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Research article

Existence of solutions to Caputo fractional differential inclusions of $1 < \alpha < 2$ with initial and impulsive boundary conditions

Ping Tong and Qunjiao Zhang*

School of Mathematical and Physical Sciences, Research Center of Nonlinear Science, Wuhan Textile University, Wuhan, China

* **Correspondence:** Email: qjzhang@wtu.edu.cn; Tel: +8615623341050.

Abstract: This paper is concerned with the existence of solutions to the Caputo fractional differential inclusion of $1 < \alpha < 2$ with initial and impulsive boundary conditions. A novel existence result is presented based on the fixed-point theorem of Dhage for multi-valued operators with some assumptions. Finally, two examples are provided to explicate the applicability of the main result.

Keywords: fractional differential inclusion; impulsive boundary conditions; fixed-point theorem of Dhage
Mathematics Subject Classification: 34A08, 34C25

1. Introduction

Nowadays, fractional differential equations play a main role in mathematics. Fractional calculus has been rapidly developed due to its important applications in more and more fields, especially in control theory, viscoelastic theory, electronic chemicals, fractal theory, and so on. The main advantage in the study of differential equations of a fractional order is that fractional derivatives provide an excellent tool for describing the memory and hereditary properties of various materials and processes. Therefore, fractional order models are more realistic and practical than classical integer-order models [1–5]. For more recent developments on fractional differential equations, see [6–18] and the references therein.

Differential equations with impulse effects were considered for the first time by Milman and Myshkis, which can describe the observed evolution processes of several real world phenomena in a natural manner. Dynamic processes with sudden changes in their states are usually governed by impulsive differential equations, such as seasonal changes or harvesting in environmental sciences, abrupt changes of prices in economics, and so on. They are widely used to model many problems in control theory, population dynamics, medicine, and economics. Recently, fractional differential equations with impulse effects have received a lot of attention, see [19–24].

Differential equations are used to describe deterministic systems. In fact, some systems, such as economics and biology, involve certain macro changes; in this case, instead of differential equations, differential inclusions are used to describe the uncertainty of the system itself. Therefore, differential inclusions have an important significance in many fields including differential variational inequalities, projected dynamical systems, dynamic Coulomb friction problems and fuzzy set arithmetic [25,26]. In the past several decades, many methods and results concerning the solvability of differential inclusions with boundary value problems were investigated. For example, in [27], M. Benchohora and S. K. Ntouyas discussed the solvability of first order differential inclusions with periodic boundary conditions. In [28], G. Grammel considered the boundary value problems for semi-continuous delayed differential inclusions on Riemannian manifolds. B. C. Dhage proved some existence theorems for hyperbolic differential inclusions in Banach algebras in [29]. N. S. Papageorgiou and V. Staicu established the method of upper-lower solutions for nonlinear second order differential inclusions in [30]. Moreover, Y. Chang and J. J. Nieto extended the study to the fractional differential inclusions with boundary conditions by using the Bohnenblust-Karlin's fixed point theorem in [31].

Various existence results of fractional differential inclusions with boundary conditions in the case of order $0 < \alpha < 1$ have been obtained [32–36]. To the best of our knowledge, the results of fractional differential inclusions of $1 < \alpha < 2$ with impulsive boundary conditions have not been obtained up to now. Motivated by the above reasons, differential inclusion problems for fractional differential inclusions of $1 < \alpha < 2$ with initial and impulsive boundary conditions are discussed in this paper. Among of existing fractional order models, the Riemann-Liouville and Caputo operators are popularly adopted by the researchers. Compared with Riemann-Liouville, the Caputo fractional operator is more suitable for modelling physical phenomena because the initial conditions are given in an classical form. As a consequence, this paper is based on the Caputo type fractional derivative. The purpose of this paper is to investigate the existence of solutions to a Caputo fractional differential inclusion with initial and impulsive boundary conditions described as follows:

$$\begin{cases} {}^C D^\alpha u(t) \in F(t,u), t \in [0,T] \backslash \{t_1, t_2, \cdots, t_p\}, \\ \Delta u(t_k) = I_k(u(t_k^-)), \Delta u'(t_k) = J_k(u(t_k^-)), \\ u(0) = a, u'(0) = b, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, ${}^C D^\alpha$ is the Caputo fractional derivative defined by

$${}^C D^\alpha u(x) = \frac{1}{\Gamma(2-\alpha)} \int_{t_k}^{x} (x-t)^{1-\alpha} u''(t) dt,$$

