^1(\Omega))^2$, $\phi(t,p,y_0)=y(t)$, $\forall(t,p,y_0)\in\mathbb{R}_+\times P\times(H^2(\Omega)\cap H_0^1(\Omega))^2$ such as $y(t)=(u(t),\alpha(t))$ and $y_0=(u_0,\alpha_0)$, where (u,α) is the unique solution of (1.1)-(1.4).

5.1. Property of the norm-to-weak continuous for a cocycle mapping

Lemma 5.1. *The cocycle defined of problem (1.1)-(1.4) is norm to weak continuous in* $(H^2(\Omega)\cap H_0^1(\Omega))^2$.

Proof. we know that the continuous injections $(H^2(\Omega)\cap H_0^1(\Omega))\hookrightarrow H_0^1(\Omega)$, $(H_0^1(\Omega))^*\hookrightarrow(H^2(\Omega)\cap H_0^1(\Omega))^*$ are dense. Then the Theorem 4.2 allows to obtain that for any $p\in P$, $t\in\mathbb{R}_+$, $\phi(t,p,y_0):(H_0^1(\Omega))^2\longrightarrow(H_0^1(\Omega))^2$ is continuous, therefore $\phi(t,p,y_0):(H_0^1(\Omega))^2\longrightarrow(H_0^1(\Omega))^2$ is the norm weak continuous.
Now let us verify that for any $p\in P$, $t\in\mathbb{R}_+$, $\phi(t,p,y_0)$ maps a compact subset of $(H^2(\Omega)\cap H_0^1(\Omega))^2$ to

be a bounded set of $(H^2(\Omega)\cap H_0^1(\Omega))^2$.

In fact, multiplying (1.2) by $\Delta^2\alpha$ and integrating over Ω, we obtain

$$\left(\frac{\partial\alpha}{\partial t},\Delta^2\alpha\right)-\left(\Delta\alpha,\Delta^2\alpha\right)=-\left(\frac{\partial u}{\partial t},\Delta^2\alpha\right)$$

which implies

$$\begin{aligned}\frac{d}{dt}\|\Delta\alpha\|^2+2\|\nabla\Delta\alpha\|^2&\leqslant 2\|\nabla\frac{\partial u}{\partial t}\|\|\nabla\Delta\alpha\|,\\&\leqslant\|\nabla\frac{\partial u}{\partial t}\|^2+\|\nabla\Delta\alpha\|^2,\\\frac{d}{dt}\|\Delta\alpha\|^2+\lambda\|\Delta\alpha\|^2&\leqslant\|\nabla\frac{\partial u}{\partial t}\|^2.\end{aligned}\tag{5.1}$$

Multiplying (1.1) by $\Delta^2 u$ and integrating over Ω, we have

$$\begin{aligned}\frac{d}{dt}\|\Delta u\|^2+2\|\Delta^2u\|^2&\leqslant-2\left(\Delta f(u),\Delta^2u\right)+2\|\Delta\frac{\partial\alpha}{\partial t}\|\|\Delta^2u\|+2\left((h(t),\Delta^2u\right),\\&\leqslant 2\|\Delta f(u)\|\|\Delta^2u\|+2\|\Delta\frac{\partial\alpha}{\partial t}\|\|\Delta^2u\|+2\|h\|\|\Delta^2u\|,\end{aligned}$$

which implies

$$\frac{d}{dt}\|\Delta u\|^2+\|\Delta^2u\|^2\leqslant 3\|\Delta f(u)\|^2+3\|\Delta\frac{\partial\alpha}{\partial t}\|^2+3\|h\|^2$$

hence

$$\frac{d}{dt}\|\Delta u\|^2+\lambda^2\|\Delta u\|^2\leqslant Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)})e^{ct}+3\|\Delta\frac{\partial\alpha}{\partial t}\|^2+3\|h\|^2,\quad .\tag{5.2}$$