$F : [0,T] \times X \to 2^X \backslash \emptyset$ is a multi-valued mapping, I_k, J_k are multi-valued mapping from X to nonempty closed, bounded and convex subset of X, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$, $u(t_k^-) = \lim_{h \to 0^-} u(t_k - h)$, and $k = 1, 2, \cdots, p$. In the following, the sign J denotes $[0,T]$ for convenience.

The outline of this paper is as follows. In Section 2, some useful results on the Banach space and preliminary facts on fractional derivative which will be used throughout this work are given. In Section 3, sufficient conditions for the existence of solutions of Eq (1.1) is established by relying on a fixed point theorem due to Dhage. In Section 4, two examples are provided to explicate the applicability of our main result. Finally, some remarks and observations regarding the obtained result are summarized in Section 5.

2. Preliminaries

For convenience for the following proof, some necessary facts about multi-valued mapping and lemmas are introduced in this section.

Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued mapping $H : X \to 2^X \backslash \emptyset$ is convex (closed) valued if $\bigcup_{x\in X} H(x)$ is convex (closed). H is bounded on bounded sets if $\cup_{x\in B} H(x)$ is bounded in X for any bounded set B of X .

H is called upper semi-continuous on X if for each $x_0 \in X$, the set $H(x_0)$ is a nonempty closed subset of X, and if for each open subset B of X containing $H(x_0)$, there exists an open neighbourhood N of x_0 such that $H(N) \subset B$. H is said to be completely continuous if $H(B)$ is relatively compact for every bounded subset B of X.

If the multi-valued mapping H is completely continuous with nonempty compact values, then H is upper semi-continuous if and only if H has a closed graph (i.e., $x_n \to \bar{x}$, $y_n \to \bar{y}$, $y_n \in H(x_n)$ implies $\bar{y} \in H(\bar{x})$).

Recall that the Pompeiu-Hausdorff distance of the closed subsets A, B of X is defined by the following:

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\},$$

where $d(x, B) = \inf_{y\in B} d(x, y)$.

In the following, $P_{cl,cv,bd}(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X, and $P_{cp,cv}(X)$ denotes all nonempty compact and convex subsets of X. H has a fixed point if there is $x \in X$ such that $x \in H(X)$. For more details on multi-valued mapping, see the books of J. P. Aubin and H. Frankowska in [36] and S. Hu and N. S. Papageogiou in [37].

Definition 2.1. *[33] $H : J \times X \to P(X)$ is called Carathéodory if $t \mapsto H(t, x)$ is measurable for each $x \in X$, and $x \mapsto H(t, x)$ upper semi-continuous for almost every $t \in [0, T]$.*

Furthermore, $H : J \times X \to P(X)$ is called L^1-Carathéodory if for each $l > 0$, there exists a function $m_l \in L^1(J, R^+)$ such that

$$\|H(t, x)\| = \sup_{v(t)\in H(t,x)} \{|v|\} \leq m_l(t),$$

for almost all $t \in J$, and for each fixed $\|x\| \leq l$.

The following lemmas are crucial in this paper.

Lemma 2.1. *[29] Let $(X, \|\cdot\|)$ be a Banach space, $\mathcal{A} : X \to P_{cl,cv,bd}(X)$, $\mathcal{B} : X \to P_{cp,cv}(X)$ are multi-valued operators satisfying*

(i) $\mathcal{A}$ is a contraction.

(ii) $\mathcal{B}$ is compact and upper semi-continuous.

If the set $E = \{u \in X | u \in \delta Au + \delta Bu, 0 \leq \delta \leq 1\}$ is bounded, then the operator $A + B$ has a fixed point.