Summing (5.1) and (5.2), thanks to (3.49), we find

$$\begin{aligned}&\frac{d}{dt}(\|\Delta u\|^2+\|\Delta\alpha\|^2)+\lambda(\|\Delta u\|^2+\|\Delta\alpha\|^2)\\&\leqslant Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)})e^{ct}+3\|\Delta\frac{\partial\alpha}{\partial t}\|^2+3\|h\|^2,\end{aligned}$$

Using gronwall's lemma, thanks to (3.54), we have

$$\begin{aligned}&\|\Delta u(t)\|^2+\|\Delta\alpha(t)\|^2\\&\leqslant(\|\Delta u_0\|^2+\|\Delta\alpha_0\|^2)e^{-\lambda t}+Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)})e^{ct}\int_0^t e^{\lambda(s-t)}ds\\&+3e^{-\lambda t}\int_0^t\|h\|^2e^{\lambda s}ds\end{aligned}$$

which implies

$$\|\Delta u(t)\|^2 + \|\Delta\alpha(t)\|^2 \leqslant (\|\Delta u_0\|^2 + \|\Delta\alpha_0\|^2)e^{ct} + (3+\lambda^{-1})Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{ct} + c'e^{ct}\int_0^t \|h(s)\|^2 ds,$$

That 's to say, for any $p \in P,\ t \in \mathbb{R}_+,\ \phi(t, p, y_0)$ maps a bounded set to be a bounded set, therefore $\phi(t, p, y_0)$ maps compact set to be a bounded set. We affirm then, thank to theorem 2.4, the cocycle mapping ϕ is the norm-to-weak continuous. □

5.2. Uniformly absorbing set relative to a cocycle mapping

Theorem 5.1. *If $h_0(t, x)$ is translation bounded in $L^2_{loc}(\mathbb{R}, L^2(\Omega))$, f(u) satisfies conditions (1.8) and (1.9) where $2 \leqslant q < +\infty$ $(n \leqslant 2)$, $2 \leqslant q \leqslant \frac{2n-2}{n-2}$, $(n \geqslant 3)$, then the cocycle $\{\phi(t, p, y)\}$ corresponding to problem (1.1)-(1.4) possesses a compact pullback attractor $A = \{A_p\}_{p\in P} = \{\omega_p(B_0)\}_{p\in P}$ where B_0 is the uniformly (w.r.t. $p \in P$) absorbing set in $(H^2(\Omega) \cap H^1_0(\Omega))^2$.*

Proof. For any $h \in P,\ |h|_b^2 = |h_0|_b^2$, using (4.10)

$$B_0 = \{y = (u, \alpha) \in (H^1_0(\Omega))^2 / \|\nabla u\|^2 + \|\nabla\alpha\|^2 \leqslant 2R_1^2\}$$

which is the uniformly absorbing set in $H^1_0(\Omega))^2$, i.e., for any $B \in B(H^1_0(\Omega))^2$, there exists $t_0 = t_0(B) \geqslant 0$ such that

$$\phi(t, p, B) \subset B_0 \ \ for\ all\ \ t \geqslant t_0,\ and\ h \in P.$$

Let

$$B_1 = \bigcup_{h\in P}\ \bigcup_{t>t_0+1} \phi(t_0 + 1, p, B_0).$$

Considering (4.26), B_1 is bounded,

$$\|\Delta u\|^2 + \|\Delta\alpha\|^2 \leqslant \rho^2 \ \ \forall (u, \alpha) \in B_1 \tag{5.3}$$

and B_1 is the uniformly absorbing set in $(H^2(\Omega) \cap H^1_0(\Omega))^2$.