Lemma 2.2. *[29] Let $(X, \|\cdot\|)$ be a Banach space, $H : J \times X \to P_{cp,cv}(X)$ is L^1-Carathéodory. Let Γ be a linear continuous operator from $L^1(J, X)$ to $C(J, X)$, then the operator*

$$\Gamma \circ S_H : C(J, X) \to P_{cp,cv}(C(J, X)), y \to (\Gamma \circ S_H)(y) = \Gamma(S_{H,y})$$

is a closed graph operator in $C(J, X) \times C(J, X)$*, where*

$$S_{H,x} := \{h(t) \in L^1(J) : h(t) \in H(t, x) \ for\ a.e.t \in J\}, \ for\ each\ fixed\ x \in X.$$

Following, this preliminary facts on the fractional derivative which will be used throughout this work are given.

Definition 2.2. *The fractional integration of order* α *is defined by*

$$I_{t_0}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{(\alpha-1)} x(\tau) d\tau,$$

where $x \in L^1[0, +\infty)$.

Definition 2.3. *The Caputo fractional derivative of order* α *is defined by*

$$D_{t_0}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau,$$

where $n-1 < \alpha < n$ *and* $x^{(n)} \in L^1[0, +\infty)$.

Lemma 2.3. *The following Caputo fractional differential equation*

$$\begin{cases} {}^C D^{\alpha} u(t) = f(t), t \in [0, T] \backslash \{t_k\}, \\ \Delta u(t_k) = I_k(u(t_k^-)), \Delta u'(t_k) = J_k(u(t_k^-)), \\ u(0) = a, u'(0) = b, \end{cases} \tag{2.1}$$

is equivalent to the integral equation as follows

$$u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds + a + bt, \ for\ t \in T_0, \\ \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds + \sum_{i=1}^{k} [\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds + I_i(u(t_i^-))] \\ + \sum_{i=1}^{k-1} (t_k - t_i) [\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s) ds + J_i(u(t_i^-))] \\ + \sum_{i=1}^{k} (t - t_k) [\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s) ds + J_i(u(t_i^-))] + a + bt, \ for\ t \in T_k. \end{cases}$$

Proof. Let $T_0 = [0, t_1], T_1 = (t_1, t_2], \cdots, T_{p-1} = (t_{p-1}, t_p], T_p = (t_p, T]$, and u is a solution of Eq (2.1). Then, for $t \in T_0$, there exist constants $c_0, c_1 \in R$ such that

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds - c_0 - c_1 t, \tag{2.2}$$

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s) ds - c_1.$$

For $t \in T_1$, there exist constants $d_0, d_1 \in R$ such that

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f(s) ds - d_0 - d_1(t - t_1),$$

$$u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_{t_1}^{t}(t-s)^{\alpha-2}f(s)ds-d_1.$$

Then, we have

$$u(t_1^-)=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s)ds-c_0-c_1t_1,u(t_1^+)=-d_0,$$

$$u'(t_1^-)=\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}f(s)ds-c_1,u'(t_1^+)=-d_1.$$

In view of $\Delta u(t_1)=I_1(u(t_1)),\Delta u'(t_1)=J_1(u(t_1))$, we further have

$$-d_0=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s)ds-c_0-c_1t_1+I_1(u(t_1^-)),$$

$$-d_1=\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}f(s)ds-c_1+J_1(u(t_1^-)).$$

Consequently, for $t\in T_1$,

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s)ds+\frac{t-t_1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}f(s)ds\\&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}f(s)ds+I_1(u(t_1))+(t-t_1)J_1(u(t_1))-c_0-c_1t.\end{aligned}$$

By a similar process, for $t\in T_k,k=1,2,\cdots,p$, we can get

$$\begin{aligned}u(t)=&\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds+\sum_{i=1}^{k}[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds+I_i(u(t_i^-))]\\&+\sum_{i=1}^{k-1}(t_k-t_i)[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds+J_i(u(t_i^-))]\\&+\sum_{i=1}^{k}(t-t_k)[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds+J_i(u(t_i^-))]-c_0-c_1t.\end{aligned}\tag{2.3}$$

Notice that for the initial conditions $u(0)=a,u'(0)=b$, one can easily find that

$$c_0=-a,c_1=-b.$$

After substituting the values of c_0 and c_1 into Eqs (2.2) and (2.3), the conclusion is derived and the proof is completed. □

3. Main results on existence of solutions

In this section, the existence result to problem (1.1) is established. We provide the following assumptions:

(A1) $F:[0,T]\times X\to P_{cp,cv}(X)$ is L^1-Carathéodory.

(A2) There exist constants a_k , b_k such that

$$H_d(I_k(x), I_k(y)) \leq a_k |y - x|,\ H_d(J_k(x), J_k(y)) \leq b_k |y - x|, \forall y, x \in X$$

and $\sum\limits_{i=1}^{p}(a_k + 2Tb_k) < 1$.

(A3) There exists a continuous nondecreasing function $\psi : [0, \infty] \to [0, \infty]$ and $p \in L^1[0, T]$ such that

$$\|F(t, x)\| = \sup_{v(t)\in F(t,x)} \{|v|\} \leq p(t)\psi(|x|),$$

for almost all $t \in [0, T]$, and for each $x \in X$, where ψ satisfies

$$\int_{c_0}^{\infty} \frac{ds}{\psi(s)} > c_1\|p\|_{L^1},$$

and

$$c_0 = \frac{\sum\limits_{i=1}^{p} (a_i |I_i(0)| + 2Tb_i |J_i(0)|) + |a + bT|}{1 - \sum\limits_{i=1}^{p} (a_i + 2Tb_i)},$$

$$c_1 = \frac{\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{2T^{\alpha-1}}{\Gamma(\alpha)}}{1 - \sum\limits_{i=1}^{p} (a_i + 2Tb_i)}.$$

The main result in this paper is presented as follows.

Theorem 3.1. *Assume conditions (A1)–(A3) hold, then problem (1.1) has at least one solution on* $[0, T]$.

Proof. Consider the following space:

$$\begin{aligned} PC(J, X) =&\{u : J \to X | u \in C(T_k), k = 0, 1, 2, \cdots , p,\\ &u(t_k^-), u(t_k^+) \text{ exist with } u(t_k) = u(t_k^-),\ k = 1, 2, \cdots , p\} \end{aligned}$$

which is a Banach space with norm $\|u\|_{PC} = sup_{t\in J} |u(t)|$. Now, let us transform the differential inclusion problem (1.1) into a fixed point problem. By (A1), for each $u \in X$, the set

$$S_{F,u} = \{v(t) \in L^1[0, T] : v(t) \in F(t, u)\ for\ a.e.t \in [0, T]\}$$

is nonempty. Multi-valued mapping $N : PC(J, X) \to 2^{PC(J,X)}\backslash\emptyset$ is defined as follows:

$$N(u) = \{h(t) \in PC(J, X)|h(t) = G(t, v), v \in S_{F,u}\},$$

where

$$\begin{aligned} G(t, v) = &\int_{t_k}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \sum_{i=1}^{k} [\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + I_i(u(t_i^-))]\\ &+ \sum_{i=1}^{k-1} (t_k - t_i)[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} v(s)ds + J_i(u(t_i^-))]\\ &+ \sum_{i=1}^{k} (t - t_k)[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} v(s)ds + J_i(u(t_i^-))] + a + bt. \end{aligned}$$

Then, the fixed-point of N is a solution of problem (1.1). Consider the multivalued operators A, B : $PC(J,X) \to 2^{PC(J,X)}\backslash\emptyset$ defined by

$$A(u) = \{h \in PC(J,X)|h(t) = \sum_{i=1}^{k} I_i(u(t_i^-)) + \sum_{i=1}^{k-1}(t_k - t_i)J_i(u(t_i^-)) + \sum_{i=1}^{k}(t - t_k)J_i(u(t_i^-))\},$$

and

$$\begin{aligned}B(u) = \{h \in PC(J,X)|h(t) = &\int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds\\ &+ \sum_{i=1}^{k-1}(t_k - t_i)\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)ds\\ &+ \sum_{i=1}^{k}(t - t_k)\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)ds + a + bt\}.\end{aligned}$$

Step 1. A has a closed, bounded and convex value. Additionally, A is a contraction.

Since I_k, J_k are multi-valued mapping from X to nonempty closed, bounded and convex subset of X, and A has a closed, bounded and convex value.