5.3. The property of pullback ω-limit compcactness for a cocycle mapping

To prove the property of the pullback ω-limit compact, we proced by verifying the pullback condition (PC). As $(-\Delta)^{-1}$ is continuous compact operator in $L^2(\Omega)$ by classic spectral theorem, there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$,

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_j \longrightarrow \infty \ \ as\ \ j \longrightarrow \infty$$

and a family of elements $\{\omega_j\}_{j=1}^{\infty}$, of $D(-\Delta)$, which are orthonormal in $L^2(\Omega)$ such that

$$-\Delta\omega_j = \lambda_j\omega_j, \quad \forall j \in \mathbb{N}.$$

Let $V_m = span\{\omega_1, \omega_2, ..., \omega_m\}$ in $H^2(\Omega) \cap H_0^1(\Omega)$ and $P_m : V \longrightarrow V_m$ an orthogonal projector.

For any $(u, \alpha) \in (D(-\Delta))^2$, write

$$y(t) = (P_m u(t), P_m \alpha(t)) + ((I - P_m)u(t), (I - P_m)\alpha(t)) = (u_1, \alpha_1) + (u_2, \alpha_2).$$

In fact, $\phi(s, \theta_{-s}(h), y_0)$ satisfies

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \frac{\partial \alpha}{\partial t} + \theta_{-s} h(t) \tag{5.4}$$

$$\frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \tag{5.5}$$

with homogenous conditions Dirichlet

$$u|_{\partial\Omega} = \alpha|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \Delta \alpha|_{\partial\Omega} = 0 \tag{5.6}$$

and initial conditions

$$u|_{t=\tau} = u_0, \quad \alpha|_{t=\tau} = \alpha_0. \tag{5.7}$$

Multiplying (5.4) by $-\Delta u_2$ and integrating over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}||\nabla u_2||^2 + 2||\nabla \Delta u_2||^2 \\
&= 2(\nabla \frac{\partial \alpha_2}{\partial t}, \nabla \Delta u_2) + 2(h(t-s), \Delta u_2) + 2(\nabla f(u), \nabla \Delta u_2) \\
&\leqslant 4||\nabla \frac{\partial \alpha_2}{\partial t}||^2 + \frac{1}{2}||\nabla \Delta u_2||^2 + 2(h(t-s), \Delta u_2) + 4||\nabla f(u)||^2.
\end{aligned} \tag{5.8}$$

We know that

$$\begin{aligned}
||(h(t-s), \Delta u_2)|| &= |\int_\Omega h(t-s)\Delta u_2 dx| \\
&\leqslant ||h(t-s)||||\Delta u_2|| \\
&\leqslant \frac{\lambda_{m+1}}{4}||\Delta u_2||^2 + 4||h(t-s)||^2 \\
&\leqslant \frac{1}{4}||\nabla \Delta u_2||^2 + 4||h(t-s)||^2
\end{aligned} \tag{5.9}$$

$$\tag{5.10}$$

and

$$\begin{aligned}
||\nabla f(u)|| &= \left| \int_\Omega f'(u) \nabla u dx \right| \\
&\leqslant \int_\Omega |f'(u)||\nabla u| dx
\end{aligned}$$

$$\leqslant \|f'(u)\|\|\nabla u\|$$
$$\leqslant \frac{\lambda_{m+1}^2}{8}\|\nabla u\|^2 + 8\beta^2\int_\Omega \left(|u|^{q-2}+1\right)^2 dx$$
$$\leqslant \frac{1}{8}\|\nabla\Delta u\|^2 + 8\beta^2|\Omega| + 8\beta^2\int_\Omega |u|^{2(q-2)}dx$$
$$\leqslant \frac{1}{8}\|\nabla\Delta u\|^2 + 8\beta^2|\Omega| + 8\beta^2\|\Delta u\|_{2(q-2)}^{2(q-2)}$$
$$\leqslant \frac{1}{8}\|\nabla\Delta u\|^2 + 8\beta^2|\Omega| + 8\beta^2C'\|\Delta u\|^{2(q-2)} \tag{5.11}$$