Then, let $u_1, u_2 \in PC(J,X)$; by (A2) we have

$$\begin{aligned}H_d(A(u_1), A(u_2)) \le &\sum_{i=1}^{k} H_d(I_i(u_1(t_i)), I_i(u_2(t_i))) + \sum_{i=1}^{k} TH_d(J_i(u_1(t_i)), J_i(u_2(t_i)))\\ &+ \sum_{i=1}^{k-1}(t_k - t_i)H_d(J_i(u_1(t_i)), J_i(u_2(t_i)))\\ \le &\sum_{i=1}^{k}(a_k + Tb_k + Tb_k)\,|u_1(t_i) - u_2(t_i)|\\ \le &\sum_{i=1}^{p}(a_k + 2Tb_k) \sup_{t\in J}|u_1(t) - u_2(t)|\\ = &\sum_{i=1}^{p}(a_k + 2Tb_k)\,\|u_1 - u_2\|\,.\end{aligned}$$

The following is the definiton of multi-valued operator: a multi-valued operator A is called a contraction if and only if there exists $0 < \lambda < 1$ such that for each $x, y \in X$.

$$H_d(A(x), A(y)) \le \lambda d(x,y),$$

From (A2), it follows that A is a contraction.

Step 2. For every $u \in PC(J,\mathbb{R})$, the multi-valued operator $B(u)$ is convex.

In fact, let $h_1, h_2 \in B(u)$, then, there exist $v_1, v_2 \in S_{F,u}$ such that for arbitrary $t \in [0,T]$,

$$\begin{aligned}h_1(t) = &\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} v_1 ds + \sum_{i=1}^{k-1}(t_k - t_i)\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} v_1(s)ds\\ &+ \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds + \sum_{i=1}^{k}(t - t_k)\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} v_1(s)ds + a + bt,\end{aligned}$$

$$h_2(t) = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds + \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} v_2(s)ds$$
$$+ \int_{t_k}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds + \sum_{i=1}^{k} (t - t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} v_2(s)ds + a + bt.$$

For each $\beta \in (0, 1)$, since F has a convex value, $S_{F,u}$ is convex, then, $\beta v_1 + (1 - \beta)v_2 \in S_{F,u}$. Denote $v_3 = \beta v_1 + (1 - \beta)v_2$, then, we get

$$\beta h_1 + (1 - \beta)h_2 = \int_{t_k}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} v_3(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} v_3(s)ds$$
$$+ \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} v_3(s)ds$$
$$+ \sum_{i=1}^{k} (t - t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} v_3(s)ds + a + bt,$$

so $\beta h_1 + (1 - \beta)h_2 \in S_{F,u}$. Consequently, $B(u)$ is convex.

Step 3. There exists a positive number l such that B maps B_l into a bounded set, where $B_l = \{x \in PC(J, X)| \, \|x\| \leq l\}$.

For each $l > 0$, there exists a function $u_l \in B_l, h_l \in B(u_l)$, such that $v_l \in S_{F,u}$, then, we can get

$$\begin{aligned}
\sup_{t\in[0,T]} |h_l(t)| &\leq |a + bT| + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \int_{t_k}^{t} |v_l(s)|ds + \frac{T^{\alpha-1}}{\Gamma(\alpha - 1)} \int_{t_0}^{t_k} |v_l(s)|ds \\
&+ \sum_{i=1}^{k-1} \frac{T^{\alpha-1}}{\Gamma(\alpha - 1)} \int_{t_{i-1}}^{t_i} |v_l(s)|ds + \sum_{i=1}^{k} \frac{T^{\alpha-1}}{\Gamma(\alpha - 1)} \int_{t_{i-1}}^{t_i} |v_l(s)|ds \\
&\leq |a + bT| + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \int_{0}^{T} |v_l(s)|ds + \frac{T^{\alpha-1}}{\Gamma(\alpha - 1)} \int_{0}^{T} |v_l(s)|ds + \frac{T^{\alpha-1}}{\Gamma(\alpha - 1)} \int_{0}^{T} |v_l(s)|ds \\
&\leq |a + bT| + (\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1}}{2\Gamma(\alpha - 1)}) \int_{0}^{T} m_l(s)ds \\
&= |a + bT| + (\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1}}{2\Gamma(\alpha - 1)}) \|m_l\|_{L^1}.
\end{aligned}$$

That's to say,

$$\|h_l\| \leq |a + bT| + (\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1}}{2\Gamma(\alpha - 1)}) \|m_l\|_{L^1}.$$

Consequently, B maps B_l into a bounded set.

Step 4. $B(B_l)$ is compact.

In fact, by Step 2, we know that $B(B_l)$ is bounded. In what follows, we only need to show that $B(B_l)$

is equi-continuous. Let $u \in B_l$, $h \in B(B_l)$, then, there exists $v \in S_{F,u}$ such that

$$\begin{aligned} h(t) = &\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds+\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds\\ &+\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds+\sum_{i=1}^{k}(t-t_k)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds+a+bt. \end{aligned}$$

$\forall\varepsilon$, $\exists\delta(\varepsilon)$, for each $|t'-t''|\le\delta$, let $t'<t''$, and $t',t''\in T_k$,

$$\begin{aligned} &|h(t')-h(t'')|\\ \le&\int_{t_k}^{t''}\frac{(t''-s)^{\alpha-1}}{\Gamma(\alpha)}m_l(s)ds-\int_{t_k}^{t'}\frac{(t'-s)^{\alpha-1}}{\Gamma(\alpha)}m_l(s)ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}m_l(s)ds\left|t^1-t^2\right|+|b|\,|t'-t''|\\ \le&|t'-t''|\,[\int_{t_k}^{\xi}\frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)}m_l(s)ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}m_l(s)ds+|b|]\\ \le&(|b|+\frac{T^{\alpha-2}}{\Gamma(\alpha-1)}\,\|m_l\|_{L^1})\,|t'-t''|\\ <&\varepsilon. \end{aligned}$$

For $t'\le t_k<t''$,

$$\begin{aligned} &h(t'')-h(t')\\ =&\int_{t_k}^{t''}\frac{(t''-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds-\int_{t_{k-1}}^{t'}\frac{(t'-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\\ &+\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds+\sum_{i=1}^{k-1}(t_k-t_{k-1})\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds\\ &+\sum_{i=1}^{k-1}(t_k-t_{k-1})\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds+\sum_{i=1}^{k-1}(t''-t')\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds\\ &+(t''-t_k)\int_{t_{k-1}}^{t_k}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}v(s)ds+b(t''-t'), \end{aligned}$$

then,

$$
\begin{aligned}
&|h(t') - h(t'')| \\
&\le \int_{t_{k-1}}^{t''} \frac{(t''-s)^{\alpha-1}}{\Gamma(\alpha)} m_l ds - \int_{t_{k-1}}^{t'} \frac{(t'-s)^{\alpha-1}}{\Gamma(\alpha)} m_l ds + |b|\,|t'-t''| \\
&+ \sum_{i=1}^{k-1} (t''-t') \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} m_l ds + (t''-t_k) \int_{t_{k-1}}^{t_k} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} m_l ds \\
&\le (t''-t') \int_{t_{k-1}}^{\xi} \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} m_l ds + \sum_{i=1}^{k-1} (t''-t') \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} m_l ds \\
&+ (t''-t') \int_{t_{k-1}}^{t_k} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} m_l ds + |b|\,|t'-t''| \\
&\le (|b| + 2\frac{T^{\alpha-2}}{\Gamma(\alpha-1)} \|m_l\|_{L^1})\,|t'-t''| \\
&< \varepsilon.
\end{aligned}
$$

Therefore, $\forall\varepsilon$, $\exists\delta(\varepsilon) < min\{t_{k+1} - t_k, \frac{\varepsilon}{(|b|+2\frac{T^{\alpha-2}}{\Gamma(\alpha-1)}\|m_l\|_{L^1})}\}$, for each $t', t'' \in [0, T]$, $|t'-t''| \le \delta$, such that

$$|h(t') - h(t'')| < \varepsilon.$$

Thus, $B(B_l)$ is equi-continuous.