Inserting (5.10) and (5.11) into (5.8), we have

$$\frac{d}{dt}\|\nabla u_2\|^2 + 2\|\nabla\Delta u_2\|^2 \leqslant 4\|\nabla\frac{\partial\alpha_2}{\partial t}\| + 8\beta^2|\Omega| + 8\beta^2C'\|\Delta u\|^{2(q-2)} + 4\|h(t-s)\|^2 \tag{5.12}$$

Multiplying (5.5) by $\Delta^2\alpha_2$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta\alpha_2\|^2 + \|\nabla\Delta\alpha_2\|^2 \leqslant \|\nabla\frac{\partial u_2}{\partial t}\|\|\nabla\Delta\alpha_2\|$$

which implies

$$\frac{d}{dt}\|\Delta\alpha_2\|^2 + \|\nabla\Delta\alpha_2\|^2 \leqslant \|\nabla\frac{\partial u_2}{\partial t}\|^2. \tag{5.13}$$

Now summing (5.12) and (5.13), we find

$$\frac{d}{dt}(\|\nabla u_2\|^2 + \|\Delta\alpha_2\|^2) + \|\nabla\Delta u_2\|^2 + \|\nabla\Delta\alpha_2\|^2$$
$$\leqslant \|\nabla\frac{\partial u_2}{\partial t}\|^2 + 4\|\nabla\frac{\partial\alpha_2}{\partial t}\|^2 + 8\beta^2|\Omega| + 8\beta^2C'\|\Delta u\|^{2(q-2)} + 4\|h(t-s)\|^2.$$

which implies

$$\frac{d}{dt}(\|\nabla u_2\|^2 + \|\Delta\alpha_2\|^2) + \lambda_{m+1}(\|\Delta u_2\|^2 + \|\Delta\alpha_2\|^2)$$
$$\leqslant 4\|\nabla\frac{\partial u_2}{\partial t}\|^2 + 4\|\nabla\frac{\partial\alpha_2}{\partial t}\|^2 + 8\beta^2|\Omega| + 8\beta^2C'\|\Delta u\|^{2(q-2)} + 4\|h(t-s)\|^2. \tag{5.14}$$

Multiplying (5.4) by $\frac{\partial u_2}{\partial t}$ and integrating over Ω, we find

$$\frac{d}{dt}\|\Delta u_2\|^2 + 2\|\frac{\partial u_2}{\partial t}\|^2 \leqslant 2\|\Delta f(u)\|^2 + \frac{1}{2}\|\frac{\partial u_2}{\partial t}\|^2 - 2(\frac{\partial u_2}{\partial t}, \Delta\frac{\partial\alpha_2}{\partial t}) + \frac{1}{2}\|\frac{\partial u_2}{\partial t}\|^2 + 2\|h(t-s)\|^2$$

which implies

$$\frac{d}{dt}\|\Delta u_2\|^2 + \|\frac{\partial u_2}{\partial t}\|^2 \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{cs} - 2(\frac{\partial u_2}{\partial t}, \Delta\frac{\partial\alpha_2}{\partial t}) + 2\|h(t-s)\|^2$$

$$\leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{cs} + \|\nabla \frac{\partial u_2}{\partial t}\|^2 + \|\nabla \frac{\partial \alpha_2}{\partial t}\|^2 + 2\|h(t-s)\|^2$$

$$\frac{d}{dt}\|\Delta u_2\|^2 \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{cs} + 2\|h(t-s)\|^2. \tag{5.15}$$

Summing (5.14) and (5.15) we find

$$\begin{aligned}&\frac{d}{dt}\left(\|\Delta u_2\|^2 + \|\nabla u_2\|^2 + 2\|\Delta \alpha_2\|^2\right) + \lambda_{m+1}(\|\Delta u_2\|^2 + 2\|\Delta \alpha_2\|^2\Big)\\ &\leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{ct} + 4(8\beta^2|\Omega| + 8\beta^2 C'\|\Delta u\|^{2(q-2)}) + 6\|h(t-s)\|^2.\end{aligned} \tag{5.16}$$

Using Gronwall's lemma, let for $\tau = t_o + 1$, we have

$$\begin{aligned}&\|\Delta u_2(s)\|^2 + \|\Delta \alpha_2(s)\|^2\\ &\leqslant (\|\Delta u(\tau)\|^2 + \|\nabla u(\tau)\|^2 + 2\|\Delta \alpha(\tau)\|^2)e^{-\lambda_{m+1}(s-\tau)}\\ &+ Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}) \int_\tau^s e^{ct-\lambda_{m+1}(s-t)}dt\\ &+ 4\int_\tau^s \left(8\beta^2|\Omega| + 8\beta^2 C'\|\Delta u\|^{2(q-2)} + 6\|h(t-s)\|^2\right) e^{-\lambda_{m+1}(s-t)}dt\\ &\leqslant (\|\Delta u(\tau)\|^2 + \|\nabla u(\tau)\|^2 + 2\|\Delta \alpha(\tau)\|^2)e^{-\lambda_{m+1}(s-\tau)}\\ &+ Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}) \int_\tau^s e^{ct-\lambda_{m+1}(s-t)}dt\\ &+ 4\int_\tau^s \left(8\beta^2|\Omega| + 8\beta^2 C'\left(\|\Delta u\|^{2(q-2)} + \|\Delta \alpha\|^{2(q-2)}\right)\right) e^{-\lambda_{m+1}(s-t)}dt\\ &+ 6\int_\tau^s \|h(t-s)\|^2 e^{-\lambda_{m+1}(s-t)}dt.\end{aligned} \tag{5.17}$$

Then we have, thanks to (5.3)

$$\begin{aligned}&\|\Delta u_2(s)\|^2 + \|\Delta \alpha_2(s)\|^2\\ &\leqslant c\rho^2 e^{-\lambda_{m+1}(\tau-t)} + e^{cs}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}) \int_\tau^s e^{-\lambda_{m+1}(s-t)}dt\\ &+ 4\int_\tau^s \left(8\beta^2|\Omega| + 8\beta^2 C'\rho^{2(q-2)}\right) e^{-\lambda_{m+1}(s-t)}dt\\ &+ 6\int_\tau^s \|h(t-s)\|^2 e^{-\lambda_{m+1}(s-t)}dt\end{aligned}$$

which implies

$$\begin{aligned}&\|\Delta u_2(s)\|^2 + \|\Delta \alpha_2(s)\|^2\\ &\leqslant c\rho^2 e^{-\lambda_{m+1}(s-\tau)} + (\lambda_{m+1})^{-1} Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)})e^{cs}\\ &+ (\lambda_{m+1})^{-1}\left(8\beta^2|\Omega| + 8\beta^2 C'\rho^{2(q-2)}\right) + 6\int_\tau^s \|h(t-s)\|^2 e^{-\lambda_{m+1}(s-t)}dt.\end{aligned} \tag{5.18}$$

The continuity of the integral allows to presice, for any $\epsilon > 0$, there exist $\eta > 0$ such that

$$\int_{s-\eta}^{s} \|h(t-s)\|^2 dt \leqslant \frac{\epsilon}{30}$$

and

$$6\int_{s-\eta}^{s} \|h(t-s)\|^2 e^{-\lambda_{m+1}(s-t)} dt \leqslant \frac{\epsilon}{5}. \tag{5.19}$$