Step 5. B has a closed graph. It is to say that if $u_n \to u_*, h_n \in B(u_n)$ and $h_n \to h_*$, then $h_* \in B(u_*)$. It is proved that there exists $v_* \in S_{F,u_*}$, such that

$$
\begin{aligned}
h_* = &\int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds + \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} v_*(s)ds \\
&+ \sum_{i=1}^{k} (t - t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} v_*(s)ds + a + bt.
\end{aligned}
$$

Consider the linear and continuous operator Γ from $L^1(J;X)$ to $C(J;X)$, and defined by

$$
\begin{aligned}
(\Gamma v)(t) = &\int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)ds \\
&+ \sum_{i=1}^{k} (t - t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)ds.
\end{aligned}
$$

From Lemma 2.2, it follows that $\Gamma \circ S_F$ is a closed graph operator, and from the definition of Γ, one has $f_n \in \Gamma \circ S_F$, $f_n \to f_*$, there exists $v_* \in S_{F,u_*}$, such that

$$
\begin{aligned}
f_* = &\int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds + + \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} v_*(s)ds \\
&+ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds + \sum_{i=1}^{k} (t - t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} v_*(s)ds,
\end{aligned}
$$

where

$$f_n = \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds + \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v_n(s)ds$$
$$+ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds + \sum_{i=1}^{k} (t - t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v_n(s)ds.$$

Then, $u_n \to u_*$, $h_n = f_n + a + bt \to f_* + a + bt = h_*$, there exists $v_* \in S_{F,u_*}$, such that

$$h_* = a + bt + \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds + \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v_*(s)ds$$
$$+ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds + \sum_{i=1}^{k} (t - t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v_*(s)ds.$$

Hence, the multi-valued operator B has a closed graph. Therefore, B is a compact multi-valued mapping and upper semi-continuous with closed convex value.

Step 6. The set $E = \{u \in PC(J, X) | u \in \delta Au + \delta Bu, 0 \leq \delta \leq 1\}$ is bounded.

$\forall u \in E$, there exists $v \in S_{F,u}$ such that

$$u(t) = \delta[\int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \sum_{i=1}^{k} [\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + I_i(u(t_i^-))]$$
$$+ \sum_{i=1}^{k-1} (t_k - t_i)[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)ds + J_i(u(t_i^-))]$$
$$+ \sum_{i=1}^{k} (t - t_k)[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)ds + J_i(u(t_i^-))] + a + bt].$$

Then, $\forall t \in [0, T]$,

$$|u(t)| \leq (\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{2T^{\alpha-1}}{\Gamma(\alpha-1)}) \int_0^t |v(s)| \, ds + \sum_{i=1}^{k} |I_i(u(t_i))| + 2T \sum_{i=1}^{k} |J_i(u(t_i))| + a + bT$$
$$\leq (\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{2T^{\alpha-1}}{\Gamma(\alpha-1)}) \int_0^t p(s)\psi(|u(s)|)ds + \sum_{i=1}^{p} (a_i |I_i(0)| + 2Tb_i |J_i(0)|)$$
$$+ \sum_{i=1}^{k} (a_i |u(t_i)| + 2Tb_i |u(t_i)|) + |a + bT| .$$

Set $c = \sum_{i=1}^{p} (a_i |I_i(0)| + 2Tb_i |J_i(0)|) + |a + bT|$, $M = \frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{2T^{\alpha-1}}{\Gamma(\alpha-1)}$.

Then, it is established that

$$|u(t)| \leq c + M \int_0^t p(s)\psi(|u(s)|)ds + \sum_{i=1}^{k} (a_i |u(t_i)| + 2Tb_i |u(t_i)|). \tag{3.1}$$

Consider the function μ be defined by

$$\mu(t) = \sup_{0\le s\le t}\{|u(s)|\}, for\ t \in J.$$

Then, by Eq (3.1),

$$\mu(t) \le c + M\int_0^t p(s)\psi(\mu(s))ds + \sum_{i=1}^{k}(a_i\mu(t) + 2Tb_i\mu(t)). \tag{3.2}$$

Thus, Eq (3.2) yields

$$(1 - \sum_{i=1}^{p}(a_i + 2Tb_i))\mu(t) \le c + M\int_0^t p(s)\psi(\mu(s))ds. \tag{3.3}$$