We know that

$$\begin{aligned}
&6\int_{\tau}^{s} \|h(t-s)\|^2 e^{-\lambda_{m+1}(s-t)} dt \\
&\leqslant 5\int_{s-\eta}^{s} \|h(t-s)\|^2 e^{-\lambda_{m+1}(s-t)} dt + 6\int_{s-\eta-1}^{s-\eta} \|h(t-s)\|^2 e^{-\lambda_{m+1}(s-t)} dt \\
&+ 6\int_{s-\eta-2}^{s-\eta-1} \|h(t-s)\|^2 e^{-\lambda_{m+1}(s-t)} dt + 6\int_{s-\eta-3}^{s-\eta-2} \|h(t-s)\|^2 e^{-\lambda_{m+1}(s-t)} dt + ... \\
&\leqslant 6\int_{s-\eta}^{s} e^{-\lambda_{m+1}(s-t)} \|h(t-s)\|^2 dt + 6e^{-\lambda_{m+1}(\eta)} \left(1 + e^{-\lambda_{m+1}} + e^{-2\lambda_{m+1}} + e^{-3\lambda_{m+1}} + ...\right) \\
&\times \sup_{l\in R} \int_{l-1}^{l} \|h(t-s)\|^2 dt \\
&\leqslant 6\int_{s-\eta}^{s} e^{-\lambda_{m+1}(s-t)} \|h(t-s)\|^2 dt + \frac{6e^{-\lambda_{m+1}(\eta)}}{1-e^{-\lambda_{m+1}}} \times \sup_{l\in\mathbb{R}} \int_{l-1}^{l} \|h(t-s)\|^2 dt.
\end{aligned} \tag{5.20}$$

For any $\epsilon > 0$, we can take $m+1$ large enough such that

$$\frac{6e^{-\lambda_{m+1}(\eta)}}{1-e^{-\lambda_{m+1}}} \times \sup_{l\in R} \int_{l-1}^{l} \|h(t-s)\| dt \leqslant \frac{\epsilon}{5} \tag{5.21}$$

and

$$(\lambda_{m+1})^{-1}\left(8\beta^2|\Omega| + 8\beta^2 C'\rho^{2(q-2)}\right) \leqslant \frac{\epsilon}{5} \tag{5.22}$$

and also

$$(\lambda_{m+1})^{-1} Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}) e^{cs} \leqslant \frac{\epsilon}{5}. \tag{5.23}$$

Let put

$$c\rho^2 e^{-\lambda_{m+1}(s-\tau)} \leqslant \frac{\epsilon}{5} \tag{5.24}$$

we obtain, for all $s \geqslant t_1$ we have

$$t_1 = \frac{1}{\lambda_{m+1}} \ln\left(\frac{5c\rho^2}{\epsilon}\right) + \tau.$$

Inserting (5.19), (5.21), (5.22), (5.23) and (5.24) into (5.18), we have

$$\|\Delta u_2(s)\|^2 + \|\Delta \alpha_2(s)\|^2 \leqslant \epsilon, \quad \forall s \geqslant t_1, \forall h \in P,$$

which allows to affirm that $\phi(s, \theta_{-s}(p), y_0)$ satisfies pullback condition in $(H^2(\Omega) \cap H_0^1(\Omega))^2$, that is to say for any $h \in P$, $B \in B(H^2(\Omega) \cap H_0^1(\Omega))^2$, there exists t_1 and a finite dimensional subspace $V_m \times V_m$ such that

$$P\left(\cup_{s \geqslant t_1} \phi(s, \theta_{-s}(p), B)\right) \;\; is \;\; bounded$$

and

$$\|(I - P)\left(\cup_{s \geqslant t_1} \phi(s, \theta_{-s}(p), y_0)\right)\| \leqslant \epsilon \quad \forall y_0 \in B.$$

The above pullback condition we have proved allows to assert thanks to Theorem 2.3 that the cocycle Φ is pullback ω-limit compact. Therefore we deduce from Lemma 5.1, Theorem 5.1, mentioning pullback condition and from Theorem 2.2 that the cocycle corresponding to the problem (1.1) − (1.4) possesses a pullback attractor. □

6. Conclusions

This manuscript clearly explains the context of a dynamical system at two temperatures, when the relative solution exists. The existence of pullback attractors, associated with the problem (1.1)-(1.4) that we demonstrated, allows us to assert that the existing solution of the problem (1.1)-(1.4) that we have shown in this work, belongs to a family of absorbing sets ensuring the existence of the pullback, for some time.

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Conflict of interest

The authors declare that there is no conflict of interests in this paper.

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