It follows from Eq (3.3) and (A3) that

$$\mu(t) \le c_0 + c_1\int_0^t p(s)\psi(\mu(s))ds. \tag{3.4}$$

Let us take the right-hand side of the above inequality as $\nu(t)$ i.e,

$$\nu(t) \le c_0 + c_1\int_0^t p(s)\psi(\mu(s))ds. \tag{3.5}$$

Then, we have

$$\mu(t) \le \nu(t)\ for\ t \in J\ and\ \nu(0) = c_0.$$

Differentiating both sides of Eq (3.5) obtains

$$\nu'(t) \le c_1 p(t)\psi(\mu(t)).$$

Using the nondecreasing character of ψ, we get

$$\nu'(t) \le c_1 p(t)\psi(\nu(t)).$$

Thus, we have

$$\frac{\nu'(t)}{\psi(\nu(t))} \le c_1 p(t). \tag{3.6}$$

Integrating both sides of Eq (3.6) from 0 to t, we get

$$\int_0^t \frac{\nu'(s)}{\psi(\nu(s))}ds \le c_1\int_0^t p(s)ds < \infty.$$

By a change of variables and (A3), we get

$$\int_{\nu(0)}^{\nu(t)} \frac{1}{\psi(s)}ds \le c_1\,\|p\|_{L^1} < \int_{c_0}^{\infty} \frac{1}{\psi(s)}ds.$$

Thus, there exists a constant K_1 such that for $t \in J$, $\mu(t) \le \nu(t) \le K_1$. Now, from the definition of μ, it follows that $|u(t)| \le K_1$.

Therefore,

$$\|u\| \le K_1.$$

That is to say, the set $E = \{u \in PC(J, X)| u \in \delta Au + \delta Bu, 0 \le \delta \le 1\}$ is bounded.

As a consequence of Lemma 2.1, we know that $A + B$ has a fixed point which is a solution of the problem (1.1). □

4. Application

In what follows, some concrete cases are discussed to describe the existence and properties of the function m_l, ψ, p.

Corollary 4.1. *(Sub-linear Growth). Suppose (A1), (A2) and the following condition hold: (H1) There exists a function* $\phi(t) \in L^1([0,T], R^+), 0 < \delta < 1$, *such that*

$$\|F(t,x)\| \leq \phi(t)|x|^{\delta}.$$

Then problem (1.1) has at least one solution on $[0,T]$.

Proof. In this case, let $m_l(t) = \phi(t)|l|^{\delta} \in L^1([0,T], R^+)$, $p(t) = \phi(t)$, and $\psi(s) = s^{\delta}$. Then,

$$\int_{c_0}^{\infty} \frac{ds}{\psi(s)} = \frac{1}{1-\delta} s^{(1-\delta)}|_{c_0}^{\infty} = \infty > c_1 \|p\|_{L^1},$$

where c_0, c_1 are defined in (A3). Hence an application of Theorem 3.1 asserts the conclusion. □

Corollary 4.2. *(Quadratic Controlled Growth). Suppose (A1), (A2) and the following condition hold: (H2) There exists a function* $\phi(t) \in L^1([0,T], R^+)$,*such that*

$$\|F(t,x)\| \leq \phi(t)|x|^2.$$

Then, problem (1.1) has at least one solution on $[0,T]$ *provided by*

$$c_0 c_1 \|\phi\|_{L^1} < 1.$$

Proof. In this case, let $m_l(t) = \phi(t)|l|^2$, $p(t) = \phi(t)$, and $\psi(s) = s^2$. Then,

$$\int_{c_0}^{\infty} \frac{ds}{\psi(s)} = -\frac{1}{s}|_{c_0}^{\infty} = \frac{1}{c_0},$$

where c_0, c_1 are defined in (A3). Theorem 3.1 claims that problem (1.1) has at least one solution provided

$$c_0 c_1 \|\phi\|_{L^1} < 1.$$

□

5. Conclusions

In recent years, the fractional differential equation has aroused a research upsurge of scholars due to its wide application. Because of the diversity of dynamical system, the usual deterministic model cannot satisfy the needs of practical problems; therefore we focus on fractional order differential inclusion. First, the Caputo fractional differential inclusion with initial and impulsive boundary conditions is established. Then, the existence of solutions is proved using the fixed-point theorem of Dhage for multi-valued operators with some assumptions. Additionally, two supportive examples are given to clarify the applicability of our presented result. This research can be extended to a more general higher-order fractional differential inclusions with impulsive boundary conditions in a similar way.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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